NEW PARTIALLY HYPERBOLIC DYNAMICAL SYSTEMS I

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ABSTRACT. We propose a new method for constructing partially hyperbolic diffeomorphisms on closed manifolds. As a demonstration of the method we show that there are simply connected closed manifolds that support partially hyperbolic diffeomorphisms. Laying aside many surgery constructions of Anosov flows (mostly in dimension three), these are the first new examples of manifolds which admit partially hyperbolic diffeomorphisms in the past 40 years.

1. INTRODUCTION

Let M be a smooth compact d-dimensional manifold. A diffeomorphism F is called *Anosov* if there exist a constant $\lambda > 1$ and a Riemannian metric along with a DF-invariant splitting $TM = E^s \oplus E^u$ of the tangent bundle of M, such that for any unit vectors, v^s and v^u in E^s and E^u, respectively, we have

$$\|DF(v^s)\| \leq \lambda^{-1}$$

$$\lambda \leq \|DF(v^u)\|$$

All known examples of Anosov diffeomorphisms are supported on manifolds which are homeomorphic to infranilmanifolds. The classification problem for Anosov diffeomorphisms is an outstanding open problem that goes back to Anosov and Smale. The great success of the theory of Anosov diffeomorphisms (and flows) [A67] motivated Hirsch-Pugh-Shub [HPS70, HPS77] and Brin-Pesin [BP74] to relax the definition as follows.

A diffeomorphism F is called *partially hyperbolic* if there exist a constant $\lambda > 1$ and a Riemannian metric along with a DF-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ of the tangent bundle of M, such that for any unit vectors, v^s, v^c, v^u in E^s, E^c, E^u, respectively, we have

$$\|DF(v^s)\| \leq \lambda^{-1}$$

$$\|DF(v^s)\| < \|DF(v^c)\| < \|DF(v^u)\|$$

$$\lambda \leq \|DF(v^u)\|$$

In recent years the dynamics of partially hyperbolic diffeomorphisms has been a popular subject, see, *e.g.,* [PS04, RHRHU06]. The pool of examples of partially

hyperbolic diffeomorphisms is larger than that of Anosov diffeomorphisms, in particular, due to the fact that extensions (*e.g.,* $F \times id_N$) of partially hyperbolic diffeomorphisms are partially hyperbolic. However, the collection of basic "building blocks" for partially hyperbolic diffeomorphisms is still rather limited. Up to homotopy, all previously known examples of irreducible[1] partially hyperbolic diffeomorphisms are either affine diffeomorphisms on homogeneous spaces or time-1 maps of Anosov flows. The affine examples go back to Brin-Pesin [BP74] and Sacksteder [S70].

Theorem 1.1 (Main Theorem)**.** *For any $d \geq 6$ there exist a closed d-dimensional simply connected manifold M that supports a volume preserving partially hyperbolic diffeomorphism $F \colon M \to M$. Moreover, F is ergodic with respect to volume.*

Remark 1.2. There are no previously known examples of partially hyperbolic diffeomorphisms on simply connected manifolds. It is easy to show that simply connected compact Lie groups do not admit partially hyperbolic automorphisms (use, *e.g.,* [HS13, Theorems 6.61, 6.63]). However, to the best of our knowledge, the possibility that some simply connected manifolds support Anosov flows is open.

Burago and Ivanov proved that simply connected 3-manifolds (*i.e.,* the sphere $\mathbb{S}^3$) do not support partially hyperbolic diffeomorphisms [BI08]. Simply connected 4-manifolds have non-zero Euler characteristic and hence do not admit line fields. Consequently simply connected 4-manifolds do not support partially hyperbolic diffeomorphisms.

Question 1.3. *Do simply connected 5-manifolds support partially hyperbolic diffeomorphisms?*

Remark 1.4. It is easy to see, for topological reasons, that the 5-sphere $\mathbb{S}^5$ does not admit partially hyperbolic diffeomorphisms. Indeed, the splitting $T\mathbb{S}^5 = E^s \oplus E^c \oplus E^u$ is either a 3-1-1 or a 2-2-1 splitting. By an old result of Eckmann [E42] (see also Whitehead [W42]) the sphere $\mathbb{S}^5$ does not admit two linearly independent vector fields and, therefore, the splitting must be a 2-2-1 splitting. Because the structure group $GL^+(2,\mathbb{R})$ retracts to $O(2,\mathbb{R})$, the 2-plane bundles over $\mathbb{S}^5$ are in one-to-one correspondence with circle bundles, which are classified by homotopy classes of maps $\mathbb{S}^4 \to \mathrm{Diff}(S^1)$ by the clutching construction. But $\mathrm{Diff}(S^1)$ is homotopy equivalent to S^1 and, hence, all 2-plane bundles over $\mathbb{S}^5$ are trivial. Therefore the existence of a 2-2-1 splitting of $T\mathbb{S}^5$ implies that $\mathbb{S}^5$ is parallelizable, which is a contradiction. It is an interesting open problem to decide whether $\mathbb{S}^7$ supports partially hyperbolic diffeomorphisms.

In the next section we briefly (and very informally) outline our approach. Then we proceed with a detailed discussion leading to the proof of the Main Theorem in Section 11. The authors would like to thank the referee for his/her careful reading.

2. Informal description of the construction

Our approach is to consider a smooth fiber bundle $M \to E \xrightarrow{p} X$, whose base X is a closed manifold and whose fiber M is a closed manifold which admits a partially hyperbolic diffeomorphism. The idea now is to equip the total space E

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[1]See Section 12.5 for our definition of "irreducible"

with a fiberwise partially hyperbolic diffeomorphism $F\colon E \to E$, which fibers over a diffeomorphism $f\colon X \to X$, *i.e.*, the following diagram commutes

$$
\begin{array}{ccc}
E & \xrightarrow{\;F\;} & E \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\
X & \xrightarrow{\;f\;} & X
\end{array}
$$

Then the diffeomorphism F is partially hyperbolic provided that f is dominated by the action (on extremal subbundles) of F along the fibers. However, for non-trivial fiber bundles the bundle map $p\colon E \to X$ intertwines the dynamics in the fiber with dynamics in the base, which makes it difficult to satisfy

1. F is fiberwise partially hyperbolic;
2. f is dominated by F;

at the same time. In particular, if X is simply connected and f is homotopic to id_X such constructions seem to be out of reach (cf. [FG14, Question 6.5]). Moreover, assuming that $f = id_X$, it was shown in [FG14] that such construction is, in fact, impossible in certain more restrictive setups. However, in this paper, we show that if $f^*\colon H^*(X) \to H^*(X)$ is allowed to be non-trivial then our method works in the setup of principal torus bundles over simply connected 4-manifolds.

3. Preliminaries on principal bundles

In this section we review some of the concepts and facts about principal fiber bundles that will be needed later. For more details consult [Hus94].

Standing assumption. *In this and further sections we will always assume that all topological spaces are connected countable CW complexes. Given a space X we will write $H_*(X;A)$ and $H^*(X;A)$ for its homology and cohomology groups with coefficients in an abelian group A. If we abbreviate to $H_*(X)$ and $H^*(X)$ then we assume that the coefficient group is $\mathbb{Z}$.*

Let X be a space and G be a topological group. Recall that a (locally trivial) principal G-bundle $\pi\colon E \to X$ is a (locally trivial) fiber bundle with fiber G and structure group G, where (the group) G acts on (the fiber) G by left multiplication. Let $\phi_\alpha\colon U_\alpha \times G \to \pi^{-1}U_\alpha$, $\alpha \in \mathcal{A}$, be a complete collection of trivializing charts of the principal bundle E. Denote by $U_{\alpha\beta}$ the intersection $U_\alpha \cap U_\beta$, $\alpha, \beta \in \mathcal{A}$. Also define $\phi_{\alpha\beta}\colon U_{\alpha\beta} \to G$ in the following way

$$
(\phi_\beta^{-1} \circ \phi_\alpha)(x, g) = (x, \phi_{\alpha\beta}(x) \cdot g), \quad x \in U_{\alpha\beta} \neq \varnothing,\ g \in G.
$$

This collection of *transition functions* $\{\phi_{\alpha\beta}\}_{U_{\alpha\beta} \neq \varnothing}$ satisfies the following *cocycle condition*

$$
\phi_{\alpha\beta}(x) \cdot \phi_{\beta\gamma}(x) \cdot \phi_{\gamma\alpha}(x) = e, \quad x \in U_\alpha \cap U_\beta \cap U_\gamma,
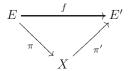$$

where e is the identity in G.

Conversely, let $\{\phi_{\alpha\beta}\}_{U_{\alpha\beta} \neq \varnothing}$ be a *cocycle of transition functions over a covering* $\{U_\alpha\}$; that is, assume that we have

1. an open covering $\{U_\alpha\}$ of the space X;
2. a collection of maps $\phi_{\alpha\beta}\colon U_{\alpha\beta} \to G$, $U_{\alpha\beta} \neq \varnothing$, that satisfy the cocycle condition.

Then we can construct a principal G-bundle $E \to X$ by gluing the spaces $U_\alpha \times G$ using the transition functions $\{\phi_{\alpha\beta}\}$. The cocycle condition ensures that the gluings are consistent.

We will need the following facts:

(3.1) Every principal G-bundle $E \to X$ has a (right) action $E \times G \to E$. This action is free and the orbits are exactly the fibers. This can be seen from the construction of E using a cocycle of transition functions: define $\phi_\alpha(x, g).h = \phi_\alpha(x, g.h)$. This is well defined because right and left translations on G commute. (There are other equivalent ways of defining principal bundles. In some of them the action is included in the definition.)

(3.2) Two G-bundles $\pi \colon E \to X$ and $\pi' \colon E' \to X$ are called *equivalent* and we write $E \cong E'$ if there exists a homeomorphism $f \colon E \to E'$ which fits into the commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\ \ f\ \ } & E' \\
{\scriptstyle \pi} \searrow & & \nearrow {\scriptstyle \pi'} \\
& X &
\end{array}
$$

and commutes with the G-action, *i.e.,* $f(y.g) = f(y).g$.

(3.3) From (3.1) we get a canonical (up to right translation) way of identifying a fiber of a principal G-bundle with G.

(3.4) For every G there is a principal G-bundle $EG \to BG$ such that for any space X and any principal G-bundle $E \to X$, there is a unique, up to homotopy, map $\rho \colon X \to BG$, such that $E \cong \rho^* EG$. We say that map ρ *classifies* the bundle $E \to X$. The space BG is called *the classifying space* of G, and the G-bundle $EG \to BG$ is called *the universal principal G-bundle.*

(3.5) The classifying space of the topological group $\mathbb{S}^1$ is $\mathbb{C}P^\infty = \cup_{n \geq 0} \mathbb{C}P^n$. The universal principal $\mathbb{S}^1$-bundle is $E\mathbb{S}^1 = \mathbb{S}^\infty \to \mathbb{C}P^\infty$. Here $\mathbb{S}^\infty = \cup_{n \geq 0} \mathbb{S}^n$. This bundle is the limit of $\mathbb{S}^1 \to \mathbb{S}^{2n-1} \to \mathbb{C}P^n$, where $\mathbb{S}^1 \subset \mathbb{C}$ acts on $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ by scalar multiplication.

(3.6) We have $B(G \times H) = BG \times BH$, provided that both BG and BH are countable CW complexes. Moreover, $E(G \times H) = EG \times EH$ and the action and projections respect the product structure. It follows that $B\mathbb{T}^k = \underbrace{\mathbb{C}P^\infty \times ... \mathbb{C}P^\infty}_{k}$,
where $\mathbb{T}^k = \underbrace{\mathbb{S}^1 \times ... \times \mathbb{S}^1}_{k}$ is the k-torus, and $E\mathbb{T}^k = \underbrace{\mathbb{S}^\infty \times ... \times \mathbb{S}^\infty}_{k}$.

4. $A(E)$ CONSTRUCTION

Let $\pi_1 \colon E_1 \to X_1$ and $\pi_2 \colon E_2 \to X_2$ be principal G-bundles. A fiber preserving map $F \colon E_1 \to E_2$, covering $f \colon X_1 \to X_2$ (i.e., $f \circ \pi_1 = \pi_2 \circ F$) is a *principal G-bundle map* if F commutes with the right action of G, that is, $F(y.g) = F(y).g$, $y \in E_1$, $g \in G$. Hence F restricted to a fiber is a left translation.

More generally, let $A \colon G \to G$ be an automorphism of the topological group G and let E_1, E_2 be as above. We say that a map $F \colon E_1 \to E_2$, covering $f \colon X_1 \to X_2$ is an *A-bundle map* (or simply an *A-map*) if $F(y.g) = F(y).A(g)$ for all $y \in E_1$, $g \in G$. Hence F restricted to a fiber is the automorphism A composed with a left translation.

Remark 4.1. Of course, an id_G-map is just a principal G-bundle map. In particular, an id_G-map that covers the identity $id_X \colon X \to X$ is a principal G-bundle equivalence.

Remark 4.2. Note that the composition of an A-map and a B-map is a BA-map.

Now let $\pi \colon E \to X$ be a principal G-bundle, and let $\{\phi_{\alpha\beta}\}$ be a cocycle of transition functions for E. Note that $\{A \circ \phi_{\alpha\beta}\}$ is also a cocycle of transition functions. This is because

$$A(\phi_{\alpha\beta}(x)) \cdot A(\phi_{\beta\gamma}(x)) \cdot A(\phi_{\gamma\alpha}(x)) = A\Big(\phi_{\alpha\beta}(x) \cdot \phi_{\beta\gamma}(x) \cdot \phi_{\gamma\alpha}(x)\Big) = A(e) = e.$$

Therefore the new cocycle of transition functions $\{A \circ \phi_{\alpha\beta}\}$ defines a principal G-bundle over X. We denote this bundle by $A(E)$. Next we show that $A(E)$ is well defined.

Proposition 4.3. *The principal G-bundle $A(E)$ does not depend on the choice of the cocycle of transition functions $\{\phi_{\alpha\beta}\}$.*

Proof. Let $\{\phi_{\alpha\beta}\}$, over the covering $\{U_\alpha\}$, and $\{\psi_{ab}\}$, over the covering $\{V_a\}$, be two cocycles of transition functions, both defining equivalent principal G-bundles. Denote the corresponding bundles by E and E', respectively.

Special case. *The cocycle $\{\psi_{ab}\}$ is a refinement of $\{\phi_{\alpha\beta}\}$. That is, the covering $\{V_a\}$ is a refinement of $\{U_\alpha\}$ (i.e every V_a is contained in some U_α), and every ψ_{ab} is the restriction of some $\phi_{\alpha\beta}$.*

Recall that in this case the principal bundle equivalence between E and E' is simply given by inclusions: the element $(x, g) \in V_a \times G$ maps to $(x, g) \in U_\alpha \times G$, where U_α is a fixed (for each a) element of $\{U_\alpha\}$ such that $V_a \subset U_\alpha$.

It is straightforward to verify that the same rule defines an equivalence between $\{A \circ \phi_{\alpha\beta}\}$ and $\{A \circ \psi_{\alpha\beta}\}$. This proves the special case.

Because of the special case we can now assume that both cocycles $\{\phi_{\alpha\beta}\}$, $\{\psi_{ab}\}$ are defined over the same covering $\{U_\alpha\}$. Then the existence of a principal bundle equivalence between E and E' is equivalent to the existence a collection of functions $\{r_\alpha\}$, $r_\alpha \colon U_\alpha \to G$ such that

$$\phi_{\alpha\beta}(x) \cdot r_\alpha(x) = r_\beta(x) \cdot \psi_{\alpha\beta}(x) \tag{1}$$

for $x \in U_{\alpha\beta}$ (see [Hus94, Chapter 5, Theorem 2.7]). Applying A to equation (1) we obtain

$$\big(A \circ \phi_{\alpha\beta}\big)(x) \cdot \big(A \circ r_\alpha\big)(x) = \big(A \circ r_\beta\big)(x) \cdot \big(A \circ \psi_{\alpha\beta}\big)(x).$$

Therefore, the collection $\{A \circ r_\alpha\}$ defines a principal bundle equivalence between $\{A \circ \phi_{\alpha\beta}\}$ and $\{A \circ \psi_{\alpha\beta}\}$. $\square$

Proposition 4.4. *Let $E \to X$ be a principal G-bundle. Also let A and B be automorphisms of G. Then*

$$(AB)(E) = A(B(E)) \quad and \quad id_G(E) = E.$$

Proof. Direct from the definition of $A(E)$. $\square$

Proposition 4.5. *Let $E \to X$ be a principal G-bundle, let A an automorphism of G and let $f \colon Z \to X$ be a map. Then*

$$f^*\big(A(E)\big) = A\big(f^*(E)\big)$$

Proof. Let $\{\phi_{\alpha\beta}\}$ be a cocycle of transition functions for E defined over a covering $\{U_\alpha\}$. Then $\{A \circ \phi_{\alpha\beta} \circ f\}$ is cocycle of transition functions over $\{f^{-1}U_\alpha\}$ for both $f^*\big(A(E)\big)$ and $A\big(f^*(E)\big)$. $\qquad\square$

Proposition 4.6. *Let $E \to X$ be a principal G-bundle and let A be an automorphism of G. Then there is an A-map $F_A \colon E \to A(E)$, covering the identity $id_X \colon X \to X$.*

Proof. Let $\{\phi_{\alpha\beta}\}$ be a cocycle of transition functions for E over a covering $\{U_\alpha\}$. Then $\{A \circ \phi_{\alpha\beta}\}$ is a cocycle of transition functions for $A(E)$ over $\{U_\alpha\}$. Define map F_A in charts as follows:

$$U_\alpha \times G \ni (x, g) \mapsto (x, A(g)) \in U_\alpha \times G,$$

where the latter copy of $U_\alpha \times G$ is a chart of $A(E)$. The map F_A is well defined because the following diagram commutes

$$
\begin{array}{ccc}
G & \xrightarrow{\ L_{\phi_{\alpha\beta}(x)}\ } & G \\
{\scriptstyle A}\big\downarrow & & \big\downarrow{\scriptstyle A} \\
G & \xrightarrow{\ L_{A(\phi_{\alpha\beta}(x))}\ } & G
\end{array}
$$

Here L_h denotes left multiplication by h. $\qquad\square$

Corollary 4.7. *Let $E \to X$ be a principal G-bundle and let A be an automorphism of G. Then there is an A-map $F_{A^{-1}} \colon A^{-1}(E) \to E$, covering the identity $id_X \colon X \to X$.*

Proof. This follows from Propositions 4.6 and Remark 4.2. $\qquad\square$

Let $EG \to BG$ be the universal principal G-bundle and let A be an automorphism of G. Then $A(EG)$ is a principal G-bundle, hence (see (3.4)) there is a map $\rho_A \colon BG \to BG$ such that

$$A(EG) \cong \rho_A^*(EG). \tag{2}$$

Moreover, this map is unique up to homotopy.

5. Principal $\mathbb{T}^k$-Bundles

We now take $G = \mathbb{T}^k = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$. Recall that by (3.6) $B\mathbb{T}^k = (\mathbb{C}P^\infty)^k$. Therefore $\pi_2 B\mathbb{T}^k$ is canonically identified with $\mathbb{Z}^k$ (by identifying i-th generator of $\mathbb{Z}^k$ with the canonical generator of the second homotopy group of the i-th copy of $\mathbb{C}P^\infty$.).

Let $A \in SL(\mathbb{Z}, k)$. The matrix A induces automorphisms $A \colon \mathbb{Z}^k \to \mathbb{Z}^k$ and $A \colon \mathbb{T}^k \to \mathbb{T}^k$ for which we use the same notation.

The next proposition is a key result and its proof occupies the rest of this section (except for the lemma at the end of this section). Recall that ρ_A is characterized by equation (2).

Proposition 5.1. *Let $g \colon B\mathbb{T}^k \to B\mathbb{T}^k$ be a map such that $\pi_2(g) = A \in SL(\mathbb{Z}, k)$. Then g is homotopic to ρ_A, that is,*

$$A(E\mathbb{T}^k) \cong g^*(E\mathbb{T}^k).$$

Proof. The proof will require some lemmas and claims.

We consider $\mathbb{C}P^\infty = \cup_n \mathbb{C}P^n$ with the usual CW-structure, *i.e.,* one cell in each even dimension. This structure induces a product CW-structure on $B\mathbb{T}^k = (\mathbb{C}P^\infty)^k$. Then the 2-skeleton of $B\mathbb{T}^k$ is the wedge $\bigvee_{i=1}^k \mathbb{S}_i^2$ of k copies of the 2-sphere $\mathbb{S}^2$. Denote by Y this 2-skeleton and by $E \to Y$ the restriction of $E\mathbb{T}^k \to B\mathbb{T}^k$ to Y. We first prove the proposition for the principal $\mathbb{T}^k$-bundle $E \to Y$.

Lemma 5.2. *Let $g_Y\colon Y \to Y$ be a map such that $\pi_2(g_Y) = A \in SL(\mathbb{Z}, k)$. Then*

$$A(E) \cong g_Y^*(E).$$

Proof. Let p be the wedge point of Y. Then we have $\mathbb{S}_i^2 \cap \mathbb{S}_j^2 = \{p\}$, $i \neq j$. We identify p with the south pole of each $\mathbb{S}_i^2$. Denote by D_i^+ and D_i^- the closed upper and lower hemispheres of $\mathbb{S}_i^2$, respectively.

Let $E_i \to \mathbb{S}_i$ be the restriction of $E \to Y$ to $\mathbb{S}_i$, $i = 1, \dots k$.

Claim 5.3. *The principal $\mathbb{T}^k$-bundle $E_i \to \mathbb{S}_i$ is obtained by identifying $D_i^- \times \mathbb{T}^k$ with $D_i^+ \times \mathbb{T}^k$ along their boundaries using the gluing map $\omega_i\colon \mathbb{S}^1 \to \mathbb{T}^k$, $\omega_i(u) = (1, \dots, 1, u, 1, \dots 1)$, that is, all coordinates of $\omega_i(u)$ are equal to $1 \in \mathbb{S}^1$, except for the i-th coordinate, which is equal to u.*

Proof. The claim follows from putting together the following two facts; see also (3.6).

1. The 2-skeleton of $B\mathbb{S}^1 = \mathbb{C}P^\infty$ is $\mathbb{C}P^1 = \mathbb{S}^2$, and the restriction of $E\mathbb{S}^1 = \mathbb{S}^\infty$ to $\mathbb{S}^2$ is the Hopf bundle $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$. Moreover, $\mathbb{S}^3$ is obtained by identifying two copies of $\mathbb{D}^2 \times \mathbb{S}^1$ along the boundaries using the identity map $id_{\mathbb{S}^1}\colon \mathbb{S}^1 \to \mathbb{S}^1$ as gluing map.
2. Let $F_1 \to E_1 \to X_1$ and $F_2 \to E_2 \to X_2$ be two fiber bundles. Consider the inclusion $X_1 \hookrightarrow X_1 \times X_2$, $x \mapsto (x, *)$, for some fixed $* \in X_2$. Then the restriction $(E_1 \times E_2)|_{X_1}$ of the product bundle $F_1 \times F_2 \to E_1 \times E_2 \to X_1 \times X_2$ to $X_1 \subset X_1 \times X_2$ is the bundle $F_1 \times F_2 \to E_1 \times F_2 \to X_1$.

□

Write $A = (a_{ij}) \in SL(\mathbb{Z}, k)$. Because $A = \pi_2(g_Y)$, after performing a homotopy, we can assume that g_Y satisfies the following property.

5.4. *For each j there are k disjoint closed 2-disks $D_{ij} \subset D_j^+$, $i = 1, \dots, k$, such that*

1. *$g_Y\colon (D_{ij}, \partial D_{ij}) \mapsto (D_i^+, \partial D_i^+)$;*
2. *the degree of $g_Y\colon (D_{ij}, \partial D_{ij}) \to (D_i^+, \partial D_i^+)$ is a_{ij}.*

Claim 5.5. *The bundle $g_Y^* E|_{\mathbb{S}_j^2}$ is obtained by gluing $D_j^- \times \mathbb{T}^k$ with $D_j^+ \times \mathbb{T}^k$ along their boundaries using the gluing map $f_j = \prod_{i=1}^k (\omega_i)^{a_{ij}}\colon \mathbb{S}^1 \to \mathbb{T}^k$. That is, $f_j(u) = (u^{a_{1j}}, \dots, u^{a_{kj}})$.*

Proof. It follows from Claim 5.3 and Property 5.4 that $g_Y^* E|_{\mathbb{S}_j^2}$ is obtained by identifying $\coprod_{i=1}^k D_{ij} \times \mathbb{T}^k$ with $(\mathbb{S}_j^2 - \bigcup_i int\, D_{ij}) \times \mathbb{T}^k$ along their boundaries (which is the union of k copies of $\mathbb{S}^1 \times \mathbb{T}^k$) via the gluing maps $\omega_i^{a_{ij}}\colon \partial D_{ij} = \mathbb{S}^1 \to \mathbb{T}^k$, $i = 1, \dots, k$. (Here we are identifying ∂D_{ij} with $\mathbb{S}^1$ using the orientation on ∂D_{ij} induced by D_{ij}.)

The claim now follows from the fact that the inclusion $\mathbb{S}^1 = \partial D_j^+ \hookrightarrow D_j^+$ is a path in $D_j^+ - \bigcup_i int\, D_{ij}$ which winds positively around each D_{ij} exactly once. □

Claim 5.6. *The principal $\mathbb{S}^1$-bundle $A(E)|_{\mathbb{S}_j^2} \to \mathbb{S}_j^2$ is obtained by identifying $D_j^-\times$ $\mathbb{T}^k$ with $D_j^+ \times \mathbb{T}^k$ along their boundaries using the gluing map $f_j\colon \mathbb{S}^1 \to \mathbb{T}^k$.*

Proof. By applying Proposition 4.5 to the inclusion map $\mathbb{S}_j^2 \hookrightarrow Y$ we obtain

$$A(E)|_{\mathbb{S}_j^2} = A(E|_{\mathbb{S}_j^2}) = A(E_j).$$

This together with Claim 5.3 and the definition of $A(E_j)$ implies that $A(E_j)$ is obtained by identifying $D_j^- \times \mathbb{T}^k$ with $D_j^+ \times \mathbb{T}^k$ along their boundaries using the gluing map $A \circ \omega_j\colon \mathbb{S}^1 \to \mathbb{T}^k$. But

$$A(\omega_j(u)) = A(1, ..., 1, u, 1, ..., 1) = (u^{a_{1j}}, ..., u^{a_{kj}}) = f_j(u).$$

$\square$

Lemma 5.2 now directly follows from Claims 5.5 and 5.6. $\square$

To finish the proof of Proposition 5.1 we need the following lemma.

Lemma 5.7. *Let $E_1 \to B\mathbb{T}^k$ and $E_2 \to B\mathbb{T}^k$ be principal $\mathbb{T}^k$-bundles. Let Z be a space and let $h\colon Z \to B\mathbb{T}^k$ be a map. Assume that $h^*\colon H^2(B\mathbb{T}^k;\mathbb{Z}) \to H^2(Z;\mathbb{Z})$ is injective. Then $h^*E_1 \cong h^*E_2$ implies $E_1 \cong E_2$.*

Proof. Recall that by (3.6) $B\mathbb{T}^k = (\mathbb{C}P^\infty)^k$. Hence $B\mathbb{T}^k$ is an Eilenberg-MacLane space of type $(\mathbb{Z}^k, 2)$, *i.e.*, $\pi_2 B\mathbb{T}^k = \mathbb{Z}^k$ and $\pi_i B\mathbb{T}^k = 0$, $i \neq 2$. Therefore, we have that for any space X the group $[X, B\mathbb{T}^k]$ of homotopy classes of maps from X to (the Eilenberg-MacLane space) $B\mathbb{T}^k$ is isomorphic to $H^2(X;\mathbb{Z}^k)$ [Hat02, Theorem 4.57]. This group splits naturally as follows

$$H^2(X;\mathbb{Z}^k) \cong H^2(X;\mathbb{Z}) \oplus ... \oplus H^2(X;\mathbb{Z}). \tag{3}$$

Indeed, the splitting $\mathbb{Z}^k = \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ induces a natural splitting of the cochain complex $C^*(X;\mathbb{Z}^k) \cong C^*(X,\mathbb{Z}) \oplus \ldots \oplus C^*(X,\mathbb{Z})$ and (3) follows directly from the definition of cohomology.

Let $h_i\colon B\mathbb{T}^k \to B\mathbb{T}^k$ classify E_i. Then $h_i \circ h\colon Z \to B\mathbb{T}^k$ classifies h^*E_i. But the map $h^*\colon [B\mathbb{T}^k, B\mathbb{T}^k] \to [Z, B\mathbb{T}^k]$, $f \mapsto f \circ h$ is the map $h^*\colon H^2(B\mathbb{T}^k;\mathbb{Z}^k) \to H^2(Z;\mathbb{Z}^k)$. This map is injective because $h^*\colon H^2(B\mathbb{T}^k;\mathbb{Z}) \to H^2(Z;\mathbb{Z})$ is injective and the splitting (3) is natural. Therefore $h_1 \circ h \simeq h_2 \circ h$ implies $h_1 \simeq h_2$. $\square$

By the Cellular Approximation Theorem [Hat02, Theorem 4.8], we can assume that $g\colon B\mathbb{T}^k \to B\mathbb{T}^k$ is a cellular map. Hence g restricts to the 2-skeleton Y.

Let $\iota\colon Y \to B\mathbb{T}^k$ be the inclusion map. Note that

$$A(E) = A(\iota^* E\mathbb{T}^k) \cong \iota^* A(E\mathbb{T}^k),$$

where the last equivalence is by Proposition 4.5. Also note that

$$(g|_Y)^* E = \iota^* g^*(E\mathbb{T}^k).$$

By Lemma 5.2, $A(E) \cong (g|_Y)^* E$. Hence, $\iota^* A(E\mathbb{T}^k) \cong \iota^* g^*(E\mathbb{T}^k)$. Now, because ι^* is an isomorphism, Lemma 5.7 applies and we conclude that $A(E\mathbb{T}^k) \cong g^*(E\mathbb{T}^k)$. This completes the proof of Proposition 5.1 . $\square$

The following is a natural question: given a homomorphism $A\colon \mathbb{Z}^k \to \mathbb{Z}^k$, is there a map $f\colon B\mathbb{T}^k \to B\mathbb{T}^k$ such that $\pi_2(f) = A$? It is well known that the answer to this question is affirmative. Moreover, the map f is unique up to homotopy. The next lemma is a bit more general, and will be needed later.

Lemma 5.8. *Let X be a simply connected space and let $A\colon \pi_2 X \to \mathbb{Z}^k = \pi_2 B\mathbb{T}^k$ be a homomorphism. Then there is a unique up to homotopy $f\colon X \to B\mathbb{T}^k$ with $\pi_2(f) = A$.*

Proof. We can equip X with a CW complex structure so that X has no 1-cells [Hat02, Corollary 4.16]. By a simple argument we can define f on the 3-skeleton of X so that $\pi_2(f) = A$ (see [Hat02, Lemma 4.31]). And since $\pi_i B\mathbb{T}^k = 0$, $i > 2$, obstruction arguments show that f can be extended cell by cell to the whole of X. The proof of the uniqueness up to homotopy is similar. $\square$

6. Principal $\mathbb{T}^k$-Bundles that admit A-Maps

Let $E \to X$ be a principal $\mathbb{T}^k$-bundle and $f\colon X \to X$. Also let $A \in SL(k, \mathbb{Z})$. In this section we answer the following question:

Question 6.1. *When does there exist an A-map $E \to E$ covering f?*

$$
\begin{array}{ccc}
E & \xrightarrow{\;A-map\;} & E \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;f\;} & X
\end{array}
$$

Recall that by (3.4) every principal $\mathbb{T}^k$-bundle over X is equivalent (as principal bundle) to the pull-back $h^* E\mathbb{T}^k$ for some $h\colon X \to B\mathbb{T}^k$. We will use the following notation:

$$E_h \overset{\text{def}}{=} h^* E\mathbb{T}^k.$$

The next result answers Question 6.1. It gives a relationship between A, f and $E = E_h$ which is equivalent to the existence of an A-map $E \to E$ covering f. The map ρ_A, characterized by equation (2), appears in the next theorem.

Theorem 6.2. *Let $A \in SL(k, \mathbb{Z})$, let X be a space and let $f\colon X \to X$ be a map. Also let $h\colon X \to B\mathbb{T}^k$. Then there exists an A-map $E_h \to E_h$ covering f*

$$
\begin{array}{ccc}
E_h & \xrightarrow{\;A-map\;} & E_h \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;f\;} & X
\end{array}
$$

if and only if $h \circ f \simeq \rho_A \circ h$. That is, the following diagram homotopy commutes

$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & X \\
h\downarrow & & \downarrow h \\
B\mathbb{T}^k & \xrightarrow{\;\rho_A\;} & B\mathbb{T}^k
\end{array}
$$

Proof. First suppose that there exists an A-map $E_h \to E_h$ covering f. We have the following diagram

$$
\begin{array}{ccccc}
A(E_h) & \xrightarrow{\;A^{-1}-map\;} & E_h & \xrightarrow{\;A-map\;} & E_h \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\;id_X\;} & X & \xrightarrow{\;f\;} & X
\end{array}
$$

where the first square comes from Corollary 4.7 (by taking A^{-1} instead of A). By composing the consecutive horizontal arrows and using Remark 4.1 we obtain a principal $\mathbb{T}^k$-bundle map $A(E_h) \to E_h$ covering f. Therefore

$$A(E_h) \cong f^* E_h \tag{4}$$

and, using Proposition 4.5 we obtain the following equivalences

$$(\rho_A \circ h)^* E\mathbb{T}^k = h^*(\rho_A^*(E\mathbb{T}^k)) \overset{(2)}{\cong} h^* A(E\mathbb{T}^k) \overset{4.5}{=} A(E_h) \overset{(4)}{\cong} f^*(E_h) = (h \circ f)^* E\mathbb{T}^k$$

and it follows that $\rho_A \circ h \simeq h \circ f$.

Conversely, suppose

$$\rho_A \circ h \simeq h \circ f \tag{5}$$

Then

$$A(E_h) = A(h^* E\mathbb{T}^k) \overset{4.5}{=} h^* A(E\mathbb{T}^k) \overset{(2)}{\cong} h^*(\rho_A^*(E\mathbb{T}^k)) = (\rho_A \circ h)^* E\mathbb{T}^k$$

$$\overset{(5)}{\cong} (h \circ f)^* E\mathbb{T}^k = f^* E_h.$$

Therefore there is a principal bundle equivalence between $A(E_h)$ and $f^* E_h$, that is there is a $id_{\mathbb{T}^k}$-map $A(E_h) \to f^* E_h$ covering id_X. This gives the second square in the diagram

The first square comes from Proposition 4.6 and the third one from the definition of pull-back bundle. By composing the consecutive horizontal arrows and using Remark 4.2 we obtain an A-map $E_h \to E_h$ covering f. This completes the proof of the theorem. $\qquad\square$

Our next result says that to verify condition $\rho_A \circ h \simeq h \circ f$ in the theorem above it is enough to verify it algebraically at the H^2 level.

Proposition 6.3. *The following are equivalent*

(1) $\rho_A \circ h \simeq h \circ f$
(2) $H^2(h) \circ H^2(\rho_A) = H^2(f) \circ H^2(h)$.

Moreover, if X is simply connected and $H_2(X)$ is free then (1) and (2) are equivalent to

(3) $H_2(\rho_A) \circ H_2(h) = H_2(h) \circ H_2(f)$.
(4) $\pi_2(\rho_A) \circ \pi_2(h) = \pi_2(h) \circ \pi_2(f)$.

This proposition follows from the following lemma.

Lemma 6.4. *Let X be a space and let $\phi, \psi : X \to B\mathbb{T}^k$ be maps. Then the following are equivalent*

(1) $\phi \simeq \psi$,
(2) $H^2(\phi) = H^2(\psi)$.

Moreover, if X is simply connected and $H_2(X)$ is free then (1) and (2) are equivalent to

(3) $H_2(\phi) = H_2(\psi)$,

(4) $\pi_2(\phi) = \pi_2(\psi)$.

Proof. Clearly (1) implies (2). Recall the splitting (3) from Lemma 5.7. Assume $H^2(\phi) = H^2(\psi)$, then, by naturality of the splitting (3), the induced maps on the cohomology with $\mathbb{Z}^k$ coefficients also coincide. Now recall that the map $H^2(\phi; \mathbb{Z}^k)\colon H^2(B\mathbb{T}^k; \mathbb{Z}^k) \to H^2(X; \mathbb{Z}^k)$ coincides with the map $\phi^*\colon [B\mathbb{T}^k, B\mathbb{T}^k] \to [X, B\mathbb{T}^k]$ given by $[\lambda] \mapsto [\lambda \circ \phi]$. Similarly for $H^2(\psi; \mathbb{Z}^k)$. Hence $\lambda \circ \phi \simeq \lambda \circ \psi$ for every λ. Taking $\lambda = id_{B\mathbb{T}^k}$ we obtain $\phi \simeq \psi$. This proves that (2) implies (1).

If X is simply connected and $H_2(X)$ is free then $H^2(X) \cong H_2(X) \cong \pi_2(X)$. (The first isomorphism is by the Universal Coefficients Theorem [Hat02, Theorem 3.2] and the second one is by the Hurewicz Theorem [Hat02, Theorem 4.37]) Therefore, $H^2(\phi) \cong H_2(\phi)^T \cong \pi_2(\phi)^T$ (the superscript ^{T} denotes the transpose). $\square$

To prove the proposition apply the above lemma to $\phi = \rho_A \circ h$ and $\psi = h \circ f$.

7. Simply Connected Principal $\mathbb{T}^k$-Bundles

Let $E \to X$ be a principal $\mathbb{T}^k$-bundle. Recall that $E \cong E_h = h^* E\mathbb{T}^k$, where the map $h\colon X \to B\mathbb{T}^k$ is unique up to homotopy. In this section we deal with the following question:

Question 7.1. *When is the total space E_h simply connected?*

Note that the fundamental group of the total space E_h surjects onto the fundamental group of X. Therefore X has to be simply connected. The next result answers Question 7.1 when X is simply connected.

Proposition 7.2. *Let X be a simply connected space, and let $h\colon X \to B\mathbb{T}^k$ be a map. Then the following are equivalent:*

(1) the total space E_h is simply connected;
(2) the homomorphism $\pi_2(h)\colon \pi_2 X \to \pi_2 B\mathbb{T}^k$ is onto;
(3) the homomorphism $H_2(h)\colon H_2 X \to H_2 B\mathbb{T}^k$ is onto.

Proof. From the homotopy exact sequence of the $\mathbb{T}^k$-bundle $\mathbb{T}^k \to E_h \to X$ and the fact that $\pi_1 X = 0$ we obtain the exact sequence
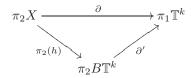
$$\to \pi_2 X \xrightarrow{\partial} \pi_1 \mathbb{T}^k \to \pi_1 E_h \to 0$$

Therefore $\pi_1 E_h = 0$ if and only if ∂ is onto. On the other hand from the homotopy exact sequence of the $\mathbb{T}^k$-bundle $\mathbb{T}^k \to E\mathbb{T}^k \to B\mathbb{T}^k$ and the fact that $E\mathbb{T}^k$ is contractible we obtain that

$$\pi_2 B\mathbb{T}^k \xrightarrow{\partial'} \pi_1 \mathbb{T}^k$$

is an isomorphism. Then the equivalence (1)⇔(2) follows from the following claim.

Claim 7.3. *The following diagram commutes*

The claim follows from the naturality of the homotopy exact sequence of a pair and the definition of the boundary map.

The equivalence (2)⇔(3) follows from the naturality of the Hurewicz map and Hurewicz Theorem. This proves the proposition. □

8. The construction

We specify to the case where X is a simply connected 4-manifold and $f\colon X \to X$ is a diffeomorphism. We make the following collection of assumptions $(*)$.

($*8.1$) Second homotopy group $\pi_2(X)$ is a free abelian group on m generators.

($*8.2$) The group $\pi_2(X)$ splits as a direct sum $\mathbb{Z}^k \oplus \mathbb{Z}^{m-k}$ in such a way that the first summand is $\pi_2(f)$-invariant, i.e., $\pi_2(f)|_{\mathbb{Z}^k}$ is an automorphism of $\mathbb{Z}^k \subset \pi_2(X)$.

($*8.3$) Let $A \in SL(k, \mathbb{Z})$ be the matrix that represents $\pi_2(f)|_{\mathbb{Z}^k}$. Then A also represents an automorphism $\mathbb{R}^k \to \mathbb{R}^k$. Assume that there exists an A-invariant splitting $\mathbb{R}^k = E_A^s \oplus E_A^c \oplus E_A^u$ and a Riemannian metric $\|\cdot\|$ on X such that the numbers

$$\lambda_\sigma = \min_{\substack{v \in E_A^\sigma, \\ \|v\|=1}} \|Av\|, \quad \mu_\sigma = \max_{\substack{v \in E_A^\sigma, \\ \|v\|=1}} \|Av\|, \ \sigma = s, c, u,$$

satisfy the following inequalities

$$\lambda_s \le \mu_s < \lambda_c \le \mu_c < \lambda_u \le \mu_u,$$
$$\mu_s < m(f),$$
$$\lambda_u > \|Df\|,$$

where $m(f)$ is minimum of the conorm $m(Df_x)$, i.e.,

$$m(f) = \min_{\substack{v \in TX, \\ \|v\|=1}} \|Df(v)\|$$

and $\|Df\|$ is the maximum of the norm $\|Df_x\|$, i.e.,

$$\|Df\| = \max_{\substack{v \in TX, \\ \|v\|=1}} \|Df(v)\|.$$

Remark 8.1. We allow E_A^c to be trivial.

Theorem 8.2. *Let X be a simply connected closed 4-manifold, let $f\colon X \to X$ be a diffeomorphism that satisfies $(*)$ and let $\pi_h\colon E_h \to X$ be a principal $\mathbb{T}^k$-bundle. Assume that E_h admits an A-map $F\colon E_h \to E_h$. Then $F\colon E_h \to E_h$ is a partially hyperbolic diffeomorphism.*

Clearly the splitting of $(*8.3)$ descends to a $\mathbb{T}^k$-invariant splitting of the tangent bundle $T\mathbb{T}^k = E_A^s \oplus E_A^c \oplus E_A^u$. Then the action of $\mathbb{T}^k$ on E_h induces a $\mathbb{T}^k$ invariant splitting of $T\mathbb{T}^k = E^s \oplus E^c \oplus E^u$; here, abusing notation, $T\mathbb{T}^k$ is the subbundle of TE_h that consists of vectors tangent to the torus fibers. Because $F\colon E_h \to E_h$ is an A-map, this splitting is DF-invariant.

Addendum 8.3 (to Theorem 8.2). *The subbundles E^s and E^u defined above are the stable and the unstable subbundles for F, respectively. The center subbundle for F has the form $E^c \oplus H'$, where H' is a certain subbundle complementary to $T\mathbb{T}^k$.*

Proof. We equip TE_h with a Riemannian metric in the following way. The flat metric on the torus induces a metric on $T\mathbb{T}^k$. Also recall that by $(*8.3)$ we have equipped X with a Riemannian metric $\|\cdot\|$. Choose a continuous horizontal subbundle $H \subset TE_h$ such that $TE_h = T\mathbb{T}^k \oplus H$. Then

$$(D\pi_h)_x \colon H(x) \to T_{\pi_h(x)}X$$

is an isomorphism for every $x \in E_h$. Set

$$\|v\| = \|D\pi_h(v)\|$$

for $v \in H$. Then extend the Riemannian metric $\|\cdot\|$ to the rest of TE_h by declaring $T\mathbb{T}^k$ and H perpendicular.

Consider the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E^s & \longrightarrow & TE_h & \longrightarrow & E^c \oplus E^u \oplus H & \longrightarrow & 0 \\
& & \Big\downarrow {\scriptstyle DF|_{E^s}} & & \Big\downarrow {\scriptstyle DF} & & \Big\downarrow {\scriptstyle DF\circ p} & & \\
0 & \longrightarrow & E^s & \longrightarrow & TE_h & \longrightarrow & E^c \oplus E^u \oplus H & \longrightarrow & 0
\end{array}
\tag{6}
$$

The horizontal rows are short exact sequences of Riemannian vector bundles and all vertical automorphisms fiber over $f \colon X \to X$. The last vertical arrow is defined as the composition of DF and the orthogonal projection p on $E^c \oplus E^u \oplus H$. Note that the diagram

$$
\begin{array}{ccc}
E^c \oplus E^u \oplus H & \xrightarrow{\;Df\circ p\;} & E^c \oplus E^u \oplus H \\
\Big\downarrow {\scriptstyle D\pi_h} & & \Big\downarrow {\scriptstyle D\pi_h} \\
TX & \xrightarrow{\quad Df \quad} & TX
\end{array}
$$

commutes and, hence, by our choice of the Riemannian metric

$$\|DF(p(v))\| \geq \min(\lambda_c, m(f))\|v\|.$$

Combining with $(*8.3)$ we obtain we following bound on the minimum of the conorm

$$m(Df \circ p) > \mu_s \tag{7}$$

Lemma 8.4 ([HPS77], Lemma 2.18). *Let*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E_1 & \xrightarrow{\;i\;} & E_2 & \xrightarrow{\;j\;} & E_3 & \longrightarrow & 0 \\
& & \Big\downarrow {\scriptstyle T_1} & & \Big\downarrow {\scriptstyle T_2} & & \Big\downarrow {\scriptstyle T_3} & & \\
0 & \longrightarrow & E_1 & \xrightarrow{\;i\;} & E_2 & \xrightarrow{\;j\;} & E_3 & \longrightarrow & 0
\end{array}
$$

be a commutative diagram of short exact sequences of Riemannian vector bundles, all over a compact metric space X, where $T_i \colon E_i \to E_i$ are bundle automorphisms over the base homeomorphism $f \colon X \to X$, $i = 1, 2, 3$. If

$$m(T_3|_{E_3(x)}) > \|T_1|_{E_1(x)}\|$$

for all $x \in X$, then $i(E_1)$ has a unique T_2-invariant complement in E_2.

Because we have (7), we can apply Lemma 8.4 to (6) and obtain a DF-invariant splitting $TE_h = E^s \oplus \widehat{E^s}$. Exchange the roles of E^s and E^u and apply the same argument to obtain a DF-invariant splitting $TE_h = \widehat{E^u} \oplus E^u$. It is easy to see that $E^c \oplus E^u \subset \widehat{E^s}$ and $E^s \oplus E^c \subset \widehat{E^u}$. Let

$$\widehat{E^c} = \widehat{E^s} \cap \widehat{E^u}.$$

Then, clearly, we have a DF-invariant splitting $TE_h = E^s \oplus \widehat{E^c} \oplus E^u$.

To see that F is partially hyperbolic with respect to this splitting pick a continuous decomposition $\widehat{E^c} = E^c \oplus H'$, and define a new Riemannian metric $\| \cdot \|'$ on TE_h in the same way $\| \cdot \|$ was defined, but using H' instead of H; i.e., we declare

1. $\|v\|' = \|v\|$ if $v \in T\mathbb{T}^k$,
2. $\|v\|' = \|D\pi_h(v)\|$ if $v \in H'$,
3. H' is orthogonal to $T\mathbb{T}^k$.

Now partial hyperbolicity (with respect to $\| \cdot \|'$) is immediate from the inequalities of $(*8.3)$. $\qquad\square$

9. The base space — the Kummer surface

A K3 surface is a simply connected complex surface whose canonical bundle is trivial. All K3 surfaces are pairwise diffeomorphic and have the same intersection form $2(-E_8) \oplus 3 \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$. In this section we recall Kummer's construction of the K3 surface and describe a holomorphic atlas on it.

Consider the complex torus

$$\mathbb{T}_{\mathbb{C}}^2 = \mathbb{C}^2/(\mathbb{Z} \oplus i\mathbb{Z})^2.$$

Also consider the involution $\iota \colon \mathbb{T}_{\mathbb{C}}^2 \to \mathbb{T}_{\mathbb{C}}^2$ given by $\iota(z_1, z_2) = (-z_1, -z_2)$. It has 16 fixed points which we call the *exceptional set* and which we denote by $\mathcal{E}(\mathbb{T}_{\mathbb{C}}^2)$. Note that $\mathbb{T}_{\mathbb{C}}^2/\iota$ is not a topological manifold because the neighborhoods of the points in the exceptional set are cones over $\mathbb{R}P^3$-s. Replace the neighborhoods of the points from the exceptional set with copies of $\overline{\mathbb{C}P}^2$ to obtain the blown up torus $\mathbb{T}_{\mathbb{C}}^2 \# 16\overline{\mathbb{C}P}^2$ (here $\overline{\mathbb{C}P}^2$ stands for the two (complex) dimensional complex projective with reversed orientation; see *e.g.,* [Sc05, p. 286] for details on complex blow up). The involution ι naturally induces a holomorphic involution ι' of $\mathbb{T}_{\mathbb{C}}^2 \# 16\overline{\mathbb{C}P}^2$. The involution ι' fixes 16 copies of $\mathbb{C}P^1$. One can check that the quotient

$$X \stackrel{\text{def}}{=} \mathbb{T}_{\mathbb{C}}^2 \# 16\overline{\mathbb{C}P}^2/\iota'$$

is a 4-dimensional manifold. This manifold is called the *Kummer surface.* Note that it comes with a map

$$\sigma \colon \mathbb{T}_{\mathbb{C}}^2 \backslash \mathcal{E}(\mathbb{T}_{\mathbb{C}}^2) \to X, \tag{8}$$

which is a double cover of its image $X \backslash \mathcal{E}(X)$, where $\mathcal{E}(X)$ is the *exceptional set* in X, *i.e.,* the union of 16 copies of $\mathbb{C}P^1$. One can also check that X is simply connected. (See [Sc05, Chapter 3.3] for more details.)

In fact, X is a complex surface and we proceed to describe the complex structure on X. For any connected open set $\mathcal{V}$ which is disjoint from the exceptional set $\mathcal{E}(X)$ and whose preimage under σ has 2 connected components, a holomorphic chart on $\mathbb{T}_{\mathbb{C}}^2$ for one of the connected components of $\sigma^{-1}(\mathcal{V})$ induces a chart on $\mathcal{V}$ by composing with σ. Hence we are left to describe the charts on a neighborhood of $\mathcal{E}(X)$.

Let $p \in \mathcal{E}(\mathbb{T}^2_\mathbb{C})$. We identify a neighborhood of p in $\mathbb{T}^2_\mathbb{C}$ with a neighborhood $\mathcal{U}$ of $(0,0)$ in $\mathbb{C}^2$. Then we blow up p, which amounts to replacing $\mathcal{U}$ with

$$\mathcal{U}' = \{(z_1, z_2, \ell(z_1, z_2)) : (z_1, z_2) \in \mathcal{U}, (z_1, z_2) \in \ell(z_1, z_2)\}.$$

Here $\ell(z_1, z_2)$ is a complex line through $(0,0)$ and (z_1, z_2). Hence, if $(z_1, z_2) \neq (0,0)$ then $\ell(z_1, z_2) = [z_1 : z_2]$ in homogeneous coordinates. Finally, note that

$$\mathcal{U}'' = \{(z_1, z_2, \ell(z_1, z_2)) \in \mathcal{U}'\}/(z_1, z_2, \ell(z_1, z_2)) \sim (-z_1, -z_2, \ell(z_1, z_2)) \quad (9)$$

is identified with a neighborhood of $\mathbb{C}P^1 \subset \mathcal{E}(X)$ in X. We will cover $\mathcal{U}''$ by two charts.

Note that the inclusion $\mathcal{U} \hookrightarrow \mathbb{C}^2$ induces the inclusion $\mathcal{U}' \hookrightarrow \mathbb{C}^2 \# \overline{\mathbb{C}P}^2$ and then the inclusion $\mathcal{U}'' \hookrightarrow \mathbb{C}^2 \# \overline{\mathbb{C}P}^2/\iota''$, where ι'' is induced by $(z_1, z_2) \mapsto (-z_1, -z_2)$. We will define charts for $\mathbb{C}^2 \# \overline{\mathbb{C}P}^2/\iota''$. Then to obtain charts for $\mathcal{U}''$ one just needs to take the restrictions of the charts for $\mathbb{C}^2 \# \overline{\mathbb{C}P}^2/\iota''$.

First note that

$$\mathbb{C}^2 \# \overline{\mathbb{C}P}^2 = \{(z_1, z_2, \ell(z_1, z_2)) : (z_1, z_2) \in \mathbb{C}^2, (z_1, z_2) \in \ell(z_1, z_2)\} \subset \mathbb{C}^2 \times \mathbb{C}P^1.$$

The projective line $\mathbb{C}P^1$ can be covered by two charts $u \mapsto [u : 1]$ and $u' \mapsto [1 : u']$. These charts extend to charts for $\mathbb{C}^2 \# \overline{\mathbb{C}P}^2$ as follows

$$\varphi_1 \colon (u_1, u_2) \mapsto (u_1 u_2, u_2, [u_1 : 1])$$

and

$$\varphi_2 \colon (u'_1, u'_2) \mapsto (u'_2, u'_1 u'_2, [1 : u'_1]).$$

Define $\xi \colon \mathbb{C}^2 \to \mathbb{C}^2$ by $\xi(u_1, u_2) = (u_1, u_2^2)$. By a direct check, we see that the following composition

$$\mathbb{C}^2 \xrightarrow{\xi^{-1}} \mathbb{C}^2 \xrightarrow{\varphi_i} \mathbb{C}^2 \# \overline{\mathbb{C}P}^2 \longrightarrow \mathbb{C}^2 \# \overline{\mathbb{C}P}^2/\iota''$$

is independent of the branch of ξ and gives a well defined chart ψ_i (homeomorphism on the image), $i = 1, 2$. It is also easy to see that the images of ψ_1 and ψ_2 cover $\mathbb{C}^2 \# \overline{\mathbb{C}P}^2$. Calculating

$$\psi_2^{-1} \circ \psi_1(v, w) = (1/v, v^2 w) \quad (10)$$

confirms that the atlas is holomorphic.

Remark 9.1. Formulas

$$\psi_1(v, w) = (v\sqrt{w}, \sqrt{w}, [v : 1]); \qquad \psi_2(v, w) = (\sqrt{w}, v\sqrt{w}, [1 : v])$$

also show that charts ψ_1 and ψ_2 are compatible with the charts induced from $\mathbb{C}^2 \backslash \{(0,0)\}$ by the double cover $\mathbb{C}^2 \backslash \{(0,0)\} \to \left(\mathbb{C}^2 \# \overline{\mathbb{C}P}^2/\iota \right) \backslash \mathbb{C}P^1$ of the complement of the exceptional set.

Remark 9.2. Consider the 2-form $dz_1 \wedge dz_2$ on $\mathbb{T}^2_\mathbb{C}$ and its pushforward $\eta = \sigma_*(dz_1 \wedge dz_2)$ to $X \backslash \mathcal{E}(X)$ (it is well defined because $dz_1 \wedge dz_2 = (-dz_1) \wedge (-dz_2)$). Calculating the latter in the chart ψ_1 yields

$$d(v\sqrt{w}) \wedge d\sqrt{w} = \frac{1}{2} dv \wedge dw.$$

Together with an analogous calculation in the chart ψ_2 this implies that η extends to a non-vanishing 2-form on X.

Remark 9.1 shows that the charts defined above for $\mathcal{E}(X)$ are compatible with the charts induced by σ from the charts for $\mathbb{T}^2_{\mathbb{C}}$. Hence we have equipped X with a holomorphic atlas.

10. THE BASE DYNAMICS — AUTOMORPHISMS OF KUMMER SURFACES

Let $B \in SL(2, \mathbb{Z})$ be a hyperbolic matrix. Then B induces an automorphism $B_{\mathbb{C}} \colon \mathbb{T}^2_{\mathbb{C}} \to \mathbb{T}^2_{\mathbb{C}}$. Note that after appropriately identifying $\mathbb{T}^2_{\mathbb{C}}$ with the real torus $\mathbb{T}^4$ the matrix that represents $B_{\mathbb{C}}$ is $B \oplus B = \left(\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix} \right)$. We use this identification $\mathbb{T}^2_{\mathbb{C}} \cong \mathbb{T}^4$ repeatedly in what follows. The automorphism $B_{\mathbb{C}}$ naturally induces an automorphism of $\mathbb{T}^2_{\mathbb{C}} \# 16\overline{\mathbb{C}P}^2$ and, hence, because the latter commutes with ι', descends to a homeomorphism $f_B \colon X \to X$. It is easy to verify that f_B is, in fact, a complex automorphism of X. The second integral cohomology group of X is $\mathbb{Z}^{22}$ and the second rational cohomology group admits a splitting

$$H^2(X; \mathbb{Q}) \cong \mathbb{Q}^6 \oplus \mathbb{Q}^{16}, \tag{11}$$

where $\mathbb{Q}^6$ is inherited from $H^2(\mathbb{T}^2_{\mathbb{C}}; \mathbb{Q})$ and the remaining 16 copies of $\mathbb{Q}$ come from the 16 copies of $\mathbb{C}P^1$ in $\mathcal{E}(X)$. See [BHPV04, Chapter VIII] for a proof of these facts.

Proposition 10.1. *The induced automorphism $f_B^* \colon H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ is represented by the matrix $\mathrm{diag}(B^2, id_{\mathbb{Z}^4}, S_{16})$, where S_{16} is a permutation matrix given by the restriction of $B_{\mathbb{C}}$ to $\mathcal{E}(\mathbb{T}^2_{\mathbb{C}})$.*

Proof. Note that, by the universal coefficients theorem, it suffices to show that the induced automorphism of the rational cohomology $f_B^* \colon H^2(X; \mathbb{Q}) \to H^2(X; \mathbb{Q})$ has the posited form. Then we can use naturality of the isomorphism (11). Under this isomorphism the restriction $f_B^*|_{H^2(\mathbb{T}^2_{\mathbb{C}}; \mathbb{Q})}$ corresponds to $B_{\mathbb{C}}^* \colon H^2(\mathbb{T}^2_{\mathbb{C}}; \mathbb{Q}) \to H^2(\mathbb{T}^2_{\mathbb{C}}; \mathbb{Q})$ given by $(B \oplus B) \wedge (B \oplus B)$. And the restriction $f_B^*|_{\mathbb{Q}^{16}}$ permutes the coordinates according to the permutation S_{16} given by the restriction of $B_{\mathbb{C}}$ to $\mathcal{E}(\mathbb{T}^2_{\mathbb{C}})$. After an (integral) change of basis we obtain that f_B^* is given by $\mathrm{diag}(B^2, id_{\mathbb{Z}^4}, S_{16})$. $\qquad \square$

Remark 10.2. Note that the basis in which the automorphism has the above diagonal form is not completely canonical because we use the eigenvectors that correspond to unit eigenvalues to write $(B \oplus B) \wedge (B \oplus B)$ as $\mathrm{diag}(B^2, id_{\mathbb{Z}^4})$.

The goal now is to perturb f_B so that the perturbation satisfies the collection of assumptions $(*)$ from Section 8.

Set $B = \left(\begin{smallmatrix} 13 & 8 \\ 8 & 5 \end{smallmatrix} \right)$. Note that because of this choice of B the automorphism $B_{\mathbb{C}}$ fixes points in $\mathcal{E}(\mathbb{T}^2_{\mathbb{C}})$.

Embed the automorphism $B \colon \mathbb{T}^2 \to \mathbb{T}^2$ into a 2-parameter family of diffeomorphisms of $\mathbb{T}^2$

$$B_{\varepsilon, d}(x, y) = (13x - h_{\varepsilon, d}(x) + 8y, 8x - h_{\varepsilon, d}(x) + 5y), \quad \varepsilon \geq 0, \ d \in \mathbb{Z}_+.$$

Here $h_{\varepsilon, 1} \colon S^1 \to S^1$ is a C^∞ smooth function that has the following properties:

1. $h_{\varepsilon, 1}(-x) = -h_{\varepsilon, 1}(x)$;
2. $\forall x \in S^1 \ |h'_{\varepsilon, 1}(x)| \leq \varepsilon$;
3. $h_{\varepsilon, 1}(x) = h_{\varepsilon, 1}(x + \frac{1}{2}) = \varepsilon x$ for $x \in U$, where U is a small symmetric neighborhood of $0 \in S^1$;

The existence of such function for sufficiently small U can be seen by standard C^∞-gluing techniques. To define $h_{\varepsilon,d}\colon S^1 \to S^1$ consider the d sheeted self cover $S^1 \to S^1$ given by $x \mapsto dx$ and let $h_{\varepsilon,d}$ be the lifting of $h_{\varepsilon,1}$ that fixes 0. It is clear that $h_{\varepsilon,d}$ also satisfies properties 1 and 2 and the following variant of 3:

$3'.$ $h_{\varepsilon,d}(x) = h_{\varepsilon,d}(x+\frac{1}{2}) = \varepsilon x$ for $x \in U_d$, where U_d is the connected component of $0 \in S^1$ of the set $\{x : dx \in U\}$;

Note that $B_{\mathbb{C}}\colon \mathbb{T}_{\mathbb{C}}^2 \to \mathbb{T}_{\mathbb{C}}^2$ embeds into the 2-parameter family $B_{\varepsilon,d} \oplus B_{\varepsilon,d}\colon \mathbb{T}_{\mathbb{C}}^2 \to \mathbb{T}_{\mathbb{C}}^2$. (Recall that we have an identification $\mathbb{T}^4 \cong \mathbb{T}_{\mathbb{C}}^2$.)

Proposition 10.3. *The diffeomorphisms $B_{\varepsilon,d} \oplus B_{\varepsilon,d}\colon \mathbb{T}_{\mathbb{C}}^2 \to \mathbb{T}_{\mathbb{C}}^2$ induce volume preserving, Bernoulli, diffeomorphisms $f_{\varepsilon,d}\colon X \to X$ for sufficiently small $\varepsilon \geq 0$ and all $d \geq 1$.*

Proof. It is easy to see that $B_{\varepsilon,d} \oplus B_{\varepsilon,d}$ fixes points from the finite set $\mathcal{E}(\mathbb{T}_{\mathbb{C}}^2)$ and that the differential at the points from $\mathcal{E}(\mathbb{T}_{\mathbb{C}}^2)$ are complex linear maps. Also, $B_{\varepsilon,d}(x,y) = B_{\varepsilon,d}(-x,-y)$, hence, $B_{\varepsilon,d} \oplus B_{\varepsilon,d}$ induces a diffeomorphism $f_{\varepsilon,d}\colon X \to X$. The fact that $f_{\varepsilon,d}$ is smooth boils down to a calculation in charts in the neighborhood of $\mathcal{E}(X)$. This is a routine calculation which we omit.

By calculating the Jacobian of $B_{\varepsilon,d}$ we see that diffeomorphism $B_{\varepsilon,d} \oplus B_{\varepsilon,d}$ preserves the volume $vol_{\mathbb{T}_{\mathbb{C}}^2}$ induced by the form $dz_1 \wedge dz_2 \wedge \overline{dz_1} \wedge \overline{dz_2}$. Remark 9.2 implies that $vol_X = \sigma_* vol_{\mathbb{T}_{\mathbb{C}}^2}$ is induced by $\eta \wedge \overline{\eta}$ and hence is indeed a smooth volume. However it is clear from the definition that $f_{\varepsilon,d}$ preserves vol_X.

For sufficiently small $\varepsilon > 0$ the diffeomorphism $B_{\varepsilon,d} \oplus B_{\varepsilon,d}$ is Anosov and, hence, Bernoulli. Because $vol_X(\mathcal{E}(X)) = 0$ the dynamical system (f, vol_X) is a measure theoretic factor of $(B_{\varepsilon,d} \oplus B_{\varepsilon,d}, vol_{\mathbb{T}_{\mathbb{C}}^2})$ and, hence, is also Bernoulli by work of Ornstein [Orn70]. $\square$

Proposition 10.4. *For any sufficiently small $\varepsilon > 0$ there exist a sufficiently large $d \geq 1$ such that the diffeomorphism $f_{\varepsilon,d}\colon X \to X$ satisfies the collection of assumptions $(*)$ from Section 8.*

The proof of this proposition requires some lemmas.

Let $C\colon \mathbb{C}^2 \to \mathbb{C}^2$ be the automorphism given by $(z_1, z_2) \mapsto (\mu z_1, \mu^{-1} z_2)$, $\mu > 1$, and let

$$C_*\colon \mathbb{C}^2 \# \overline{\mathbb{C}P}^2/\iota'' \to \mathbb{C}^2 \# \overline{\mathbb{C}P}^2/\iota''$$

be the automorphism induced by C on the quotient of the blow up. (Recall that ι'' is induced by $(z_1, z_2) \mapsto (-z_1, -z_2)$.) It is easy to see that C_* leaves the projective line $\mathbb{C}P^1 \subset \mathbb{C}^2 \# \overline{\mathbb{C}P}^2/\iota''$ over $(0,0)$ invariant.

Lemma 10.5. *There exists a Riemannian metric k on $\mathbb{C}^2 \# \overline{\mathbb{C}P}^2/\iota''$ such that for any $x \in \mathbb{C}P^1$ and any $u \in T_x(\mathbb{C}^2 \# \overline{\mathbb{C}P}^2/\iota'')$ we have*

$$\mu^{-2}\|u\|_k \leq \|D_x C_*(u)\|_k \leq \mu^2 \|u\|_k,$$

where $\| \cdot \|_k = \sqrt{k(\cdot,\cdot)}$.

Proof. Clearly it is enough to define k on $\mathbb{C}P^1 \subset \mathbb{C}^2 \# \overline{\mathbb{C}P}^2/\iota''$. (Then we extend it in an arbitrary way.)

Recall that in Section 9 we covered $\mathbb{C}^2 \# \overline{\mathbb{C}P}^2/\iota''$ by two charts ψ_1 and ψ_2. Note that in both charts $\mathbb{C}P^1$ is given by $w = 0$. We use Remark 9.1 to calculate C_* in

charts

$$C_{*,\psi_1}: (v,w) \xrightarrow{\psi_1} (v\sqrt{w}, \sqrt{w}, [v:1])$$
$$\xrightarrow{C_*} (\mu v\sqrt{w}, \mu^{-1}\sqrt{w}, [\mu^2 v:1]) \xrightarrow{\psi_1^{-1}} (\mu^2 v, \mu^{-2}w)$$

$$C_{*,\psi_2}: (v,w) \xrightarrow{\psi_2} (\sqrt{w}, v\sqrt{w}, [1:v])$$
$$\xrightarrow{C_*} (\mu\sqrt{w}, \mu^{-1}v\sqrt{w}, [1:\mu^{-2}v]) \xrightarrow{\psi_2^{-1}} (\mu^{-2}v, \mu^2 w)$$

Let us define a Hermitian metric in the chart ψ_1. Given a point $(v,0)$ define

$$h_{(v,0)} = Q(v)dv\overline{dv} + Q(v)^{-1}dw\overline{dw}, \tag{12}$$

where

$$Q(v) = \left(\frac{1}{1+|v|^2}\right)^2.$$

Define a Hermitian metric in the chart ψ_2 by the same formula (12). The fact that these definitions are consistent can be seen from the following calculation that uses the transition formula (10)

$$(\psi_2^{-1} \circ \psi_1)^* h_{(v,0)} = Q(1/v)d(1/v)\overline{d(1/v)} + Q(1/v)^{-1}dv^2 w\overline{dv^2 w}$$
$$= Q(1/v)\frac{1}{|v|^4}dv\overline{dv} + Q(1/v)^{-1}|v|^4 dw\overline{dw} = h_{(v,0)},$$

where the second equality follows from

$$d(v^2 w) = v^2 dw + 2vw dv = v^2 dw$$

when $w = 0$; and the last equality follows from the following identity

$$Q(1/v) = |v|^4 Q(v).$$

Therefore, (12) gives a well-defined Hermitian metric h on $\mathbb{C}P^1 \subset \mathbb{C}^2\#\overline{\mathbb{C}P}^2/\iota''$. Define the Riemannian metric k as real part of h

$$k = \frac{h + \bar{h}}{2}$$

Notice that in charts k is a warped product. Thus, we only need to prove the posited inequalities for the real parts of dual vectors e_v and e_w. We check the inequality in the chart ψ_1. The calculation in the chart ψ_2 is completely analogous.

$$\frac{k_{C_{*,\psi_1}(v,0)}(DC_{*,\psi_1}(e_v), DC_{*,\psi_1}(e_v))}{k_{(v,0)}(e_v, e_v)} = \frac{k_{(\mu^2 v,0)}(\mu^2 e_v, \mu^2 e_v)}{k_{(v,0)}(e_v, e_v)}$$
$$= \frac{Q(\mu^2 v)\mu^4}{Q(v)} = \left(\frac{\mu^2 + |\mu v|^2}{1 + |\mu^2 v|^2}\right)^2;$$

$$\frac{k_{C_{*,\psi_1}(v,0)}(DC_{*,\psi_1}(e_w), DC_{*,\psi_1}(e_w))}{k_{(v,0)}(e_w, e_w)} = \frac{k_{(\mu^2 v,0)}(\mu^{-2}e_w, \mu^{-2}e_w)}{k_{(v,0)}(e_w, e_w)}$$
$$= \frac{Q(\mu^2 v)^{-1}\mu^{-4}}{Q(v)^{-1}} = \left(\frac{\mu^2 + |\mu v|^2}{1 + |\mu^2 v|^2}\right)^{-2};$$

Finally, the posited inequalities follow from the following elementary estimate

$$\mu^{-2} \leq \frac{\mu^2 + |\mu v|^2}{1 + |\mu^2 v|^2} \leq \mu^2.$$

$\square$

Let $g_{\mathbb{T}^2_{\mathbb{C}}} = \mathrm{Re}(dz_1\overline{dz_1} + dz_2\overline{dz_2})$ be the standard flat metric on $\mathbb{T}^2_{\mathbb{C}}$ and let

$$g_{d,\mathbb{T}^2_{\mathbb{C}}} = d^2 g_{\mathbb{T}^2_{\mathbb{C}}} \quad \text{for } d \geq 1.$$

We will write $\|\cdot\|_{d,\mathbb{T}^2_{\mathbb{C}}}$ for the induced norms.

Let $\lambda > 1$ be the larger eigenvalue of B. The following lemma follows immediately from property 2 of $h_{\varepsilon,d}$ and the definition of $B_{\varepsilon,d}$.

Lemma 10.6. *There exist a function λ_ε, $\varepsilon \geq 0$, such that $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to 0$ and*

$$\lambda_\varepsilon^{-1}\|u\|_{d,\mathbb{T}^2_{\mathbb{C}}} \leq \|D(B_{\varepsilon,d} \oplus B_{\varepsilon,d})(u)\|_{d,\mathbb{T}^2_{\mathbb{C}}} \leq \lambda_\varepsilon\|u\|_{d,\mathbb{T}^2_{\mathbb{C}}}$$

for all $d \geq 1$.

For each $d \geq 1$ consider the open set

$$\mathcal{U}_d = \left(U_d \cup \left(U_d + \frac{1}{2}\right)\right)^4 \subset \mathbb{T}^2_{\mathbb{C}}$$

(Recall that U_d is $(const/d)$-neighborhood of 0 in S^1 defined in the statement of property $3'$ of $h_{\varepsilon,d}$.) Clearly $\mathcal{U}_d$ is a neighborhood of $\mathcal{E}(\mathbb{T}^2_{\mathbb{C}})$ which has 16 connected components. We will write $\mathcal{U}_d(p)$ for the connected component of $p \in \mathcal{E}(\mathbb{T}^2_{\mathbb{C}})$.

Remark 10.7. By definition the neighborhoods $(\mathcal{U}_d(p), g_{d,\mathbb{T}^2_{\mathbb{C}}})$ are all pairwise isometric for all $p \in \mathcal{E}(\mathbb{T}^2_{\mathbb{C}})$ and $d \geq 1$.

Let $\mu_\varepsilon > 1$ be the larger eigenvalue of the matrix $\left(\begin{smallmatrix} 13-\varepsilon & 8 \\ 8-\varepsilon & 5 \end{smallmatrix}\right)$. We have

$$\mu_\varepsilon < \lambda \quad \text{for } \varepsilon > 0. \tag{13}$$

The following lemma follows immediately from our definition of $B_{\varepsilon,d} \oplus B_{\varepsilon,d}$.

Lemma 10.8. *Let $d \geq 1$ and let $p \in \mathcal{E}(\mathbb{T}^2_{\mathbb{C}})$. Identify $\mathcal{U}_d(p)$ with a neighborhood of $(0,0)$ in $\mathbb{C}^2$ in the obvious way. Then the restriction $B_{\varepsilon,d} \oplus B_{\varepsilon,d}|_{\mathcal{U}_d(p)}$ is a complex-linear map, which is given by*

$$(z_1, z_2) \mapsto (\mu_\varepsilon z_1, \mu_\varepsilon^{-1} z_2)$$

in the basis of eigenvectors.

Proof of Proposition 10.4. Start by fixing a sufficiently small $\varepsilon > 0$ such that $\mu_\varepsilon \in (1, \lambda)$ and

$$\lambda_\varepsilon < \lambda^2, \tag{14}$$

where λ_ε comes from Lemma 10.6. Consider diffeomorphism $B_{\varepsilon,d} \oplus B_{\varepsilon,d}$ and open sets $\mathcal{U}_d$ and $\mathcal{U}_d(p)$ as described above. By Remark 10.7 each $(\mathcal{U}_d(p), g_{d,\mathbb{T}^2_{\mathbb{C}}})$ is isometric to $(\mathcal{U}, g)$, where $\mathcal{U}$ is a neighborhood of $(0,0)$ is $\mathbb{C}^2$ and $g = \mathrm{Re}(dz_1\overline{dz_1} + dz_2\overline{dz_2})$. Using Lemma 10.8 and the fact that the basis of eigenvectors for $B_{\varepsilon,d} \oplus B_{\varepsilon,d}|_{\mathcal{U}_d(p)}$ is orthogonal, we can precompose with a rotation and obtain another isometric identification $\mathcal{U}_d(p) = \mathcal{U}$ under which $B_{\varepsilon,d} \oplus B_{\varepsilon,d}|_{\mathcal{U}_d(p)}$ becomes

$$C \colon (z_1, z_2) \mapsto (\mu_\varepsilon z_1, \mu_\varepsilon^{-1} z_2);$$

that is, the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{U}_d(p) & \xrightarrow{B_{\varepsilon,d} \oplus B_{\varepsilon,d}} & \mathcal{U}_d(p) \\
\| & & \| \\
\mathcal{U} & \xrightarrow{\quad C \quad} & \mathcal{U}
\end{array}
$$

This diagram induces the commutative diagram

$$
\begin{array}{ccc}
\mathcal{U}''_d(p) & \xrightarrow{\ f_{\varepsilon,d}\ } & \mathcal{U}''_d(p) \\
\Big\| & & \Big\| \\
\mathcal{U}'' & \xrightarrow{\ C_*\ } & \mathcal{U}''
\end{array}
\tag{15}
$$

where $\mathcal{U}''$ is the quotient of the blow up (9) and $\mathcal{U}''_d(p)$ are the corresponding neighborhoods of the 16 copies of $\mathbb{C}P^1$ in X. (Note that the identification $\mathcal{U}''_d(p) = \mathcal{U}''$ is not isometric yet.)

Applying Lemma 10.5 to C_* yields a Riemannian metric k on a neighborhood of $\mathbb{C}P^1 \subset \mathcal{U}''$. Extend k to $\mathcal{U}''$ in an arbitrary way. By (13) we can pick a number $\bar{\mu}_\varepsilon \in (\mu_\varepsilon, \lambda)$. Then, by continuity, Lemma 10.5 implies that for a sufficiently small neighborhood $\mathcal{V}_1 \subset \mathcal{U}''$ of $\mathbb{C}P^1$ we have

$$
(\bar{\mu}_\varepsilon)^{-2}\|u\|_k \le \|D_x C_*(u)\|_k \le \bar{\mu}_\varepsilon^2 \|u\|_k
\tag{16}
$$

for $x \in \mathcal{V}_1$ and $u \in T_x(\mathcal{U}'')$.

Next choose a neighborhood $\mathcal{V}_2 \supset \mathcal{V}_1$ such that the collar $\mathcal{V}_2\backslash\mathcal{V}_1$ has the following properties:

1. any orbit of C_* visits the collar $\mathcal{V}_2\backslash\mathcal{V}_1$ at most twice;
2. any orbit of C_* that visits $\mathcal{V}_1$ also visits the collar $\mathcal{V}_2\backslash\mathcal{V}_1$ exactly twice — once when entering and once when leaving $\mathcal{V}_1$; in particular, for any $x \in \mathcal{U}''$ $(x, f(x)) \notin (\mathcal{V}_1 \times \mathcal{U}''\backslash\mathcal{V}_2) \cup (\mathcal{U}''\backslash\mathcal{V}_2 \times \mathcal{V}_1)$.

Such choice of $\mathcal{V}_2$ is possible due to hyperbolicity of C. Also choose a smooth function $\rho \colon \mathcal{U}'' \to [0,1]$ such that $\rho|_{\mathcal{V}_1} = 1$ and $\rho|_{\mathcal{U}''\backslash\mathcal{V}_2} = 0$. Define Riemannian metric $\tilde{g}$ on $\mathcal{U}''$ by

$$
\tilde{g} = \rho k + (1 - \rho)(\sigma_{\mathcal{U}})_* g.
$$

Here $\sigma_{\mathcal{U}} \colon \mathcal{U}\backslash(0,0) \to \mathcal{U}''$ is $(z_1, z_2) \mapsto (z_1, z_2, \ell(z_1, z_2))$.

Finally, for each $d \ge 1$ decompose X as the union of 16 neighborhoods $\mathcal{U}''_d(p)$ and the complement $X\backslash\mathcal{U}''_d$ and define the sequence of Riemannian metrics

$$
g_{d,X} = \begin{cases} \tilde{g} & \text{on } \mathcal{U}''_d(p) \\ \sigma_* g_d & \text{on } X\backslash\mathcal{U}''_d \end{cases}
$$

In this definition we used the identifications $\mathcal{U}''_d(p) = \mathcal{U}''$ and the push-forward $\sigma_* g_d$ by σ (8) is well defined on the complement because the involution ι is an isometry of $(\mathbb{T}^2_{\mathbb{C}}, g_{d,\mathbb{T}^2_{\mathbb{C}}})$. Because $\tilde{g} = (\sigma_{\mathcal{U}})_* g$ near the boundary of $\mathcal{U}''$ this definition, indeed, gives a smooth Riemannian metric on X.

Denote by $\mathcal{V}_d$ the union of 16 copies of $\mathcal{V}_1$ in $(X, g_{d,X})$, denote by $\mathcal{B}_d$ the union of 16 copies of the collar $\mathcal{V}_2\backslash\mathcal{V}_1$ in $(X, g_{d,X})$ and let $\mathcal{G}_d = X\backslash(\mathcal{V}_d \cup \mathcal{B}_d)$.

We write $\|\cdot\|_{\varepsilon,d}$ for the norm induced by $g_{d,X}$. We have the following estimates:

1. if $\{x, f_{\varepsilon,d}(x)\} \subset \mathcal{G}_d$ then

$$
\lambda_\varepsilon^{-1}\|u\|_{d,X} \le \|D_x f_{\varepsilon,d}(u)\|_{d,X} \le \lambda_\varepsilon \|u\|_{d,X};
$$

2. if $\{x, f_{\varepsilon,d}(x)\} \subset \mathcal{V}_d$ then

$$
\bar{\mu}_\varepsilon^{-2}\|u\|_{d,X} \le \|D_x f_{\varepsilon,d}(u)\|_{d,X} \le \bar{\mu}_\varepsilon^2 \|u\|_{d,X};
$$

3. otherwise

$$
K^{-1}\|u\|_{d,X} \le \|D_x f_{\varepsilon,d}(u)\|_{d,X} \le K\|u\|_{d,X};
$$

where K is a constant which is independent of d. Property 1 follows from Lemma 10.6. Property 2 follows from (16). Property 3 is due to the fact that in the collars both the dynamics (C_*) and the metric $(\tilde{g})$ do not depend on d. Properties 1 and 2 together with our choice of $\bar{\mu}_\varepsilon$ and (14) imply that

$$\lambda^{-2}\|u\|_{d,X} < \|D_x f_{\varepsilon,d}(u)\|_{d,X} < \lambda^2\|u\|_{d,X}$$

holds whenever $\{x, f_{\varepsilon,d}(x)\} \subset \mathcal{V}_d \cup \mathcal{G}_d$.

Hence, the only region without effective control on $Df_{\varepsilon,d}$ is $\mathcal{B}_d$, i.e., when a point enters a collar or leaves a collar. However, by our construction the neighborhoods $\mathcal{U}_d''$ of 16 copies of $\mathbb{C}P^1$ in X are nested, moreover,

$$\bigcap_{d \geq 1} \mathcal{U}_d''(p) = \mathbb{C}P^1(p),$$

where $\mathbb{C}P^1(p)$ is the projective line above $p \in \mathcal{E}(\mathbb{T}_\mathbb{C}^2)$. It follows that for large d the neighborhood $\mathcal{U}_d''$ is (topologically) small and it takes a lot of time for an orbit of $f_{d,X}$ to travel from a neighborhood $\mathcal{U}_d''(p_1)$ to another neighborhood $\mathcal{U}_d''(p_2)$. When an orbit travels through a neighborhood $\mathcal{U}_d''(p)$ it meets $\mathcal{B}_d$ at most twice and the rest of the time it spends in $\mathcal{V}_d \cup \mathcal{G}_d$. Hence, when an orbit travels through a neighborhood $\mathcal{U}_d''(p)$ we may have only up to four iterates when the differential is pinched between K^{-1} and K. These observations together with the standard adapted metric construction (see e.g., [Math68]) imply that there exists $d = d(K)$ and an adapted metric $g_{d,X}^{\text{adapted}}$ such that

$$\lambda^{-2}\|u\| < \|D_x f_{d,X}(u)\| < \lambda^2\|u\|, \tag{17}$$

for all $x \in X$ and $u \in T_x X$, where $\|\cdot\|$ is the norm induced by $g_{d,X}^{\text{adapted}}$.

We can check now that $(X, \|\cdot\|)$ and $f_{d,X}$ satisfy assumption $(*)$ of Section 8. Indeed, $\pi_2(X) \cong H_2(X; \mathbb{Z}) \cong \mathbb{Z}^{22}$ verifying $(*8.1)$. By Proposition 10.1, $\pi_2(f_{d,X}) = \pi_2(f_B) = (B^2, Id_{\mathbb{Z}^{20}})$ verifying $(*8.2)$ with $k \geq 2$. Finally, the inequalities of $(*8.3)$ also hold true because B^2 has eigenvalues λ^{-2}, λ^2 and we have verified (17). $\square$

11. Proof of the Main Theorem 1.1

Let X be the Kummer surface and let $B = \left(\begin{smallmatrix} 13 & 8 \\ 8 & 5 \end{smallmatrix}\right)$. Then by Propositions 10.1, 10.3 and 10.4 there exists a volume preserving, Bernoulli diffeomorphism $f \colon X \to X$ which verifies the collection of assumptions $(*)$ of Section 8. Moreover, because $\pi_2(f) = (B^2, id_{\mathbb{Z}^{20}})$ by Proposition 10.1, we can take any k in $[2, 20]$ and the splitting $\mathbb{Z}^{22} = \mathbb{Z}^k \oplus \mathbb{Z}^{22-k}$ will verify $(*8.2)$ and $(*8.3)$. The matrix $A \in SL(k, \mathbb{Z})$ from $(*8.3)$ is given by

$$A = diag(B^2, id_{\mathbb{Z}^{k-2}}).$$

By Lemma 5.8 there exist a map $h \colon X \to B\mathbb{T}^k$ such that $\pi_2(h) \colon \mathbb{Z}^k \oplus \mathbb{Z}^{m-k} \to \mathbb{Z}^k$ is the projection onto the first summand $\mathbb{Z}^k$. Let $\pi_h \colon E_h \to X$ be the pullback bundle $h^* E\mathbb{T}^k$. By Proposition 7.2 the total space E_h is simply connected. Also consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow{h} & & \downarrow{h} \\ B\mathbb{T}^k & \xrightarrow{\rho_A} & B\mathbb{T}^k \end{array}$$

Recall that, by Proposition 5.1, $\pi_2(\rho_A) = A$. Together with ($*8.2$), this implies that the above diagram commutes on the level of π_2, and hence homotopy commutes by Proposition 6.3. Then Theorem 6.2 applies and yields an A-map $F\colon E_h \to E_h$. By Theorem 8.2 the diffeomorphism F is partially hyperbolic. Because F is an A-map over a volume preserving diffeomorphism, Fubini's Theorem implies that F is also volume preserving.

To establish ergodicity start by removing the 3-skeleton of X and all its iterates under f. We obtain a subset $\bar{X} \subset X$ of full volume. Over $\bar{X}$ the bundle trivializes and the A-map F takes the form

$$F(x, y_1, y_2) = (f(x), B^2(y_1) + \alpha(x), y_2 + \beta(x)),$$

where $(y_1, y_2) \in \mathbb{T}^2 \times \mathbb{T}^{k-2} = \mathbb{T}^k$. After making the coordinate change $(x, y_1, y_2) \mapsto (x, y_1 + u(x), y_2)$, where $u(x) = (Id - B^2)^{-1}\alpha(x)$, F takes the form

$$F(x, y_1, y_2) = (f(x), B^2(y_1), y_2 + \beta(x))$$

Recall that f is Bernoulli, $B^2\colon \mathbb{T}^2 \to \mathbb{T}^2$ is Anosov and, hence, is also Bernoulli. Because the product of two Bernoulli automorphisms is also Bernoulli we can write

$$F(z, y_2) = (T(z), y_2 + \beta(z)),$$

where $z = (x, y_1)$, $\beta(z) = \beta(x)$ and T is Bernoulli. Note that this already solves the case $k = 2$. Now consider an F-invariant L_2 function and use Fourier decomposition with respect to the y_2-coordinate to see that F is ergodic (*i.e.*, the invariant function must be constant) if and only if the cohomological equation

$$\xi(Tz) - \xi(z) = \beta(z)$$

has a non-trivial solution ξ. Thus F is ergodic if $\int \beta(z)dvol \neq 0$.

Recall that $\mathbb{T}^k$ acts on E_h on the right by translation on the fiber. It is easy to see that $\rho \circ F$, $\rho \in \mathbb{T}^k$ is still an A-map and hence is volume preserving and partially hyperbolic. If $\int \beta(z)dvol \neq 0$ then consider

$$F' = \rho \circ F,$$

where $\rho = (0, \omega) \in \mathbb{T}^2 \times \mathbb{T}^{k-2}$, $\omega \neq 0$. In (z, y_2)-coordinates F' takes the form

$$F'(z, y_2) = (T(z), y_2 + \beta(z) + \omega).$$

Because $\int (\beta(z) + \omega)dvol = \omega \neq 0$ the diffeomorphism F' is ergodic.

We have constructed partially hyperbolic diffeomorphisms on simply connected manifolds of dimension 6 to 26. To obtain higher dimensional examples one can couple these examples or couple them with sufficiently slow ergodic diffeomorphisms of spheres.

12. Final remarks

12.1. **The six dimensional example.** Note that our 6 dimensional example is in fact Bernoulli. It is also easy to see that it is stably non dynamically coherent. Indeed, a center leaf would cover X, hence, would be a trivial one-to-one cover and give a section of the bundle, but the bundle E_h is non-trivial and, hence, does not admit sections.

12.2. **Real analytic version.** We believe that our examples can be made real analytic by modifying the base diffeomorphism. More specifically one only needs to change the definition of $B_{\varepsilon,d}$ in the following way

$$B_{\varepsilon,d}(x,y) = (13x - \varepsilon\sin(4d\pi x) + 8y, 8x - \varepsilon\sin(4d\pi x) + 5y), \quad \varepsilon \geq 0, \ \ d \in \mathbb{Z}_+.$$

One then has to work out a version of Lemma 10.5. Note that calculations become tedious; in particular, because the cubic term of $B_{\varepsilon,d}$ at $(0,0)$ affects the dynamics on $\mathbb{C}P^1$.

12.3. **Bunching.** By a more careful construction of the base diffeomorphism $f\colon X \to X$ one can obtain similar examples F that are also $(2-\varepsilon)$-bunched; that is, for any $\varepsilon > 0$ there exist a Riemannian metric $\|\cdot\|$ and $\lambda > 1$ such that for any unit vectors, v^s, v^c, v^u respectively in E^s, E^c, E^u we have that

$$\|DF(v^s)\| \leq \lambda^{-2}$$

$$\lambda^{1+\varepsilon}\|DF(v^s)\| < \|DF(v^c)\| < \lambda^{-1-\varepsilon}\|DF(v^u)\|$$

$$\lambda^2 \leq \|DF(v^u)\|$$

12.4. **2-connected example.** It is easy to see from long exact sequence of the fiber bundle that, when $k = 22$, our construction yields a partially diffeomorphism $F\colon E_h \to E_h$ of a simply connected, 2-connected, 26-dimensional manifold, *i.e.,* $\pi_1(E_h) = \pi_2(E_h) = 0$.

12.5. **Irreducibility.** We recall a definition introduced in [FG14, Section 7]. A partially hyperbolic diffeomorphism $F\colon N \to N$ is called *irreducible* if it verifies the following conditions:

1. the diffeomorphism F does not fiber over a (topologically) partially hyperbolic (or Anosov) diffeomorphism $\hat{F}\colon \hat{N} \to \hat{N}$ of a lower dimensional manifold $\hat{N}$; that is, one cannot find a fiber bundle $p\colon N \to \hat{N}$ and a (topologically) partially hyperbolic (or Anosov) diffeomorphism $\hat{F}\colon \hat{N} \to \hat{N}$ such that $p \circ F = \hat{F} \circ p$;
2. if F' is homotopic to F then F' also verifies 1;
3. if $\tilde{F}$ is a finite cover of F then $\tilde{F}$ also verifies 1 and 2.

Conjecture 12.1. *Our 6-dimensional example is irreducible.*

12.6. **A partially hyperbolic branched self-covering of $\mathbb{S}^3$.** Our construction can be applied to the Hopf bundle $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$. Namely, consider the Lattès map of $\mathbb{S}^2$ induced by multiplication by n on $\mathbb{T}^2$, $n \geq 2$. This is a rational map of degree n^2, which is self-covering outside of the ramification locus that consists of 4 points (see [M99, §7] for a detailed description). Then, by working through the A-map machinery, one obtains a self map of $\mathbb{S}^3$ that covers the Lattès map and which is given by multiplication by n^2 in the $\mathbb{S}^1$ fibers. Further, by slowing down the Lattès map at the ramification points, one can obtain a partially hyperbolic branched self-covering of $\mathbb{S}^3$ of degree n^4. In fact, we can use a rational (non-Lattès) map of the base coming from Theorem 1 of [BE14]. This map does not require further perturbation.

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