

# Hybrid Systems and Control With Fractional Dynamics (I): Modeling and Analysis

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**Abstract**—No mixed research of hybrid and fractional-order systems into a cohesive and multifaceted whole can be found in the literature. This paper focuses on such a synergistic approach of the theories of both branches, which is believed to give additional flexibility and help to the system designer. It is part I of two companion papers and introduces the fundamentals of fractional-order hybrid systems, in particular, modeling and stability analysis of two kinds of such systems, i.e., fractional-order switching and reset control systems. Some examples are given to illustrate the applicability and effectiveness of the developed theory. Part II will focus on fractional-order hybrid control.

## I. INTRODUCTION

Hybrid systems (HS) are heterogeneous dynamic systems whose behaviour is determined by interacting continuous-variable and discrete-event dynamics, and they arise from the use of finite-state logic to govern continuous physical processes or from topological and networks constraints interacting with continuous control [1], [2], [3]. It is worth mentioning that, among them, we focus on two kinds of HS in this work: switching and reset control systems. Switching systems, a class of HS consisting of several subsystems and a switching rule indicating the active subsystem at each instant of time, have been the subject of interest for the past decades, for their wide application areas. Likewise, reset control systems are standard control systems endowed with a reset mechanism, i.e., a strategy that resets to zero the controller state (or part of it) when some condition holds. The hybrid behaviour comes from the instantaneous jump due to resets of whole or part of system states [4], [5].

Many real dynamic systems are better characterized using a fractional-order dynamic model based on differentiation and integration of non-integer-order. The concept of fractional calculus has tremendous potential to change the way we see, model, and control the nature around us. Denying fractional derivatives is like saying that zero, fractional, or irrational numbers do not exist. From the control engineering point of view, improving and developing the control is the major concern (see e.g. [6], [7]).

Recently, the wide applicability of both HS and systems with fractional-order dynamics has inspired a great deal of research and interest in both fields. Unfortunately, in general there are many difficulties in mixing different mathematical

domains. The case of combining the theories of such systems is no exception. Given this motivation, this paper arises from the idea of coupling two different distinct branches of research, fractional calculus and HS, into a synergistic way, which is believed to give additional flexibility and help to the system designer, taking advantage of the potentialities of both worlds. To this respect, a mathematical framework of fractional-order hybrid systems (FHS), including modeling, stability analysis, control and simulation, is required to be developed. Accordingly, the objective of part I of these two companion papers is to introduce the mentioned framework of HS with fractional-order dynamics, namely, modeling and analysis issues.

The remainder of part I of this paper is organized as follows. In Section II, modeling of FHS is presented through differential inclusions (DI); two special examples of switching and reset control systems are studied. Section III addresses stability analysis of such systems. Three stability examples, again for switching and reset control systems, are given to show the applicability of the developed theory. The concluding remarks are drawn in Section IV.

## II. MODELING OF FRACTIONAL-ORDER HYBRID SYSTEMS

This section deals with fundamentals of two kinds of fractional-order hybrid systems, i.e., switching systems and reset control systems based on fractional-order differential inclusions (FDI). Then, two special HS are modeled.

### A. Fractional-order differential inclusions

A widely used model of a continuous-time dynamical system is the first-order differential equation  $\dot{x} = f(x, u)$ , with  $x$  and  $u$  belonging to an  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ . This model can be expanded in two directions that are relevant for HS. First, we can consider differential equations with state constraints, that is,  $\dot{x} = f(x, u)$  and  $x \in C$ ,  $u \in C_u$ , where flow sets  $C$  and  $C_u$  are subsets of  $\mathbb{R}^n$ . Second, we can consider the situation where the right-hand side of the DI is replaced by a set that may depend on  $x$ . Both situations lead to the DI  $\dot{x} \in F(x)$ , where  $F$  is a set-valued mapping. Likewise, the combination of the two generalizations leads to constrained DI as follows:  $\dot{x} \in F(x, u)$ ,  $x \in C$ ,  $u \in C_u$ .

A typical model of a discrete-time dynamical system is the first-order equation  $x^+ = g(x, u)$ , with  $x, u \in \mathbb{R}^n$ . The notation  $x^+$  indicates that the next value of the state is given as a function of the current state  $x$  through the value  $g(x)$ . As for differential equations, it is a natural extension to consider constrained difference equations and difference inclusions,

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which leads to the model  $x^+ \in G(x, u)$ ,  $x \in D$ ,  $u \in D_u$ , where  $G$  is a set-valued mapping and jump sets  $D$  and  $D_u$  are subsets of  $\mathbb{R}^n$ . Since a model of a hybrid dynamical system requires a description of the time driven dynamics, the event driven dynamics, and the regions on which these dynamics apply, we include both a constrained DI and a constrained difference inclusion in a general model of a HS in the form

$$\begin{aligned} \dot{x} &\in F(x, u), x \in C, u \in C_u, \\ x^+ &\in G(x, u), x \in D, u \in D_u. \end{aligned} \quad (1)$$

Taking into account integer-order DI described by (1), its generalization to fractional-order can be expressed as

$$\begin{aligned} D^\alpha x &\in F(x, u), x \in C, u \in C_u, \\ x^+ &\in G(x, u), x \in D, u \in D_u, \end{aligned} \quad (2)$$

where  $D^\alpha$  is the fractional-order operator with  $\alpha \in \mathbb{R}$ .

### B. Switching systems

Switching system is a hybrid dynamical system consisting of a family of continuous-time subsystems and a rule that orchestrates the switching among them [8]. A general formulation of the switching systems with fractional-order is:

$$D^\alpha x = Ax, A \in \text{co}\{A_1, \dots, A_L\}. \quad (3)$$

where  $\text{co}$  denotes the convex combination and  $A_i$ ,  $i = 1, \dots, L$ , is the switching subsystem. A primary motivation for studying such systems came partly from the fact that switching systems and switching multi-controller systems have numerous applications in control of mechanical systems, process control, automotive industry, power systems, traffic control, and so on. Let us now model switching system of multi-controller by means of FDI in the following example.

*Example 1:* Modelling of a fractional-order multi-controller system

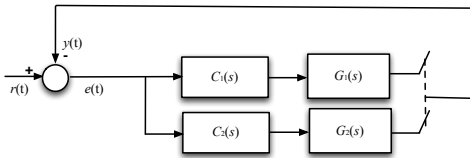


Fig. 1

CLOSED-LOOP SYSTEM WITH TWO CONTROLLERS

Let us consider a first-order system with two different dynamics as follows (see Fig. 1):

$$G_i(s) = \frac{K_i}{s + \tau_i}, i = \{1, 2\}, \quad (4)$$

controlled by the following fractional-order PI controllers:

$$C_i(s) = k_{p_i} + \frac{k_{i_i}}{s^{\alpha_i}}, i = \{1, 2\}. \quad (5)$$

Then, the closed-loop transfer function of the system can be written as:

$$\frac{Y(s)}{R(s)} = \frac{a_i s^{\alpha_i} + b_i}{s^{\alpha_i+1} + (\tau_i + a_i) s^{\alpha_i} + b_i}, i = \{1, 2\}, \quad (6)$$

where  $a_i = K_i k_{p_i}$  and  $b_i = K_i k_{i_i}$ . Assuming  $\alpha_i = \frac{q_i}{p_i}$ , the state space form of (6) is given by:

$$\begin{bmatrix} D^{\frac{1}{q_i}} x_1 \\ D^{\frac{1}{q_i}} x_2 \\ \vdots \\ D^{\frac{1}{q_i}} x_{p_i+1} \\ \vdots \\ D^{\frac{1}{q_i}} x_{p_i+q_i-1} \\ D^{\frac{1}{q_i}} x_{p_i+q_i} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \dots & 1 \\ -b_i & 0 & 0 & \dots & -(\tau_i + a_i) & \dots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{p_i+1} \\ \vdots \\ x_{p_i+q_i-1} \\ x_{p_i+q_i} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} U(r(t)), i = \{1, 2\}, \quad (7)$$

where  $U(r(t)) = a_i D^{\alpha_i} r(t) + b_i r(t)$ . It is obvious that the closed-loop system can be written in a general form as:

$$D^{\alpha_i} x = A_i x + B_i U_i. \quad (8)$$

Now assume that the controller one  $C_1(e)$  will be activated if  $e = r(t) - y(t) > -\varepsilon$ , whereas the controller  $C_2(e)$  will be activated if  $e = r(t) - y(t) < \varepsilon$ . Thus, the FDI are taken to be:

$$\begin{bmatrix} D^{\alpha_i} x \\ D^{\alpha_i} i \end{bmatrix} = \begin{bmatrix} A_i x + B_i U_i \\ 0 \end{bmatrix}, \quad (9)$$

The flow set and jump set are respectively taken as:

$$C := \{(x, i) \in \mathbb{R}^{\alpha_i+1} \times \{1, 2\} \mid i = 1 \& y(t) < r(t) + \varepsilon \\ i = 2 \& y(t) > r(t) - \varepsilon\}, \quad (10)$$

and

$$D := \{(x, i) \in \mathbb{R}^{\alpha_i+1} \times \{1, 2\} \mid i = 1 \& y(t) = r(t) + \varepsilon \\ i = 2 \& y(t) = r(t) - \varepsilon\}. \quad (11)$$

In what concerns the jump map, since the role of jump changes is to toggle the logic mode and the state component  $x$  does not change during jumps, the jump map will be:

$$\begin{bmatrix} x \\ i \end{bmatrix}^+ = \begin{bmatrix} x \\ 3 - i \end{bmatrix}. \quad (12)$$

### C. Fractional-order reset control systems

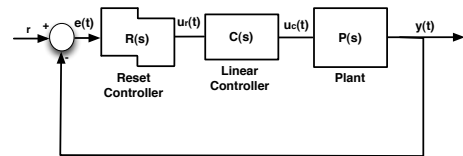


Fig. 2

BLOCK DIAGRAM OF A RESET CONTROL SYSTEM

Let us now model reset control systems by means of FDI. The block diagram of a general reset control system is shown

in Fig. 2. It can be observed that the dynamics of the reset controller can be described by the FDI equation as:

$$\begin{aligned} D^\alpha x_r(t) &= A_r x_r(t) + B_r e(t), \quad e(t) \neq 0, \\ x_r(t^+) &= A_{R_r} x_r(t), \quad e(t) = 0, \\ u_r(t) &= C_r x_r(t) + D_r e(t), \end{aligned} \quad (13)$$

where  $0 < \alpha \leq 1$  is the order of differentiation,  $x_r(t) \in \mathbb{R}^{n_r}$  is the reset controller state vector and  $u_r(t) \in \mathbb{R}$  is its output. The matrix  $A_{R_r} \in \mathbb{R}^{n_r \times n_r}$  identifies that subset of states  $x_r$  that are reset (the last  $\mathcal{R}$  states) and use the structure  $A_{R_r} = \begin{bmatrix} I_{n_{\bar{\mathcal{R}}}} & 0 \\ 0 & 0_{n_{\mathcal{R}}} \end{bmatrix}$  and  $n_{\bar{\mathcal{R}}} = n_r - n_{\mathcal{R}}$ .

The linear controller  $C(s)$  and plant  $P(s)$  have, respectively, state space representations as follows:

$$\begin{aligned} D^\alpha x_c(t) &= A_c x_c(t) + B_c u_r(t), \\ u_c(t) &= C_c x_c(t), \end{aligned} \quad (14)$$

and

$$\begin{aligned} D^\alpha x_p(t) &= A_p x_p(t) + B_p u_c(t), \\ y(t) &= C_p x_p(t), \end{aligned} \quad (15)$$

where  $A_p \in \mathbb{R}^{n_p \times n_p}$ ,  $B_p \in \mathbb{R}^{n_p \times 1}$ ,  $C_p \in \mathbb{R}^{1 \times n_p}$ ,  $A_c \in \mathbb{R}^{n_c \times n_c}$ ,  $B_c \in \mathbb{R}^{n_c \times 1}$  and  $C_c \in \mathbb{R}^{1 \times n_c}$ .

The closed-loop reset control system can be then described by the following FDI:

$$\begin{aligned} D^\alpha x(t) &= A_{cl} x(t) + B_{cl} r, \quad x(t) \notin \mathcal{M} \\ x(t^+) &= A_R x(t), \quad x(t) \in \mathcal{M} \\ y(t) &= C_{cl} x(t) \end{aligned} \quad (16)$$

where  $x = \begin{bmatrix} x_p \\ x_c \\ x_r \end{bmatrix}$ ,  $A_{cl} = \begin{bmatrix} A_p & B_p C_c & 0 \\ -B_c D_r C_p & A_c & B_c C_r \\ -B_r C_p & 0 & A_r \end{bmatrix}$ ,

$A_R = \begin{bmatrix} I_{n_p} & 0 & 0 \\ 0 & I_{n_c} & 0 \\ 0 & 0 & A_{R_r} \end{bmatrix}$ ,  $B_{cl} = [0 \quad B_c D_r \quad B_r]^T$  and  $C_{cl} = [C_p \quad 0 \quad 0]$ . The reset surface  $\mathcal{M}$  is defined by:

$$\mathcal{M} = \{x \in \mathbb{R}^n : C_{cl} x = r, (I - A_R)x \neq 0\}. \quad (17)$$

where  $n = n_r + n_c + n_p$ . In absence of the linear controller  $C(s)$ , the state space realization of the closed-loop system can be also stated as (16) with  $x = \begin{bmatrix} x_p \\ x_r \end{bmatrix}$ ,

$A_{cl} = \begin{bmatrix} A_p - B_p D_r C_p & B_p C_r \\ -B_r C_p & A_r \end{bmatrix}$ ,  $A_R = \begin{bmatrix} I_{n_p} & 0 \\ 0 & A_{R_r} \end{bmatrix}$ ,  $B_{cl} = [B_p D_r \quad B_r]^T$ ,  $C_{cl} = [C_p \quad 0]$ .

*Example 2:* Modeling of a servomotor controlled by a fractional-order proportional-Clegg integrator (FPCI)

Consider the control scheme shown in see Fig. 3, where the servomotor is given by

$$G_s(s) = \frac{K}{Ts+1} = \frac{0.93}{0.61s+1}, \quad (18)$$

and the FPCI by

$$R(s) = K_p + K_i \text{CI}^\alpha(s) = 0.067 + 13.4 \text{CI}^{0.75}(s), \quad (19)$$

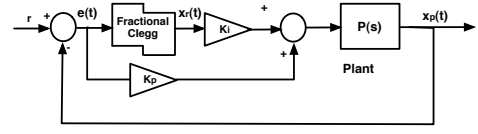


Fig. 3

BLOCK DIAGRAM OF A SYSTEM CONTROLLED BY A FPCI

being  $\text{CI}^\alpha$  a fractional Clegg integrator (FCI) (refer to part II [9] for design details). Denote the state vector as  $x(t) = (x_p(t), x_r(t))^T$ , being  $x_p(t)$  and  $x_r(t)$  the plant and the controller states, respectively. Thus, the controlled system can be expressed of the form of (16) as follows:

$$\begin{aligned} \begin{bmatrix} \dot{x}_p(t) \\ D^\alpha x_r(t) \end{bmatrix} &= A_{cl} x(t) = \begin{bmatrix} -\frac{1+KK_p}{\tau} & \frac{KK_i}{\tau} \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} \frac{KK_p}{\tau} \\ 1 \end{bmatrix} r = \\ &= \begin{bmatrix} -1.7415 & 20.4295 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.1021 \\ 1 \end{bmatrix} r, \end{aligned} \quad (20)$$

$$x(t^+) = A_R x(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(t), \quad y(t) = C_{cl} x(t) = [1 \quad 0] x(t). \quad (21)$$

Taking into account that  $\alpha = 0.75 = \frac{3}{4}$ , let consider  $\mathcal{X}_{p_i}(t) = D^{\frac{i-1}{4}} x_p(t)$ ,  $i = 1, \dots, 4$  and  $\mathcal{X}_{r_i}(t) = D^{\frac{i-1}{4}} x_r(t)$ ,  $i = 1, \dots, 3$  and define the state vector of the augmented system as  $\mathcal{X}(t) = (\mathcal{X}_{p_1}(t), \dots, \mathcal{X}_{p_4}(t), \mathcal{X}_{r_1}(t), \mathcal{X}_{r_2}(t), \mathcal{X}_{r_3}(t))$ , the augmented system can be represented as:

$$D^{\frac{1}{4}} \mathcal{X}(t) = A \mathcal{X}(t) + Br =$$

$$\begin{bmatrix} O_{3,1} & I_{3,3} & O_{3,1} & O_{3,2} \\ -1.7415 & O_{1,3} & 20.4295 & O_{1,2} \\ O_{2,1} & O_{2,3} & O_{2,1} & I_{2,2} \\ -1 & O_{1,3} & 0 & O_{1,2} \end{bmatrix} \mathcal{X}(t) + \begin{bmatrix} O_{3,1} \\ 1.5246 \\ O_{2,1} \\ 1 \end{bmatrix} r, \quad (22)$$

$$\mathcal{X}(t^+) = \begin{bmatrix} I_{6,6} & O_{6,1} \\ O_{1,6} & 0 \end{bmatrix} \mathcal{X}(t), \quad y(t) = [1 \quad O_{(1,6)}] \mathcal{X}(t), \quad (23)$$

where  $O_{l,m}$  denotes a matrix of zeros with dimension of  $l \times m$ .

### III. STABILITY OF FRACTIONAL-ORDER HYBRID SYSTEMS

Although stability of hybrid systems is typically analysed by Lyapunov's theory (see e.g. [10], [11], [12], [13]), recently a frequency domain method equivalent to the common Lyapunov was proposed in [14] to analyse the stability of a particular class of such systems. This section provides the stability conditions for two kinds of fractional-order hybrid systems, namely, switching and reset control systems, based on Lyapunov's theory and its frequency domain equivalence. Two examples of application are also given.

### A. Fractional-order switching systems

The developed theory for fractional-order switching systems can be found in [15], [16], [17]. Firstly, let us to recall the stability of fractional-order switching systems by common Lyapunov functions and its equivalence in frequency domain as preliminaries.

*Theorem 1:* ([15], [16]) A fractional system described by (3) with order  $\alpha$ ,  $1 \leq \alpha < 2$ , is stable if and only if there exists a matrix  $P = P^T > 0$ ,  $P \in \mathbb{R}^{n \times n}$ , such that

$$\begin{bmatrix} (A_i^T P + PA_i) \sin \phi & (A_i^T P - PA_i) \cos \phi \\ (-A_i^T P + PA_i) \cos \phi & (A_i^T P + PA_i) \sin \phi \end{bmatrix} < 0, \forall i = 1, \dots, L, \quad (24)$$

where  $\phi = \frac{\alpha\pi}{2}$ .

*Theorem 2:* ([15], [16]) A fractional system given by (3) with order  $\alpha$ ,  $0 < \alpha \leq 1$ , is stable if and only if there exists a matrix  $P = P^T > 0$ ,  $P \in \mathbb{R}^{n \times n}$ , such that

$$\mathcal{A}_i^T P + P \mathcal{A}_i < 0, \quad \forall i = 1, \dots, L. \quad (25)$$

Next, frequency domain stability conditions will be given for fractional-order switching systems based on results in [14]. Consider a stable pseudo-polynomial of order  $n\alpha$  of system (3) as

$$d(s) = s^{n\alpha} + d_{n-1}s^{(n-1)\alpha} + \dots + d_1s^\alpha + d_0, \quad (26)$$

and a polynomial of order  $n$  of system  $\dot{\tilde{x}} = \tilde{A}\tilde{x}$  as

$$c(s) = s^n + c_{n-1}s^{(n-1)} + \dots + c_1s + c_0. \quad (27)$$

In the following, the necessary and sufficient condition for the stability for fractional-order switching systems is given.

*Theorem 3:* ([15]) Consider  $d_1(s)$  and  $d_2(s)$ , two stable pseudo-polynomials of order  $n$  corresponding to the subsystems  $D^\alpha x = A_1 x$  and  $D^\alpha x = A_2 x$  with order  $\alpha$ ,  $1 \leq \alpha < 2$ , respectively, then the following statements are equivalent:

- 1)  $|\arg(\det((A_1^2 - \omega^2 I) - 2j\omega A_1 \sin \phi)) - \arg(\det((A_2^2 - \omega^2 I) - 2j\omega A_2 \sin \phi))| < \frac{\pi}{2}, \forall \omega$ , being  $I$  the identity matrix with proper dimensions.
- 2)  $A_1$  and  $A_2$  are stable, which means that  $\exists P = P^T > 0 \in \mathbb{R}^{n \times n}$  such that

$$\begin{bmatrix} (A_i^T P + PA_i) \sin \phi & (A_i^T P - PA_i) \cos \phi \\ (-A_i^T P + PA_i) \cos \phi & (A_i^T P + PA_i) \sin \phi \end{bmatrix} < 0, \forall i = 1, 2.$$

*Theorem 4:* ([15]) Consider two stable fractional-order subsystems  $D^\alpha x = A_1 x$  and  $D^\alpha x = A_2 x$  with order  $\alpha$ ,  $0 < \alpha \leq 1$ , then the following statements are equivalent:

- 1)  $|\arg(\det(\mathcal{A}_1 - j\omega I)) - \arg(\det(\mathcal{A}_2 - j\omega I))| < \frac{\pi}{2}, \forall \omega$ .
- 2)  $A_1$  and  $A_2$  are stable, which means that  $\exists P = P^T > 0 \in \mathbb{R}^{n \times n}$  such that

$$\mathcal{A}_i^T P + P \mathcal{A}_i < 0, \forall i = 1, 2.$$

Although the theory developed in the frequency domain does not necessarily prove the strictly positive realness, a

relation equivalent to the stability was obtained. See [18] for the switching systems more than two subsystems.

*Example 3:* Stability of a fractional-order switching system with two subsystems

Consider the switching system (3) with  $L = 2$  with the following parameters:  $A_1 = \begin{bmatrix} -0.1 & 0.1 \\ -2.0 & -0.1 \end{bmatrix}$ ,  $A_2 = \begin{bmatrix} -0.01 & 2.0 \\ -0.1 & -0.01 \end{bmatrix}$  and order  $\alpha$ ,  $0 < \alpha \leq 1$ . Applying Theorem 4, the phase difference condition should be satisfied for all  $\alpha$ ,  $0 < \alpha \leq 1$ , to guarantee the stability –this condition is depicted in Fig. 4 for  $0 < \alpha \leq 1$  with increments of 0.1. As can be seen, the fractional-order system is stable for  $\alpha \in (0, 0.6]$ . The phase differences when  $\alpha \in [0.7, 1]$  are bigger than  $\pi/2$  which indicates unknown stability status, i.e., the system may be stable or unstable. For more details see [15].

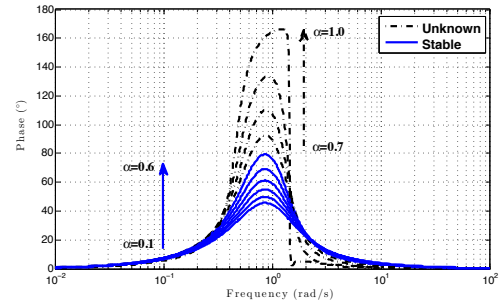


Fig. 4

PHASE DIFFERENCES OF CHARACTERISTIC POLYNOMIALS OF SYSTEM IN EXAMPLE 3 FOR DIFFERENT VALUES OF ITS ORDER  $\alpha$ ,  $0 < \alpha \leq 1$

*Example 4:* Stability analysis of the SmartWheel controlled by fractional-order gain scheduled controller

In the literature, it is widely noticed that systems controlled through networks exhibit high switching behaviours and thus their design and analysis within the switching system framework are highly desirable (refer to e.g. [19], [20], [21]). The case of study to be considered is the Internet-based control of a platform, called SmartWheel, placed at the Center for Self-Organizing and Intelligent Systems (CSOIS), Utah State University, USA, from the University of Extremadura, Spain. Thus, the existence of network time-varying delays together with the application of gain scheduling result in the transformation of the closed-loop system into a switching system with finite number of subsystems as follows (the full description can be found in [22], [23]):

$$G_j(s) = \frac{0.1484}{0.045s + 1} e^{-(0.592 + \tau_j)s}, \quad (28)$$

$$C_j(s) = \beta_j \left( 2.1586 + \frac{5.9853}{s^{1.1}} \right), \quad j = 1, 2, \dots, 13, \quad (29)$$

where  $\tau_j$  refers to the network delay  $\tau_{network}$  and  $\beta_j$  is the gain scheduler with the switching parameters given in Table I. Hence, there are 13 subsystems to be considered.

In order to apply Theorem 4 the controlled system has to be described in the form of commensurate-order system. Therefore, assuming Padé approximation of delay is

$$\text{Pade}(e^{-(0.592+\tau_j)s}) = \frac{P_n(s)_j}{P_d(s)_j},$$

the closed-loop pseudo characteristic polynomials can be represented as follows:

$$d_j(s) = P_d(s) (s^{2.1} + 22.22s^{1.1}) + \beta_j P_n(s) (7.12s + 19.74). \quad (30)$$

Defining  $\lambda = s^{0.1}$ , the characteristic polynomials of the system can be obtained as

$$c_j(\lambda^{10}) = P_d(\lambda^{10}) (\lambda^{21} + 22.22\lambda^{11}) + \beta_j P_n(\lambda^{10}) (7.12\lambda^{10} + 19.74). \quad (31)$$

Suppose  $c_j(\lambda) = \mathbf{c}_j [\lambda^{21+m} \ \lambda^{m+20} \ \dots \ 1]$ , where  $\mathbf{c}_j = [1 \ \mathbf{c}_{m+20}^j \ \dots \ \mathbf{c}_0^j]$  is a vector with  $m+22$  elements and  $m$  is order of Padé approximation. Hence, the commensurate fractional-order system can be realised as

$$D^{0.1}x = A_j x = \begin{bmatrix} -\mathbf{c}_{m+20}^j & \dots & -\mathbf{c}_1^j & -\mathbf{c}_0^j \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix} x, x \in \mathbb{R}^{m+21}. \quad (32)$$

Now, we can easily apply Theorem 4 to analyse the stability of the system. The following 12 conditions should be satisfied to guarantee the stability of the controlled system:

$$\left| \arg(\det((A_1^2 - \omega^2 I) - 2j\omega A_1 \sin \phi)) - \arg(\det((A_2^2 - \omega^2 I) - 2j\omega A_2 \sin \phi)) \right| < \frac{\pi}{2}, \forall \omega, \quad (33)$$

$$\left| \arg(\det((A_2^2 - \omega^2 I) - 2j\omega A_2 \sin \phi)) - \arg(\det((A_3^2 - \omega^2 I) - 2j\omega A_3 \sin \phi)) \right| < \frac{\pi}{2}, \forall \omega, \quad (34)$$

⋮

$$\left| \arg(\det((A_{j-1}^2 - \omega^2 I) - 2j\omega A_{j-1} \sin \phi)) - \arg(\det((A_j^2 - \omega^2 I) - 2j\omega A_j \sin \phi)) \right| < \frac{\pi}{2}, \forall \omega, \quad (35)$$

where  $\phi = \frac{\alpha\pi}{2}$ . The simulation of conditions (33)–(35) is shown in Fig. 5. It can be observed that the maximum phase difference is less than  $90^\circ$  and, consequently, the system is stable.

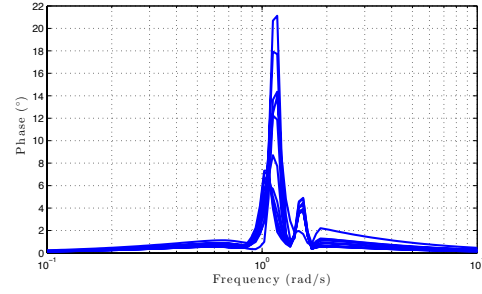


Fig. 5

PHASE DIFFERENCES OF CHARACTERISTIC POLYNOMIALS OF SYSTEM IN EXAMPLE 4 GIVEN BY CONDITIONS (33)–(35)

## B. Fractional-order reset control systems

In this section, stability of fractional-order reset control systems is analysed using the Lyapunov-like method presented previously. This theory was proposed in [24].

*Definition 1:* Reset control system (16) is said to satisfy the  $H_\beta$ -condition if there exists a  $\beta \in \mathbb{R}^{n_{\mathcal{R}}}$  and a positive-definite matrix  $P_{\mathcal{R}} \in \mathbb{R}^{n_{\mathcal{R}} \times n_{\mathcal{R}}}$  such that

$$H_\beta(s) = [\beta C_p \ 0_{n_{\mathcal{R}}} \ P_{\mathcal{R}}] (sI - \mathcal{A})^{-1} \begin{bmatrix} 0 \\ 0_{\mathcal{R}}^T \\ I_{\mathcal{R}} \end{bmatrix}, \quad (36)$$

where  $\mathcal{A} = \left( -(-A_{cl})^{\frac{1}{2-\alpha}} \right)$ .

In accordance with [25], it is obvious that the  $H_\beta(s)$  is strictly positive real (SPR) if

$$\left| \arg(H_\beta(j\omega)) \right| < \frac{\pi}{2}, \forall \omega. \quad (37)$$

*Theorem 5:* ([24]) The closed-loop fractional-order reset control system (16) is asymptotically stable if and only if it satisfies the  $H_\beta$ -condition (36) or its phase equivalence (37).

An example of application is given next.

*Example 5:* Stability of a fractional-order reset control system

Let us consider the same feedback system as in [26] with the following system, base controller and reset controller transfer functions:  $P(s) = \frac{1}{s^2+0.2s}$ ,  $C(s) = s+1$  and  $R(s) = \frac{1}{s^{\alpha+b}}$ , respectively. The system stability will be analysed for different reset controllers: the first-order reset element (FORE) controller, with  $b \neq 0$  and  $\alpha = 1$ , the CI, with  $b = 0$  and  $\alpha = 1$ , and the FCI, with  $b = 0$  and  $\alpha = 0.5$ . For FORE controller, the integer-order closed-loop system can be given

TABLE I  
SYSTEM AND CONTROLLER PARAMETERS IN EXAMPLE 4 FOR EACH SWITCHING

$j$	1	2	3	4	5	6	7	8	9	10	11	12	13
$\tau$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
$\beta$	1.6	1.35	1.3	1.15	1	0.9	0.8	0.7	0.65	0.6	0.55	0.5	0.45

by:

$$\begin{cases} \dot{x}(t) = A_{cl}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.2 & 1 \\ -1 & -1 & -b \end{bmatrix} x(t) \\ x(t^+) = A_R x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) \\ y = C_{cl}x = [1 \ 1 \ 0] x(t) \end{cases}$$

where  $x(t) = [x_{p1}(t), x_{p2}(t), x_r(t)]^T$ . And, the closed-loop system using FCI can be stated as

$$\begin{cases} D^{0.5} \mathcal{X}(t) = \mathbf{A}_{cl} \mathcal{X}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -0.2 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 \end{bmatrix} \mathcal{X}(t) \\ \mathcal{X}(t^+) = \mathbf{A}_R \mathcal{X}(t) = \begin{bmatrix} I_4 & 0_{4,1} \\ 0_{1,4} & 0 \end{bmatrix} \mathcal{X}(t) \\ y = \mathbf{C}_{cl} \mathcal{X}(t) = [1 \ 0 \ 1 \ 0 \ 0] \mathcal{X}(t) \end{cases}$$

where  $\mathcal{X}(t) = [\mathcal{X}_{p1}(t), \dots, \mathcal{X}_{p4}(t), x_r(t)]^T$ ,  $\mathcal{X}_{p1}(t) = x_{p1}(t)$ ,  $\mathcal{X}_{p3}(t) = x_{p2}(t)$ . According to (36),  $H_\beta$ -conditions corresponding to FORE and FCI controllers are, respectively, given by (for both cases,  $n_{\mathcal{X}} = 1$  and, then,  $P_{\mathcal{X}} = 1$ ):

$$H_\beta^{FORE}(s) = [\beta \ 0 \ 1] (sI - A_{cl})^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{s^2 + 0.2s + 0.8\beta}{s^3 + (b + 0.2)s^2 + (1 + 0.2b)s + 1}, \quad (38)$$

$$H_\beta^{FCI}(s) = [\beta \ 0 \ \beta \ 0 \ 1] \left( sI - \left( -(-\mathbf{A}_{cl})^{\frac{2}{5}} \right) \right)^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \quad (39)$$

Using Theorem 5, the closed-loop systems controlled by FORE and FCI are asymptotically stable if  $H_\beta^{FORE}(s)$  and  $H_\beta^{FCI}(s)$  are SPR. Substituting  $b = 1$  in (38), the FORE reset system is asymptotically stable for all  $0.42 < \beta \leq 1.46$ . With respect to CI (similarly to FORE but with  $b = 0$ ), stability cannot be guaranteed with this theorem. And applying FCI, it can be easily stated that the system is asymptotically stable for  $\beta \leq 0.62$ . In addition, the phase equivalences corresponding to (38) and (39) are shown in Fig. 6 for  $\beta = 0.5$  and  $b = 1$ . It can be seen that both phases verifies condition (37), which has concordance with the theoretical results.

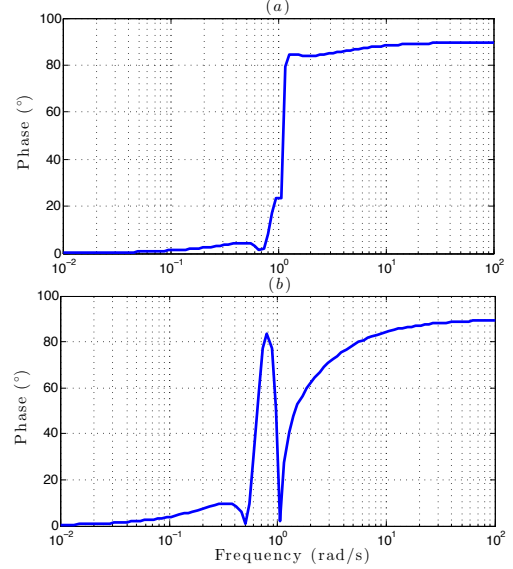


Fig. 6  
PHASE EQUIVALENCE OF  $H_\beta$  IN EXAMPLE 5: (a) APPLYING FCI (b) APPLYING FORE

#### IV. CONCLUSIONS

In part I of this paper, modeling of fractional-order hybrid systems (FHS) was introduced based on fractional-order differential inclusions, especially for two special kinds of them, i.e., switching and reset control systems. Moreover, stability of such FHS was also analysed based on Lyapunov's theory and its frequency domain equivalence. Some examples were given to show the way of modeling and the applicability of the developed stability theory.

Since there is no a general agreement of the interpretation of state space representation of fractional-order systems, mainly concerning initial values (see e.g. [27]), a further study should be carried out for fractional-order reset control taking into account this issue in future work.

#### V. REFERENCES

- [1] A. Gollu and P. Varaiya, "Hybrid dynamical systems," in *Proceedings of the 28th IEEE Conference on Decision and Control*. IEEE, 1989, pp. 2708–2712.
- [2] A. J. van der Schaft and J. M. Schumacher, *Introduction to hybrid dynamical systems*. Springer-Verlag, 1999.

- [3] R. Goebel, R. Sanfelice, and A. Teel, "Hybrid dynamical systems," *Control Systems Magazine, IEEE*, vol. 29, no. 2, pp. 28–93, 2009.
- [4] A. Baños and A. Barreiro, *Reset Control Systems*. Springer Verlag, 2011.
- [5] B. D. Schutter, W. Heemels, J. Lunze, and C. Prieur, *Handbook of Hybrid Systems Control—Theory, Tools, Applications*. Cambridge University Press, 2009, pp. 31–35.
- [6] C. A. Monje, Y. Q. Chen, B. M. Vinagre, D. Xue, and V. Feliu, *Fractional-order Systems and Controls. Fundamentals and Applications*. Springer, 2010.
- [7] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications*. Academic Press, San Diego - New York - London, 1999.
- [8] Z. Sun and S. S. Ge, *Switched Linear Systems: Control and Design*. Springer-Verlag, 2005.
- [9] S. H. HosseinNia, I. Tejado, and B. M. Vinagre, "Hybrid systems and control with fractional dynamics (II): Control," in *Proceedings of the 2014 International Conference in Fractional Differentiation and its Applications (ICFDA'14)*, 2014.
- [10] D. Liberzon, *Switching in Systems and Control*. Birkäuser, 2003.
- [11] K. S. Narendra and J. Balakrishnan, "A common Lyapunov function for stable LTI system with commuting A-matrices," *IEEE Transactions on Automatic Control*, vol. 39, no. 12, pp. 2469–2471, 1994.
- [12] Y. Mori, T. Mori, and Y. Kuroe, "On a class of linear constant systems which have a common quadratic lyapunov function," in *Proceedings of the 37th IEEE Conference on Decision and Control*, 1998.
- [13] H. Shim, D. Noh, , and J. Seo, "Common lyapunov function for exponentially stable nonlinear systems," in *Proceedings of the 4th SIAM Conference on Control and its Applications*, 1998.
- [14] M. Kunze, A. Karimi, and R. Longchamp, "Frequency domain controller design by linear programming guaranteeing quadratic stability," in *Proceedings of the 47th Conference on Decision and Control (CDC'08)*, 2008, pp. 345–350.
- [15] S. H. HosseinNia, I. Tejado, and B. M. Vinagre, "Stability of fractional order switching systems," *Computer & Mathematics with Applications*, vol. 66, no. 5, pp. 585–596, 2013.
- [16] —, "Stability of fractional order switching systems," in *Proceedings of the 5th Workshop on Fractional Differentiation and Its Applications (FDA'12)*, 2012.
- [17] —, "Basic properties and stability of fractional order reset control systems," in *Proceedings of the 12th European Control Conference (ECC'13)*, 2013.
- [18] S. H. HosseinNia, "Fractional hybrid control systems: Modeling, analysis and applications to mobile robotics and mechatronics," Ph.D. dissertation, University of Extremadura, 2013.
- [19] F.-Y. Wang and D. Liu, Eds., *Networked Control Systems: Theory and Applications*. Springer-Verlag, 2008.
- [20] M. S. Branicky, "Introduction to hybrid systems," in *Handbook of Networked and Embedded Control Systems*, D. Hristu-Varsakelis and W. Levine, Eds. Birkhäuser Boston, 2005, pp. 91–116.
- [21] R. Alur, K.-E. Arzen, J. Baillieul, T. Henzinger, D. Hristu-Varsakelis, and W. S. Levine, *Handbook of networked and embedded control systems*. Birkhäuser Boston, 2005.
- [22] I. Tejado, S. H. HosseinNia, B. M. Vinagre, and Y. Q. Chen, "Efficient control of a SmartWheel via internet with compensation of variable delays," *Mechatronics*, vol. 23, pp. 821–827, 2013.
- [23] I. Tejado, "Some contributions in networked control systems based on fractional calculus," Ph.D. dissertation, University of Extremadura, Spain, 2011.
- [24] S. H. HosseinNia, I. Tejado, and B. M. Vinagre, "Fractional-order reset control: Application to a servomotor," *Mechatronics*, vol. 23, no. 7, pp. 781–788, 2013.
- [25] P. Ioannou and G. Tao, "Frequency domain conditions for strictly positive real functions," *IEEE Transactions on Automatic Control*, vol. 32, no. 1, pp. 53–54, 1987.
- [26] C. Hollot, O. Beker, Y. Chait, and Q. Chen, "On stabilizing classic performance measures for reset control systems," *Perspectives in robust control*, pp. 123–147, 2001.
- [27] J. Sabatier, C. Farges, and J.-C. Trigeassou, "Fractional systems state space description: Some wrong ideas and proposed solutions," *Journal of Vibration and Control*, 2013.