Last zero time or Maximum time of the winding number of Brownian motions

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Abstract

In this paper we consider the winding number, $\theta(s)$, of planar Brownian motion and study asymptotic behavior of the process of the maximum time, the time when $\theta(s)$ attains the maximum in the interval $0 \le s \le t$. We find the limit law of its logarithm with a suitable normalization factor and the upper growth rate of the maximum time process itself. We also show that the process of the last zero time of $\theta(s)$ in $[0, t]$ has the same law as the maximum time process.

1 Introduction and Main results

In this paper we seek for an analogue of the arcsine law of the linear Brownian motion for the argument of a complex Brownian motion $\{W(t) = W_1(t) + iW_2(t) : t \ge 0\}$ started at $W(0) = (1, 0)$. Skew-product representation tells us that there exist two independent linear Brownian motions $\{B(t) : t \ge 0\}$ and $\{\hat{B}(t) : t \ge 0\}$ such that

$$W(t) = \exp(\hat{B}(H(t)) + iB(H(t))) \text{ for all } t \ge 0, \tag{1}$$

where

$$H(t) = \int_0^t \frac{ds}{|W(s)|^2} = \inf\{u \ge 0 : \int_0^u \exp(2\hat{B}(s))ds > t\},$$

which entails that B is independent of $|W|$ and hence of H, while $\log |W|$ is time change of $\hat{B}$ (cf. e.g., [5], Theorem 7.26).

We let $\theta(t) = B(H(t))$ so that $\theta(t) = \arg W(t)$, which we call the winding number. Without loss of generality we suppose $\theta(0) = 0$. The well-known result of Spitzer [9] states the convergence of $2\theta(t)/\log t$ in law:

$$\lim_{t\to\infty} P\Big(\frac{2\theta(t)}{\log t} \le a\Big) = \frac{1}{\pi}\int_{-\infty}^a \frac{dx}{1+x^2}.$$

It is shown in [1] that for any increasing function $f : (0, \infty) \to (0, \infty)$

$$\limsup_{t\to\infty} \frac{\theta(t)}{f(t)} = 0 \text{ or } \infty \quad \text{a.s.} \tag{2}$$

according as the integral $\int^\infty \frac{1}{f(t)t}dt$ converges or diverges and

$$\liminf_{t\to\infty} \frac{1}{f(t)} \sup\{\theta(s), 1 \le s \le t\} = 0 \text{ or } \infty \quad \text{a.s.}$$

according as the integral $\int^\infty \frac{f(t)}{t(\log t)^2}dt$ diverges or converges; moreover, it is shown that the square root of the random time $H(t)$ is subjected to the same growth law as of θ in (2) and the lim inf behavior of $H(t)$ is also given. Another proof of (2) is given in [8]. Also, it is shown in [7]

$$\liminf_{t\to\infty} \frac{\log\log\log t}{\log t} \sup\{|\theta(s)|, 1 \le s \le t\} = \frac{\pi}{4} \quad \text{a.s..}$$

Before advancing our result we recall the two arcsine laws whose analogues are studied in this paper. Let $\{B(t) : t \geq 0\}$ be a standard linear Brownian motion started at zero and denote by Z_t the time when the maximum of B_s in the interval $0 \leq s \leq t$ is attained. Then, the process Z_t and the process $\sup\{s \in [0,t] : B(s) = 0\}$, the last zero of Brownian motion in the time interval $[0,t]$, are subject to the same law, and according to Lévy's arcsine law the scaled variable Z_t/t is subject to the arcsin law. (cf. e.g., [5] Theorem 5.26 and 5.28)

In order to state the results of this paper we set

$$V(a) = \frac{4}{\pi^2} \iint_{0 \leq y \leq ax} \frac{dx}{1+x^2} \frac{dy}{1+y^2}. \tag{3}$$

We also define a random variable $M_t \in [0,t]$ by

$$\theta(M_t) = \max_{s \in [0,t]} \theta(s),$$

the time when $\theta(s)$ attains the maximum in the interval $0 \leq s \leq t$, and a random variable L_t by

$$L_t = \sup\{s \in [0,t] : \theta(s) = 0\},$$

the last zero of $\theta(s)$ in $[0,t]$. According to Theorem 2.11 of [5] a linear Brownian motion attains its maximum at a single point on each finite interval with probability one. In view of the representation $\theta(t) = B(H(t))$, it therefore follows that the maximiser M_t is uniquely determined for all t with probability one.

Theorem 1.1. *(a) For every* $0 < a < 1$

$$\lim_{t \to \infty} P\Big(\frac{\log M_t}{\log t} \leq a\Big) = V\Big(\frac{a}{1-a}\Big).$$

(b) It holds that

$$\{L_t : t \geq 0\} =_d \{M_t : t \geq 0\}.$$

Theorem 1.2. *Let* $\alpha(t)$ *be a positive function that is non-increasing, tends to zero as* $t \to \infty$ *and satisfies*

$$2\alpha(t^e) \geq \alpha(t), \tag{4}$$

and put

$$I\{\alpha\} = \int^{\infty} \frac{\alpha(t)|\log \alpha(t)|}{t \log t} dt.$$

Then, with probability one

$$\liminf_{t \to \infty} \frac{M_t}{t^{\alpha(t)}} = \infty \ \ or \ \ 0$$

according as the integral $I\{\alpha\}$ *converges or diverges.*

It may be worth noting that the distribution function $V(a/(1-a))$ $(0 \leq a \leq 1)$ is expressed as

$$V\Big(\frac{a}{1-a}\Big) = \int_0^a \frac{1}{2u-1} \log \frac{u}{1-u} du.$$

Indeed,

$$V'(c) = \int_0^{\infty} \frac{x dx}{(1+x^2)(1+c^2x^2)} = \frac{\log c}{c^2-1} \quad (c \neq 1),$$

where

$$\frac{d}{da} V\Big(\frac{a}{1-a}\Big) = \frac{1}{(1-a)^2} V'\Big(\frac{a}{1-a}\Big) \quad (a \neq \frac{1}{2}),$$

and we find the density asserted above.

2 Proofs

2.1 Proof of Theorem 1.1

Let $\{N(t) : t \geq 0\}$ be the maximum process of a winding number $\{\theta(t) : t \geq 0\}$, i.e. the process defined by

$$N(t) = \max_{s\in[0,t]} \theta(s).$$

Lemma 2.1. *If $a > 0$, then $P(N(t) > a) = 2P(\theta(t) > a) = P(|\theta(t)| > a)$.*

Proof. By reflection principle [5], (Theorem 2.21) it holds that for any $t > 0$

$$\max_{0\leq l\leq t} B(l) =_d |B(t)|.$$

By Skew-product representation $B(t)$ is independent of $|W(t)|$, hence since $B(l)$ is independent of $H(t) = \int_0^t \frac{dm}{|W(m)|^2}$, it holds

$$\max_{0\leq l\leq t} B(H(l)) =_d |B(H(t))|,$$

showing the assertion of the lemma. □

Lemma 2.2. $\{N(t) - \theta(t) : t \geq 0\} =_d \{|\theta(t)| : t \geq 0\}$.

Proof. According to Lévy's representation of the reflecting Brownian motion [5], (Theorem 2.34) we have

$$\{\max_{0\leq l\leq t} B(l) - B(t) : t \geq 0\} =_d \{|B(t)| : t \geq 0\}.$$

Hence as in the preceding proof,

$$\{\max_{0\leq l\leq t} B(H(l)) - B(H(t)) : t \geq 0\} =_d \{|B(H(t))| : t \geq 0\},$$

as desired. □

Proof of Theorem 1.1. Lemma 2.2 together with Lemma 2.1 show that the process $\{M_s : s \geq 0\}$ has the same law as $\{L_s : s \geq 0\}$, being nothing but the last zero of the process $\{N(t) - \theta(t) : 0 \leq t \leq s\}$ for any s. So it remains to prove part (a). Fix $a \in (0, 1)$. Set $T_c = \inf\{l \geq 0 : |W(l)| = c\}$, for which we sometimes write $T(c)$ for typographical reasons. We first prove the upper bound. By (1) it holds that

$$\begin{aligned} P(M_t < t^a) =& P(\max_{0\leq u\leq t^a} B(H(u)) > \max_{t^a\leq u\leq t} B(H(u))) \\ =& P(\max_{0\leq u\leq t^a} B(H(u)) - B(H(t^a)) > \max_{t^a\leq u\leq t} B(H(u)) - B(H(t^a))) \\ =& P(\max_{0\leq u\leq t^a} B(H(u)) - B(H(t^a)) > \max_{t^a\leq u\leq t} \tilde{B}(H(u)) - \tilde{B}(H(t^a))), \end{aligned} \tag{5}$$

where $\tilde{B}$ is a linear Brownian motion started at zero which is independent of W. Corresponding to (1) we can write $\tilde{W}(0) = (1, 0)$, $\arg \tilde{W}(l) = \tilde{B}(\tilde{H}(l))$, $\tilde{H}(l) = \int_0^l \frac{dm}{|\tilde{W}(m)|^2}$ with $\tilde{W}$ independent of W, and put $\tilde{T}_c = \inf\{l \geq 0 : |\tilde{W}(l)| = c\}$. By Lemma 2.1 and Lemma 2.2 we have $\max_{0\leq u\leq t^a} B(H(u)) - B(H(t^a)) =_d \max_{0\leq u\leq t^a} B(H(u))$, and therefore

$$\begin{aligned} &P(\max_{0\leq u\leq t^a} B(H(u)) - B(H(t^a)) > \max_{t^a\leq u\leq t} \tilde{B}(H(u)) - \tilde{B}(H(t^a))) \\ =& P(\max_{0\leq u\leq t^a} B(H(u)) > \max_{t^a\leq u\leq t} \tilde{B}(H(u)) - \tilde{B}(H(t^a))). \end{aligned} \tag{6}$$

By standard large deviation result (cf. e.g., [4], (11) and (12)), given $\epsilon > 0$, it holds that for all sufficiently large t

$$P(t^a \leq T_{t^{\frac{a+\epsilon}{2}}}, T_{t^{\frac{1-\epsilon}{2}}} \leq t) \geq 1 - \epsilon.$$

Therefore, we get

$$\begin{aligned}&P(\max_{0\le u\le t^a} B(H(u)) > \max_{t^a\le u\le t} \tilde{B}(H(u)) - \tilde{B}(H(t^a)))\\ \le &P(\max_{0\le u\le T(t^{\frac{a+\epsilon}{2}})} B(H(u)) > \max_{T(t^{\frac{a+\epsilon}{2}})\le u\le T(t^{\frac{1-\epsilon}{2}})} \tilde{B}(H(u)) - \tilde{B}(H(T_{t^{\frac{a+\epsilon}{2}}}))) + \epsilon.\end{aligned} \tag{7}$$

Also, strong Markov property tells us

$$\int_{T_{t^{\frac{a+\epsilon}{2}}}}^{T_{t^{\frac{1-\epsilon}{2}}}} \frac{dm}{|W(m)|^2} =_d \int_0^{\tilde{T}_{t^{\frac{1-a-2\epsilon}{2}}}} \frac{dm}{|\tilde{W}(m)|^2},$$
and $H(T_{t^{\frac{1-\epsilon}{2}}}) - H(T_{t^{\frac{a+\epsilon}{2}}})$ is independent of $H(T_{t^{\frac{a+\epsilon}{2}}})$.

So, if we set for $a, b < \infty$

$$Q(a,b) = P(\max_{0\le u\le T(a)} B(H(u)) > \max_{0\le u\le \tilde{T}(b)} \tilde{B}(\tilde{H}(u))),$$

it holds that

$$P(\max_{0\le u\le T(t^{\frac{a+\epsilon}{2}})} B(H(u)) > \max_{T(t^{\frac{a+\epsilon}{2}})\le u\le T(t^{\frac{1-\epsilon}{2}})} \tilde{B}(H(u)) - \tilde{B}(H(T_{t^{\frac{a+\epsilon}{2}}}))) = Q(t^{\frac{a+\epsilon}{2}}, t^{\frac{1-a-2\epsilon}{2}}). \tag{8}$$

Note that by Skew-product representation $B(t)$(resp. $\tilde{B}(t)$) is independent of $H(T_{t^{\frac{a+\epsilon}{2}}})$(resp. $\tilde{H}(\tilde{T}_{t^{\frac{a+\epsilon}{2}}})$). Then, if $\tilde{\theta}(l) = \tilde{B}(\tilde{H}(l))$, by reflection principle we get

$$\begin{aligned}Q(t^{\frac{a+\epsilon}{2}}, t^{\frac{1-a-2\epsilon}{2}}) &= P(|B(H(T_{t^{\frac{a+\epsilon}{2}}}))| > |\tilde{B}(\tilde{H}(\tilde{T}_{t^{\frac{1-a-2\epsilon}{2}}}))|)\\ &= P(|\theta(T_{t^{\frac{a+\epsilon}{2}}})| > |\tilde{\theta}(\tilde{T}_{t^{\frac{1-a-2\epsilon}{2}}})|).\end{aligned} \tag{9}$$

Moreover, since $\theta(T_r)$ follows the Cauchy distribution with parameter $|\log r|$ (cf. e.g., [6], Section 5, Exercise 2.16, [11], Proposition 2.3, and [12]), we get

$$Q(t^{\frac{a+\epsilon}{2}}, t^{\frac{1-a-2\epsilon}{2}}) = P(|\theta(T_{t^{\frac{a+\epsilon}{2}}})| > |\tilde{\theta}(\tilde{T}_{t^{\frac{1-a-2\epsilon}{2}}})|) = V(\frac{a+\epsilon}{1-a-2\epsilon}). \tag{10}$$

Therefore, since ϵ is arbitrary, this gives the desired upper bound.

Next, we prove the lower bound. By standard large deviation result (cf. e.g., [4], (11) and (12)), given $\epsilon > 0$, it holds that for all sufficiently large t

$$P(T_{t^{\frac{a-\epsilon}{2}}} \le t^a, t \le T_{t^{\frac{1+\epsilon}{2}}}) \ge 1 - \epsilon. \tag{11}$$

Moreover, by repeating the argument in (7) and (8), we get

$$\begin{aligned}&P(\max_{0\le u\le t^a} B(H(u)) > \max_{t^a\le u\le t} \tilde{B}(H(u)) - \tilde{B}(H(t^a)))\\ \ge &Q(t^{\frac{a-\epsilon}{2}}, t^{\frac{1-a+2\epsilon}{2}}) - \epsilon.\end{aligned}$$

Therefore, repeating the arguments in (5), (6), (9) and (10), we get

$$\begin{aligned}P(M_t < t^a) =&P(\max_{0\le u\le t^a} B(H(u)) > \max_{t^a\le u\le t} \tilde{B}(H(u)) - \tilde{B}(H(t^a)))\\ \ge &Q(t^{\frac{a-\epsilon}{2}}, t^{\frac{1-a+2\epsilon}{2}}) - \epsilon\\ =&V(\frac{a-\epsilon}{1-a+2\epsilon}) - \epsilon,\end{aligned}$$

yielding the lower bound. □

2.2 Proof of Theorem 1.2

Proof of Theorem 1.2. We first prove $\liminf_{t\to\infty} M_t/t^{\alpha(t)} = \infty$ if $I\{\alpha\} < \infty$. We may replace $\alpha(t)$ by $\alpha(t) \vee (\log\log t)^{-2}$. Indeed, if we set

$$\tilde{\alpha}(t) = \alpha(t)1\{\alpha(t) > (\log\log t)^{-2}\} + (\log\log t)^{-2}1\{\alpha(t) \le (\log\log t)^{-2}\},$$

$I\{\tilde{\alpha}\} < \infty$. By standard large deviation result (cf. e.g., [4], (11) and (12)) for any $q < \infty$ there exist $0 < c_1$, $c_2 < \infty$ such that

$$P(qt^{4\alpha(t)} \le T(t^{4\alpha(t)}), T(t^{\frac{1}{2}-\alpha(t)}) \le t) \ge 1 - c_1 \exp(-t^{c_2\alpha(t)}). \tag{12}$$

Therefore, by the same arguments as made for (5), (6), (7), (8), (9) and (10) we infer that for any $q < \infty$

$$\begin{aligned} P(M_t < qt^{4\alpha(t)}) =& P(\max_{0\le u\le qt^{4\alpha(t)}} B(H(u)) - B(H(qt^{4\alpha(t)})) > \max_{qt^{4\alpha(t)}\le u\le t} \tilde{B}(H(u)) - \tilde{B}(H(qt^{4\alpha(t)}))) \\ \le& Q(t^{4\alpha(t)}, t^{\frac{1}{2}-5\alpha(t)}) + c_1 \exp(-t^{c_2\alpha(t)}) \\ =& V(\frac{4\alpha(t)}{\frac{1}{2} - 5\alpha(t)}) + c_1 \exp(-t^{c_2\alpha(t)}). \end{aligned}$$

We set $t_n = \exp(e^n)$. Then, noting that $V(\alpha(n)) \asymp \alpha(n)|\log\alpha(n)|$, we deduce from (12) that for some $C < \infty$

$$P(M_{t_n} < t_n^{4\alpha(t_n)}) \le C\alpha(t_n)|\log\alpha(t_n)| + c_1 \exp(-t_n^{c_2\alpha(t_n)}).$$

The sum of the right-hand side over n is finite since $\sum_{n=1}^{\infty} \alpha(t_n)|\log\alpha(t_n)| < \infty$ if $I\{\alpha\} < \infty$, and $\alpha(t) \ge (\log\log t)^{-2}$ according to our assumption. Thus, by Borel-Cantelli lemma for any $q < \infty$, with probability one

$$\frac{M_{t_n}}{t_n^{4\alpha(t_n)}} > q \quad \text{for almost all } n. \tag{13}$$

Note that if we choose t such that $t_n < t \le t_{n+1}$, then $t_n^{4\alpha(t_n)} > t^{\alpha(t)}$ and from (13) it follows that $M_t > M_{t_n} > qt^{\alpha(t)}$ for all sufficiently large n. Hence,

$$\liminf_{t\to\infty} \frac{M_t}{t^{\alpha(t)}} > q \quad a.s..$$

Since $q < \infty$ is arbitrary, this concludes the proof.

Next, we prove $\liminf_{t\to\infty} M_t/t^{\alpha(t)} = 0$ assuming that $I\{\alpha\} = \infty$. For any $a < b < \infty$, we set

$$\theta^*[a,b] = \max\{\theta(t) : T_a \le t \le T_b\},$$

and define $\overline{M}[a,b]$ via

$$\theta(\overline{M}[a,b]) = \theta^*[a,b] \quad \text{and } T_a \le \overline{M}[a,b] \le T_b.$$

Recall we have set $t_n = \exp(e^n)$. For $q > 0$, denote by A_n the event

$$\overline{M}[qt_n^{\alpha(t_n)}, t_n] < T(qt_n^{2\alpha(t_n)}).$$

Bringing in the set $D = \{n \in \mathbb{N} : \alpha(t_n) > \frac{1}{(\log\log t_n)^2}\}$, we shall prove $\sum_{n=1,n\in D}^{\infty} P(A_n) = \infty$ and

$$\liminf_{n\in D, n\to\infty} \frac{\sum_{j=1,j\in D}^{n} \sum_{k=1,k\in D}^{n} P(A_j \cap A_k)}{(\sum_{j=1,j\in D}^{n} P(A_j))^2} < \infty, \tag{14}$$

which together imply $P(\limsup_{n\in D, n\to\infty} A_n) = 1$ according to the Borel-Cantelli lemma (cf. [10], p.319 or [3]) and Kolmogorov's $0-1$ law. First we prove $\sum_{n=1,n\in D}^{\infty} P(A_n) = \infty$. Note that it holds that for $0 < a < b < c$

$$P(\theta^*[a,b] > \theta^*[b,c]) = P(\theta^*[1,\frac{b}{a}] > \theta^*[\frac{b}{a},\frac{c}{a}]).$$

Thus,

$$P(\theta^*[qt^{\alpha(t)}, qt^{2\alpha(t)}] > \theta^*[qt^{2\alpha(t)}, t]) = P(\theta^*[1, t^{\alpha(t)}] > \theta^*[t^{\alpha(t)}, \frac{1}{q}t^{1-\alpha(t)}]).$$

Therefore, we get by the same argument as employed for (5), (6), (7), (8), (9) and (10)

$$\begin{aligned}
&P(\overline{M}[qt^{\alpha(t)}, t] < T(qt^{2\alpha(t)})) \\
=&P(\theta^*[1, t^{\alpha(t)}] > \theta^*[t^{\alpha(t)}, \frac{1}{q}t^{1-\alpha(t)}]) \\
=&P(\max_{u\le T(t^{\alpha(t)})} B(H(u)) - B(H(T(t^{\alpha(t)}))) > \max_{T(t^{\alpha(t)})\le u\le T(\frac{1}{q}t^{1-\alpha(t)})} \tilde{B}(H(u)) - \tilde{B}(H(T(t^{\alpha(t)})))) \\
=&Q(t^{\alpha(t)}, \frac{1}{q}t^{1-2\alpha(t)}) \\
=&V(\frac{\alpha(t)}{1-2\alpha(t)-(\log t\log q)^{-1}}).
\end{aligned} \tag{15}$$

Moreover, using $V(\alpha(n)) \asymp \alpha(n)|\log\alpha(n)|$ again, we get for some $C > 0$

$$P(A_n) \ge C\alpha(t_n)|\log\alpha(t_n)|.$$

It holds that $\sum_{n\in D}\alpha(t_n)|\log\alpha(t_n)| = \infty$ if $I\{\alpha\} = \infty$, since $\sum_{n\notin D}\alpha(t_n)|\log\alpha(t_n)| < \infty$. So we get $\sum_{n\in D} P(A_n) = \infty$.

Next we prove (14). We only need to consider $\sum_{j=1,j\in D}\sum_{k<j,k\in D} P(A_j\cap A_k)$. First we consider $\sum_{j=1,j\in D}^{n}\sum_{k\in R_{k,j},k\in D} P(A_j\cap A_k)$ where $R_{k,j} = \{k : qt_j^{\alpha(t_j)} \ge t_k\}$. Note that for $a < b \le c < d < \infty$

$$\overline{M}[a,b] - T_a \text{ is independent of } \overline{M}[c,d] - T_c. \tag{16}$$

Then, since $qt_k^{\alpha(t_k)} < t_k \le qt_j^{\alpha(t_j)} < t_j$ when k is satisfied with $qt_j^{\alpha(t_j)} \ge t_k$, it holds that

$$P(A_j\cap A_k) = P(A_j)P(A_k). \tag{17}$$

So, next we consider the case $qt_j^{\alpha(t_j)} < t_k$. We denote by $A'_{k,j}$ the event $\overline{M}[qt_k^{\alpha(t_k)}, qt_j^{\alpha(t_j)}] < T(qt_k^{2\alpha(t_k)})$. Note that when k is satisfied with $qt_j^{\alpha(t_j)} < t_k$, we have $A_k \subset A'_{k,j}$, and by (16) $P(A_j\cap A'_{k,j}) = P(A_j)P(A'_{k,j})$. Then, since by the same argument for (15) $P(A'_{k,j}) = V(\frac{e^k\alpha(t_k)}{e^j\alpha(t_j)-e^k\alpha(t_k)})$, we get

$$P(A_j\cap A_k) \le P(A_j\cap A'_{k,j}) = P(A_j)P(A'_{k,j}) = P(A_j)V(\frac{e^k\alpha(t_k)}{e^j\alpha(t_j)-e^k\alpha(t_k)}). \tag{18}$$

Furthermore, since $\alpha(t_k) \le 2\alpha(t_{k+1})$ due to the assumption (4), we get

$$\begin{aligned}
\sum_{k\in R_{k,j}^c, k<j, k\in D} P(A'_{k,j}) &= \sum_{k\in R_{k,j}^c, k<j, k\in D} V(\frac{e^k\alpha(t_k)}{e^j\alpha(t_j)-e^k\alpha(t_k)}) \\
&\le \sum_{k=1}^{\infty} V(\frac{2^k}{e^k-2^k}) \le C\sum_{k=1}^{\infty}(\frac{e}{2})^{-k} \le C',
\end{aligned} \tag{19}$$

where $R^c_{k,j} = \{k : qt_j^{\alpha(t_j)} < t_k\}$. So, by (18) and (19) we get $\sum_{j=1,j\in D}^n \sum_{k\in R^c_{k,j},k\in D} P(A_j \cap A_k) \le C \sum_{j=1,j\in D}^n P(A_j)$. Combined with (17) this shows

$$\sum_{j=1,j\in D}^{n} \sum_{k\le j,k\in D}^{n} P(A_j \cap A_k) \le \sum_{j=1,j\in D}^{n} \sum_{k\le j,k\in D}^{n} P(A_j)P(A_k) + C' \sum_{j=1,j\in D}^{n} P(A_j),$$

completing the proof of (14). Therefore, we can conclude that with probability one

$$\overline{M}[qt_n^{\alpha(t_n)}, t_n] < T(qt_n^{2\alpha(t_n)}) \quad \text{infinitely often for } n \in D. \tag{20}$$

On the other hand, by standard large deviation result (cf. e.g., [4], (11) and (12)) there exist $0 < c_3, c_4 < \infty$ such that

$$P(T(qt^{2\alpha(t)}) \le qt^{5\alpha(t)}, t^{\frac{1}{4}} \le T_t) \ge 1 - c_3 \exp(-c_4 t^{\alpha(t)}).$$

Moreover, $\sum_{n\in D} c_3 \exp(-c_4 t_n^{\alpha(t_n)}) < \infty$. Then, by Borel-Cantelli lemma it holds that with probability one

$$T(qt_n^{2\alpha(t_n)}) \le qt_n^{5\alpha(t_n)}, \quad M_{t_n^{\frac{1}{4}}} \le \overline{M}[qt_n^{\alpha(t_n)}, t_n], \quad \text{for almost all } n \in D. \tag{21}$$

So, by (20) and (21) it holds that

$$\liminf_{t\to\infty} \frac{M_t}{qt^{20\alpha(t)}} \le \liminf_{n\in D,n\to\infty} \frac{M_{t_n}}{qt_n^{20\alpha(t_n)}} \le \liminf_{n\in D,n\to\infty} \frac{M_{t_n^{\frac{1}{4}}}}{qt_n^{5\alpha(t_n)}} \le \liminf_{n\in D,n\to\infty} \frac{\overline{M}[qt_n^{\alpha(t_n)}, t_n]}{T(qt_n^{2\alpha(t_n)})} < 1 \quad a.s..$$

The proof finishes since $q > 0$ is arbitrary by replacing $\alpha(t)$ by $\frac{\alpha(t)}{20}$. □

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