

On the Kurzweil-Henstock Integral in Probability

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Abstract

By using the method in [5], the aim of the present note is to generalize the Riemann integral in probability introduced in [7], to Kurzweil-Henstock integral in probability. Properties of the new integral are proved.

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1 Introduction

Let (E, B, P) be a field of probability, where E is a nonempty set, B a field of parts on E and P a composite probability on B . Let us denote by $L(E, B, P)$ the set of all real random variables defined on E and a.e. finite.

It is well-known the following concept :

Definition 1.1. (see e.g. [7], p. 50) We say that the random function $f : [a, b] \rightarrow L(E, B, P)$ (where $a, b \in \mathbb{R}, a < b$) is Riemann integrable in probability on $[a, b]$, if there exists a random variable $I = I(\omega) \in L(E, B, P)$ satisfying : for all $\varepsilon > 0, \eta > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that for all divisions $d : a = x_0 < x_1 < \dots < x_n = b$ with the norm $\nu(d) < \delta$ and all $\xi_i \in [x_i, x_{i+1}]$, $i \in \{0, \dots, n-1\}$, we have

$$P(\{\omega \in E; |S(f; d, \xi_i)(\omega) - I(\omega)| \geq \varepsilon\}) < \eta,$$

where $\nu(d) = \max\{x_{i+1} - x_i; i = 0, 1, \dots, n-1\}$ and

$$S(f; d, \xi_i)(\omega) = \sum_{i=0}^{n-1} f(\xi_i, \omega)(x_{i+1} - x_i).$$

In this case, $I(\omega)$ is called the Riemann integral in probability of f on $[a, b]$ and it is denoted by $I(\omega) = (P) \int_a^b f(t, \omega) dt$.

Remark. As it was proved in [7, p. 50], if $I_1(\omega), I_2(\omega)$ are Riemann integrals in probability of f on $[a, b]$, then $P(\{\omega \in E; I_1(\omega) \neq I_2(\omega)\}) = 0$.

Using the method in [5], in Section 2 we introduce the so-called Kurzweil-Henstock (Riemann generalized) integral in probability. Section 3 contains basic properties of this generalized integral.

2 The Kurzweil-Henstock Integral in Probability

Firstly, we recall some concepts in [5] we need for our purpose.

A tagged division of $[a, b]$ is of the form

$$d_S : a = x_0 \leq \xi_0 \leq x_1 < \dots < x_i \leq \xi_i \leq x_{i+1} < \dots < x_{n-1} \leq \xi_{n-1} \leq x_n = b.$$

A gauge on $[a, b]$ is an open interval valued function γ defined on $[a, b]$, such that $t \in \gamma(t)$, $t \in [a, b]$. A tagged division d_S of $[a, b]$ is called γ -sharp if $[x_i, x_{i+1}] \subset \gamma(\xi_i)$, for all $i \in \{0, \dots, n-1\}$.

Now, we are in position to introduce the following.

Definition 2.1. Let $f : [a, b] \rightarrow L(E, B, P)$. A random variable $I(\omega) \in L(E, B, P)$ is called a Kurzweil-Henstock (shortly (KH)) integral in probability of f on $[a, b]$, if : for all $\varepsilon > 0, \eta > 0$, there exists $\gamma_{\varepsilon, \eta}$ -gauge on $[a, b]$, such that for any tagged division d_S which is $\gamma_{\varepsilon, \eta}$ -sharp, we have

$$P(\{\omega \in E; |S(f; d_S, \xi_i)(\omega) - I(\omega)| \geq \varepsilon\}) < \eta,$$

where $S(f; d_S, \xi_i)(\omega) = \sum_{i=0}^{n-1} f(\xi_i, \omega)(x_{i+1} - x_i)$.

In this case, we write $I(\omega) = (KH) \int_a^b f(t, \omega) dt$.

Theorem 2.2. If $I_1(\omega), I_2(\omega)$ are (KH) integrals in probability of f on $[a, b]$, then $P(\{\omega \in E; I_1(\omega) \neq I_2(\omega)\}) = 0$.

Proof. We will prove that $P(\{\omega \in E; |I_1(\omega) - I_2(\omega)| > 0\}) = 0$. Let $\varepsilon, \eta > 0$. There exists the gauges $\gamma_{\varepsilon, \eta}^{(1)}, \gamma_{\varepsilon, \eta}^{(2)}$ on $[a, b]$, such that for any tagged divisions of $[a, b]$, $d_S^{(1)}, d_S^{(2)}$ which are $\gamma_{\varepsilon, \eta}^{(1)}$ -sharp and $\gamma_{\varepsilon, \eta}^{(2)}$ -sharp, respectively, we have

$$P(\{\omega \in E; |S(f; d_S^{(1)}, \xi_i)(\omega) - I_1(\omega)| \geq \varepsilon/2\}) < \eta/2,$$

$$P(\{\omega \in E; |S(f; d_S^{(2)}, \xi_i)(\omega) - I_2(\omega)| \geq \varepsilon/2\}) < \eta/2.$$

Let us define a new gauge on $[a, b]$ by $\gamma(t) = \gamma_{\varepsilon, \eta}^{(1)}(t) \cap \gamma_{\varepsilon, \eta}^{(2)}(t)$, $t \in [a, b]$. By [5, Section 1.8], there exists a γ -sharp tagged division d_S of $[a, b]$.

Since $\gamma(t) \subset \gamma_{\varepsilon, \eta}^{(1)}(t)$, $\gamma(t) \subset \gamma_{\varepsilon, \eta}^{(2)}(t)$, $t \in [a, b]$, obviously that d_S is $\gamma_{\varepsilon, \eta}^{(1)}$ -sharp and $\gamma_{\varepsilon, \eta}^{(2)}$ -sharp too.

We have

$$|I_1(\omega) - I_2(\omega)| \leq |I_1(\omega) - S(f; d_S, \xi_i)(\omega)| + |S(f; d_S, \xi_i)(\omega) - I_2(\omega)|,$$

which immediately implies

$$\{\omega \in E; |I_1(\omega) - I_2(\omega)| \geq \varepsilon\} \subset \{\omega \in E; |I_1(\omega) - S(f; d_S, \xi_i)(\omega)| \geq \varepsilon/2\}$$

$$\bigcup\{\omega \in E; |S(f; d_S, \xi_i)(\omega) - I_2(\omega)| \geq \varepsilon/2\}$$

and

$$\begin{aligned} P(\{\omega \in E; |I_1(\omega) - I_2(\omega)| \geq \varepsilon\}) &\leq P(\{\omega \in E; |I_1(\omega) - S(f; d_S, \xi_i)(\omega)| \geq \varepsilon/2\}) \\ &\quad + P(\{\omega \in E; |S(f; d_S, \xi_i)(\omega) - I_2(\omega)| \geq \varepsilon/2\}) < \eta/2 + \eta/2 = \eta. \end{aligned}$$

Now, considering $\varepsilon > 0$ fixed and passing to limit with $\eta \rightarrow 0$, we get $P(\{\omega \in E; |I_1(\omega) - I_2(\omega)| \geq \varepsilon\}) = 0$.

For $\varepsilon = \frac{1}{n}$, let us denote $A_n = \{\omega \in E; |I_1 - I_2| \geq 1/n\}$. Obviously $A_n \subset A_{n+1}$ and $\bigcup_{n=1}^{\infty} A_n = \{\omega \in E; |I_1(\omega) - I_2(\omega)| > 0\}$. Then,

$$P(\{\omega \in E; |I_1(\omega) - I_2(\omega)| > 0\}) = \lim_{n \rightarrow \infty} P(A_n) = 0,$$

which proves the theorem. \square

As in the case of usual real functions, another definition for the (KH) integral can be the following.

Definition 2.3. Let $f : [a, b] \rightarrow L(E, B, P)$. We say that f is Kurzweil-Henstock integrable in probability on $[a, b]$, if there exists $I \in L(E, B, P)$ with the property : for all $\varepsilon > 0$ and $\eta > 0$, there exists $\delta_{\varepsilon, \eta} : [a, b] \rightarrow \mathbb{R}_+$, such that for any division $d_S : a = x_0 < x_1 < \dots < x_n = b$ and any $\xi_i \in [x_i, x_{i+1}]$ with $x_{i+1} - x_i < \delta_{\varepsilon, \eta}(\xi_i)$, $i = 0, \dots, n-1$, we have

$$P(\{\omega \in E; |S(f; d_S, \xi_i)(\omega) - I(\omega)| \geq \varepsilon\}) < \eta.$$

Remarks. 1) The Definitions 2.1 and 2.3 are equivalent. Indeed, this easily follows from the fact that any function $\delta_{\varepsilon, \eta} : [a, b] \rightarrow \mathbb{R}_+$, generates the gauge $\gamma_{\varepsilon, \eta}(t) = (t - \delta_{\varepsilon, \eta}(t)/2, t + \delta_{\varepsilon, \eta}(t)/2)$, $t \in [a, b]$ and conversely, any gauge $\gamma_{\varepsilon, \eta}$ on $[a, b]$ (which obviously can be written in the form $\gamma_{\varepsilon, \eta}(t) = (t - \alpha(t), t + \beta(t))$, $\alpha(t), \beta(t) > 0$, $t \in [a, b]$) generates the function $\delta_{\varepsilon, \eta}(t) = \alpha(t) + \beta(t)$, $t \in [a, b]$, such that the (KH)-integrability which uses the function $\delta_{\varepsilon, \eta}$ is equivalent with the (KH)-integrability which uses the gauge $\gamma_{\varepsilon, \eta}$.

2) If $\delta_{\varepsilon, \eta}$ is a constant function, Definition 2.3 reduces to Definition 1.1.

3 Properties of the (KH)-Integral in Probability

In this section, we will prove some properties of the (KH)-integral in probability. Firstly, we need the following.

Definition 3.1. (see e.g. [3, p. 82], [4]). We say that $\varphi : [a, b] \rightarrow \mathbb{R}$ is Kurzweil-Henstock integrable on $[a, b]$, if there exists $I \in \mathbb{R}$, such that for all $\varepsilon > 0$, there exists $\delta_\varepsilon : [a, b] \rightarrow \mathbb{R}$, such that for any division $d : a = x_0 < x_1 < \dots < x_n = b$ and any $\xi_i \in [x_i, x_{i+1}]$ with $x_{i+1} - x_i < \delta(\xi_i)$, we have $|I - \sum_{i=0}^{n-1} \varphi(\xi_i)(x_{i+1} - x_i)| < \varepsilon$. We write $I = (KH) \int_a^b \varphi(t) dt$.

The following result holds.

Theorem 3.2. If $f : [a, b] \rightarrow L(E, B, P)$ is of the form $f(t, \omega) = \sum_{k=1}^p C_k(\omega) \cdot \varphi_k(t)$, where $C_k \in L(E, B, P)$ and φ_k are Kurzweil-Henstock integrable on $[a, b]$,

$k = 1, \dots, p$, then f is Kurzweil-Henstock integrable in probability on $[a, b]$ and we have

$$(KH) \int_a^b f(t, \omega) dt = \sum_{k=1}^p C_k(\omega) \cdot (KH) \int_a^b \varphi_k(t) dt.$$

Proof. Obviously that it is sufficient to consider only the case when $f(t, \omega) = C(\omega) \cdot \varphi(t)$, $\omega \in E$, $t \in [a, b]$.

If $d : a = x_0 < \dots < x_n = b$, $\xi_i \in [x_i, x_{i+1}]$, $i = 0, \dots, n-1$, then it is easy to see that $S(f; d, \xi_i)(\omega) = C(\omega) \cdot \sum_{i=0}^{n-1} \varphi(\xi_i) \cdot (x_{i+1} - x_i)$. Let us denote $I = (KH) \int_a^b \varphi(t) dt$ and $A_m = \{\omega \in E; |C(\omega)| \geq m\}$. Obviously, $A_{m+1} \subset A_m$, $m \in \mathbb{N}$. Denoting $A = \bigcap_{m=1}^{\infty} A_m$, since $C \in L(E, B, P)$ we get $P(A) = 0$ and $\lim_{m \rightarrow \infty} P(A_m) = P(A) = 0$. Consequently, if $\eta > 0$, there exists $N(\eta) \in \mathbb{N}$, such that

$$P(\{\omega \in E; |C(\omega)| \geq m\}) < \eta, m \in \mathbb{N}, m \geq N(\eta).$$

For fixed $m \geq N(\eta)$, let us consider $\varepsilon > 0$, such that $1/\varepsilon \geq m$. Now, for $\varepsilon^2 > 0$, since φ is Kurzweil-Henstock integrable on $[a, b]$, by Definition 3.1, there exists $\delta_{\varepsilon^2} : [a, b] \rightarrow \mathbb{R}$, such that for any division $d : a = x_0 < \dots < x_n = b$ and any $\xi_i \in [x_i, x_{i+1}]$ with $x_{i+1} - x_i < \delta_{\varepsilon^2}(\xi_i)$, $i = 0, \dots, n-1$, we have $|I - \sum_{i=0}^{n-1} \varphi(\xi_i)(x_{i+1} - x_i)| < \varepsilon^2$.

We have

$$\begin{aligned} & \{\omega \in E; |S(f; d, \xi_i)(\omega) - C(\omega) \cdot I| \geq \varepsilon\} \\ &= \{\omega \in E; |C(\omega)| \cdot |I - \sum_{i=0}^{n-1} \varphi(\xi_i)(x_{i+1} - x_i)| \geq \varepsilon\} \\ &\subset \{\omega \in E; |C(\omega)| \geq 1/\varepsilon\} \subset \{\omega \in E; |C(\omega)| \geq m\}, \end{aligned}$$

i.e. $P(\{\omega \in E; |S(f; d, \xi_i)(\omega) - C(\omega) \cdot I| \geq \varepsilon\}) < \eta$, for any division $d : a = x_0 < \dots < x_n = b$ and any $\xi_i \in [x_i, x_{i+1}]$ with $x_{i+1} - x_i < \delta_{\varepsilon^2}(\xi_i)$ (in fact, ε depends on m , which depends on η , therefore $\delta_{\varepsilon^2}(\xi_i)$ depends on η too).

Then, by Definition 2.3, we get

$$(KH) \int_a^b f(t, \omega) dt = C(\omega) \cdot (KH) \int_a^b \varphi(t) dt,$$

which proves the theorem. \square

Remark. Since the Kurzweil-Henstock integrability of a function $\varphi : [a, b] \rightarrow \mathbb{R}$ is more general than the Riemann integrability (in fact, it is equivalent with the so-called Denjoy-Perron integrability, see [2], [8]), Theorem 3.2 gives examples of random functions which are Kurzweil-Henstock integrable in probability on $[a, b]$ but are not Riemann integrable in probability in the sense of Definition 1.1.

For $p \geq 1$, let us consider

$$L^p(E, B, P) = \{g \in L(E, B, P); \int_E |g(\omega)|^p dP(\omega) < +\infty\},$$

where $\int_E |g(\omega)|^q dP(\omega)$ represents the q -th moment of the random variable g .

The following Fubini-type result holds.

Theorem 3.3. *Let $f : [a, b] \rightarrow L(E, B, P)$ be Kurzweil-Henstock integrable in probability on $[a, b]$ and such that there exists $A \in L^1(E, B, P)$, $A(\omega) \geq 0$, a.e. $\omega \in E$ with $P(\{\omega \in E; |f(t, \omega)| \leq A(\omega)\}) = 1$, for all $t \in [a, b]$. Then, $\varphi(t) = \int_E f(t, \omega) dP(\omega)$, $t \in [a, b]$, is Kurzweil-Henstock integrable on $[a, b]$ and*

$$(KH) \int_a^b \left[\int_E f(t, \omega) dP(\omega) \right] dt = \int_E \left[(KH) \int_a^b f(t, \omega) dt \right] dP(\omega).$$

Proof. Let us denote $I(\omega) = (KH) \int_a^b f(t, \omega) dt \in L(E, B, P)$. Since f is Kurzweil-Henstock integrable on $[a, b]$, for $\varepsilon > 0$ and $\eta = 1/m$, $m \in \mathbb{N}$, there exists $\delta_{\varepsilon, m} : [a, b] \rightarrow \mathbb{R}$, such that for any division $d_m : a = x_0^{(m)} < x_1^{(m)} < \dots < x_{n_m}^{(m)} = b$ and any $\xi_i^{(m)} \in [x_i^{(m)}, x_{i+1}^{(m)}]$, with $x_{i+1}^{(m)} - x_i^{(m)} < \delta_{\varepsilon, m}(\xi_i^{(m)})$, $i = 0, \dots, n_m - 1$, we have

$$P(\{\omega \in E; |S(f; d_m, \xi_i^{(m)}) (\omega) - I(\omega)| \geq \varepsilon\}) < 1/m, m \in \mathbb{N}.$$

This means that $S(f; d_m, \xi_i^{(m)}) (\omega) \rightarrow I(\omega)$ in probability, as $m \rightarrow \infty$.

On the other hand,

$$|S(f; d_m, \xi_i^{(m)}) (\omega)| = \left| \sum_{i=0}^{n_m-1} f(\xi_i^{(m)}, \omega) (x_{i+1}^{(m)} - x_i^{(m)}) \right| \leq A(\omega) \cdot (b-a), \text{ a.e. } \omega \in E$$

and taking into account the well-known property of the integral with respect to P , we immediately get

$$\begin{aligned} \int_E I(\omega) dP(\omega) &= \lim_{m \rightarrow \infty} \int_E S(f; d_m, \xi_i^{(m)}) (\omega) dP(\omega) \\ &= \lim_{m \rightarrow \infty} \sum_{i=0}^{n_m-1} \left[\int_E f(\xi_i^{(m)}, \omega) dP(\omega) \right] \cdot (x_{i+1}^{(m)} - x_i^{(m)}). \end{aligned}$$

Now, reasoning exactly as in the case of the definitions of Riemann integrability (see e.g. [6, p. 379-380 and p. 383-384]), it is easy to obtain that the Kurzweil-Henstock integrability in Definition 3.1 is equivalent with the fact that there exists a sequence $\delta_m : [a, b] \rightarrow \mathbb{R}$, $m \in \mathbb{N}$, such that for any sequence of divisions $(d_m)_{m \in \mathbb{N}}$, $d_m : a = x_0^{(m)} < x_1^{(m)} < \dots < x_{n_m}^{(m)} = b$, and any sequence $(\xi_i^{(m)})_{i=0, n_m-1}$ with $\xi_i^{(m)} \in [x_i^{(m)}, x_{i+1}^{(m)}]$, $x_{i+1}^{(m)} - x_i^{(m)} < \delta_m(\xi_i^{(m)})$, $i = 0, \dots, n_m - 1$, we have

$$\lim_{m \rightarrow \infty} S(\varphi, d_m, \xi_i^{(m)}) = I = (KH) \int_a^b \varphi(t) dt.$$

But, denoting $\varphi(t) = \int_E f(t, \omega) dP(\omega)$, $t \in [a, b]$, by the previous reasonings we immediately get that φ is Kurzweil-Henstock integrable on $[a, b]$ and

$$(KH) \int_a^b \left[\int_E f(t, \omega) dP(\omega) \right] dt = \int_E I(\omega) dP(\omega)$$

$$= \int_E \left[(KH) \int_a^b f(t, \omega) dt \right] dP(\omega),$$

which proves the theorem. \square

Remarks. 1) Theorem 3.3 is an analogue of Theorem III.8 in [7, p. 55].

2) Let $f, F : [a, b] \rightarrow \mathbb{R}$ be such that $F'(x) = f(x)$, $x \in (a, b)$. It is known (see e.g. [1]) that in this case f is (KH)-integrable on $[a, b]$ and $(KH) \int_a^b f(x) dx = F(b) - F(a)$.

Now, let $f, F : [a, b] \rightarrow L(E, B, P)$ be such that in each $t_0 \in (a, b)$, f is the derivative in probability of $F(t_0, \omega)$, i.e. for all $\varepsilon, \eta > 0$, there exists $\delta(\varepsilon, \eta) > 0$, such that for all $t \in [a, b]$, $t \neq t_0$, $|t - t_0| < \delta(\varepsilon, \eta)$, we have

$$P(\{\omega \in E; |[F(t_0, \omega) - F(t, \omega)] / (t - t_0) - f(t_0, \omega)| \geq \varepsilon\}) < \eta,$$

holds.

Then, the following question arises : in what conditions $f(t, \omega)$ is (KH)-integrable in probability on $[a, b]$ and

$$(KH) \int_a^b f(t, \omega) dP(\omega) = F(b, \omega) - F(a, \omega), \text{ a.e. } \omega \in E.$$

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