

A Generalization of the Petrov Strong Law of Large Numbers

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Abstract

In 1969 V.V. Petrov found a new sufficient condition for the applicability of the strong law of large numbers to sequences of independent random variables. He proved the following theorem: let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with finite variances and let $S_n = \sum_{k=1}^n X_k$. If $\text{Var}(S_n) = O(n^2/\psi(n))$ for a positive non-decreasing function $\psi(x)$ such that $\sum 1/(n\psi(n)) < \infty$ (Petrov's condition) then the relation $(S_n - ES_n)/n \rightarrow 0$ a.s. holds.

In 2008 V.V. Petrov showed that under some additional assumptions Petrov's condition remains sufficient for the applicability of the strong law of large numbers to sequences of random variables without the independence condition.

In the present work, we generalize Petrov's results (for both dependent and independent random variables), using an arbitrary norming sequence in place of the classical normalization.

Keywords: strong law of large numbers, sequences of independent random variables, dependent random variables.

1. Introduction

Following [8], we denote by Ψ_c (or, respectively, Ψ_d) the set of functions $\psi(x)$ such that $\psi(x)$ is positive and non-decreasing in the interval $x > x_0$ for some x_0 and the series $\sum \frac{1}{n\psi(n)}$ converges (respectively, diverges). The value x_0 is not assumed to be the same for different functions ψ . Examples of functions of the class Ψ_c are the functions x^δ and $(\log x)^{1+\delta}$ for any $\delta > 0$. The functions $\log x$ and $\log \log x$ belong to the class Ψ_d .

The next result is classical Kolmogorov's theorem:

Theorem A. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with finite variances $\text{Var}(X_n)$ and let $S_n = \sum_{k=1}^n X_k$. If*

$$\sum_{n=1}^{\infty} \frac{\text{Var}(X_n)}{n^2} < \infty \quad (1)$$

then

$$\frac{S_n - ES_n}{n} \rightarrow 0 \quad a.s. \quad (2)$$

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Another sufficient condition for the applicability of the strong law of large numbers to sequences of independent random variables was founded by Petrov [7] (see also [8]).

Theorem B (Petrov). *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with finite variances. If*

$$\text{Var}(S_n) = O\left(\frac{n^2}{\psi(n)}\right) \quad \text{for some function } \psi \in \Psi_c, \quad (3)$$

then relation (2) holds.

Relation (3) will be called Petrov's condition. It is known [7] (see also [3]) that condition (3) in Theorem B is optimal in the following sense: it is impossible to replace condition (3) by the weaker assumption that corresponds to the replacement of $\psi \in \Psi_c$ by some function $\psi \in \Psi_d$.

If the random variables X_1, X_2, \dots are independent, then Petrov's condition is equivalent to the requirement that

$$\sum_{k=1}^n \text{Var}(X_k) = O\left(\frac{n^2}{\psi(n)}\right) \quad \text{for some function } \psi \in \Psi_c. \quad (4)$$

It is proved ([5, Theorem 1]) that (4) implies (1). It follows that theorem B is a consequence of Kolmogorov's theorem (Theorem A). Nevertheless, Petrov proved [9, 10] that under some additional assumptions Petrov's condition is sufficient for the applicability of the strong law of large numbers to sequences of random variables without any independence assumptions.

Theorem C (Petrov [9]). *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of non-negative random variables with finite variances. Suppose that conditions (3) is satisfied and*

$$E(S_n - S_m) \leq C(n - m) \quad \text{for all sufficiently large } n - m, \quad (5)$$

where C is a constant. Then relation (2) holds.

It is proved in [6] the next generalization of Theorem C:

Theorem D (Petrov and Korchevsky). *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of non-negative random variables with finite absolute moments of some order $p \geq 1$. Suppose that condition (5) is satisfied and*

$$E|S_n - ES_n|^p = O\left(\frac{n^p}{\psi(n)}\right) \quad \text{for some function } \psi \in \Psi_c.$$

Then relation (2) holds.

(Theorem C corresponds to the case $p = 2$).

The aim of present work is to generalize Theorems B and D using an arbitrary norming sequence in place of the classical normalization. Also we present a generalization of Theorem 1 in [5].

To prove the theorems of the work we use methods developed by Petrov [9, 10], Chandra and Goswami [1], and Csörgő, Tandori, and Totik [2].

2. Main results

Theorem 1. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of non-negative random variables with finite absolute moments of some order $p \geq 1$. Assume that $\{a_n\}_{n=1}^{\infty}$ is non-decreasing unbounded sequence of positive numbers. If*

$$ES_n = O(a_n) \tag{6}$$

and

$$E|S_n - ES_n|^p = O\left(\frac{a_n^p}{\psi(a_n)}\right) \quad \text{for some function } \psi \in \Psi_c, \tag{7}$$

then

$$\frac{S_n - ES_n}{a_n} \rightarrow 0 \quad \text{a.s.} \tag{8}$$

Theorem 1 generalizes Theorem D, which corresponds to the case $a_n = n$ for all $n \geq 1$. Moreover, in the case $a_n = n$ for all $n \geq 1$, condition (6) is less restrictive than assumption (5).

Let us indicate two consequences of Theorem 1.

Theorem 2. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of non-negative random variables with finite variances. Assume that $\{a_n\}_{n=1}^{\infty}$ is non-decreasing unbounded sequence of positive numbers. If condition (6) is satisfied and*

$$\text{Var}(S_n) = O\left(\frac{a_n^2}{\psi(a_n)}\right) \quad \text{for some function } \psi \in \Psi_c,$$

then relation (8) holds.

We arrive at this proposition putting $p = 2$ in Theorem 1.

Theorem 3. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of non-negative random variables with finite absolute moments of some order $p \geq 1$. Assume that $\{w_n\}_{n=1}^{\infty}$ is a sequence of positive numbers,*

$$W_n = \sum_{k=1}^n w_k, \quad T_n = \sum_{k=1}^n w_k X_k.$$

Suppose that $W_n \rightarrow \infty$ ($n \rightarrow \infty$),

$$ET_n = O(W_n), \tag{9}$$

$$E|T_n - ET_n|^p = O\left(\frac{W_n^p}{\psi(W_n)}\right) \quad \text{for some function } \psi \in \Psi_c.$$

Then

$$\frac{T_n - ET_n}{W_n} \rightarrow 0 \quad \text{a.s.}$$

Theorem 3 is a generalization of Theorem 1 in [6] which includes condition

$$\sum_{k=m}^n w_k EX_k \leq C \sum_{k=m}^n w_k \quad \text{for all sufficiently large } n - m,$$

instead assumption (9). To prove Theorem 3 we can put $a_n = W_n$, $Y_n = w_n X_n$ for all $n \geq 1$ and apply Theorem 1 to the sequence of random variables $\{Y_n\}_{n=1}^{\infty}$.

The next theorem generalizes Theorem B, which corresponds to the case $a_n = n$ for all $n \geq 1$.

Theorem 4. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with finite variances. Assume that $\{a_n\}_{n=1}^{\infty}$ is non-decreasing unbounded sequence of positive numbers such that*

$$\frac{a_{2n}}{a_n} \leq Q \quad \text{for all sufficiently large } n, \quad (10)$$

where Q is a constant. If

$$\text{Var}(S_n) = O\left(\frac{a_n^2}{\psi(n)}\right) \quad \text{for some function } \psi \in \Psi_c, \quad (11)$$

then relation (8) holds.

Remark 1. *We cannot omit condition (10) in Theorem 4 (See Example 1 below).*

As mentioned above, in [5] was proved that condition (4) implies (1). The next theorem generalizes this result.

Theorem 5. *Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of non-negative numbers. Assume that $\{a_n\}_{n=1}^{\infty}$ is non-decreasing unbounded sequence of positive numbers such that condition (10) is satisfied. If*

$$\sum_{k=1}^n b_k = O\left(\frac{a_n^2}{\psi(n)}\right) \quad \text{for some function } \psi \in \Psi_c, \quad (12)$$

then

$$\sum_{n=1}^{\infty} \frac{b_n}{a_n^2} < \infty. \quad (13)$$

Remark 2. *We cannot omit condition (10) in Theorem 5.*

Indeed, let $b_1 = b_2 = 1$, $b_n = 2^n/n - 2^{n-1}/(n-1)$ for all $n \geq 3$. Then

$$\sum_{k=1}^n b_k = \frac{2^n}{n} \quad \text{for all } n \geq 3.$$

Thus, the sequence $\{b_n\}_{n=1}^{\infty}$ satisfies condition (12) with $a_n = 2^{n/2}$, $n \geq 1$ and function $\psi(x) = x$, $x > 0$ (belonging to Ψ_c). But relation (13) does not hold since

$$\sum_{n=3}^{\infty} \frac{b_n}{a_n^2} = \sum_{n=3}^{\infty} \frac{2^n/n - 2^{n-1}/(n-1)}{2^n} = \sum_{n=3}^{\infty} \frac{n-2}{2n(n-1)} = \infty.$$

3. Proofs

To prove Theorems 1 and 4 we need the following proposition.

Lemma 1 (see [9]). *If $\psi(x) \in \Psi_c$, then the series $\sum 1/\psi(b^n)$ converges for every $b > 1$.*

Proof of Theorem 1. By assumption (6) there is a constant A such that inequality

$$ES_n/a_n \leq A$$

is satisfied for each $n \geq 1$. Let $\alpha > 1$, $\varepsilon > 0$ and $L = [A/\varepsilon]$, the integer part of A/ε . Put

$$m_1 = \inf\{m \geq 0 : \alpha^m \leq a_n < \alpha^{m+1} \text{ for some } n\},$$

$$m_l = \inf\{m > m_{l-1} : \alpha^m \leq a_n < \alpha^{m+1} \text{ for some } n\} \quad \text{for } l \geq 2.$$

We recall that $a_n \uparrow \infty$, so $\{m_l\}_{l=1}^{\infty}$ is a subsequence of integers satisfying $0 \leq m_1 < m_2 < \dots$ and $m_l \rightarrow \infty$ ($l \rightarrow \infty$). For each pair of integers l and s such that $l \geq 1$, $s = 0, 1, \dots, L$, put

$$A_s(l) = \{k : \alpha^{m_l} \leq a_k < \alpha^{m_l+1}, \frac{ES_k}{a_k} \in [s\varepsilon, (s+1)\varepsilon]\}.$$

Let $k_s^-(l) = \inf A_s(l)$, $k_s^+(l) = \sup A_s(l)$, if the set $A_s(l)$ is not empty, and let $k_s^-(l) = k_s^+(l) = \inf\{k : \alpha^{m_l} \leq a_k < \alpha^{m_l+1}\}$ otherwise.

By the definition of $k_s^{\pm}(l)$ for any $l \geq 1$ and $s = 0, 1, \dots, L$ we have

$$a_{k_s^{\pm}(l)} \geq \alpha^{m_l}.$$

Hence, using assumption (7) and Lemma 1, by Chebyshev's inequality for any $s = 0, 1, \dots, L$ and $\lambda > 0$ we obtain

$$\begin{aligned} \sum_{l=1}^{\infty} P\left(\left|\frac{S_{k_s^{\pm}(l)} - ES_{k_s^{\pm}(l)}}{a_{k_s^{\pm}(l)}}\right| > \lambda\right) &\leq \frac{1}{\lambda^p} \sum_{l=1}^{\infty} \frac{E|S_{k_s^{\pm}(l)} - ES_{k_s^{\pm}(l)}|^p}{(a_{k_s^{\pm}(l)})^p} \leq \\ &\leq C\lambda^{-p} \frac{1}{\psi(a_{k_s^{\pm}(l)})} \leq C\lambda^{-p} \frac{1}{\psi(\alpha^{m_l})} < \infty. \end{aligned}$$

The application of Borel–Cantelli lemma yields to

$$\frac{S_{k_s^{\pm}(l)} - ES_{k_s^{\pm}(l)}}{a_{k_s^{\pm}(l)}} \rightarrow 0 \quad \text{a.s.} \quad (l \rightarrow \infty) \quad (14)$$

for any $s = 0, 1, \dots, L$.

Now for any natural number n there exists $l = l(n)$ and $s = s(n)$, $\lim_{n \rightarrow \infty} l(n) = \infty$, $0 \leq s(n) \leq L$ such that

$$\alpha^{m_l} \leq a_n < \alpha^{m_l+1}, \quad \frac{ES_n}{a_n} \in [s\varepsilon, (s+1)\varepsilon).$$

By the definition of $k_s^\pm(l)$ we have

$$k_s^-(l) \leq n \leq k_s^+(l), \quad \left| \frac{ES_{k_s^\pm(l)}}{a_{k_s^\pm(l)}} - \frac{ES_n}{a_n} \right| < \varepsilon,$$

and so

$$\begin{aligned} -\varepsilon - \left(1 - \frac{1}{\alpha}\right)A + \frac{1}{\alpha} \frac{1}{a_{k_s^-(l)}} (S_{k_s^-(l)} - ES_{k_s^-(l)}) &\leq \\ &\leq -\varepsilon - \left(1 - \frac{1}{\alpha}\right) \frac{ES_{k_s^-(l)}}{a_{k_s^-(l)}} + \frac{1}{\alpha} \frac{1}{a_{k_s^-(l)}} (S_{k_s^-(l)} - ES_{k_s^-(l)}) = \\ &= -\varepsilon - \frac{ES_{k_s^-(l)}}{a_{k_s^-(l)}} + \frac{1}{\alpha} \frac{S_{k_s^-(l)}}{a_{k_s^-(l)}} \leq -\varepsilon - \frac{ES_{k_s^-(l)}}{a_{k_s^-(l)}} + \frac{S_{k_s^-(l)}}{a_n} \leq \frac{S_{k_s^-(l)}}{a_n} - \frac{ES_n}{a_n} \leq \\ &\leq \frac{S_n - ES_n}{a_n} \leq \frac{S_{k_s^+(l)}}{a_n} - \frac{ES_n}{a_n} \leq \frac{S_{k_s^+(l)}}{a_n} - \frac{ES_{k_s^+(l)}}{a_{k_s^+(l)}} + \varepsilon \leq \alpha \frac{S_{k_s^+(l)}}{a_{k_s^+(l)}} - \frac{ES_{k_s^+(l)}}{a_{k_s^+(l)}} + \varepsilon = \\ &= \alpha \frac{(S_{k_s^+(l)} - ES_{k_s^+(l)})}{a_{k_s^+(l)}} + (\alpha - 1) \frac{ES_{k_s^+(l)}}{a_{k_s^+(l)}} + \varepsilon \leq \alpha \frac{(S_{k_s^+(l)} - ES_{k_s^+(l)})}{a_{k_s^+(l)}} + (\alpha - 1)A + \varepsilon. \end{aligned}$$

Thus, using (14), we obtain

$$-\varepsilon - \left(1 - \frac{1}{\alpha}\right)A \leq \liminf_{n \rightarrow \infty} \frac{S_n - ES_n}{a_n} \leq \limsup_{n \rightarrow \infty} \frac{S_n - ES_n}{a_n} \leq (\alpha - 1)A + \varepsilon \quad (15)$$

almost surely. Since (15) is true for any $\alpha > 1$ and $\varepsilon > 0$, we get relation (8). \square

Proof of Theorem 4. Without loss of generality it can be assumed that $EX_n = 0$ for all $n \geq 1$. By Chebyshev's inequality, using (11) and Lemma 1, for any $\varepsilon > 0$, we get

$$\sum_{n=1}^{\infty} P\left(\left|\frac{S_{2^n}}{a_{2^n}}\right| > \varepsilon\right) \leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{ES_{2^n}^2}{a_{2^n}^2} \leq C\varepsilon^{-2} \sum_{n=1}^{\infty} \frac{1}{\psi(2^n)} < \infty.$$

The application of Borel–Cantelli lemma yields to

$$\frac{S_{2^n}}{a_{2^n}} \rightarrow 0 \quad \text{a.s.}$$

To complete the proof it is sufficiently to show that

$$\lim_{n \rightarrow \infty} \max_{2^n < k \leq 2^{n+1}} \left| \frac{S_k}{a_k} \right| = 0 \quad \text{a.s.}$$

We have

$$\begin{aligned} \max_{2^n < k \leq 2^{n+1}} \left| \frac{S_k}{a_k} \right| &= \max_{2^n < k \leq 2^{n+1}} \left| \frac{S_k - S_{2^n} + S_{2^n}}{a_k} \right| \leq \\ &\leq \left| \frac{S_{2^n}}{a_{2^n}} \right| + \max_{1 \leq k \leq 2^n} \left| \frac{\sum_{i=2^n+1}^{2^n+k} X_i}{a_{2^{n+1}}} \right| \frac{a_{2^{n+1}}}{a_{2^n}} \end{aligned} \quad (16)$$

The first summand in the right-hand side of (16) converges to zero almost surely. Taking into account assumption (10), it is sufficient to prove that

$$\lim_{n \rightarrow \infty} \max_{1 \leq k \leq 2^n} \left| \frac{\sum_{i=2^n+1}^{2^n+k} X_i}{a_{2^{n+1}}} \right| = 0 \quad \text{a.s.} \quad (17)$$

By Kolmogorov's inequality (see [4]), for any $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} P \left(\max_{1 \leq k \leq 2^n} \left| \frac{\sum_{i=2^n+1}^{2^n+k} X_i}{a_{2^{n+1}}} \right| > \varepsilon \right) \leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{\sum_{i=2^n+1}^{2^{n+1}} EX_i^2}{a_{2^{n+1}}^2} \leq C\varepsilon^{-2} \sum_{n=2}^{\infty} \frac{1}{\psi(2^n)} < \infty.$$

Thus, (17) follows from Borel–Cantelli lemma. \square

Proof of Theorem 5. Suppose that conditions of Theorem 5 are satisfied for sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ of non-negative numbers, nevertheless the series $\sum_{n=1}^{\infty} b_n/a_n^2$ diverges. Then there is a sequence of independent random variables $\{X_n\}_{n=1}^{\infty}$ such that $EX_n = 0$, $Var(X_n) = b_n$ for all $n \geq 1$, but relation (8) does not hold (see, for example, [8]). The sequence $\{X_n\}_{n=1}^{\infty}$ satisfies the conditions of Theorem 4, so (8) has to hold. This contradiction concludes the proof. \square

The next example shows that assumption (10) in Theorem 4 cannot be dropped.

Example 1. Let $a_n = 2^{n/2}$, $n \geq 1$. We consider the sequence of independent random variables $\{X_n\}_{n=1}^{\infty}$ such that

$$P(X_n = 1) = P(X_n = -1) = \frac{1}{2} \quad \text{for } n = 1 \text{ or } 2,$$

and

$$P(X_n = 2^{n/2}) = P(X_n = -2^{n/2}) = \frac{n-2}{4n(n-1)},$$

$$P(X_n = 0) = 1 - \frac{n-2}{2n(n-1)}$$

for all $n \geq 3$. Then $EX_n = 0$ for all $n \geq 1$, $Var(X_1) = Var(X_2) = 1$ and

$$Var(X_n) = \frac{2^n}{n} - \frac{2^{n-1}}{n-1} \quad \text{for all } n \geq 3.$$

We have

$$\text{Var}(S_n) = \sum_{k=1}^n \text{Var}(X_k) = \frac{2^n}{n} \quad \text{for all } n \geq 3.$$

Thus, the sequence of random variables $\{X_n\}_{n=1}^{\infty}$ satisfies condition (11) with $a_n = 2^{n/2}$, $n \geq 1$ and function $\psi(x) = x$, $x > 0$ (belonging to Ψ_c). Moreover

$$\sum_{n=3}^{\infty} P(|X_n| = a_n) = \sum_{n=3}^{\infty} \frac{n-2}{2n(n-1)} = \infty.$$

Application of Borel–Cantelli lemma yields to

$$P(|X_n| = a_n \text{ i.o.}) = 1. \tag{18}$$

We shall suppose that relation (8) holds. Then we have

$$\frac{X_n}{a_n} = \frac{S_n}{a_n} - \frac{a_{n-1}}{a_n} \cdot \frac{S_{n-1}}{a_{n-1}} \rightarrow 0 \quad \text{a.s.,}$$

which contradicts (18).

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