# Equivariant crystalline cohomology and base change ELMAR GROSSE-KLÖNNE

#### Abstract

Given a perfect field k of characteristic p > 0, a smooth proper k-scheme Y, a crystal E on Y relative to W(k) and a finite group G acting on Y and E, we show that, viewed as virtual k[G]-module, the reduction modulo p of the crystalline cohomology of E is the de Rham cohomology of E modulo p. On the way we prove a base change theorem for the virtual G-representions associated with G-equivariant objects in the derived category of W(k)-modules.

# 1 The Theorem

Let k be a perfect field of characteristic p > 0, let W denote its ring of Witt vectors, let  $K = \operatorname{Quot}(W)$ . Let Y be a proper and smooth k-scheme and suppose that the finite group G acts (from the right) on Y. Let E be a locally free, finitely generated crystal of  $\mathcal{O}_{Y/W}$ -modules and suppose that for each  $g \in G$  we are given an isomorphism of crystals  $\tau_g : E \to g^*E$  (where  $g^*E$  denotes the pull back of E via  $g : Y \to Y$ ) such that  $g_2^*(\tau_{g_1}) \circ \tau_{g_2} = \tau_{g_2g_1}$  (equality as maps  $E \to (g_2g_1)^*E = g_2^*g_1^*E)$  for any two  $g_1, g_2 \in G$ . For  $s \in \mathbb{Z}$  let  $H_{crys}^s(Y/W, E)$  denote the s-th crystalline cohomology group (relative to  $\operatorname{Spf}(W)$ ) of the crystal E, a finitely generated W-module which is zero if  $s \notin [0, 2 \dim(Y)]$ (see [1]). On the other hand, the reduction modulo p of the crystal E is equivalent with a locally free  $\mathcal{O}_Y$ -module  $E_k$  with connection  $E_k \to E_k \otimes_{\mathcal{O}_Y} \Omega_Y^1$ ; here  $\Omega_Y^1$  denotes the  $\mathcal{O}_Y$ module of differentials of Y/k. Let  $\Omega_Y^\bullet \otimes E_k$  denote the corresponding de Rham complex. The cohomology group  $H^s(Y, \Omega_Y^\bullet \otimes E_k)$  is a finite dimensional k-vector space which is zero if  $s \notin [0, 2 \dim(Y)]$ . The isomorphisms  $\tau_g$  for  $g \in G$  provide each  $H_{crys}^s(Y/W, E)$ , each  $H^s(Y, \Omega_Y^\bullet \otimes E_k)$  and each  $H^s(Y, \Omega_Y^t \otimes E_k)$  with an action of G (from the left). By

<sup>2000</sup> Mathematics Subject Classification. 14F30, 13D

Key words and phrases. crystalline cohomology, base change, virtual representation

definition, the reduction modulo p of the K[G]-module  $H^s_{crys}(Y/W, E) \otimes_W K$  is the k[G]module obtained by reducing modulo p the G-stable W-lattice  $H^s_{crys}(Y/W, E)/(\text{torsion})$ in  $H^s_{crys}(Y/W, E) \otimes_W K$ .

**Theorem 1.1.** For any j, the following three virtual k[G]-modules are the same: (i) the reduction modulo p of the virtual K[G]-module  $\sum_{s}(-1)^{s}H_{crys}^{s}(Y/W, E) \otimes_{W} K$ (ii)  $\sum_{s}(-1)^{s}H^{s}(Y, \Omega_{Y}^{\bullet} \otimes E_{k})$ (iii)  $\sum_{s,t}(-1)^{s+t}H^{s}(Y, \Omega_{Y}^{t} \otimes E_{k})$ .

An obvious variant of Theorem 1.1 holds in logarithmic crystalline cohomology, for crystals E on the logarithmic crystalline site of Y/W with respect to a log structure defined by a normal crossings divisor on Y. Similarly, the proof which we give below also shows the analog of Theorem 1.1 for the  $\ell$ -adic cohomology ( $\ell \neq p$ ) of constructible  $\ell$ -adic sheaves on Y, even if Y/k is not proper. Of course, the result in the  $\ell$ -adic case (even for non-poper Y/k is well known; it has been used for investigating the reduction modulo  $\ell$ of the Deligne-Lusztig characters of groups  $G = \mathbb{G}(\mathbb{F})$ , where  $\mathbb{G}$  is a reductive group over a finite field  $\mathbb{F}$  of characteristic p. In [3] we use the variant of Theorem 1.1 in logarithmic crystalline cohomology to show that these Deligne-Lusztig characters, usually defined via  $\ell$ -adic cohomology of certain  $\mathbb{F}$ -varieties which are non-proper in general, can also be expressed through the log crystalline cohomology of suitable log crystals on suitable proper and smooth F-varieties with a normal crossings divisor. Unfortunately, the (more geometric) proof of the  $\ell$ -adic analog of Theorem 1.1 (due to Deligne and Lusztig, see for example [2] Lemma 12.4 and A3.15) breaks down for crystalline cohomology. On the other hand, our proof of Theorem 1.1 contains a result (Theorem 2.1) on G-actions on strictly perfect complexes in the derived category which should be of independent interest.

### 2 The Proof

PROOF OF THEOREM 1.1: (ii)=(iii) is clear. By [1] we know that the total crystalline cohomology  $\mathbb{R}\Gamma_{crys}(Y/W, E)$ , as an object in the derived category D(W) of the category of W-modules, is represented by a complex of W-modules of finite tor-dimension and with finitely generated cohomology; by functoriality, G acts on  $\mathbb{R}\Gamma_{crys}(Y/W, E)$ . Also from [1] we know that the total crystalline cohomology commutes with base change, i.e. that  $\mathbb{R}\Gamma_{crys}(Y/W, E) \otimes_W^{\mathbb{L}} k$  is the total crystalline cohomology of the reduction modulo p of E (as a crystal relative to  $\operatorname{Spec}(k)$ ). But the latter is known (see [1] Corollary 7.4) to be the de Rham cohomology of  $E_k$ , i.e. its s-th cohomology group is  $H^s(Y, \Omega_Y^{\bullet} \otimes E_k)$ . Hence (i)=(ii) follows from Theorem 2.1 below. Let A be a complete discrete valuation ring with perfect residue field k of characteristic p > 0 and fraction field K of characteristic 0. Let  $L^{\bullet}$  be a complex of A-modules of finite tor-dimension and with finitely generated cohomology; by [1] Lemma 7.15 this is equivalent with saying that  $L^{\bullet}$  is quasiisomorphic to a strictly perfect complex, i.e. a bounded complex of finitely generated projective A-modules. Suppose the finite group G acts on  $L^{\bullet}$  when  $L^{\bullet}$  is viewed as an object in the derived category D(A) of the category of A-modules. Then each cohomology group  $H^{i}(L^{\bullet} \otimes_{A} K) = H^{i}(L^{\bullet}) \otimes_{A} K$  (resp. each cohomology group  $H^{i}(L^{\bullet} \otimes_{A} k)$ ) becomes a representation of G on a finite dimensional K-vector space (resp. k-vector space).

**Theorem 2.1.** The virtual k[G]-module  $\sum_i (-1)^i H^i(L^{\bullet} \otimes_A^{\mathbb{L}} k)$  is the reduction (modulo the maximal ideal of A) of the virtual K[G]-module  $\sum_i (-1)^i H^i(L^{\bullet}) \otimes_A K$ . Equivalently, the restriction of the character of  $\sum_i (-1)^i H^i(L^{\bullet}) \otimes_A K$  to the subset of p-regular elements of G is the Brauer character of  $\sum_i (-1)^i H^i(L^{\bullet} \otimes_A^{\mathbb{L}} k)$ .

We say that the automorphism  $\gamma$  of the finitely generated A-module M is prime to p if and only if the following holds. For any finite extension  $A' \supset A$  with a discrete valuation ring A' and for any two  $\gamma \otimes_A A'$ -stable submodules N, N' of  $M \otimes_A A'$  with  $N' \subset N$  and such that N/N' is a cyclic A'-module, the endomorphism which  $\gamma \otimes_A A'$  induces on N/N'is of finite order prime to p.

**Lemma 2.2.** Let  $\gamma$  be an automorphism of the finitely generated A-module M. (a) If M is free then  $\gamma$  is prime to p if and only if the roots of the characteristic polynomial of  $\gamma$  are roots of unity of order prime to p. In particular,  $\gamma|_N : N \to N$  is prime to p for each submodule N of M with  $\gamma(N) = N$ .

(b) Let  $M_1 \subset M$  be a submodule with  $\gamma(M_1) = M_1$  and such that  $M_2 = M/M_1$  is free. Let  $\gamma_1$ , resp.  $\gamma_2$ , be the induced automorphism of  $M_1$ , resp. of  $M_2$ . If  $\gamma_1$  and  $\gamma_2$  are prime to p, then  $\gamma$  is prime to p.

PROOF: Statement (a) is clear. (b) Let  $N' \subset N \subset M \otimes_A A'$  be as in the definition. If  $N \subset M_1 \otimes_A A'$  the hypothesis on  $\gamma_1$  applies. Otherwise, since  $M_2 \otimes_A A'$  is free over A'and N/N' is cyclic, N/N' maps injectively to  $M_2 \otimes_A A'$  and the hypothesis on  $\gamma_2$  applies.

PROOF OF THEOREM 2.1: The problem is of course that the  $H^i(L^{\bullet})$  may have torsion, i.e.  $H^i(L^{\bullet}) \otimes_A k \neq H^i(L^{\bullet} \otimes_A^{\mathbb{L}} k)$  in general. Similarly, the task would be easy if we knew that there is a strictly perfect complex  $K^{\bullet}$  quasiisomorphic to  $L^{\bullet}$  such that the action of G on  $L^{\bullet}$  in D(A) is given by the action of G on  $K^{\bullet}$  by true morphisms of complexes (not just by morphisms in D(A)). We introduce some notations. For an automorphism  $\gamma: L^{\bullet} \to L^{\bullet}$  in D(A) let  $\epsilon_1^i, \ldots, \epsilon_{n(i)}^i$  (with  $n(i) = \dim_k H^i(L^{\bullet} \otimes_A^{\mathbb{L}} k)$ ) denote the roots of the characteristic polynomial of  $\gamma$  acting on  $H^i(L^{\bullet} \otimes_A^{\mathbb{L}} k)$  and let  $\tilde{\epsilon}_1^i, \ldots, \tilde{\epsilon}_{n(i)}^i$  denote their Teichmüller liftings. On the other hand let  $\xi_1^i, \ldots, \xi_{n'(i)}^i$  (with  $n'(i) = \dim_K H^i(L^{\bullet}) \otimes_A K$ ) denote the roots of the characteristic polynomial of  $\gamma$  acting on  $H^i(L^{\bullet}) \otimes_A K$ . Then let

$$Br(\gamma, H^{\heartsuit}(L^{\bullet} \otimes_{A}^{\mathbb{L}} k)) = \sum_{i} (-1)^{i} \sum_{j=1}^{n(i)} \widetilde{\epsilon}_{j}^{i},$$
$$Tr(\gamma, H^{\heartsuit}(L^{\bullet}) \otimes_{A} K) = \sum_{i} (-1)^{i} \sum_{j=1}^{n'(i)} \xi_{j}^{i}.$$

What we must show is that for all *p*-regular elements  $g \in G$  (those whose order in G is not divisible by p), if  $\gamma : L^{\bullet} \to L^{\bullet}$  denotes the corresponding automorphism of  $L^{\bullet}$  in D(A), then

$$Br(\gamma, H^{\heartsuit}(L^{\bullet} \otimes_{A}^{\mathbb{L}} k)) = Tr(\gamma, H^{\heartsuit}(L^{\bullet}) \otimes_{A} K).$$

Clearly it is enough to show the following statement. For any strictly perfect complex  $L^{\bullet}$  of *A*-modules (not necessarily endowed with a *G*-action in D(A)) and for any automorphism  $\gamma: L^{\bullet} \to L^{\bullet}$  in D(A) which on the cohomology modules induces automorphisms prime to *p* we have

$$Br(\gamma, H^{\heartsuit}(L^{\bullet} \otimes_{A}^{\mathbb{L}} k)) = Tr(\gamma, H^{\heartsuit}(L^{\bullet}) \otimes_{A} K).$$

We use induction on the minimal  $m \in \mathbb{Z}_{\geq 0}$  with the following property: after a suitable degree shift we have  $L^i = 0$  for all  $i \notin [0, m]$ . For m = 0 the statement is clear from Lemma 2.2 (a). Now let  $m \ge 1$ ; shifting degrees we may assume  $L^i = 0$  for all  $i \notin [0, m]$ . Let  $d^m : L^{m-1} \to L^m$  denote the differential. Choose a sub-k-vector space  $N_k^{m-1}$  of  $L^{m-1} \otimes k$  which under  $d^m \otimes k$  maps isomorphically to the kernel of  $L^m \otimes k \to H^m(L^{\bullet} \otimes k)$  $k = H^m(L^{\bullet}) \otimes k$ . Then  $N_k^{m-1} = N^{m-1} \otimes k$  for a direct summand  $N^{m-1}$  of  $L^{m-1}$ . By construction,  $N^{m-1}$  maps isomorphically to its image  $N^m$  in  $L^m$ . Thus, setting  $N^i = 0$ if  $i \notin \{m-1, m\}$ , the subcomplex  $N^{\bullet}$  of  $L^{\bullet}$  is acyclic. Dividing it out we may therefore assume  $L^m \otimes k = H^m(L^{\bullet} \otimes k)$ . Since the functor  $K^-(\text{proj}-A) \to D(A)$  from the homotopy category of complexes of projective A-modules bounded above to D(A) is fully faithful, the action of  $\gamma$  on  $L^{\bullet}$  in D(A) is in fact represented by a true morphism of complexes  $\gamma^{\bullet}: L^{\bullet} \to L^{\bullet}$ . Base changing to a finite extension of A by a discrete valuation ring (this does not affect the numbers Br and Tr) we may suppose that the characteristic polynomial of  $\gamma^m : L^m \to L^m$  splits in A (we remark that  $\gamma^m$  is bijective: this follows from  $L^m \otimes k = H^m(L^{\bullet} \otimes k)$  and the fact that  $\gamma$  acts bijectively on  $H^m(L^{\bullet} \otimes k)$ ). We therefore find a  $\gamma^m$ -stable filtration

(1) 
$$(0) = F^0 \subset F^1 \subset \ldots \subset F^s = L^m \qquad (s = \operatorname{rk}(L^m))$$

such that  $G^e = F^e/F^{e-1}$  is free of rank one, for any  $1 \le e \le s$ . The cyclic A-module

$$\frac{F^{e}}{(F^{e} \cap \operatorname{im}(d^{m})) + F^{e-1}}$$

is a  $\gamma^m$ -stable subquotient of  $H^m(L^{\bullet})$  (it is non-zero because of  $L^m \otimes k = H^m(L^{\bullet} \otimes k)$ ), hence  $\gamma^m$  acts on it by multiplication with a root of unity of order prime to p. Let  $\xi_e \in A^{\times}$ denote its Teichmüller lifting. Choose  $\ell_e \in F^e$  which represents a basis element of  $G^e$ ; then  $\ell_1, \ldots, \ell_s$  is a basis of  $L^m$ . Modulo  $F^{e-1}$  the class of  $\xi_e \ell_e - \gamma^m(\ell_e) \in F^e$  lies in  $\operatorname{im}(d^m)$ . Choose a  $t_e \in L^{m-1}$  with  $d^m(t_e) = \xi_e \ell_e - \gamma^m(\ell_e)$  modulo  $F^{e-1}$ . Let  $t: L^m \to L^{m-1}$  denote the A-linear map which sends  $\ell_e$  to  $t_e$ , for each  $1 \leq e \leq s$ . Using t we see that we may modify  $\gamma^{\bullet}$  within its homotopy class to achieve that the filtration (1) is still  $\gamma^m$ -stable and such that  $\gamma^m$  acts on each  $G^e$  by multiplication with a root of unity of prime-to-*p*-order in  $A^{\times}$ . Therefore we may assume that  $\gamma^m : L^m \to L^m$  is prime to *p*. Let  $L_1^m = L^m$  and  $L_1^i = 0$  for  $i \neq m$ . Then  $L_1^{\bullet}$  is a  $\gamma^{\bullet}$ -stable subcomplex of  $L^{\bullet}$  and since  $Br(\gamma)$  and  $Tr(\gamma)$  are additive in exact  $\gamma^{\bullet}$ -equivariant sequences of complexes it suffices to show  $Br(\gamma) = Tr(\gamma)$  for the complexes  $L_1^{\bullet}$  and  $L^{\bullet}/L_1^{\bullet}$ . Since these complexes are shorter than  $L^{\bullet}$  this follows from the induction hypothesis. Indeed, the prime to p hypothesis is clearly satisfied for  $L_1^{\bullet}$  so it remains to show that  $\gamma^{\bullet}$  induces automorphisms prime to p on the cohomology modules of  $L^{\bullet}/L_{1}^{\bullet}$ . In degrees smaller than m-1 this is clear from the corresponding hypothesis on  $L^{\bullet}$ , only  $H^{m-1}(L^{\bullet}/L_1^{\bullet})$  is critical. But  $H^{m-1}(L^{\bullet})$ is a submodule of  $H^{m-1}(L^{\bullet}/L_1^{\bullet})$  and the quotient  $Q = H^{m-1}(L^{\bullet}/L_1^{\bullet})/H^{m-1}(L^{\bullet})$  maps isomorphically to a submodule of  $L_1^m = L^m$ . By Lemma 2.2 (b) it suffices to show that  $\gamma^{\bullet}$  induces automorphisms prime to p on  $H^{m-1}(L^{\bullet})$  and on Q. For  $H^{m-1}(L^{\bullet})$  this holds by hypothesis, for Q this follows from Lemma 2.2 (a). 

## References

- P. Berthelot, A. Ogus, Notes on crystalline cohomology. Princeton University Press (1978)
- M. Cabanes, M. Enguehard, Representation theory of finite reductive groups. New Mathematical Monographs, 1. Cambridge University Press, Cambridge (2004)
- [3] E. Grosse-Klönne, On the crystalline cohomology of Deligne-Lusztig varieties, preprint