CALABI FLOW ON TORIC VARIETIES WITH BOUNDED SOBOLEV CONSTANT, I

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ABSTRACT. Let (X, P) be a toric variety. In this note, we show that the C^0-norm of the Calabi flow $\varphi(t)$ on X is uniformly bounded in $[0, T)$ if the Sobolev constant of $\varphi(t)$ is uniformly bounded in $[0, T)$. We also show that if (X, P) is uniform K-stable, then the modified Calabi flow converges exponentially fast to an extremal Kähler metric if the Ricci curvature and the Sobolev constant are uniformly bounded. At last, we discuss an extension of our results to a quasi-proper Kähler manifold.

1. INTRODUCTION

In [9, 10], Chen and He study the Calabi flow on toric surfaces with bounded Sobolev constant. The study of Calabi flow with bounded Sobolev constant has been also elaborated by Li and Zheng [19]. Related work can be found in [20, 21, 12, 11, 23, 24, 25]. Interested readers are encouraged to read these papers and the references therein.

In this note, we study the Calabi flow on toric varieties with bounded Sobolev constant and on general Kähler manifolds which are quasi-proper. Our first result is:

Theorem 1.1. *Let X be a toric variety with Delzant polytope P. Let $\varphi(t)$, $0 \leq t < T < \infty$ be a one parameter family toric invariant relative Kähler potentials satisfying the Calabi flow equation. Suppose that the Sobolev constant of $\varphi(t)$ is uniformly bounded. Then*

$$|\varphi(t)|_{L^\infty} < C, \ \forall \ t \in [0, T),$$

where C is some constant independent of t.

Suppose (X, P) is uniform K-stable, we would like to understand the global convergence of the (modified) Calabi flow. The following result can be compared to the global convergence results in [16, 17, 22].

Theorem 1.2. *If (X, P) is uniform K-stable, then the modified Calabi flow introduced in [18] will converge to an extremal Kähler metric exponentially fast if*

- *The Calabi flow starts from a toric invariant Kähler metric.*
- *The Sobolev constant is uniformly bounded along the flow.*
- *The Ricci curvature is uniformly bounded along the flow.*

For a general manifold, we have:

Theorem 1.3. *Let (X, J, ω) be a quasi-proper Kähler manifold in the sense of Chen [7], i.e., there exists a small constant $\delta > 0$ and a constant C such*

that the Mabuchi energy

$$Ma(\varphi) \geq \delta \int_X \log \frac{\omega_\varphi^n}{\omega^n} \, \omega_\varphi^n - C.$$

Then $|\varphi(t)|_{L^\infty}$ is uniformly bounded for $t \in [0, T)$ if the Sobolev constant is uniformly bounded in $[0, T)$.

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2. NOTATIONS AND SETUP

2.1. **Kähler geometry.** Let (X, J, ω) be a Kähler manifold with complex dimension n, where locally

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

$\left(g_{i\bar{j}} \right)$ is a positive definite Hermitian matrix. The Kähler metric is (locally):

$$g = g_{i\bar{j}} dz^i \otimes d\bar{z}^j.$$

The set of Kähler metrics can be identified with $\mathcal{H}/\mathbb{R}$, where

$$\mathcal{H} = \{ \varphi \in C^\infty(X) \mid \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \}.$$

We call $\varphi \in \mathcal{H}$ is a relative Kähler potential. The corresponding Kähler metric is

$$g_\varphi = \left(g_{i\bar{j}} + \varphi_{i\bar{j}} \right) dz^i \otimes d\bar{z}^j.$$

Its volume form is

$$\omega_\varphi^n = \frac{\left(\sqrt{-1} \right)^n}{n!} \det \left(g_{i\bar{j}} \right) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n,$$

where n is the complex dimension of X. Its Ricci and scalar curvature are:

$$Ric_\varphi = -\partial \bar{\partial} \log \det(g_{\varphi, i\bar{j}}), \quad R_\varphi = -\triangle_\varphi \log \det(g_{\varphi, i\bar{j}}).$$

The Calabi flow [3, 4] starting from ω_φ is defined as:

$$\frac{\partial \varphi}{\partial t} = R_\varphi - \underline{R},$$

where $\underline{R} = \frac{\int_X R_\varphi \, \omega_\varphi^n}{\int_X \omega_\varphi^n}$ is a topological constant. The short time existence of the Calabi flow is established in [8].

Now we introduce three functionals I, J, D. The I and J functionals are introduced by Aubin [2]: for any $\varphi \in \mathcal{H}$,

$$I(\varphi) = \int_X \varphi(\omega^n - \omega_\varphi^n) = \sqrt{-1} \sum_{i=0}^{n-1} \int_X \partial\varphi \wedge \bar\partial\bar\varphi \wedge \omega^i \wedge \omega_\varphi^{n-1-i},$$

$$J(\varphi) = \sqrt{-1} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_X \partial\varphi \wedge \bar\partial\bar\varphi \wedge \omega^i \wedge \omega_\varphi^{n-1-i}.$$

We have

$$\frac{1}{n+1}I \le J \le \frac{n}{n+1}I.$$

The third functional is the D functional defined as follows: (c.f. [13])

$$D(\varphi) = \int_X \varphi\, \omega^n - J(\varphi).$$

Direct calculations show that

$$\delta\left(D(\varphi)\right)(f) = \int_X f\, \omega_\varphi^n.$$

Thus $D(t) = const$ along the Calabi flow.

2.2. **Toric geometry.** Let X be a toric manifold and ω be a toric invariant Kähler metric. We obtain the Delzant polytope P of X through the moment map. On P, we have the standard Lebesgue measure $d\mu$. On each facet P_i of P, the measure $d\sigma$ equals to $\frac{1}{|\vec{n}_i|}$ times the standard Lebesgue measure, where $\vec{n}_i$ is an inward normal vector associated to P_i. Also for any vertex v of P, there exists exactly n facets $P_1, \ldots, P_n$ intersecting at v and $(\vec{n}_1, \ldots, \vec{n}_n)$ is a basis of $\mathbb{Z}^n$.

Suppose that P has d facets. Each facet P_i can be represented by

$$l_i(x) = 0,$$

where $l_i(x) = \langle x, \vec{n}_i \rangle + c_i$. A symplectic potential u of P satisfies the following Guillemin boundary conditions [15]:

- u is a smooth, strictly convex function on P.
- The restriction of u to each facet of P is also a smooth, strictly convex function.
-

$$u(x) = \frac{1}{2} \sum_{i=1}^{d} l_i(x) \ln l_i(x) + f(x),\ x \in P,$$

where $f(x)$ is a smooth function on $\bar{P}$.

On the open orbit of the $(\mathbb{C}^*)^n = \mathbb{R}^n \times \mathbb{T}^n$ action of X, a toric invariant Kähler metric ω can be express as

$$\omega = \psi_{ij} dz^i \wedge d\bar{z}^j,$$

where ψ is a real, smooth, strictly convex function on $\mathbb{R}^n$ and $z_i = \xi_i + \sqrt{-1}\theta_i$, $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $(\theta_1, \ldots, \theta_n) \in \mathbb{T}^n$. In fact, the Legendre dual of ψ is a symplectic potential u on P. Thus

$$\psi(\xi) + u(x) = \sum_{i=1}^{n} \xi_i x_i, \quad x = \nabla \psi(\xi).$$

The following Proposition is due to Donaldson [14]:

Proposition 2.1. *For any toric invariant $\varphi \in \mathcal{H}$, we obtain a symplectic potential u_φ through the Legendre dual of $\psi + \varphi$. The map from φ to u_φ is one to one and onto.*

Using the Legendre transform, Abreu [1] shows that the scalar curvature equation of a toric invariant metric ω_φ on the open $(\mathbb{C}^*)n$ orbit

$$R_\varphi = \triangle_\varphi \log \det(D^2(\psi + \varphi))$$

can be transformed to

$$R_{u_\varphi} = -\sum_{i,j=1}^{n} u_{\varphi,\ ij}^{ij}$$

in the symplectic side. Thus the Calabi flow equation on P is

$$\frac{\partial u}{\partial t} = \underline{R} - R_u.$$

3. Controlling $\max \varphi$

Let $u(t,x)$ be a sequence of symplectic potentials satisfying the Calabi flow equation on P. Our first lemma is

Lemma 3.1.
$$\|u(t,x)\|_{L^2} < C(P,T,u_0),$$
where $C(P,T,u_0)$ is a constant depending on P,T and $u(0,x)$.

Proof. Since the Calabi flow decreases the distance [5], for any $t \in [0,T)$, we have

$$\|u(\frac{t}{2},x) - u(0,x)\|_{L^2} \geq \|u(t,x) - u(\frac{t}{2},x)\|_{L^2}.$$

By the triangle inequality, we have

$$2\|u(\frac{t}{2},x)\|_{L^2} + \|u(0,x)\|_{L^2} \geq \|u(t,x)\|_{L^2}.$$

Similarly,

$$2\|u(\frac{t}{4},x)\|_{L^2} + \|u(0,x)\|_{L^2} \geq \|u(\frac{t}{2},x)\|_{L^2}.$$

Thus,

$$4\|u(\frac{t}{4}, x)\|_{L^2} + 3\|u(0, x)\|_{L^2} \geq \|u(t, x)\|_{L^2}.$$

It is easy to see that there exists a constant $C(P, T, u_0)$ depending on P, T and $u(0, x)$ such that

$$\|u(t, x)\|_{L^2} < C(P, T, u_0),$$

$\square$

An immediate corollary is:

Corollary 3.2.
$$\min_{x \in \bar{P}} u(t, x) < C_1(P, T, u_0),$$
where $C_1(P, T, u_0)$ is some constant depending on P, T and $u(0, x)$.

Since the L^2 norm of $u(t, x)$ is bounded by $C(P, T, u_0)$, our next lemma shows that $\min_{x \in \bar{P}} u(x)$ is bounded from below by some constant $C_2(P, T, u_0)$ which depends on $u(0, x), P$ and T.

Lemma 3.3. *If $\|u(t, x)\|_{L^2} < C(P, T, u_0)$ for any $t < T$, then*
$$\min_{x \in \bar{P}} u(x, t) > C_2(P, T, u_0)$$

Proof. To simplify our notation, we write $u(t, x)$ as $u(x)$. Let o be the barycenter of P. By shrinking the vertices by $\frac{1}{2}$ around o, we get $P_{\frac{1}{2}}$. Our first observation is that there exists a constant $C_3(P, T, u_0)$ depending on P, T and $u(0, x)$ such that for any $x \in P_{\frac{1}{2}}$, we have

$$u(x) < C_3(P, T, u_0).$$

This is because:
- $u(x)$ is a convex function.
- For any $x \in P_{\frac{1}{2}}$, any hyperplane l passing through x will cut P as P_1 and P_2. There exists a positive constant $C_4(P)$ depending on P such that the Euclidean volume of P_1 and P_2 are both greater than $C_4(P)$.

Let $x_0 \in \bar{P}$ be a point such that $u(x_0) = \min_{x \in \bar{P}} u(x)$. Then there exists a constant $C_5(P)$ depending on P such that the follwing holds:
- At least one facet of $P_{\frac{1}{2}}$, say $Q \subset \partial P_{\frac{1}{2}}$ such that the Euclidean distance between x_0 and Q is greater than $C_5(P)$.

By shrinking the vertices of Q by $\frac{1}{2}$ around x_0, we obtain $Q_{\frac{1}{2}}$. We denote $\tilde{P}$ as the convex hull of x_0 and $Q_{\frac{1}{2}}$. Then for any point $x \in \tilde{P}$, we have

$$u(x) \leq \frac{u(x_0) + C_3(P, T, u_0)}{2}.$$

Since we know

$$\int_{\tilde{P}} u^2(x) \, d\mu \leq \int_P u^2(x) \, d\mu < C_1(P, T, u_0)^2,$$

it is clear that there exists a constant $C_2(P, T, u_0)$ depending on P, T and $u(0, x)$ such that

$$\min_{x \in \bar{P}} u(x, t) > C_2(P, T, u_0).$$

$\square$

Translating our result to the complex side, we have the following proposition:

Proposition 3.4. *There exists a constant $C_3(P, T, u_0)$ depending on P, T and $u(0, x)$ such that for any $t < T$ we have*

$$\max_{z \in X} \varphi(t, z) < C_3(P, T, u_0).$$

Proof. We write $x = \nabla(\psi(t, \xi))$, $\xi \in \mathbb{R}^n$. Then we have

$$
\begin{aligned}
\varphi(t, \xi) &= \psi(t, \xi) - \psi(0, \xi) \\
&= \left(\sum_{i=1}^n \xi x - u(t, x) \right) - \max_{y \in P} \left(\sum_{i=1}^n \xi y - u(0, y) \right) \\
&\leq u(0, x) - u(t, x) \\
&< C_3(P, T, u_0).
\end{aligned}
$$

$\square$

A conjecture of Donaldson, proved by Chen [6], saying that

Lemma 3.5.

$$d(0, \varphi) \geq \frac{1}{\sqrt{C(P)}} \left(\max \left(\int_{\varphi > 0} \varphi \, \omega_\varphi^n, \, - \int_{\varphi < 0} \varphi \, \omega^n \right) \right),$$

where $C(P) = (2\pi)^n Vol(P)$.

In the toric case, we can have a stronger result:

Lemma 3.6. *For any toric invariant Kähler metric*

$$\omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi = \sqrt{-1} \partial \bar{\partial} (\psi + \varphi) \text{ on } \mathbb{R}^n.$$

Let the Legendre dual of $\psi, \psi + \varphi$ be u, u_φ respectively. We have

$$\frac{(2\pi)^n}{n!} \int_P (u_\varphi - u)^2 \, d\mu \geq \left(\max \left(\int_{\varphi > 0} \varphi^2 \, \omega_\varphi^n, \, \int_{\varphi < 0} \varphi^2 \, \omega^n \right) \right).$$

Proof. Let $x = \nabla(\psi + \varphi)(\xi)$, $\xi \in \mathbb{R}^n$. Again we have

$$\varphi(\xi) \leq u(x) - u_\varphi(x).$$

It implies that

$$\frac{(2\pi)^n}{n!} \int_P (u_\varphi - u)^2 \, d\mu \geq \int_{\varphi > 0} \varphi^2 \, \omega_\varphi^n.$$

On the other hand, let $x = \nabla \psi(\xi)$, $\xi \in \mathbb{R}^n$. Similarly we have

$$\varphi(\xi) = (\psi + \varphi)(\xi) - \psi(\xi)$$

$$= \left(\max_{y \in P} \sum_{i=1}^{n} \xi y - u_\varphi(y) \right) - \left(\sum_{i=1}^{n} \xi x - u(x) \right)$$

$$\geq u(x) - u_\varphi(x).$$

Then we have

$$\frac{(2\pi)^n}{n!} \int_P (u_\varphi - u)^2 \, d\mu \geq \int_{\varphi < 0} \varphi^2 \, \omega^n.$$

$\square$

As a consequence, we have

Corollary 3.7. *There exists a constant $C_4(P, T, u_0)$ depending on P, T and $u(0, x)$ such that for any $t < T$ we have*

$$\max_{z \in X} \varphi(t, z) > C_4(P, T, u_0).$$

4. L^1-NORM

It is easy to see that for any $t < T$, we have

Lemma 4.1.

$$\int_X |\varphi(t)| \, \omega^n < C_5(P, T, u_0),$$

for some constant $C_5(P, T, u_0)$ depending on P, T and u_0.

In fact, it is well known that the bound of

$$\int_X |\varphi| \, \omega^n$$

can be derived from the bound of $\max_{z \in X} \varphi(z)$. The arguments go as follows:

Let $z_0 \in X$ be the point where $\varphi(z)$ reaches its maximum. By the Green's formula, we have

$$\varphi(z_0) = (Vol(X))^{-1} \int_X \varphi \, \omega^n - (Vol(X))^{-1} \int_X \triangle_\omega \varphi(\tilde{z}) G(z_0, \tilde{z}) \, \omega^n.$$

Since $\omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0$, taking trace with respect to ω, we have

$$n + \triangle_\omega \varphi > 0.$$

Thus we obtain

$$\varphi(z_0) - (Vol(X))^{-1} \int_X \varphi \, \omega^n < C(\omega),$$

where $C(\omega)$ is a constant depending on ω.

- If $\varphi(z_0) > 0$, then

$$C(\omega) > \varphi(z_0) - Vol(X)^{-1} \int_{\varphi>0} \varphi \ \omega^n + Vol(X)^{-1} \int_{\varphi<0} -\varphi \ \omega^n$$

$$> Vol(X)^{-1} \int_{\varphi<0} -\varphi \ \omega^n.$$

Thus we have

$$(1) \qquad \int_X |\varphi| \ \omega^n < C(\omega, \varphi(z_0)),$$

where $C(\omega, \varphi(z_0))$ depending on ω and $\varphi(z_0)$.

- If $\varphi(z_0) \leq 0$, then it is straightforward to see inequality (1).

Our next lemma shows that we can also bound the L^1-norm of $\varphi(t)$ with respect to $\omega(t)$.

Lemma 4.2. *Suppose that $D_\omega(\varphi) = 0$, then*

$$(2) \qquad \int_X |\varphi| \ \omega_\varphi^n < C\left(\max_{z\in X} \varphi(z), |\varphi|_{L^1(\omega)}\right),$$

where $C\left(\max_{z\in X} \varphi(z), |\varphi|_{L^1(\omega)}\right)$ is some constant depending on $\max_{z\in X} \varphi(z)$ and $|\varphi|_{L^1(\omega)}$.

Proof. Recall that

$$D(\varphi) = \int_X \varphi \ \omega^n - J(\varphi).$$

The bound of $L^1_\omega(\varphi)$ give us the bound of $J(\varphi)$. Since

$$\frac{1}{n+1} I(\varphi) \leq J(\varphi) \leq \frac{n}{n+1} I(\varphi).$$

We obtain the bound of $I(\varphi)$ which also gives us the bound of

$$\int_X \varphi \ \omega_\varphi^n.$$

Hence we obtain the inequality (2).

$\square$

5. L^∞-NORM

First, we will try to control the L^2-norm of φ with respect of ω_φ:

Proposition 5.1. *Let φ be a smooth relative Kähler potential with $|\varphi(z)| \geq c_0 > 0$, $\forall z \in X$. Then*

$$\|\varphi\|_{L^2_{\omega_\varphi}} \leq C(n, c_0, C_s, J_\omega(\varphi), L^1_{\omega_\varphi}(\varphi)),$$

where $C(n, c_0, C_s, J_\omega(\varphi), L^1_{\omega_\varphi}(\varphi))$ is a constant depending on n, c_0, the Sobolev constant of ω_φ, $J_\omega(\varphi)$ and the L^1-norm of φ with respect to ω_φ.

Proof. Notice that

$$\int_X |\nabla \sqrt{|\varphi|}|^2 \, \omega_\varphi^n$$

$$= \int_X \frac{|\nabla \varphi|^2}{4|\varphi|} \, \omega_\varphi^n$$

$$\leq \frac{1}{4c_0} \int_X |\nabla \varphi|^2 \, \omega_\varphi^n$$

$$\leq \frac{n+1}{4c_0} J_\omega(\varphi).$$

The Sobolev inequality shows that

$$\|\sqrt{|\varphi|}\|_{L^{\frac{2n}{n-1}}_{\omega_\varphi}} \leq C_s \left(\|\nabla \sqrt{|\varphi|}\|_{L^2_{\omega_\varphi}} + \|\sqrt{|\varphi|}\|_{L^2_{\omega_\varphi}} \right)$$

It is clear that

$$\|\varphi\|_{L^{\frac{n}{n-1}}_{\omega_\varphi}} \leq C(n, c_0, C_s, J_\omega(\varphi), L^1_{\omega_\varphi}(\varphi)).$$

If the complex dimension of X is 2, then we are done. If $n > 2$, then we let $f = |\varphi|^{\frac{n}{2(n-1)}}$. Notice that

$$|\nabla f| = \frac{n}{2(n-1)} \frac{|\nabla \varphi|}{|\varphi|^{\frac{n-2}{2(n-1)}}} \leq C(n, c_0)|\nabla \varphi|$$

$$\|f\|_{L^2_{\omega_\varphi}} = \left(\|\varphi\|_{L^{\frac{n}{n-1}}_{\omega_\varphi}} \right)^{\frac{n}{2(n-1)}} \leq C(n, c_0, C_s, J_\omega(\varphi), L^1_{\omega_\varphi}(\varphi)).$$

Applying the Sobolev inequality again, we have

$$\|\varphi\|_{L^{\frac{n^2}{(n-1)^2}}_{\omega_\varphi}} = \left(\|f\|_{L^{\frac{2n}{n-1}}_{\omega_\varphi}} \right)^{\frac{2(n-1)}{n}} \leq C(n, c_0, C_s, J_\omega(\varphi), L^1_{\omega_\varphi}(\varphi)).$$

Repeating the steps, we get the conclusion. $\qquad \square$

Once we have the L^2 estimate, we could get the L^∞ estimate by using the De Giorgi-Nash-Moser iteration:

Proposition 5.2. *For a smooth relative Kähler potential φ,*

$$|\varphi|_{L^\infty} \leq C \left(n, C_s, \max_{z \in X} \varphi(z), L^2_{\omega_\varphi}(\varphi) \right),$$

where $C \left(n, C_s, \max_{z \in X} \varphi(z), L^2_{\omega_\varphi}(\varphi) \right)$ is a constant depending on n, the Sobolev constant of ω_φ, $\max_{z \in X} \varphi(z)$ and $L^2_{\omega_\varphi}(\varphi)$.

Proof. Let $\varphi_1 = -\varphi + \max_{z \in X} \varphi(z) + 1$. Since $\omega = \omega_\varphi + \sqrt{-1}\partial\bar{\partial}\varphi_1 > 0$, we have

$$n + \triangle_\varphi \varphi_1 > 0.$$

Thus for any $p \geq 1$,

$$\int_X n\varphi_1^p \; \omega_\varphi^n \geq \int_X -\varphi_1^p \triangle_\varphi \varphi_1 \; \omega_\varphi^n$$
$$= \frac{p}{(\frac{p+1}{2})^2} \int_X |\nabla \varphi_1^{\frac{p+1}{2}}|_\varphi^2 \; \omega_\varphi^n$$

Thus by the Sobolev inequality, we have

$$\left(\int_X \varphi_1^{(p+1)\frac{n}{n-1}} \; \omega_\varphi^n \right)^{\frac{n-1}{n}} \leq C(n, C_s)(p+1) \int_X \varphi_1^{p+1} \; \omega_\varphi^n.$$

Then

$$\|\varphi_1\|_{L_{\omega_\varphi}^{\frac{(p+1)n}{n-1}}} \leq (C(p+1))^{\frac{1}{p+1}} \|\varphi_1\|_{L_{\omega_\varphi}^{p+1}}.$$

By iterations, we get the conclusion. $\square$

6. Proofs of the theorems

Proof of Theorem (1.1). By Proposition (3.4) and Corollary (3.7), we uniformly control $\max_{z \in X} \varphi(t, z)$ for any $t \in [0, T)$. Then Lemma (4.1) and Lemma (4.2) provide us the uniform L^1-norm bound of $\varphi(t)$ with respect to ω and $\omega(t)$ respectively. Hence Proposition (5.1) gives us the L^2-norm of $\varphi(t)$ with respect to $\omega(t)$ uniformly and Proposition (5.2) gives us the L^∞-norm of $\varphi(t)$ uniformly. $\square$

6.1. **Global convergence.**

Proof of Theorem (1.2). Let us fix $t_0 > 0$ and write $\varphi(t)$ as φ_0 and $u(t)$ as u_0. Let u be the normalized symplectic potential of u_0 at some point $x_0 \in P$. Since (X, P) is uniformly K-stable, we have $\int_P u \; d\mu < C$ where C is some constant independent of t by Proposition 5.1.8 and Lemma 5.1.3 in [14].

We shall consider the corresponding Kähler potential ψ of u under the Legendre transformation and the relative Kähler potential $\varphi = \psi - \psi_\omega$. As in Proposition (3.4) and Corollary (3.7), we obtain the upper and lower bound of $\max_{z \in X} \varphi(z)$. By using the arguments of Lemma (3.6), we obtain the bounds of

$$\int_{\varphi > 0} \varphi \; \omega_\varphi^n, \quad \int_{\varphi < 0} -\varphi \; \omega^n.$$

Lemma 2.1 in [26] provides the bound of $J(\varphi)$. Thus we obtain the bound of

$$\int_X |\varphi| \; \omega_\varphi^n.$$

As before, we obtain the L^∞ estimate of φ. The rest of the proof is identical to the proof of Theorem 1.6 of [17]. $\square$

6.2. **General case.**

Proof of Theorem (1.3). By the proof of Theorem 1.4 in [7], we obtain uniform upper bounds of

$$\max_{z \in X} \varphi(t, z)$$

and

$$\int_X |\varphi| \; \omega^n.$$

Then Lemma (4.2) provide us the L^1-norm bound of $\varphi(t)$ with respect to $\omega(t)$ uniformly. Hence Proposition (5.1) gives us the L^2-norm of $\varphi(t)$ with respect to $\omega(t)$ uniformly and Proposition (5.2) gives us the L^∞-norm of $\varphi(t)$ uniformly. $\qquad\square$

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