Inverse problems for hyperbolic equations.

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1   Formulation of the problem and the main theorem.

Let Ω be a smooth bounded domain in $\mathbf{R}^n, n \geq 2$. Consider in the cylinder $\Omega \times (0, T_0)$ the following hyperbolic equation:

$$(1.1) \qquad Lu \overset{def}{=} \left(-i\frac{\partial u}{\partial t} + A_0(x,t)\right)^2 u(x,t)$$

$$-\sum_{j,k=1}^n \frac{1}{\sqrt{g(x)}}\left(-i\frac{\partial}{\partial x_j} + A_j(x,t)\right)\sqrt{g(x)}g^{jk}(x)\left(-i\frac{\partial}{\partial x_k} + A_k(x,t)\right)u$$

$$-V(x,t)u = 0,$$

where $A_j(x,t), 0 \leq j \leq n, V(x,t)$ are $C^\infty(\overline{\Omega} \times [0,T_0])$ functions, analytic in $t, \|g^{jk}(x)\|^{-1}$ is the metric tensor in $\overline{\Omega}, g(x) = \det \|g^{jk}\|^{-1}$. We consider the initial-boundary value problem for (1.1) in $\Omega \times (0, T_0)$:

$$(1.2) \qquad u(x,0) = u_t(x,0) = 0, \quad x \in \Omega,$$

$$(1.3) \qquad u(x,t)\big|_{\partial\Omega \times (0,T_0)} = f(x,t).$$

The following operator is called the Dirichlet-to-Neumann (D-to-N) operator:
(1.4)

$$\Lambda f \overset{def}{=} \sum_{j,k=1}^n g^{jk}(x)\left(\frac{\partial u}{\partial x_j} + iA_j(x,t)u\right)\nu_k \left(\sum_{p,r=1}^n g^{pr}(x)\nu_p\nu_r\right)^{-\frac{1}{2}}\big|_{\partial\Omega \times (0,T_0)},$$

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where $u(x,t)$ is the solution of the initial-boundary value problem (1.1), (1.2), (1.3), $\nu = (\nu_1, ..., \nu_n)$ is the unit exterior normal vector at $x \in \partial\Omega$ with respect to the Euclidian metric. If $F(x) = 0$ is the equation of $\partial\Omega$ in some neighborhood of a point $x_0 \in \partial\Omega$ then Λf has the following form in this neighborhood:

$$(1.5) \qquad \Lambda f = \sum_{j,k=1}^{n} g^{jk}(x) \left(\frac{\partial u}{\partial x_j} + iA_j(x,t)u \right) F_{x_j}(x)$$

$$\cdot \left(\sum_{p,r=1}^{n} g^{pr}(x) F_{x_p} F_{x_r} \right)^{-\frac{1}{2}} \Big|_{F(x)=0, 0<t<T_0}.$$

Let Γ_0 be an open subset of $\partial\Omega$. We shall consider smooth $f(x,t)$ such that supp $f \subset \Gamma_0 \times (0, T_0]$. The inverse problem consists of recovering the coefficients of (1.1) knowing the restriction of Λf to $\Gamma_0 \times (0, T_0)$ for all smooth f with supports in $\Gamma_0 \times (0, T_0]$.

There is a built-in nonuniqueness of this inverse problem:

a) Let $y = \varphi(x)$ be a diffeomorphism of $\overline{\Omega}$ onto $\overline{\Omega}_0 \stackrel{def}{=} \varphi(\overline{\Omega})$ such that $\Gamma_0 \subset \partial\Omega_0$ and $\varphi = I$ on Γ_0.

Let $\hat{L}\hat{u} = 0$ be the equation (1.1) in y-coordinates and let $\hat{\Lambda}$ be the new D-to-N operator. It follows from (1.5) that $\hat{\Lambda} = \Lambda$ on $\Gamma_0 \times (0, T)$, i.e. $\hat{\Lambda} f|_{\Gamma_0 \times (0,T_0)} = \Lambda f|_{\Gamma_0 \times (0,T_0)}$ for all f, supp $f \subset \Gamma_0 \times (0, T_0]$, i.e. the D-to-N operator on $\Gamma_0 \times (0, T_0)$ cannot distinguish between $Lu = 0$ in $\Omega \times (0, T_0)$ and $\hat{L}\hat{u} = 0$ in $\Omega_0 \times (0, T_0)$.

b) Let $G_0(\Omega \times [0, T_0])$ be a group of $C^\infty(\overline{\Omega} \times [0, T_0])$ complex-valued functions $c(x,t)$ such that $c(x,t) \neq 0$ in $\overline{\Omega} \times [0, T_0]$, $c(x,t) = 1$ on $\overline{\Gamma}_0 \times [0, T_0]$. We say that potentials $A(x,t) = (A_0(x,t), A_1(x,t), ..., A_n(x,t))$ and $A'(x,t) = (A'_0(x,t), A'_1(x,t), ..., A'_n(x,t))$ are gauge equivalent if there exists $c(x,t) \in G_0(\Omega \times [0, T_0])$ such that

$$(1.6) \qquad A'_0(x,t) = A_0(x,t) - ic^{-1}(x,t)\frac{\partial c}{\partial t},$$

$$A'_j(x,t) = A_j(x,t) - ic^{-1}(x,t)\frac{\partial c}{\partial x_j}, \quad 1 \le j \le n.$$

Note that if $Lu = 0$ and $u' = c(x,t)u$ then $L'u' = 0$ where L' is an operator of the form (1.1) with $A_j(x,t)$, $0 \le j \le n$, replaced by $A'_j(x,t)$, $0 \le j \le n$. We shall write for brevity

$$L' = c \circ L.$$

It is easy to show that if Λ' is the D-to-N operator for L' then $\Lambda' = \Lambda$ on $\Gamma_0 \times (0, T_0)$, i.e. all potentials $A(x, t)$ in the same gauge equivalence class correspond to the same D-to-N operator on $\Gamma_0 \times (0, T_0)$. Note that if we consider real-valued potentials only then the gauge group G_0 should be reduced to $c(x, t)$ such that $|c(x, t)| = 1$. If Ω is simply-connected then any $c(x, t) \in G_0$ has a form $c(x, t) = e^{i\varphi(x, t)}$ where $\varphi(x, t) \in C^\infty(\overline{\Omega} \times [0, T])$. Also if coefficients of L are independent of t it is natural that the group G_0 consists of $c(x)$ independent of t. Then $A_0'(x) = A_0(x)$ (see (1.6)). Denote

$$T_* = \max_{x \in \overline{\Omega}} d(x, \Gamma_0),$$

where $d(x, \Gamma_0)$ is the distance in $\overline{\Omega}$ with respect to the metric $\|g^{jk}(x)\|^{-1}$ from $x \in \overline{\Omega}$ to Γ_0. We shall assume L and Γ_0 satisfy the BLR-condition (see [BLR92]) for $t = T_{**}$. This means roughly speaking that any null-bicharacteristic of L in $(\overline{\Omega} \times [0, T_{**}]) \times (\mathbf{R}^{n+1} \setminus \{0\})$ intersects $(\Gamma_0 \times [0, T_{**}]) \times (\mathbf{R}^{n+1} \setminus \{0\})$. It was proven in [BLR92] that the BLR-condition implies that the (bounded) map of $f \in H_0^1(\Gamma_0 \times (0, T_0))$ to $(u(x, T_{**}), u_t(x, T_{**})) \in H^1(\Omega) \times L^2(\Omega)$ is onto. Here $H_0^1(\Gamma_0 \times (0, T_{**})$ is the subspace of $H^1(\partial\Omega \times (0, T_{**}))$ such that $f|_{t=0} = 0$ and supp $f \subset \overline{\Gamma_0} \times (0, T_{**}]$, $u(x, t)$ is the solution of (1.1), (1.2), (1.3).

The following theorem was proven in [E06]:

Theorem 1.1. *Let L and L_0 be two operators of the form (1.1) in domains Ω and Ω_0, respectively, with coefficients $A(x, t), V(x, t)$ and $A_0(x, t), V_0(x, t)$ analytic in t and real-valued. Suppose $\Gamma_0 \subset \partial\Omega \cap \partial\Omega_0$ and suppose that L and Γ_0 satisfy the BLR-condition when $t = T_{**}$. Suppose that D-to-N operators Λ and Λ_0, corresponding to L and L_0, respectively, are equal on $\Gamma_0 \times (0, T_0)$ for all smooth f with supports on $\Gamma_0 \times (0, T_0]$. Let $T_0 > 2T_* + T_{**}$. Then there exists a diffeomorphism φ of $\overline{\Omega}$ onto $\overline{\Omega_0}$, $\varphi = I$ on Γ_0, and there exists a gauge transformation $c_0(x, t) \in G_0(\overline{\Omega} \times [0, T_0])$ such that*

$$c_0 \circ \varphi^{-1} \circ L_0 = L$$

on $\Omega \times (0, T_0)$.

Denote by L^* the formally adjoint operator to L. Note that L^* has the form (1.1) with $A_j(x, t)$, $0 \le j \le n, V(x, t)$ replaced by $\overline{A_j(x, t)}$, $0 \le j \le n, \overline{V(x, t)}$. To prove Theorem 1.1 we need to know also the D-to-N operator Λ_* corresponding to L^*. If $L^* = L$ then obviously $\Lambda_* = \Lambda$. In the case when

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$A_0 = 0$ and potentials $A_j(x)$, $1 \leq j \leq n$, $V(x)$ are independent of t one can show that Λ determines Λ_* on $\Gamma_0 \times (0, T_0)$ (c.f. [KL00] and §2 below) even when $A_j(x)$, $1 \leq j \leq n$, $V(x)$ are complex-valued. Therefore Theorem 1.1 holds in this case and gives a new proof of the correponding result in [KL00], [KL2 97]. When $A_0 = 0$, $A_j(x)$, $1 \leq j \leq n$, $V(x)$ are real-valued and independent of t, i.e. in the self-adjoint case, the BLR-condition is not needed. In this case Theorem 1.1 is true with $T_{**} = 0$. This result was first obtained by BC (Boundary Control) method (see [B97]) (see also [B1 02], [B2 93], [KKL01], [K93], [KK98]). In [E1 06] we gave a new proof for time-independent self-adjoint case. The proof in [E06] is based on the new approach in [E1 06]. The inverse problems for the wave equations with time-dependent potentials in the case when $\Gamma_0 = \partial\Omega$ was considered in [St89], [RS91] (see also [I98]).

A crucial step of the proof of Theorem 1.1 uses the unique continuation theorem by Tataru [T95]. This theorem requires that $A_j(x, t)$, $0 \leq j \leq n, V(x, t)$ depend analytically on t.

The proof of Theorem 1.1 consists of two steps : the local step and the global step. In the local step we recover the coefficients of L (up to a diffeomorphism and a gauge transformation) in the domain $\Gamma_\delta \times [0, T_0]$ where Γ is an open connected subset of Γ_0 and Γ_δ in a small neighborhood of Γ in $\overline{\Omega}$. The main novelty of the proof here is the study of the restrictions of the solutions of $Lu = 0$ to the characteristic surfaces instead of the restrictions to the hyperplanes $t = $ constant as in BC-method.

The main part of the global step is the following lemma that reduced the inverse problem in the domain to the inverse problem in a smaller domain (c.f. [KKL1 04]):

Lemma 1.1. *Let $L^{(p)}, p = 1, 2$ be two operators of the form (1.1) in domains $\Omega_p, p = 1, 2$, respectively, satisfying the initial-boundary conditions (1.2), (1.3). We assume that $\Gamma_0 \subset \partial\Omega_1 \cap \partial\Omega_2$, supp $f \subset \Gamma_0 \times (0, T_0]$ and $\Lambda_1 = \Lambda_2$ on $\Gamma_0 \times (0, T_0)$ where Λ_p are the D-to-N operators corresponding to $L^{(p)}$, $p = 1, 2$. Let $B \subset \Omega_1 \cap \Omega_2$ be such that the domains $\Omega_p \setminus \overline{B}$ are smooth, $L^{(1)} = L^{(2)}$ in $\overline{B}$ and $S_1 \stackrel{def}{=} \partial B \cap \partial\Omega_p \subset \Gamma_0$, $p = 1, 2$. Let $\delta = \max_{x \in \overline{B}} d(x, \Gamma_0)$ where $d(x, \Gamma_0)$ is the distance in $\overline{B}$ from $x \in \overline{B}$ to Γ_0. Denote by $\hat{\Lambda}_p$ the D-to-N operators corresponding to $L^{(p)}$ in domains $(\Omega_p \setminus \overline{B}) \times (\delta, T_0 - \delta)$, $p = 1, 2$. Let $S_2 = \partial B \setminus \overline{S_1}$ and let $\Gamma_1 = (\Gamma_0 \setminus S_1) \cup S_2$. Then $\hat{\Lambda}_1 = \hat{\Lambda}_2$ on $\Gamma_1 \times (\delta, T_0 - \delta)$.*

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2 Hyperbolic systems with Yang-Mills potentials and domains with obstacles.

Consider in $\Omega \times (0, T_0)$ a system of the form (c.f. [E2 05])

$$(2.1) \qquad Lu \stackrel{def}{=} \left(-i\frac{\partial u}{\partial t} I_m + A_0(x,t) \right)^2 u(x,t)$$

$$-\sum_{j,k=1}^{n} \frac{1}{\sqrt{g(x)}} \left(-i\frac{\partial}{\partial x_j} I_m + A_j(x,t) \right) \sqrt{g(x)} g^{jk}(x) \left(-i\frac{\partial}{\partial x_k} I_m + A_k(x,t) \right) u$$

$$-V(x,t)u = 0,$$

where $u(x,t)$, $A_j(x,t)$, $0 \le j \le n$, $V(x,t)$ are $m \times m$ matrices, I_m is the identity $m \times m$ matrix. Assume that the initial-boundary conditions (1.2), (1.3) are satisfied. Let $\Gamma_0 \subset \partial\Omega$ and let $G_0(\overline{\Omega} \times [0,T])$ be the gauge group of non-singular C^∞ $m \times m$ matrices $C(x,t)$ in $\overline{\Omega} \times [0,T_0]$ analytic in t and such that $C(x,t) = I_m$ on $\Gamma_0 \times [0,T_0]$. Matrices $A(x,t) = (A_0(x,t),...,A_n(x,t))$, $V(x,t)$ are called Yang-Mills potentials. We say that $(A(x,t), V(x,t))$ and $(A'(x,t), V'(x,t))$ are gauge equivalent if there exists $C(x,t) \in G_0(\Omega_0 \times [0,T_0])$ such that

$$(2.2) \quad A_0'(x,t) = C^{-1}(x,t)A_0(x,t)C(x,t) - iC^{-1}(x,t)\frac{\partial C(x,t)}{\partial t},$$

$$A_j'(x,t) = C^{-1}A_j(x,t)C - iC^{-1}\frac{\partial C}{\partial x_j}, \quad 1 \le j \le n,$$

$$V'(x,t) = C^{-1}V(x,t)C.$$

When we consider self-adjoint operators of the form (2.1) , i.e. when matrices $A_j(x,t)$, $0 \le j \le n$, $V(x,t)$ are self-adjoint, the group $G_0(\Omega \times [0,T_0])$ consists of unitary matrices $C(x,t)$.

A generalization of the proof of Theorem 1.1 leads to the following result (c.f. [E2 05]:

Theorem 2.1. *Theorem 1.1 holds for the equations of the form (2.1) with Yang-Mills potentials.*

Consider now the system of the form (2.1) when the Yang-Mills potentials

are independent of t but not necessary self-adjoint matrices:

$$(2.3) \qquad Lu \stackrel{def}{=} \left(-i\frac{\partial u}{\partial t}I_m + A_0(x) \right)^2 u(x,t)$$

$$-\sum_{j,k=1}^{n} \frac{1}{\sqrt{g(x)}} \left(-i\frac{\partial}{\partial x_j}I_m + A_j(x) \right) \sqrt{g(x)}g^{jk}(x) \left(-i\frac{\partial}{\partial x_k}I_m + A_k(x) \right) u(x,t)$$

$$-V(x)u(x,t) = 0.$$

We also assume that $T_0 = +\infty$, i.e. (2.3) and the boundary condition (1.3) hold for $t \in (0, +\infty)$. Let L^* be formally adjoint to L, i.e. when $A_j(x)$, $0 \leq j \leq n$, $V(x)$ are replaced by the adjoint matrices $A_j^*(x)$, $0 \leq j \leq n$, $V^*(x)$.

Consider the initial-boundary value problem adjoint to (2.3), (1.2), (1.3) on some interval $(0, T)$:

$$(2.4) \qquad\qquad L^*v = 0 \quad \text{on} \quad \Omega \times (0, T),$$

$$(2.5) \qquad\qquad v|_{t=T} = \frac{\partial v}{\partial t}\Big|_{t=T} = 0, \quad v|_{\partial\Omega \times (0,T)} = g,$$

where supp $g \subset \Gamma_0 \times (0, T]$. Let Λ^* be the D-to-N operator corresponding to (2.4), (2.5). We have

$$0 = (Lu, v) - (u, L^*v) = (\Lambda f, g) - (f, \Lambda^* g)$$

for any smooth f and g, supp $f \subset \Gamma_0 \times (0, T]$, supp $g \subset \Gamma_0 \times [0, T_0)$. Therefore Λ^* is an adjoint operator to Λ and we can determine Λ^* on $\Gamma_0 \times [0, T)$ if we know Λ on $\Gamma_0 \times (0, T)$. Change in (2.4), (2.5) t to $T - t$. Then we get an initial-boundary value problem

$$(2.6) \qquad\qquad L_1^*w = 0 \quad \text{on} \quad \Omega \times (0, T),$$

$$(2.7) \qquad\qquad w(x, 0) = w_t(x, 0) = 0, \quad w|_{\partial\Omega \times (0,T)} = g_1(x, t),$$

where $w(x, t) = v(x, T - t)$, $g_1(x, t) = g(x, T - t)$, $0 < t < T$, L_1^* is obtained from L^* by changing $A_0^*(x)$ to $-A_0^*(x)$. It is clear that the D-to-N operator Λ_{1*} on $\Gamma_0 \times (0, T)$ corresponding to (2.6), (2.7) is determined by Λ^*.

Consider also the initial-boundary value problem

$$(2.8) \qquad\qquad L^*u = 0 \quad \text{on} \quad \Omega \times (0, T),$$

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(2.9) $u(x, 0) = u_t(x, 0) = 0, \quad u|_{\partial\Omega \times (0,T)} = f(x, t).$

Denote by Λ_* the D-to-N operator correspondin to (2.8), (2.9). Here $T > 0$ is arbitrary, i.e. (2.6), (2.7) and (2.8), (2.9) hold on $(0, +\infty)$. We assume that $f(x, t)$ and $g_1(x, t)$ belong to $C_0^\infty(\Gamma \times (0, +\infty))$. Performing the Fourier-Laplace transform in t in (2.6), (2.7) and in (2.8), (2.9) when $T = +\infty$ we get:

(2.10) $L^*(k)\tilde{u}(x, k) = 0, \quad x \in \Omega,$

(2.11) $\tilde{u}(x, k)|_{\partial\Omega} = \tilde{f}(x, k),$

and

(2.12) $L^*(-k)\tilde{w}(x, k) = 0, \quad x \in \Omega,$

(2.13) $\tilde{w}(x, k)|_{\partial\Omega} = \tilde{g}_1(x, k),$

where $\tilde{u}(x, k), \tilde{w}(x, k)$ are analytic in k for $\Im k < -C_0$ for some $C_0 > 0$, $L^*(k)$ is obtained from L^* by replacing $-i\frac{\partial}{\partial t}$ by k. Let $\Lambda_*(k)$ be the D-to-N operator on Γ corresponding to the boundary value problem (2.10), (2.11), depending on parameter k. Note that $\Lambda_*(k)$ is the Fourier-Laplace transform in t of the D-to-N operator Λ_* corresponding to (2.8), (2.9) on $(0, +\infty)$. Since Ω is a bounded domain $\Lambda_*(k)$ has an analytic continuation from $\Re k \leq -C_0$ to $\mathbf{C} \setminus Z$ where Z is a discrete set. Note that the Fourier-Laplace transform of Λ_{1*} is $\Lambda_*(-k)$. Since $\Lambda_*(k)$ is analytic in $\mathbf{C} \setminus Z$, $\Lambda_*(-k)$ determines $\Lambda_*(k)$. Therefore when $T_0 = +\infty$ we get that the D-to-N operator Λ on $\Gamma_0 \times (0, +\infty)$ determines the D-to-N operator Λ_* on $\Gamma_0 \times (0, +\infty)$. Therefore the proof of Theorem 2.1 applies and we have the following result (c.f. [KL1 00], [KL2 97]:

Theorem 2.2. *Let L_p be two operators of the form (2.3) in domains $\Omega_p \times (0, +\infty)$, $p = 1, 2$. Suppose $\Gamma_0 \subset \partial\Omega_1 \cap \partial\Omega_2$ and $\Lambda_1 = \Lambda_2$ on $\Gamma_0 \times (0, +\infty)$ where Λ_p are the D-to-N operators corresponding to $L_p, p = 1, 2$. Suppose that L_1 and Γ_0 satisfy the BLR-condition for some $t = T_{**}$. Then there exists a diffeomorphism $y = \varphi(x)$ of $\overline{\Omega_1}$ onto $\overline{\Omega_2}$ and a gauge transformation $c_0(x) \in G_0(\overline{\Omega_1})$ such that*

$$c_0 \circ \varphi^{-1} \circ L_2 = L_1, \quad x \in \Omega.$$

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We do not assume here that L_p, $p = 1, 2$, are formally self-adjoint.

Note that domains Ω can be multi-connected and $\Gamma_0 \subset \partial\Omega$ can be not connected. An important example of inverse problems with the boundary data prescribed on a part of the boundary are the inverse problems in domains with obstacles. In this case $\Omega = \Omega_0 \setminus (\cup_{j=1}^r \overline{\Omega_j})$, where $\Omega_1, ..., \Omega_r$ are nonintersecting domains inside Ω_0, called obstacles, $\Gamma_0 = \partial\Omega_0$ and the zero Dirichlet boundary conditions are prescribed on $\partial\Omega_j$, $1 \leq j \leq r$, i.e. we have

$$(2.14) \qquad\qquad Lu = 0 \quad \text{on} \quad \Omega \times (0, T_0),$$

$$(2.15) \qquad\qquad u(x, 0) = u_t(x, 0) = 0 \quad \text{on} \quad \Omega,$$
$$u|_{\partial\Omega_0 \times (0, T_0)} = f(x, t), \quad u|_{\partial\Omega_j \times (0, T_0)} = 0, \quad 1 \leq j \leq r.$$

Unfortunately, the BLR-condition is not satisfied for domains with more than one smooth obstacle. Therefore we shall assume that L is a formally self-adjoint operator of the form (2.3), i.e. when $A_j(x)$, $0 \leq j \leq n$, $V(x)$ are self-adjoint matrices, and initial-boundary conditions (2.15) are satisfied. In this case Theorem 2.1 holds for any $T_0 > 2T_*$ and for any number of obstacles.

Finally consider the following particular case: $T_0 = +\infty$, $g^{jk}(x) = \delta_{jk}$, $A_0(x) = 0$, $A_j(x)$, $1 \leq j \leq n$, $V(x)$ are self-adjoint. Making the Fourier-Laplace transform in (2.14) we get the Schrödinger equation with Yang-Mills potentials in Ω:

$$(2.16) \qquad \sum_{j=1}^n \left(-i\frac{\partial}{\partial x_j} I_m + A_j(x) \right)^2 w(x) + V(x)w(x) - k^2 w(x) = 0,$$

where we omitted the dependence of w on k in (2.16). When $m = 1$ we have the Schrödinger equation with electromagnetic potentials. The boundary conditions for (2.16) have the form:

$$(2.17) \qquad w|_{\partial\Omega_0} = h(x), \quad w|_{\partial\Omega_j} = 0, \quad , 1 \leq j \leq r.$$

The D-to-N operator for (2.16), (2.17) has the form:

$$(2.18) \qquad\qquad \Lambda(k)h = \left(\frac{\partial w}{\partial \nu} + i(A \cdot \nu)w \right)|_{\partial\Omega_0},$$

where ν is the exterior unit normal vector to $\partial\Omega_0$. Knowing the hyperbolic D-to-N operator for (2.14), (2.15) for the arbitrary $T_0 > 0$ we can find $\Lambda(k)$

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for all $k \in \mathbf{C} \setminus Z$, and vice versa. Since $\Lambda(k)$ is analytic, knowing $\Lambda(k)$ on any interval $(k_0 - \varepsilon, k_0 + \varepsilon)$ of analyticity determines $\Lambda(k)$ for all $k \in \mathbf{C} \setminus Z$. Therefore Theorem 2.1 implies that $\Lambda(k)$ given on $(k_0 - \varepsilon, k_0 + \varepsilon)$, $k_0 > 0$, $\varepsilon > 0$, determines the location of all obstacles Ω_j, $1 \leq j \leq r$, since the metric is fixed, and determines potentials $A_j(x)$, $1 \leq j \leq n$, $V(x)$ in $\overline{\Omega}$ up to a gauge transformation $C(x) \in G_0(\overline{\Omega})$, i.e $C(x) = I_m$ on $\partial\Omega_0$, $C(x)$ is an unitary matrix in $\overline{\Omega}$.

The interest of considering multi-connected domains with obstacles was spurred by the Aharonov-Bohm effect. It was shown by Aharonov and Bohm [AB59] that the presence of distinct gauge equivalence classes of potentials can be detected in an experiment and this phenomenon is called the Aharonov-Bohm effect. As it was shown above the D-to-N $\Lambda(k)$ on $\partial\Omega_0$ given for all $k \in (k_0 - \varepsilon, k_0 + \varepsilon)$ allows to detect the gauge equivalent class of Yang-Mills (or electroctromagnetic) potentials.

3   A geometric optic approach.

Consider the Schrödinger equation with electromagnetic potentials in the domain $\Omega = \Omega_0 \setminus (\cup_{j=1}^{r}\overline{\Omega_j})$ with obstacles, i.e. consider (2.16) when $m = 1$, with boundary conditions (2.17).

Assume that the D-to-N operator $\Lambda(k)$ on $\partial\Omega_0$ (see (2.18)) is given for all $k \in \mathbf{C} \setminus K$. Another approach to the inverse problem for (2.16), (2.17) is based on geometric optics constructions and the reduction to the integral geometry (tomography) problems.

We say that $\gamma = \gamma_1 \cup \gamma_2 \cup ... \cup \gamma_N$ is a broken ray with legs $\gamma_1.\gamma_2, ..., \gamma_N$ if γ_k, $1 \leq k \leq N$, are geodesics, γ starts at point $x_0 \in \partial\Omega_0$, γ has $N - 1$ nontangential points of reflection at the obstacles and γ ends at a point $x_N \in \partial\Omega_0$. One can construct geometric optics solutions supported in a small neighborhood of γ (c.f. [E3 04], [E2 05])

Consider two Schrödinger equations with electro-magnetic potentials $A^{(p)}(x)$, $V^{(p)}(x)$, $p = 1, 2$, with the Euclidian metric $g^{jk} = \delta_{jk}$ in a plane domain with convex obstacles. Let $\Lambda_p(k)$ be the corresponding D-to-N operators, $p = 1, 2$.

Using the geometric optics solutions one can prove that if the D-to-N operators are equal on $\partial\Omega_0$ then

$$(3.1) \qquad \exp(i \int_{\gamma} A^{(1)}(x) \cdot dx) = \exp(i \int_{\gamma} A^{(2)}(x) \cdot dx),$$

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$$(3.2) \qquad \int_\gamma V^{(1)}(x)ds = \int_\gamma V^{(2)}(x)ds$$

for any broken ray (c.f. [E3 04], [E2 05]). The geometric optics construction and equalities (3.1), (3.2) hold in any dimension $n \geq 2$ and for any broken ray even when the broken rays are passing through generic caustics. Having (3.1), (3.2) we reduce the inverse problem for the Schrödinger equation to the inverse problem of the integral geometry of broken rays, i.e. the recovery of potentials from integrals over broken rays. This is a difficult problem.

Some results in this direction were obtained in [E3 04] for $n = 2$ under the geometric restriction that there is no trapped rays. This condition is not satisfied when one has more than one smooth obstacle. However, there are piecewise smooth convex obstacles that satisfy these conditions. In this case it was shown in [E3 04] that if (3.1), (3.2) hold for all broken rays in Ω_0 then $V^{(1)} = V^{(2)}$ and $A^{(1)}$ and $A^{(2)}$ are gauge equivalent. Despite that this approach is much more restrictive than the hyperbolic equations approach it has an advantage that it allows to prove the stability results in some cases. It also does not require the BLR-condition in the nonself-adjoint case.

Consider the following example:

Let $\Omega_1 \subset \Omega_0$ be the only convex obstacle in Ω_0 and let $f(x)$ be a smooth function in $\overline{\Omega_0} \setminus \Omega_1$. It is well-known (c.f. [He80]) that if $\int_\gamma f(x)ds = 0$ for all lines γ not intersecting $\overline{\Omega_1}$ then $f(x) = 0$. This problem is severly ill-posed. If one uses the broken rays, i.e. if one compute $\int_\gamma f ds$ for all broken rays γ, then the inverse problem is well-posed and there is a stability estimate. More precisely, let $\gamma_{x,\theta}$ be the broken ray starting on $\partial\Omega_0$ and ending at $x \in \overline{\Omega_0} \setminus \Omega_1$. Here θ is the direction of the ray at the endpoint x. We assume that $w(x,\theta)$ is known when $x \in \partial\Omega_0$, $\forall\theta \in S^1$. The following staility estimate holds (c.f. [E3 04] and [M77] in the case of no obstacles) :

$$\int_{\Omega_0\setminus\Omega_1} |f(x)|^2 dx \leq C \int_0^{l_0} \int_{S^1} \left(\left|\frac{\partial w(x(s),\theta)}{\partial s}\right| + \left|\frac{\partial w}{\partial \theta}\right|^2 \right) ds d\theta,$$

where $x = x(s)$ is the equation of $\partial\Omega_0$, l_0 is the arclength of $\partial\Omega_0$.

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