

# On the spectral radius of a class of non-odd-bipartite even uniform hypergraphs\*

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**Abstract:** In order to investigate the non-odd-bipartiteness of even uniform hypergraphs, starting from a simple graph  $G$ , we construct a generalized power of  $G$ , denoted by  $G^{k,s}$ , which is obtained from  $G$  by blowing up each vertex into a  $k$ -set and each edge into a  $(k-2s)$ -set, where  $s \leq k/2$ . When  $s < k/2$ ,  $G^{k,s}$  is always odd-bipartite. We show that  $G^{k,\frac{k}{2}}$  is non-odd-bipartite if and only if  $G$  is non-bipartite, and find that  $G^{k,\frac{k}{2}}$  has the same adjacency (respectively, signless Laplacian) spectral radius as  $G$ . So the results involving the adjacency or signless Laplacian spectral radius of a simple graph  $G$  hold for  $G^{k,\frac{k}{2}}$ . In particular, we characterize the unique graph with minimum adjacency or signless Laplacian spectral radius among all non-odd-bipartite hypergraphs  $G^{k,\frac{k}{2}}$  of fixed order, and prove that  $\sqrt{2+\sqrt{5}}$  is the smallest limit point of the non-odd-bipartite hypergraphs  $G^{k,\frac{k}{2}}$ . In addition we obtain some results for the spectral radii of the weakly irreducible nonnegative tensors.

**Keywords:** Hypergraph; non-odd-bipartiteness; adjacency tensor; signless Laplacian tensor; spectral radius

## 1 Introduction

Hypergraphs are a generalization of simple graphs. They are really handy to show complex relationships found in the real world. A *hypergraph*  $G = (V(G), E(G))$  is a set of vertices say  $V(G) = \{v_1, v_2, \dots, v_n\}$  and a set of edges, say  $E(G) = \{e_1, e_2, \dots, e_m\}$  where  $e_j \subseteq V(G)$ . If  $|e_j| = k$  for each  $j = 1, 2, \dots, m$ , then  $G$  is called a *k-uniform* hypergraph. In particular, the 2-uniform hypergraphs are exactly the classical simple graphs. The *degree*  $d_v$  of a vertex  $v \in V(G)$  is defined as  $d_v = |\{e_j : v \in e_j \in E(G)\}|$ . A *walk*  $W$  of length  $l$  in  $G$  is a sequences of alternate vertices and edges:  $v_0, e_1, v_1, e_2, \dots, e_l, v_l$ , where  $\{v_i, v_{i+1}\} \subseteq e_i$  for  $i = 0, 1, \dots, l-1$ . If  $v_0 = v_l$ , then  $W$  is called a *circuit*. A walk in  $G$  is called a *path* if no vertices or edges are repeated. A circuit in  $G$  is called a *cycle* if no vertices or edges are repeated. The hypergraph  $G$  is said to be

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*connected* if every two vertices are connected by a walk. A hypergraph  $H$  is a *sub-hypergraph* of  $G$  if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ , and  $H$  is a *proper sub-hypergraph* of  $G$  if  $V(H) \subsetneq V(G)$  or  $E(H) \subsetneq E(G)$ .

In recent years spectral hypergraph theory has emerged as an important field in algebraic graph theory. Let  $G$  be a  $k$ -uniform hypergraph. The *adjacency tensor*  $\mathcal{A} = \mathcal{A}(G) = (a_{i_1 i_2 \dots i_k})$  of  $G$  is a  $k$ th order  $n$ -dimensional symmetric tensor, where

$$a_{i_1 i_2 \dots i_k} = \begin{cases} \frac{1}{(k-1)!} & \text{if } \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\} \in E(G); \\ 0 & \text{otherwise.} \end{cases}$$

Let  $\mathcal{D} = \mathcal{D}(G)$  be a  $k$ th order  $n$ -dimensional diagonal tensor, where  $d_{i \dots i} = d_{v_i}$  for all  $i \in [n] := \{1, 2, \dots, n\}$ . Then  $\mathcal{L} = \mathcal{L}(G) = \mathcal{D}(G) - \mathcal{A}(G)$  is the *Laplacian tensor* of the hypergraph  $G$ , and  $\mathcal{Q} = \mathcal{Q}(G) = \mathcal{D}(G) + \mathcal{A}(G)$  is the *signless Laplacian tensor* of  $G$ .

Qi [15] showed that  $\rho(\mathcal{L}(G)) \leq \rho(\mathcal{Q}(G))$ , and posed a question of identifying the conditions under which the equality holds. Hu et al. [9] proved that if  $G$  is connected, then the equality holds if and only if  $k$  is even and  $G$  is odd-bipartite. Here an even uniform hypergraph  $G$  is called *odd-bipartite* if  $V(G)$  has a bipartition  $V(G) = V_1 \cup V_2$  such that each edge has an odd number of vertices in both  $V_1$  and  $V_2$ . Such partition will be called an *odd-bipartition* of  $G$ . Shao et al. [17] proved a stronger result that the Laplacian  $H$ -spectrum (respectively, Laplacian spectrum) and signless Laplacian  $H$ -spectrum (respectively, Laplacian spectrum) of a connected  $k$ -uniform hypergraph  $G$  are equal if and only if  $k$  is even and  $G$  is odd-bipartite. They also proved that the adjacency  $H$ -spectrum of  $G$  (respectively, adjacency spectrum) is symmetric with respect to the origin if and only if  $k$  is even and  $G$  is odd-bipartite. So, the non-odd-bipartite even uniform hypergraphs are more interesting on distinguishing the Laplacian spectrum and signless Laplacian spectrum and studying the non-symmetric adjacency spectrum.

Hu, Qi and Shao [10] introduced the *cored hypergraphs* and the *power hypergraphs*, where the cored hypergraph is one such that each edge contains at least one vertex of degree 1, and the  $k$ -th power of a simple graph  $G$ , denoted by  $G^k$ , is obtained by replacing each edge (a 2-set) with a  $k$ -set by adding  $k - 2$  new vertices. These two kinds of hypergraphs are both odd-bipartite.

Peng [13] introduced  $s$ -path and  $s$ -cycle. Suppose  $1 \leq s \leq k - 1$ . An  $s$ -path  $P$  of length  $d$  is a  $k$ -uniform hypergraph on  $s + d(k - s)$  vertices, say  $v_1, v_2, \dots, v_{s+d(k-s)}$ , such that  $\{v_{1+j(k-s)}, v_{2+j(k-s)}, \dots, v_{s+(j+1)(k-s)}\}$  is an edge of  $P$  for  $j = 0, \dots, d - 1$ . An  $s$ -cycle  $C$  of length  $d$  is a  $k$ -uniform hypergraph on  $d(k - s)$  vertices, say  $v_1, v_2, \dots, v_{d(k-s)}$ , such that  $\{v_{1+j(k-s)}, v_{2+j(k-s)}, \dots, v_{s+(j+1)(k-s)}\}$  is an edge of  $C$  for  $j = 0, \dots, d - 1$ , where  $v_{d(k-s)+j} = v_j$  for  $j = 1, \dots, s$ . When  $1 \leq s < \frac{k}{2}$ , an  $s$ -path or  $s$ -cycle is a cored hypergraph and hence it is odd-bipartite.

Up to now, the construction of non-odd-bipartite hypergraphs has rarely appeared. In Section 2 we proved that an  $s$ -path is always odd-bipartite. But this does not hold for  $s$ -cycles. However, when  $s = k/2$  for  $k$  being even, an  $s$ -cycle is odd-bipartite if and only if its length is even, which is consistent with the result on the bipartiteness of a simple cycle. Motivated by the discussion of  $s$ -cycles, we introduce a class of  $k$ -uniform hypergraphs, which is obtained from a simple graph by blowing up vertices and/or edges.

**Definition 1.1** Let  $G = (V, E)$  be a simple graph. For any  $k \geq 3$  and  $1 \leq s \leq k/2$ , the generalized power of  $G$ , denoted by  $G^{k,s}$ , is defined as the  $k$ -uniform hypergraph with the vertex set  $\{\mathbf{v} : v \in V\} \cup \{\mathbf{e} : e \in E\}$ , and the edge set  $\{\mathbf{u} \cup \mathbf{v} \cup \mathbf{e} : e = \{u, v\} \in E\}$ , where  $\mathbf{v}$  is an  $s$ -set containing  $v$  and  $\mathbf{e}$  is a  $(k - 2s)$ -set corresponding to  $e$ .

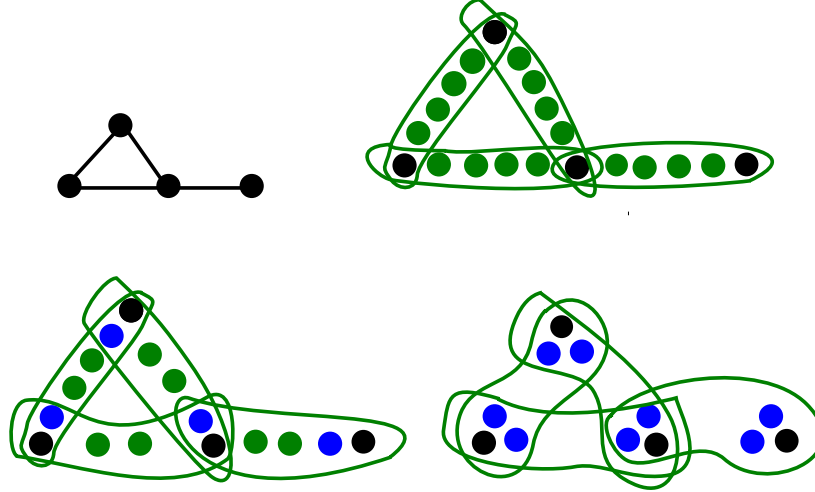


Fig. 1.1 Constructing power hypergraphs  $G^6$  (right upper),  $G^{6,2}$  (left below) and  $G^{6,3}$  (right below) from a simple graph  $G$  (left upper), where a closed curve represents an edge

Intuitively,  $G^{k,s}$  is obtained from  $G$  by replacing each vertex  $v$  by an  $s$ -subset  $\mathbf{v}$  and each edge  $\{u, v\}$  by a  $k$ -set obtained from  $\mathbf{u} \cup \mathbf{v}$  by adding  $(k - 2s)$  new vertices; see Fig. 1.1 for illustration. If  $s = 1$ , then  $G^{k,s}$  is exactly the  $k$ -th power hypergraph of  $G$ . When  $G$  is a path or a cycle, then  $G^{k,s}$  is an  $s$ -path or  $s$ -cycle for  $s \leq k/2$ . So the notion  $G^{k,s}$  is a generalization of the above hypergraphs.

Note that if  $s < k/2$ , then  $G^{k,s}$  is a cored hypergraphs and hence is odd-bipartite. If  $s = k/2$ , then  $G^{k,s}$  is obtained from  $G$  by only blowing up its vertices. In this case,  $\{u, v\}$  is an edge of  $G$  if and only  $\mathbf{u} \cup \mathbf{v}$  is an edge of  $G^{k, \frac{k}{2}}$ , where we use the black font  $\mathbf{v}$  to denote the blowing-up of the vertex  $v$  in  $G$ . For simplicity, we write  $uv$  rather than  $\{u, v\}$ ,  $\mathbf{uv}$  rather than  $\mathbf{u} \cup \mathbf{v}$ , and call  $\mathbf{u}$  a *half edge* of  $G^{k, \frac{k}{2}}$ . In Section 2, we show that  $G^{k, \frac{k}{2}}$  is non-odd-bipartite if and only if  $G$  is non-bipartite. So, we here give an explicit construction of non-odd-bipartite hypergraphs.

Another problem is how to apply the spectral theory of simple graphs to that of hypergraphs. In Section 3, we find that  $G^{k, \frac{k}{2}}$  has the same adjacency (respectively, signless Laplacian) spectral radius as  $G$ . So the results involving the adjacency or signless Laplacian spectral radius of a simple graph  $G$  hold for  $G^{k, \frac{k}{2}}$ . Here we concern two problems: the minimum adjacency or signless Laplacian spectral radius and the smallest limit point of the graphs  $G^{k, \frac{k}{2}}$ , which are addressed in Section 4 respectively.

In the paper [11] the authors proved that the smallest limit point of the adjacency spectral radii of the connected  $k$ -uniform hypergraphs is  $\rho_k = (k - 1)! \sqrt[k]{4}$ . (Note that if using our definition for the adjacency tensor, the limit point would be  $\sqrt[k]{4}$ .) They also classified all

connected  $k$ -uniform hypergraphs with spectral radii at most  $\rho_k$ , which are all cored hypergraphs for  $k \geq 5$ . (They used the notion of “reducible hypergraphs” instead of cored hypergraphs.) Even for  $k = 4$ , those graphs are not cored hypergraphs but still odd-bipartite hypergraphs. So, the next problem is to investigate the smallest limit point of the adjacency spectral radii of the connected  $k$ -uniform non-odd-bipartite hypergraphs. We start this problem by considering the class of hypergraphs  $G^{k, \frac{k}{2}}$  where  $G$  is non-bipartite.

It is known that a uniform hypergraph is connected if and only if its adjacency tensor is weakly irreducible. There are many results on the spectral theory of irreducible or weakly irreducible nonnegative tensor, e.g. [1, 6, 18, 19, 20]. However, to investigate the spectral radius of the adjacency tensor (or signless Laplacian tensor), we still need more results on the weakly irreducible nonnegative tensors. This will be discussed in Section 3.

## 2 Odd-bipartiteness of hypergraphs

We first discuss the odd-bipartiteness of  $s$ -paths and  $s$ -cycles.

**Lemma 2.1** *An  $s$ -path is always odd-bipartite where  $\frac{k}{2} \leq s \leq k - 1$ .*

**Proof.** Let  $P$  be an  $s$ -path of length  $d$ . If  $d = 1$ , the assertion holds clearly. Assume the assertion holds for all  $s$ -paths of length  $d < m$ . We prove it by induction on the length. Consider an  $s$ -path  $P$  of length  $m$ . Let  $e_m$  be the last edge of  $P$ . Note that  $P - e_m$  is an  $s$ -path, say  $P'$  of length  $m - 1$ , together with  $k - s$  isolated vertices. By induction,  $P'$  is odd-bipartite, which has an odd-bipartition  $V(P') = V_1 \cup V_2$ . Now, if  $|V_1 \cap e_m|$  is odd, put all vertices of  $e_m \setminus V(P')$  into  $V_2$ . Otherwise, take one vertex from  $e_m \setminus V(P')$  and put it into  $V_1$ , and put the remaining into  $V_2$ . Then we get an odd-bipartition of  $P$ . ■

What about the odd-bipartiteness of  $s$ -cycles when  $\frac{k}{2} \leq s \leq k - 1$ ? We first discuss the case of  $s = \frac{k}{2}$ . In this case, we use the notation  $C_m^{k, \frac{k}{2}}$  instead, where  $C_m$  denote a simple cycle of length  $m$ .

**Lemma 2.2** *The cycle  $C_m^{k, \frac{k}{2}}$  is odd-bipartite if and only if  $m$  is even.*

**Proof.** Let  $C := C_m^{k, \frac{k}{2}}$ . We have a partition of  $V(C) = V_1 \cup V_2 \cup \dots \cup V_m$  such that  $e_i := V_i \cup V_{i+1}$  is an edge of  $C$  for  $i = 1, 2, \dots, m$ , where  $V_{m+1} = V_1$ . Suppose that  $C$  is odd-bipartite, which has an odd-bipartition. We color the vertices in one part of the bipartition with red, and color the vertices in the other part with blue. Note that  $e_1 = V_1 \cup V_2$  contains an odd number of red vertices. Without loss of generality,  $V_1$  contains an odd number of red vertices. So  $V_2$  contains an even number of red vertices, and then  $V_3$  contains an odd number of red vertices by considering the edge  $e_2$ . Repeating the above discussion, we get that  $V_m$  contains an odd number of red vertices if  $m$  is odd, and even number of red vertices otherwise. However, if  $m$  is odd, then the edge  $e_m = V_m \cup V_1$  would contain an even number of red vertices, a contradiction. So  $m$  is necessarily even. On the other hand, if  $m$  is even, it is easy to give an odd-bipartition of  $C$ . ■

For general case, it may not be easy to determine under which conditions an  $s$ -cycle is odd-bipartite when  $\frac{k}{2} < s \leq k - 1$ . For example, let  $k = 4$ , a 3-cycle of length 8 is odd-bipartite, but a 3-cycle of length 6 is non-odd-bipartite. We will not investigate this problem further in this paper. By Lemma 2.2,  $C_m^{k, \frac{k}{2}}$  is non-odd-bipartite if and only if  $C_m$  is non-bipartite. We generalize this fact as follows.

**Theorem 2.3** *The hypergraph  $G^{k, \frac{k}{2}}$  is non-odd-bipartite if and only if  $G$  is non-bipartite.*

**Proof.** We prove an equivalent assertion:  $G^{k, \frac{k}{2}}$  is odd-bipartite if and only if  $G$  is bipartite. Assume that  $G^{k, \frac{k}{2}}$  is odd-bipartite. If  $G$  is a forest, surely it is bipartite. Otherwise, any cycle of  $G^{k, \frac{k}{2}}$  must have the form  $C_m^{k, \frac{k}{2}}$  for some positive integer  $m$ . Then  $C_m^{k, \frac{k}{2}}$  is also odd-bipartite and hence  $C_m$  is bipartite by Lemma 2.2. So,  $G$  is bipartite.

On the contrary, assume that  $G$  is bipartite, with a bipartition  $(V_1, V_2)$ . Extend this bipartition to a bipartition  $(\mathbf{V}_1, \mathbf{V}_2)$  of  $G^{k, \frac{k}{2}}$ , that is,  $\mathbf{V}_1$  (respectively,  $\mathbf{V}_2$ ) is obtained by replacing each vertex in  $V_1$  (respectively  $V_2$ ) by the corresponding half edge. Choosing an arbitrary vertex from each half edge in  $\mathbf{V}_1$  and forming a new set  $U_1$ , then  $(U_1, V(G^{k, \frac{k}{2}}) \setminus U_1)$  is an odd-bipartition of  $G^{k, \frac{k}{2}}$ . ■

### 3 Spectral radii and eigenvectors of hypergraphs

For integers  $k \geq 3$  and  $n \geq 2$ , a real *tensor* (also called *hypermatrix*)  $\mathcal{T} = (t_{i_1 \dots i_k})$  of order  $k$  and dimension  $n$  refers to a multidimensional array with entries  $t_{i_1 \dots i_k}$  such that  $t_{i_1 \dots i_k} \in \mathbb{R}$  for all  $i_j \in [n]$  and  $j \in [k]$ . The tensor  $\mathcal{T}$  is called *symmetric* if its entries are invariant under any permutation of their indices. Given a vector  $x \in \mathbb{R}^n$ ,  $\mathcal{T}x^k$  is a real number, and  $\mathcal{T}x^{k-1}$  is an  $n$ -dimensional vector, which are defined as follows:

$$\mathcal{T}x^k = \sum_{i_1, i_2, \dots, i_k \in [n]} t_{i_1 i_2 \dots i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, \quad (\mathcal{T}x^{k-1})_i = \sum_{i_2, \dots, i_k \in [n]} t_{i i_2 \dots i_k} x_{i_2} \cdots x_{i_k} \text{ for } i \in [n].$$

Let  $\mathcal{I}$  be the *identity tensor* of order  $k$  and dimension  $n$ , that is,  $i_{i_1 i_2 \dots i_k} = 1$  if and only if  $i_1 = i_2 = \cdots = i_k \in [n]$  and zero otherwise.

**Definition 3.1** [16] *Let  $\mathcal{T}$  be a  $k$ th order  $n$ -dimensional real tensor. For some  $\lambda \in \mathbb{C}$ , if the polynomial system  $(\lambda \mathcal{I} - \mathcal{T})x^{k-1} = 0$ , or equivalently  $\mathcal{T}x^{k-1} = \lambda x^{[k-1]}$ , has a solution  $x \in \mathbb{C}^n \setminus \{0\}$ , then  $\lambda$  is called an *eigenvalue* of  $\mathcal{T}$  and  $x$  is an *eigenvector* of  $\mathcal{T}$  associated with  $\lambda$ , where  $x^{[k-1]} := (x_1^{k-1}, x_2^{k-1}, \dots, x_n^{k-1}) \in \mathbb{C}^n$ .*

If  $x$  is a real eigenvector of  $\mathcal{T}$ , surely the corresponding eigenvalue  $\lambda$  is real. In this case,  $x$  is called an *H-eigenvector* and  $\lambda$  is called an *H-eigenvalue*. Furthermore, if  $x \in \mathbb{R}_+^n$  (the set of nonnegative vectors of dimension  $n$ ), then  $\lambda$  is called an *H<sup>+</sup>-eigenvalue* of  $\mathcal{T}$ ; if  $x \in \mathbb{R}_{++}^n$  (the set of positive vectors of dimension  $n$ ), then  $\lambda$  is said to be an *H<sup>++</sup>-eigenvalue* of  $\mathcal{T}$ . The *spectral radius* of  $\mathcal{T}$  is defined as

$$\rho(\mathcal{T}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{T}\}.$$

To generalize the classical Perron-Frobenius Theorem from nonnegative matrices to nonnegative tensors, we need the definition of the irreducibility of tensor. Chang et al. [1] introduced the irreducibility of tensor. A tensor  $\mathcal{T} = (t_{i_1 \dots i_k})$  of order  $k$  and dimension  $n$  is called *reducible* if there exists a nonempty proper subset  $I \subsetneq [n]$  such that  $t_{i_1 i_2 \dots i_k} = 0$  for any  $i_1 \in I$  and any  $i_2, \dots, i_k \notin I$ . If  $\mathcal{T}$  is not reducible, then it is called *irreducible*.

Friedland et al. [6] proposed a weak version of irreducible nonnegative tensors  $\mathcal{T}$ . The graph associated with  $\mathcal{T}$ , denoted by  $G(\mathcal{T})$ , is the directed graph with vertices  $1, \dots, n$  and an edge from  $i$  to  $j$  if and only if  $t_{i_1 i_2 \dots i_k} > 0$  for some  $i_l = j$ ,  $l = 2, \dots, k$ . The tensor  $\mathcal{T}$  is called *weakly irreducible* if  $G(\mathcal{T})$  is strongly connected. Surely, an irreducible tensor is always weakly irreducible. Pearson and Zhang [12] proved that the adjacency tensor of a uniform hypergraph  $G$  is weakly irreducible if and only if  $G$  is connected. Clearly, this shows that if  $G$  is connected, then  $\mathcal{A}(G)$ ,  $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$  are all weakly irreducible.

**Theorem 3.2 (The Perron-Frobenius Theorem for Nonnegative Tensors)**

1. (Yang and Yang 2010 [18]) *If  $\mathcal{T}$  is a nonnegative tensor of order  $k$  and dimension  $n$ , then  $\rho(\mathcal{T})$  is an  $H^+$ -eigenvalue of  $\mathcal{T}$ .*
2. (Friedland, Gaubert and Han 2011 [6]) *If furthermore  $\mathcal{T}$  is weakly irreducible, then  $\rho(\mathcal{T})$  is the unique  $H^{++}$ -eigenvalue of  $\mathcal{T}$ , with the unique eigenvector  $x \in \mathbb{R}_{++}^n$ , up to a positive scaling coefficient.*
3. (Chang, Pearson and Zhang 2008 [1]) *If moreover  $\mathcal{T}$  is irreducible, then  $\rho(\mathcal{T})$  is the unique  $H^+$ -eigenvalue of  $\mathcal{T}$ , with the unique eigenvector  $x \in \mathbb{R}_+^n$ , up to a positive scaling coefficient.*

**Theorem 3.3** [18, 20] *Let  $\mathcal{B}, \mathcal{C}$  be order  $k$  dimension  $n$  tensors satisfying  $|\mathcal{B}| \leq \mathcal{C}$ , where  $\mathcal{C}$  is weakly irreducible. Let  $\beta$  be an eigenvalue of  $\mathcal{B}$ . Then*

- (1)  $|\beta| \leq \rho(\mathcal{C})$ .
- (2) *if  $\beta = \rho(\mathcal{C})e^{i\varphi}$  and  $y$  is corresponding eigenvector, then all entries of  $y$  are nonzero, and  $\mathcal{C} = e^{-i\varphi} \mathcal{B} \cdot D^{-(k-1)} \cdot \overbrace{D \dots D}^{k-1}$ , where  $D = \text{diag}(\frac{y_1}{|y_1|}, \frac{y_2}{|y_2|}, \dots, \frac{y_n}{|y_n|})$ .*

**Corollary 3.4** *Suppose  $0 \leq \mathcal{B} \leq \mathcal{C}$ , where  $\mathcal{C}$  is weakly irreducible. Then  $\rho(\mathcal{B}) < \rho(\mathcal{C})$ .*

**Proof.** By Theorem 3.2(1),  $\rho(\mathcal{B})$  is an eigenvalue of  $\mathcal{B}$ , with a nonnegative eigenvector  $y$ . By Theorem 3.3,  $\rho(\mathcal{B}) \leq \rho(\mathcal{C})$ . If  $\rho(\mathcal{B}) = \rho(\mathcal{C})$ , then, also by Theorem 3.3,  $y > 0$ , and hence  $\mathcal{B} = \mathcal{C}$ ; a contradiction. ■

**Corollary 3.5** *Suppose  $G$  is a connected  $k$ -uniform hypergraph and  $H$  is a proper sub-hypergraph of  $G$ . Then  $\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(G))$  and  $\rho(\mathcal{Q}(H)) < \rho(\mathcal{Q}(G))$ .*

**Proof.** We only consider the adjacency tensor. The other case can be discussed in a similar manner. Observe that  $\mathcal{A}(G)$  is weakly irreducible. First assume  $V(H) = V(G)$ . Then  $\mathcal{A}(H) \leq \mathcal{A}(G)$ , which implies the result by Corollary 3.4. Secondly assume  $V(H) \subsetneq V(G)$ . Add the isolated vertices of  $V(G) \setminus V(H)$  to  $H$  such that the resulting hypergraph, say  $H'$ , has the same

vertex set as  $G$ . Then  $\rho(\mathcal{A}(H)) = \rho(\mathcal{A}(H'))$ ; or see [21, Theroem 3.2]. Noting that  $\mathcal{A}(H') \preceq \mathcal{A}(G)$ , we also get the result.  $\blacksquare$

Let  $\mathcal{B} \geq 0$ . Let  $x \in \mathbb{R}_{++}^n$ . Denote

$$r_i(\mathcal{B}) = \sum_{i_2, \dots, i_k=1}^n b_{ii_2 \dots i_k}, \quad s_i(\mathcal{B}, x) = \frac{(\mathcal{B}x^{k-1})_i}{x_i^{k-1}}, \quad \text{for } i = 1, 2, \dots, n.$$

The following two results give bounds for the spectral radius  $\rho(\mathcal{B})$  of a general nonnegative tensor  $\mathcal{B}$ . Here we impose an additional condition on  $\mathcal{B}$ , that is,  $\mathcal{B}$  is weakly irreducible, and characterize the equality cases.

**Lemma 3.6** [18, Lemma 5.2] *Let  $\mathcal{B} \geq 0$ . Then*

$$\min_{1 \leq i \leq n} r_i(\mathcal{B}) \leq \rho(\mathcal{B}) \leq \max_{1 \leq i \leq n} r_i(\mathcal{B}). \quad (3.1)$$

**Lemma 3.7** [18, Lemma 5.3] *Let  $\mathcal{B} \geq 0$ , and let  $x \in \mathbb{R}_{++}^n$ . Then*

$$\min_{1 \leq i \leq n} s_i(\mathcal{B}, x) \leq \rho(\mathcal{B}) \leq \max_{1 \leq i \leq n} s_i(\mathcal{B}, x). \quad (3.2)$$

**Lemma 3.8** *Let  $\mathcal{B} \geq 0$  and  $x \in \mathbb{R}_{++}^n$ . Suppose that  $\mathcal{B}$  is weakly irreducible. Then either equality in (3.1) holds if and only if  $r_1(\mathcal{B}) = r_2(\mathcal{B}) = \dots = r_n(\mathcal{B})$ ; either equality in (3.2) holds if and only if  $\mathcal{B}x^{k-1} = \rho(\mathcal{B})x^{[k-1]}$ .*

**Proof.** For completeness we restate the proof of (3.1) and (3.2) as in [18]. We first consider the equality cases of (3.1). Let  $\alpha = \min_{1 \leq i \leq n} r_i$ . If  $\alpha = 0$ , surely  $\rho(\mathcal{B}) \geq \alpha = 0$ , and  $\rho(\mathcal{B}) = 0$  if and only if  $\mathcal{B} = 0$  (see [21, Theorem 3.1]). So we assume that  $\alpha > 0$ . Let  $\mathcal{C}$  be a tensor with the same order and dimension as  $\mathcal{B}$  whose entries are defined as  $c_{i_1 i_2 \dots i_k} = \frac{\alpha}{r_{i_1}(\mathcal{B})} b_{i_1 i_2 \dots i_k}$ . Then  $0 \leq \mathcal{C} \leq \mathcal{B}$ , and by Theorem 3.3(1),  $\rho(\mathcal{B}) \geq \rho(\mathcal{C})$ . In addition,  $r_i(\mathcal{C}) = \alpha$  for each  $i = 1, 2, \dots, n$ , which implies  $\rho(\mathcal{C}) = \alpha$  (see [18, Lemma 5.1]). So we get  $\rho(\mathcal{B}) \geq \rho(\mathcal{C}) = \alpha$ . If  $\rho(\mathcal{B}) = \alpha$ , then  $\rho(\mathcal{B}) = \rho(\mathcal{C})$ , and then  $\mathcal{B} = \mathcal{C}$  by Corollary 3.4. This implies that  $r_i(\mathcal{B}) = \alpha$  for each  $i = 1, 2, \dots, n$ , and the necessity holds. The sufficiency is easily verified by [18, Lemma 5.1]. For the right equality of (3.1), the proof is similar.

Next we consider the equality cases of (3.2). Let  $D = \text{diag}(x_1, x_2, \dots, x_n)$ , and let  $\mathcal{E} = \mathcal{B} \cdot D^{-(k-1)} \cdot \overbrace{D \dots D}^{k-1}$ . Then  $\mathcal{E}$  and  $\mathcal{B}$  have the same eigenvalues ([20, Theorem 2.7]), which yields  $\rho(\mathcal{B}) = \rho(\mathcal{E})$ . In addition,  $\mathcal{E}$  is also weakly irreducible. Noting that  $r_i(\mathcal{E}) = s_i(\mathcal{B}, x)$  for  $i = 1, 2, \dots, n$ . By (3.1),

$$\min_{1 \leq i \leq n} s_i(\mathcal{B}, x) = \min_{1 \leq i \leq n} r_i(\mathcal{E}) \leq \rho(\mathcal{E}) \leq \max_{1 \leq i \leq n} r_i(\mathcal{E}) = \max_{1 \leq i \leq n} s_i(\mathcal{B}, x).$$

If the right equality holds, then all  $r_i(\mathcal{E})$ , and hence all  $s_i(\mathcal{B}, x)$ , have the same value, i.e.  $\mathcal{B}x^{k-1} = \rho(\mathcal{B})x^{[k-1]}$ . The proof for the right equality is similar.  $\blacksquare$

**Corollary 3.9** *Suppose that  $\mathcal{B}$  is a weakly irreducible nonnegative tensor. If there exists a vector  $y \succeq 0$  such that  $\mathcal{B}y^{k-1} \preceq \mu y^{[k-1]}$  (respectively,  $\mathcal{B}y^{k-1} \succeq \mu y^{[k-1]}$ ), then  $\rho(\mathcal{B}) < \mu$  (respectively,  $\rho(\mathcal{B}) > \mu$ ).*

**Proof.** Assume that  $\mathcal{B}y^{k-1} \preceq \mu y^{[k-1]}$ . By a similar discussion as in the proof of Theorem 1.4 (1) of [1], we get  $y > 0$ . From the inequality and by Lemma 3.7,

$$\rho(\mathcal{B}) \leq \max_{1 \leq i \leq n} s_i(\mathcal{B}, y) \leq \mu.$$

If  $\rho(\mathcal{B}) = \mu$ , then  $\rho(\mathcal{B}) = \max_{1 \leq i \leq n} s_i(\mathcal{B}, y)$ , which implies that  $\mathcal{B}y^{k-1} = \rho(\mathcal{B})y^{[k-1]}$ , a contradiction to the assumption.

Next we assume that  $\mathcal{B}y^{k-1} \succeq \mu y^{[k-1]}$ . By Theorem 5.3 of [18],

$$\rho(\mathcal{B}) = \max_{x \geq 0} \min_{x_i > 0} s_i(\mathcal{B}, x) \geq \mu.$$

If  $\rho(\mathcal{B}) = \mu$ , then  $\mathcal{B}y^{k-1} \succeq \rho(\mathcal{B})y^{[k-1]}$ , which implies that  $\mathcal{B}y^{k-1} = \rho(\mathcal{B})y^{[k-1]}$  by Lemma 3.5 of [20] as  $\mathcal{B}$  is weakly irreducible; a contradiction.  $\blacksquare$

For the adjacency tensor of a  $k$ -uniform hypergraph  $G$ , the eigenvector equation  $\mathcal{A}(G)x^{k-1} = \lambda x^{[k-1]}$  could be interpreted as

$$\lambda x_u^{k-1} = \sum_{\{u, u_2, u_3, \dots, u_k\} \in E(G)} x_{u_2} x_{u_3} \cdots x_{u_k}, \text{ for each } u \in V(G). \quad (3.3)$$

The eigenvector equation  $\mathcal{Q}(G)x^{k-1} = \lambda x^{[k-1]}$  could be interpreted as

$$[\lambda - d(u)]x_u^{k-1} = \sum_{\{u, u_2, u_3, \dots, u_k\} \in E(G)} x_{u_2} x_{u_3} \cdots x_{u_k}, \text{ for each } u \in V(G). \quad (3.4)$$

A hypergraph  $G$  is *isomorphic* to a hypergraph  $H$ , if there exists a bijection  $\sigma : V(G) \rightarrow V(H)$  such that  $\{v_1, v_2, \dots, v_k\} \in E(G)$  if and only if  $\{\sigma(v_1), \sigma(v_2), \dots, \sigma(v_k)\} \in E(H)$ . The bijection  $\sigma$  is called an *isomorphism* of  $G$  and  $H$ . If  $G = H$ , then  $\sigma$  is called an *automorphism* of  $G$ . Let  $x$  be a vector defined on  $V(G)$ . Denote  $x_\sigma$  to be the vector such that  $(x_\sigma)_u = x_{\sigma(u)}$  for each  $u \in V(G)$ .

**Lemma 3.10** *Let  $G$  be a  $k$ -uniform hypergraph and  $\sigma$  be an automorphism of  $G$ . Let  $x$  be an eigenvector of  $\mathcal{A}(G)$  (respectively,  $\mathcal{L}(G)$ ,  $\mathcal{Q}(G)$ ) associated with an eigenvalue  $\lambda$ . Then  $x_\sigma$  is also an eigenvector of  $\mathcal{A}(G)$  (respectively,  $\mathcal{L}(G)$ ,  $\mathcal{Q}(G)$ ) associated with  $\lambda$ .*

**Proof.** Let  $u \in V(G)$  be an arbitrary but fixed vertex. By Eq. (3.3), we have

$$\begin{aligned} (\mathcal{A}(G)x_\sigma^{k-1})_u &= \sum_{\{u, u_2, \dots, u_k\} \in E(G)} (x_\sigma)_{u_2} (x_\sigma)_{u_3} \cdots (x_\sigma)_{u_k} \\ &= \sum_{\{u, u_2, \dots, u_k\} \in E(G)} x_{\sigma(u_2)} x_{\sigma(u_3)} \cdots x_{\sigma(u_k)} \\ &= \sum_{\{\sigma(u), \sigma(u_2), \dots, \sigma(u_k)\} \in E(G)} x_{\sigma(u_2)} x_{\sigma(u_3)} \cdots x_{\sigma(u_k)} \\ &= \lambda x_{\sigma(u)}^{k-1} \\ &= \lambda (x_\sigma)_u^{k-1}, \end{aligned}$$

where the fourth equality is obtained from the eigenvector equation. Hence  $x_\sigma$  is also an eigenvector of  $\mathcal{A}(G)$  associated with the eigenvalue  $\lambda$ . The proof for  $\mathcal{L}(G)$  and  $\mathcal{Q}(G)$  is similar by the fact  $d_u = d_{\sigma(u)}$  for each  $u \in V(G)$ .  $\blacksquare$



**Lemma 3.11** *Let  $G$  be a connected simple graph, and let  $x > 0$  be an eigenvector of  $\mathcal{A}(G^{k, \frac{k}{2}})$  (respectively,  $\mathcal{Q}(G^{k, \frac{k}{2}})$ ). If  $u$  and  $v$  are the vertices in the same half edge of  $G^{k, \frac{k}{2}}$ , then  $x_u = x_v$ .*

**Proof.** Let  $\sigma$  be a permutation of  $V(G^{k, \frac{k}{2}})$  such that it interchanges  $u$  and  $v$  and fix all other vertices. It is easily seen that  $\sigma$  is an automorphism of  $G^{k, \frac{k}{2}}$ . Then by Lemma 3.10,  $x_\sigma$  is also an eigenvector of  $\mathcal{A}(G^{k, \frac{k}{2}})$  (respectively,  $\mathcal{Q}(G^{k, \frac{k}{2}})$ ). By Theorem 3.2(2),  $\mathcal{A}(G^{k, \frac{k}{2}})$  (respectively,  $\mathcal{Q}(G^{k, \frac{k}{2}})$ ) has a unique  $H^{++}$ -eigenvector up to a multiple, so  $x_\sigma = x$ , which implies the result. ■

Let  $G, x$  be defined as in Lemma 3.11. We will use  $x_{\mathbf{v}}$  to denote the common value of the vertices in the half edge  $\mathbf{v}$ .

**Lemma 3.12** *Let  $G$  be a connected simple graph, and let  $x > 0$  be vector defined on  $V(G)$ . Let  $\mathbf{x} > 0$  be a vector defined on  $V(G^{k, \frac{k}{2}})$  such that  $\mathbf{x}_u = x_{\frac{2}{v}^k}$  for each vertex  $u \in \mathbf{v}$ . Then  $x$  is an eigenvector of  $A(G)$  (respectively,  $\mathcal{Q}(G)$ ) corresponding to the spectral radius  $\rho$  if and only if  $\mathbf{x}$  is an eigenvector of  $A(G^{k, \frac{k}{2}})$  (respectively,  $\mathcal{Q}(G^{k, \frac{k}{2}})$ ) corresponding to the spectral radius  $\rho$ . Hence  $\rho(A(G)) = \rho(\mathcal{A}(G^{k, \frac{k}{2}}))$  and  $\rho(\mathcal{Q}(G)) = \rho(\mathcal{Q}(G^{k, \frac{k}{2}}))$ .*

**Proof.** Let  $\mathbf{x}_{\mathbf{v}}$  be the common value of the vertices in  $\mathbf{v}$  given by  $\mathbf{x}$ . The result follows by the following equivalent equations:

$$\begin{aligned} \rho x_u &= \sum_{uv \in E(G)} x_v \Leftrightarrow \rho \mathbf{x}_{\mathbf{u}}^{\frac{k}{2}} = \sum_{\mathbf{uv} \in E(G^{k, \frac{k}{2}})} \mathbf{x}_{\mathbf{v}}^{\frac{k}{2}} \Leftrightarrow \rho \mathbf{x}_{\mathbf{u}}^{k-1} = \sum_{\mathbf{uv} \in E(G^{k, \frac{k}{2}})} \mathbf{x}_{\mathbf{u}}^{\frac{k}{2}-1} \mathbf{x}_{\mathbf{v}}^{\frac{k}{2}}, \\ [\rho - d(u)]x_u &= \sum_{uv \in E(G)} x_v \Leftrightarrow [\rho - d(u)]\mathbf{x}_{\mathbf{u}}^{\frac{k}{2}} = \sum_{\mathbf{uv} \in E(G^{k, \frac{k}{2}})} \mathbf{x}_{\mathbf{v}}^{\frac{k}{2}} \Leftrightarrow [\rho - d(u)]\mathbf{x}_{\mathbf{u}}^{k-1} = \sum_{\mathbf{uv} \in E(G^{k, \frac{k}{2}})} \mathbf{x}_{\mathbf{u}}^{\frac{k}{2}-1} \mathbf{x}_{\mathbf{v}}^{\frac{k}{2}}, \end{aligned}$$

■

Lemma 3.12 establishes a relationship between the adjacency or signless Laplacian spectral radii of the simple graphs  $G$  and those of a class of hypergraphs  $G^{k, \frac{k}{2}}$ . So, the results involving the adjacency or signless Laplacian spectral radii of the simple graphs hold for such kind of hypergraphs.

## 4 Minimum spectral radius and smallest limit point

Let  $P_n, C_n$  be the (simple) path and cycle of order  $n$ , respectively. Denote  $\mathcal{G}_n$  (respectively,  ${}^{\text{nb}}\mathcal{G}_n$ ) the class of simple connected graphs (respectively, non-bipartite graphs) of order  $n$ . Denote  $\mathcal{G}_n^{k, \frac{k}{2}} = \{G^{k, \frac{k}{2}} : G \in \mathcal{G}_n\}$  and  ${}^{\text{nob}}\mathcal{G}_n^{k, \frac{k}{2}} = \{G^{k, \frac{k}{2}} : G \in {}^{\text{nb}}\mathcal{G}_n\}$ . By Lemma 3.12, for a connected graph  $G$ ,  $\rho(A(G)) = \rho(\mathcal{A}(G^{k, \frac{k}{2}}))$  and  $\rho(\mathcal{Q}(G)) = \rho(\mathcal{Q}(G^{k, \frac{k}{2}}))$ . So, the problem of finding the hypergraphs with minimal spectral radius of the adjacency or signless Laplacian tensor among all graphs in  $\mathcal{G}_n^{k, \frac{k}{2}}$  (respectively,  ${}^{\text{nob}}\mathcal{G}_n^{k, \frac{k}{2}}$ ) is equivalent to that of finding the simple graphs with minimal spectral radius of the adjacency or signless Laplacian matrix among all graphs in  $\mathcal{G}_n$  (respectively,  ${}^{\text{nb}}\mathcal{G}_n$ ). The results on the limit points of the adjacency or signless Laplacian spectral radii of simple graphs  $G$  also hold for the hypergraphs  $G^{k, \frac{k}{2}}$ .

Feng et.al [5] showed that among all graphs in  ${}^{\text{nb}}\mathcal{G}_n$ , the minimum adjacency spectral radius is achieved by  $C_n$  for odd  $n$ , and by  $C_{n-1}+e$  for even  $n$ , where  $C_{n-1}+e$  denotes the graph obtained from  $C_{n-1}$  by appending a pendant edge at some vertex. Similar result holds for the minimum signless Laplacian spectral radius. The proof technique is involved with the perturbation of the spectral radius of a graph after one of its edges is subdivided.

Let  $G$  be a simple graph containing an edge  $uw$ . Denote by  $G_{u,w}$  the graph obtained from  $G$  by *subdividing the edge  $uw$* , that is, by inserting a new vertex say  $v$  and forming two new edges  $uv$  and  $vw$  instead of the original edge  $uw$ . An *internal path*  $P$  of  $G$  is a sequence of edges  $u_1, u_2, \dots, u_l$ , such that all  $u_i$  are distinct (except possibly  $u_1 = u_l$ ),  $u_i u_{i+1}$  is an edge of  $G$  for  $i = 1, 2, \dots, l-1$ ,  $d(u_1) \geq 3$ ,  $d(u_2) = \dots = d(u_{l-1}) = 2$  (unless  $l = 2$ ), and  $d(u_l) \geq 3$ . Hoffman and Smith [7] gave the following result.

**Lemma 4.1** [7] *Let  $G$  be a simple connected graph of order  $n$ . If  $uw$  is an edge of  $G$  not on any internal path, and  $G \neq C_n$ , then  $\rho(A(G_{u,w})) > \rho(A(G))$ . If  $uw$  is an edge of  $G$  on an internal path, and  $G \neq T_n$ , then  $\rho(A(G_{u,w})) < \rho(A(G))$ , where  $T_n$  is obtained from  $P_{n-4}$  by appending two pendant edges at each of its two end points.*

With respect to the signless Laplacian matrix of a graph, Cvetković and Simć [2], and Feng, Li and Zhang [4] obtained the following similar result.

**Lemma 4.2** [2, 4] *Let  $G$  be a simple connected graph of order  $n$ . If  $uw$  is an edge of  $G$  not on any internal path, and  $G \neq C_n$ , then  $\rho(Q(G_{u,w})) > \rho(Q(G))$ . If  $uw$  is an edge of  $G$  on an internal path, then  $\rho(Q(G_{u,w})) < \rho(Q(G))$ .*

By Lemma 3.12, combining with Lemmas 4.1 and 4.2 we will have a parallel result for the graphs  $G^{k, \frac{k}{2}}$ . We will call  $P^{k, \frac{k}{2}}$  an *internal path of the hypergraph  $G^{k, \frac{k}{2}}$* , where  $P$  is an internal path of  $G$ . We use  $G_{\mathbf{u}, \mathbf{w}}^{k, \frac{k}{2}}$  to denote the hypergraph  $(G_{u,w})^{k, \frac{k}{2}}$ . Equivalently,  $G_{\mathbf{u}, \mathbf{w}}^{k, \frac{k}{2}}$  is obtained from  $G^{k, \frac{k}{2}}$  by *subdividing the edge  $\mathbf{uw}$* , that is, by inserting a half edge say  $\mathbf{v}$  and forming two new edges  $\mathbf{uv}$  and  $\mathbf{vw}$  instead of the original edge  $\mathbf{uw}$ .

**Corollary 4.3** *Let  $G$  be a connected simple graph of order  $n$ . If  $uw$  is an edge of  $G$  not on any internal path and  $G \neq C_n$ , then  $\rho(\mathcal{A}(G^{k, \frac{k}{2}})) < \rho(\mathcal{A}(G_{\mathbf{u}, \mathbf{w}}^{k, \frac{k}{2}}))$  and  $\rho(\mathcal{Q}(G^{k, \frac{k}{2}})) < \rho(\mathcal{Q}(G_{\mathbf{u}, \mathbf{w}}^{k, \frac{k}{2}}))$ .*

*If  $uw$  is an edge of  $G$  on an internal path, then  $\rho(\mathcal{Q}(G^{k, \frac{k}{2}})) < \rho(\mathcal{Q}(G_{\mathbf{u}, \mathbf{w}}^{k, \frac{k}{2}}))$ . If, in addition,  $G \neq T_n$ , then  $\rho(\mathcal{A}(G^{k, \frac{k}{2}})) > \rho(\mathcal{A}(G_{\mathbf{u}, \mathbf{w}}^{k, \frac{k}{2}}))$ .*

We can also prove Corollary 4.3 by a direct discussion following the approaches of Hoffman and Smith [7], Cvetković and Simć [2], and Feng, Li and Zhang [4], together with the using of Corollary 3.9.

It is known that the path  $P_n$  is the unique graph with the minimum adjacency or signless Laplacian spectral radius among all graphs in  $\mathcal{G}_n$ . So, by Lemma 3.12  $P_n^{k, \frac{k}{2}}$  is the unique one with the minimum adjacency or signless Laplacian spectral radius among all hypergraphs in

$\mathcal{G}_n^{k, \frac{k}{2}}$ . Applying Lemma 3.12 and the result of [5, Theorem 3.5] on the adjacency spectral radii of simple graphs, or using Corollaries 3.5 and 4.3, we get the following result on the minimizing hypergraphs in  $\text{nob}\mathcal{G}_n^{k, \frac{k}{2}}$ .

**Theorem 4.4** *Among all hypergraphs in  $\text{nob}\mathcal{G}_n^{k, \frac{k}{2}}$ , the minimum spectral radius of the adjacency tensor (respectively, the signless Laplacian tensor) is achieved uniquely by  $C_n^{k, \frac{k}{2}}$  for odd  $n$ , and achieved uniquely by  $(C_{n-1} + e)^{k, \frac{k}{2}}$  for even  $n$ .*

Hoffman [8] observed if a simple graph  $G$  properly contains a cycle, then  $\rho(A(G)) > \tau^{1/2} + \tau^{-1/2} = \tau^{3/2} = \sqrt{2 + \sqrt{5}}$ , where  $\tau = (\sqrt{5} + 1)/2$  is the golden mean. He proved that  $\tau^{3/2}$  is a limit point, and found all limit points of the adjacency spectral radii less than  $\tau^{3/2}$ . The work of Hoffman was extended by Shearer [14] to show that every real number  $r \geq \tau^{3/2}$  is the limit point of the adjacency spectral radii of simple graphs. Furthermore, Doob [3] proved that for each  $r \geq \tau^{3/2}$  (respectively,  $r \leq -\tau^{3/2}$ ) and for any  $k$ , there exists a sequences of graphs whose  $k$ th largest eigenvalue (respectively,  $k$ th smallest eigenvalues) converge to  $r$ . By Lemma 3.12, we get the following result on the hypergraphs  $\mathcal{G}_n^{k, \frac{k}{2}}$ , which correspond to the results of Hoffman [8] and Shearer [14] respectively. Denote  $\mathcal{G}^{k, \frac{k}{2}} = \cup_{n \in \mathbb{N}} \mathcal{G}_n^{k, \frac{k}{2}}$  and  $\text{nob}\mathcal{G}^{k, \frac{k}{2}} = \cup_{n \in \mathbb{N}} \text{nob}\mathcal{G}_n^{k, \frac{k}{2}}$ .

**Theorem 4.5** *For  $n = 1, 2, \dots$ , let  $\beta_n$  be the positive root of  $P_n(x) = x^{n+1} - (1 + x + x^2 + \dots + x^{n-1})$ . Let  $\alpha_n = \beta_n^{1/2} + \beta_n^{-1/2}$ . Then  $2 = \alpha_1 < \alpha_2 < \dots$  are all limit points of the hypergraphs in  $\mathcal{G}^{k, \frac{k}{2}}$  smaller than  $\tau^{1/2} + \tau^{-1/2} = \lim_n \alpha_n$ .*

**Theorem 4.6** *For any  $r \geq \tau^{3/2}$ , there exists a sequences of hypergraphs  $\mathcal{G}_{n_t}^{k, \frac{k}{2}}$  whose spectral radii converge to  $r$ .*

The smallest limit point of the adjacency spectral radii of simple graphs is 2, which is realized by a sequence of path. If  $r < \tau^{3/2}$  is a limit point, it suffices to consider the trees by Hoffman's observation. The construction of graphs whose adjacency spectral radii converge to  $r \geq \tau^{3/2}$  in [3, 14] are trees  $T(n_1, n_2, \dots, n_k)$  called *caterpillars*, which is obtained from a path on vertices  $v_1, v_2, \dots, v_k$  by attaching  $n_j \geq 0$  pendant edges at the vertex  $v_j$  for each  $j = 1, 2, \dots, k$ . We could not find any known sequence of non-bipartite graphs whose adjacency spectral radii converge. However, motivated by an example in Hoffman's work [8], we get the following result.

**Lemma 4.7**

$$\lim_{n \rightarrow \infty} \rho(A(C_{2n+1} + e)) = \tau^{3/2}.$$

**Proof.** Label the vertices of  $C_{2n+1} + e$  as follows: the pendant vertex is labeled by  $v_0$ , starting from the vertex of degree 3 the vertices of the cycle is labeled by  $v_1, v_2, \dots, v_{2n+1}$  clockwise. Note that now  $e = v_0v_1$ . Let  $x$  be a unit Perron vector of  $A(C_{2n+1} + e)$ , and let  $\rho := \rho(A(C_{2n+1} + e))$ . We assert that  $x_{v_1} > x_{v_2} > \dots > x_{v_{n+1}}$ . By symmetry,  $x_{v_k} = x_{v_{2n+3-k}}$  for  $k = 2, 3, \dots, n+1$ . By the eigenvector equation on the vertex  $v_{n+1}$ , noting that  $\rho > \tau^{3/2}$ , we get

$$x_{v_n} = \rho x_{v_{n+1}} - x_{v_{n+2}} = \rho x_{v_{n+1}} - x_{v_{n+1}} = (\rho - 1)x_{v_{n+1}} > x_{v_{n+1}}.$$

Assume that  $x_{v_{n-k+1}} > x_{v_{n-k+2}}$  for  $k \geq 1$ . Then

$$x_{v_{n-k}} = \rho x_{v_{n-k+1}} - x_{v_{n-k+2}} > (\rho - 1)x_{v_{n-k+1}} > x_{v_{n-k+1}}.$$

So we prove the assertion by induction. Note that

$$1 = \sum_{i=0}^{2n+1} x_{v_i}^2 > x_{v_1}^2 + 2(x_{v_2}^2 + \cdots + x_{v_{n+1}}^2) > (2n+1)x_{v_{n+1}}^2.$$

So  $2x_{v_{n+1}}^2 < \frac{2}{2n+1}$ . Noting that  $x_{v_{n+1}} = x_{v_{n+2}}$ , we have

$$\begin{aligned} \rho(A(C_{2n+1} + e)) &= \sum_{uv \in E(C_{2n+1} + e)} 2x_u x_v \\ &= x^T A(C_{2n+1} + e - v_{n+1}v_{n+2})x + 2x_{v_{n+1}}x_{v_{n+2}} \\ &< \rho(A(C_{2n+1} + e - v_{n+1}v_{n+2})) + \frac{2}{2n+1}. \end{aligned}$$

As  $\rho(A(C_{2n+1} + e))$  is decreasing in  $n$  by Lemma 4.1 and  $\rho(A(C_{2n+1} + e)) > \tau^{3/2}$ , the limit  $\lim_{n \rightarrow \infty} \rho(A(C_{2n+1} + e))$  exists. By the above inequality, we have

$$\tau^{3/2} \leq \lim_{n \rightarrow \infty} \rho(A(C_{2n+1} + e)) \leq \lim_{n \rightarrow \infty} \rho(A(C_{2n+1} + e - v_{n+1}v_{n+2})) = \tau^{3/2},$$

where the last equality follows from Proposition 3.6 of [8]. ■

By Theorem 2.3 and Lemma 3.12, we get the smallest limit point of the adjacency spectral radii of the non-odd-bipartite hypergraphs in  $\text{nob}\mathcal{G}^{k, \frac{k}{2}}$ .

**Corollary 4.8** *The value  $\tau^{3/2}$  is the smallest limit point of the spectral radii of the adjacency tensors of the non-odd-bipartite hypergraphs in  $\text{nob}\mathcal{G}^{k, \frac{k}{2}}$ .*

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