STAR-CUMULANTS OF FREE UNITARY BROWNIAN MOTION

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Abstract. We study joint free cumulants of u_t and u_t^*, where u_t is a free unitary Brownian motion at time t. We determine explicitly some special families of such cumulants. On the other hand, for a general joint cumulant of u_t and u_t^*, we "calculate the derivative" for $t \to \infty$, when u_t approaches a Haar unitary. In connection to the latter calculation we put into evidence an "infinitesimal determining sequence" which naturally accompanies an arbitrary R-diagonal element in a tracial $*$-probability space.

1. Introduction

Let $(u_t)_{t\geq 0}$ be a free unitary Brownian motion in the sense of [3], [4] — that is, every u_t is a unitary element in some tracial $*$-probability space $(\mathcal{A}_t, \varphi_t)$, with $\varphi_t(u_t) = e^{-t/2}$, and where the rescaled element $v_t := e^{t/2} u_t$ has S-transform given by

$$S_{v_t}(z) = e^{tz}, \quad z \in \mathbb{C}. \tag{1.1}$$

Closely related to Equation (1.1), one has a nice formula for the free cumulants of u_t, i.e. for the sequence of numbers $\big(\kappa_n(u_t, \ldots, u_t) \big)_{n=1}^{\infty}$, where $\kappa_n : \mathcal{A}_t^n \to \mathbb{C}$ is the n-th free cumulant functional of the space $(\mathcal{A}_t, \varphi_t)$. Indeed, these numbers are the coefficients of the R-transform R_{u_t}. By using the relation between the S-transform and the compositional inverse of the R-transform (which simply says that $zS(z) = R^{<-1>}(z)$), one finds that

$$R_{v_t}(z) = \frac{1}{t} W(tz), \quad t > 0, \tag{1.2}$$

where

$$W(y) = y - y^2 + \frac{3}{2} y^3 - \frac{8}{3} y^4 + \cdots + \frac{(-n)^{n-1}}{n!} y^n + \cdots$$

is the Lambert series. Extracting the coefficient of z^n in (1.2) gives the value of $\kappa_n(v_t, \ldots, v_t)$, then rescaling back gives

$$\kappa_n(u_t, \ldots, u_t) = e^{-nt/2} \kappa_n(v_t, \ldots, v_t) = e^{-nt/2} \frac{(-n)^{n-1}}{n!} \cdot t^{n-1}, \ n \in \mathbb{N}, \ t \geq 0. \tag{1.3}$$

In this paper we study joint free cumulants of u_t and u_t^*, that is, quantities of the form

$$\kappa_n\big(u_t^{\omega(1)}, \ldots, u_t^{\omega(n)} \big), \text{ where } n \in \mathbb{N} \text{ and } \omega = (\omega(1), \ldots, \omega(n)) \in \{1, *\}^n.$$

The motivation for paying attention to these joint free cumulants comes from looking at the limit $t \to \infty$, when u_t approximates in distribution a Haar unitary. Recall that a unitary u in a $*$-probability space $(\mathcal{A}, \varphi)$ is said to be a Haar unitary when it has the property that $\varphi(u^n) = 0$ for every $n \in \mathbb{Z} \setminus \{0\}$. This property trivially implies $\kappa_n(u, \ldots, u) = 0$ for every $n \in \mathbb{N}$, thus the free cumulants of u alone do not look too exciting. However, things become interesting upon considering the larger family of joint free cumulants of u and u^*. There we

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get the following non-trivial formula, first found in [10]: for $\omega = (\omega(1), \ldots, \omega(n)) \in \{1, *\}^n$ one has

$$(1.4) \qquad \kappa_n\big(u^{\omega(1)}, \ldots, u^{\omega(n)}\big) = \begin{cases} (-1)^{k-1} C_{k-1}, & \text{if } n \text{ is even, } n = 2k, \text{ and} \\ & \quad \omega = (1, *, 1, *, \ldots, 1, *) \\ & \quad \text{or } \omega = (*, 1, *, 1, \ldots, *, 1), \\ 0, & \text{otherwise,} \end{cases}$$

with $C_{k-1} = (2k-2)!/(k-1)!k!$, the $(k-1)$-th Catalan number. Formula (1.4) leads to the combinatorial approach to R-diagonal elements — these are elements in a $*$-probability space which display, in some sense, free independence in their polar decomposition ([8], see also Lecture 15 of [9]; a brief review appears in Remark 2.6 below). For an R-diagonal element a in a tracial $*$-probability space $(\mathcal{A}, \varphi)$, the sequence $\big(\kappa_{2n}(a, a^*, \ldots, a, a^*) \big)_{n=1}^{\infty}$ is called "determining sequence of a" (and does indeed determine the joint distribution of a and a^*); from this point of view, Equation (1.4) says that the determining sequence of a Haar unitary consists of signed Catalan numbers.

Returning to the point of view that, for $t \to \infty$, the free unitary Brownian motion u_t is an approximation of u, it then becomes natural to ask what can be said about the joint free cumulants of u_t and u_t^*. The expressions for these joint cumulants are far more involved than what is on the right-hand side of (1.4), but still turn out to have some tractable features. In order to discuss them, it is convenient to start from the fact (easily obtained from the general formula connecting free cumulants to moments in a noncommutative probability space) that for every fixed $\omega \in \{1, *\}^n$, the cumulant $\kappa_n\big(u_t^{\omega(1)}, \ldots, u_t^{\omega(n)} \big)$ is a quasi-polynomial in $-t/2$; more precisely, there exists a polynomial $Z_\omega \in \mathbb{Q}[x, y]$, uniquely determined, such that

$$(1.5) \qquad \kappa_n\big(u_t^{\omega(1)}, \ldots, u_t^{\omega(n)} \big) = Z_\omega(t, e^{-t/2}), \quad \forall\, t \in [0, \infty).$$

It is moreover easy to see that the y-degree of $Z_\omega(x, y)$ is at most n, and that all the powers of y that appear in Z_ω have exponents of same parity as n. In other words, we can write

$$(1.6) \qquad Z_\omega(x, y) = Z_\omega^{(n)}(x) \cdot y^n + Z_\omega^{(n-2)}(x) \cdot y^{n-2} + \cdots,$$

with $Z_\omega^{(n)}, Z_\omega^{(n-2)}, \ldots$ in $\mathbb{Q}[x]$. Note that the formula (1.3) which describes the cumulants of u_t (without u_t^*) fits here, and can be read as

$$(1.7) \qquad Z_{(\underbrace{1, 1, \ldots, 1}_{n})}(x, y) = \frac{(-n)^{n-1}}{n!}\, x^{n-1} y^n, \quad n \in \mathbb{N}.$$

A less obvious fact about the polynomials Z_ω is that the number of relevant terms (counting from the top) in the expansion (1.6) is limited by how many times one switches between the symbols '1' and '$*$', while going around the string ω. Thus for $\omega = (1, 1, \ldots, 1)$ we have $Z_\omega(x, y) = Z_\omega^{(n)}(x) \cdot y^n$ (as just seen above), then for ω of the form $(1, \ldots, 1, *, \ldots, *)$ we have $Z_\omega(x, y) = Z_\omega^{(n)}(x) \cdot y^n + Z_\omega^{(n-2)}(x) \cdot y^{n-2}$, and so on. This fact is stated precisely in Section 3 of the paper, and proved in Theorem 3.8 of that section. It is significant because it gives information on the speed of decay of $\kappa_n\big(u_t^{\omega(1)}, \ldots, u_t^{\omega(n)} \big)$ when $t \to \infty$, in the case (covering "most" strings $\omega \in \{1, *\}^n$) when the right-hand side of Equation (1.4) is equal to 0.

Here are some more details about what we do in this paper, and about how it is organized. Besides the present introduction, we have five sections. After a review of background and notations in Section 2, some general basic properties of the polynomials Z_ω are established in Section 3. Then Sections 4 and 5 study two special types of ω's, as follows.

• In Section 4 we look at strings of the form $\omega = (1, \ldots, 1, *, \ldots, *)$, with k occurrences of '1' followed by ℓ occurrences of '$*$'. We retrieve by direct calculation the fact mentioned above, that the expansion from Equation (1.6) is in this case reduced to its top two terms, and we show moreover how the two polynomials $Z_{\omega}^{(k+\ell)}(x)$ and $Z_{\omega}^{(k+\ell-2)}(x)$ can be written explicitly as Laplace transform integrals.

• In Section 5 we look at the case when ω is an alternating string of even length; in other words, we pay attention (as suggested by formula (1.4)) to free cumulants

$$\xi_n(t) := \kappa_{2n}(u_t, u_t^*, \ldots, u_t, u_t^*), \quad n \in \mathbb{N}, \ t \in [0, \infty).$$

The main point of this section is to observe a recursive formula for $\frac{d}{dt}\xi_n(t)$, which amounts to the fact that the generating function

$$H(t,z) := \frac{1}{2} + \sum_{n=1}^{\infty} \xi_n(t) z^n$$

satisfies a quasi-linear partial differential equation of Burgers type,

$$\partial_t H + 2zH\, \partial_z H = z, \quad \text{with initial condition } H(0,z) = 1/2.$$

We also show how examining the characteristic curves of the above partial differential equation gives further information on $\xi_n(t)$.

Finally, in Section 6 we look at a general string ω, and we study the behaviour of the corresponding joint cumulant of u_t and u_t^* when $t \to \infty$. We look at the limit

$$\lim_{t\to\infty} \frac{\kappa_n\big(u_t^{\omega(1)}, \ldots, u_t^{\omega(n)}\big) - \kappa_n\big(u^{\omega(1)}, \ldots, u^{\omega(n)}\big)}{e^{-t/2}},$$

where u is a Haar unitary. This limit turns out to always exist, and to have a very pleasing form, which suggests some kind of "infinitesimal determining sequence" for a Haar unitary. In Section 6 we also show how the idea of infinitesimal determining sequence can be extended to the framework of a general R-diagonal distribution — this is done by considering products $u_t q$ where $q = q^*$ is free from $\{u_t, u_t^*\}$, and then by taking the same kind of "derivative at $t = \infty$" as above.

2. Background and Notation

This section gives a very concise review, intended mostly for setting notations, of free cumulants on a noncommutative probability space. We follow the terminology from [9] and, for the various definitions and facts stated below, we give specific page references to that monograph.

We start with the structure lying at the basis of the combinatorics of free probability, the lattices $NC(n)$ of non-crossing partitions. We will assume the reader to be familiar with these objects, and we merely list below some basic notation that we will use in connection to them.

Notation 2.1. *[$NC(n)$-terminology.]* Let n be a positive integer, and let us consider the set $NC(n)$ of all non-crossing partitions of $\{1, \ldots, n\}$.

1^o Partitions in $NC(n)$ will be denoted by letters like $\pi, \rho, \ldots$. Typical explicit notation for a $\pi \in NC(n)$ is $\pi = \{V_1, \ldots, V_k\}$, where $V_1, \ldots, V_k$ are called the *blocks* of π. We sometimes simply write $V \in \pi$ to mean that "V is one of the blocks of π."

2^o On $NC(n)$ we consider the partial order given by *reverse refinement*, where for $\pi, \rho \in NC(n)$ we have $\pi \leq \rho$ if and only if every block of ρ is a union of blocks of π. The partially ordered set $(NC(n), \leq)$ turns out to be a *lattice* — that is, every $\pi, \rho \in NC(n)$ have a smallest common upper bound $\pi \vee \rho$ and a greatest common lower bound $\pi \wedge \rho$. (See [9], pp. 144-146.)

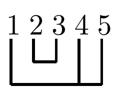
The minimal and maximal element of $(NC(n), \leq)$ are denoted as 0_n (the partition of $\{1, \ldots, n\}$ into n blocks of 1 element each) and respectively as 1_n (the partition of $\{1, \ldots, n\}$ into one block with n elements).

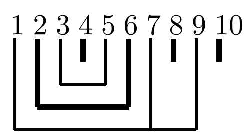
3^o $(NC(n), \leq)$ has a special anti-automorphism called the *Kreweras complementation map*, which will be denoted as $\mathrm{Kr} : NC(n) \to NC(n)$ (or as Kr_n, if we need to clarify what n we are working with). The definition of Kr_n is made by using partitions of $\{1, \ldots, 2n\}$; we take a moment to review how this goes, because it illuminates a construction of the same kind which we introduce in Section 6.

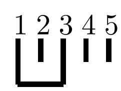
Let π and ρ be in $NC(n)$. We will denote by $\pi^{(\mathrm{odd})} \sqcup \rho^{(\mathrm{even})}$ the partition of $\{1, \ldots, 2n\}$ which is obtained when one turns π into a partition of $\{1, 3, \ldots, 2n-1\}$ and one turns ρ into a partition of $\{2, 4, \ldots, 2n\}$, in the natural way. That is, $\pi^{(\mathrm{odd})} \sqcup \rho^{(\mathrm{even})}$ has blocks of the form $\{2i - 1 \mid i \in V\}$ where V is a block of π, and has blocks of the form $\{2j \mid j \in W\}$ where W is a block of ρ. Note that $\pi^{(\mathrm{odd})} \sqcup \rho^{(\mathrm{even})}$ may not belong to $NC(2n)$, due to crossings between its odd and even blocks. If we fix $\pi \in NC(n)$, then it actually turns out that the set $\{\rho \in NC(n) \mid \pi^{(\mathrm{odd})} \sqcup \rho^{(\mathrm{even})} \in NC(2n)\}$ has a largest element with respect to reverse refinement order; this largest element is, by definition, the Kreweras complement of π. That is, $\mathrm{Kr}(\pi)$ is defined via the requirement that for $\rho \in NC(n)$ we have:

$$\pi^{(\mathrm{odd})} \sqcup \rho^{(\mathrm{even})} \in NC(2n) \ \Leftrightarrow \ \rho \leq \mathrm{Kr}(\pi).$$

Here is a concrete example, considered for $n = 5$, which also illustrates a standard way of drawing non-crossing partitions.

$\pi =$ 

$\Rightarrow \pi^{(\mathrm{odd})} \sqcup \mathrm{Kr}(\pi)^{(\mathrm{even})} =$ 

$\Rightarrow \mathrm{Kr}(\pi) =$ 

For more details on the Kr map, see [9], pp. 147-148. A neat fact which will be used in Section 5 below is that one has

$$|\,\mathrm{Kr}_n(\pi)\,| \ = \ n + 1 - |\pi|, \quad \forall\, \pi \in NC(n),$$

where $|\pi|$, $|\,\mathrm{Kr}_n(\pi)\,|$ denote the numbers of blocks of the partitions in question.

4^o The Möbius function of $NC(n)$ will be denoted as Moeb (or as Moeb_n, if we need to clarify what n we are working with). This function is defined on $\{(\pi, \rho) \mid \pi, \rho \in NC(n),$ $\pi \leq \rho\}$. We will actually only use two special cases of Moeb. The first case is when $\pi = 0_n$; here we simply get (see [9], pp. 162-164)

$$\mathrm{Moeb}(0_n, \rho) = \prod_{W \in \rho} (-1)^{|W|-1} C_{|W|-1},$$

where for $k \in \mathbb{N} \cup \{0\}$ we denote

$$C_k := \frac{(2k)!}{k!(k+1)!} \quad \text{(the } k\text{-th Catalan number).}$$

The second case we will encounter is the one having $\rho = 1_n$, which reduces to the above via the immediate observation that $\mathrm{Moeb}(\pi, 1_n) = \mathrm{Moeb}(0_n, \mathrm{Kr}(\pi))$.

Remark 2.2. In the description of Kr we used tacitly the fact that one can talk about the lattice of non-crossing partitions $NC(X)$ for any finite totally ordered set X (in particular for $X = \{2, 4, \ldots, 2n\}$). The lattice $NC(X)$ can be of course canonically identified to $NC(n)$ for $n = |X|$, upon labelling the elements of X as $1, 2, \ldots, n$ in increasing order.

Another natural convention used in Notation 2.1.3 was that if X and Y are two disjoint finite sets, then we can put together a partition π of X with a partition ρ of Y in order to obtain a partition denoted as "$\pi \sqcup \rho$" of $X \cup Y$. If $X \cup Y$ (hence each of X, Y as well) is totally ordered and if we start with $\pi \in NC(X)$ and $\rho \in NC(Y)$, then it may or may not be that $\pi \sqcup \rho \in NC(X \cup Y)$ — the definition of the Kreweras complementation map is actually built around this fact.

We now move to the review of free cumulants.

Notation 2.3. *[Restrictions of n-tuples.]*
In order to write more concisely various formulas that will appear in the paper, it is convenient to use the following natural convention of notation. Let $\mathcal{X}$ be a non-empty set, let n be a positive integer, and let $(x_1, \ldots, x_n)$ be an n-tuple in $\mathcal{X}^n$. For a subset $V = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$, with $1 \leq m \leq n$ and $1 \leq i_1 < \cdots < i_m \leq n$, we denote

$$(x_1, \ldots, x_n) \mid V := (x_{i_1}, \ldots, x_{i_m}) \in \mathcal{X}^m.$$

We will use this notation in two ways: one of them (already appearing in the next definition) is when $\mathcal{X}$ is an algebra $\mathcal{A}$ of noncommutative random variables, and the other is when $\mathcal{X} = \{1, *\}$ and we talk about the restriction $\omega \mid V \in \{1, *\}^m$ of a string $\omega \in \{1, *\}^n$.

Definition 2.4. *[Free cumulant functionals and R-transforms.]*
Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space.

1^o For every $n \in \mathbb{N}$, the n-th *moment functional* of $(\mathcal{A}, \varphi)$ is the multilinear functional $\varphi_n : \mathcal{A}^n \to \mathbb{C}$ defined by $\varphi_n(a_1, \ldots, a_n) := \varphi(a_1 \cdots a_n), \quad a_1, \ldots, a_n \in \mathcal{A}$.

2^o For every $n \in \mathbb{N}$, the n-th *free cumulant functional* of $(\mathcal{A}, \varphi)$ is the multilinear functional $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ defined by

$$\kappa_n(a_1, \ldots, a_n) = \sum_{\pi \in NC(n)} \Big(\mathrm{Moeb}(\pi, 1_n) \cdot \prod_{V \in \pi} \varphi_{|V|}\big((a_1, \ldots, a_n) \mid V \big) \Big). \tag{2.1}$$

Equation (2.1) is referred to as the *moment–cumulant formula*.

3^o Let a be an element of $\mathcal{A}$. The formal power series $R_a(z) := \sum_{n=1}^{\infty} \kappa_n(a, \ldots, a)\, z^n$ is called the *R-transform* of a.

Remark 2.5. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and consider its free cumulant functionals $\kappa_n : \mathcal{A}^n \to \mathbb{C}$, as above.

1^o Let $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ be unital subalgebras which are freely independent. The fundamental property of the κ_n's is that $\kappa_n(a_1, \ldots, a_n) = 0$ whenever $n \geq 2$, $a_1, \ldots, a_n \in \mathcal{B} \cup \mathcal{C}$, and there are elements from both $\mathcal{B}$ and $\mathcal{C}$ among $a_1, \ldots, a_n$. We also record here a consequence of this fact — a formula (presented in [9] on pp. 226-227) which expresses an alternating moment $\varphi(b_1 c_1 \cdots b_n c_n)$ in terms of "free cumulants of the b's and moments of the c's":

$$\varphi(b_1 c_1 \cdots b_n c_n) = \sum_{\pi \in NC(n)} \prod_{V \in \pi} \kappa_{|V|}\big((b_1, \ldots, b_n) \mid V \big) \cdot \prod_{W \in \mathrm{Kr}(\pi)} \varphi_{|W|}\big((c_1, \ldots, c_n) \mid W \big). \tag{2.2}$$

2^o We will make essential use of a result of Krawczyk and Speicher [7] (presented in [9] on pp. 178-181), which gives a structured summation formula for "free cumulants with products as entries", as follows. Let $\sigma = \{J_1, \ldots, J_k\} \in NC(n)$ be a partition where every block is an interval: $J_1 = \{1, \ldots, j_1\}, J_2 = \{j_1 + 1, \ldots, j_2\}, \ldots, J_k = \{j_{k-1} + 1, \ldots, j_k\}$ for some $1 \leq j_1 < j_2 < \cdots < j_k = n$. Then for every $a_1, \ldots, a_n \in \mathcal{A}$ one has

$$\kappa_k\big(a_1 \cdots a_{j_1},\ a_{j_1+1} \cdots a_{j_2},\ \ldots,\ a_{j_{k-1}+1} \cdots a_{j_k}\big)$$

$$= \sum_{\substack{\pi \in NC(n) \ such \\ that \ \pi \vee \sigma = 1_n}} \prod_{V \in \pi} \kappa_{|V|}\big((a_1, \ldots, a_n) \mid V \big). \tag{2.3}$$

In the special case when $\sigma = 1_n$, Equation (2.3) becomes a formula expressing the moment $\varphi(a_1 \cdots a_n)$ in terms of free cumulants; this special case turns out to be equivalent to (2.1), and also goes (same as (2.1)) under the name of "moment–cumulant formula".

3^o We also record some useful properties of free cumulants which follow immediately from their definition, by taking into account obvious symmetries of the lattices $NC(n)$.

(a) Invariance under cyclic permutations of entries:

$$\kappa_n(a_1, \ldots, a_n) = \kappa_n(a_m, \ldots, a_n, a_1, \ldots, a_{m-1}), \quad \forall\, 1 \leq m \leq n \text{ and } a_1, \ldots, a_n \in \mathcal{A}.$$

(b) Left-right symmetry: if $\mathcal{C} \subseteq \mathcal{A}$ is a commutative subalgebra, then

$$\kappa_n(c_1, c_2, \ldots, c_n) = \kappa_n(c_n, \ldots, c_2, c_1), \quad \forall\, n \geq 1 \text{ and } c_1, \ldots, c_n \in \mathcal{C}.$$

(c) Left-right symmetry in $*$-probability framework: suppose $(\mathcal{A}, \varphi)$ is a $*$-probability space, then one has $\kappa_n(a_n^*, \ldots, a_2^*, a_1^*) = \overline{\kappa_n(a_1, a_2, \ldots, a_n)}$, $\forall\, n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$.

4^o Suppose again that $(\mathcal{A}, \varphi)$ is a $*$-probability space. Then by using the Cauchy-Schwarz inequality for the functional φ, one immediately sees that every unitary $u \in \mathcal{A}$ has $|\varphi(u)| \leq 1$. As a consequence, it follows that

$$\kappa_n(u_1, \ldots, u_n) \leq 16^n, \quad \forall\, n \geq 1 \text{ and } u_1, \ldots, u_n \in \mathcal{A} \text{ unitaries.} \tag{2.4}$$

The constant 16 in (2.4) appears upon writing cumulants in terms of moments as in Equation (2.1), then by using estimates on the Möbius function — see the discussion on p. 219 of [9].

From the bound (2.4) it is clear that for every unitary $u \in \mathcal{A}$, the R-transform $R_u(z)$ (which was introduced in Definition 2.4.3 as a formal power series) can also be viewed as an analytic function on the disc $\{z \in \mathbb{C} \mid |z| < 1/16\}$.

Remark 2.6. *[R-diagonal elements.]*

Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and suppose that $u, q \in \mathcal{A}$ are such that u is Haar unitary, $q = q^*$, and q is free from $\{u, u^*\}$. The element $a := uq \in \mathcal{A}$ is said to be *R-diagonal*. The motivation for this name (introduced in [8]) is that there exists a sequence $(\alpha_k)_{k=1}^{\infty}$, called the *determining sequence* of a, such that for $\omega = (\omega(1), \ldots, \omega(n)) \in \{1, *\}^n$ one has:

$$\kappa_n\big(a^{\omega(1)}, \ldots, a^{\omega(n)}\big) = \begin{cases} \alpha_{n/2}, & \text{if } n \text{ even and } \omega = (1, *, 1, *, \ldots, 1, *) \\ & \quad \text{or } \omega = (*, 1, *, 1, \ldots, *, 1), \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

The α_k's can be written in terms of the free cumulants of q^2 via a formula which looks similar to Equation (2.1):

$$\alpha_k = \sum_{\pi \in NC(k)} \Big(\text{Moeb}(\pi, 1_k) \cdot \prod_{V \in \pi} \kappa_{|V|}(q^2, \ldots, q^2) \Big), \quad k \in \mathbb{N}. \tag{2.6}$$

The derivation of these facts is presented in [9] on pp. 241-244 of Lecture 15.

Note that Equation (1.4) from the Introduction corresponds to the special case $q = 1_{\mathcal{A}}$ of the above formulas. Indeed, in this case the sum on the right-hand side of (2.6) has only one non-vanishing term, corresponding to $\pi = 0_k$, and we get $\alpha_k = \text{Moeb}(0_k, 1_k) = (-1)^{k-1} C_{k-1}$, as stated in (1.4).

3. The polynomials Z_ω

Proposition and Notation 3.1. *Let $\omega = (\omega(1), \ldots, \omega(n))$ be a string in $\{1, *\}^n$, for some $n \geq 1$. There exists a polynomial $Z_\omega \in \mathbb{Q}[x, y]$, uniquely determined, such that*

$$\kappa_n\big(u_t^{\omega(1)}, \ldots, u_t^{\omega(n)} \big) = Z_\omega(t, e^{-t/2}), \quad \forall\, t \in [0, \infty). \tag{3.1}$$

Moreover, the polynomial $Z_\omega(x, y)$ has the form

$$Z_\omega(x, y) = \sum_{0 \leq j \leq n/2} Z_\omega^{(n-2j)}(x) \cdot y^{n-2j}, \tag{3.2}$$

where $Z_\omega^{(n-2j)} \in \mathbb{Q}[x]$ for $0 \leq j \leq n/2$.

Proof. We will give an explicit formula for the polynomial Z_ω. In order to state it, we introduce some preliminary items of notation. We first recall that Lemma 1 on page 4 of [4] says that the moments of u_t are $\varphi_t(u_t^n) = Q_n(t) e^{-nt/2}$, $n \geq 1$, where

$$Q_n(t) = -\sum_{j=0}^{n-1} \frac{(-n)^{j-1}}{j!} \binom{n}{j+1} t^j.$$

For a string ω in $\{1, *\}^n$ which has k occurrences of the symbol "1" and $\ell = n - k$ occurrences of "$*$", we then introduce a polynomial $M_\omega \in \mathbb{Q}[x, y]$ defined by

$$M_\omega(x, y) := \begin{cases} Q_{|k-\ell|}(x)\, y^{|k-\ell|}, & \text{if } k \neq \ell \\ 1, & \text{if } k = \ell. \end{cases} \tag{3.3}$$

[For instance, if $n = 7$ and $\omega = (1, 1, *, 1, 1, *, 1)$ then $M_\omega(x, y) = Q_3(x)\, y^3 = (1 - 3x + \frac{3}{2} x^2) y^3$.]

Based on (3.3), we define the Z_ω's as follows: for every $\omega \in \{1, *\}^n$ we put

$$Z_\omega := \sum_{\pi \in NC(n)} \text{Moeb}(\pi, 1_n) \cdot \Big(\prod_{V \in \pi} M_{\omega|V} \Big), \tag{3.4}$$

where the notations related to $NC(n)$ and its Möbius function are as in Section 2.

[A concrete example: if $n = 3$ and $\omega = (1, *, 1)$, then

$$Z_{(1,*,1)} := M_{(1,*,1)} - M_{(1)} M_{(*,1)} - M_{(1,1)} M_{(*)} - M_{(1,*)} M_{(1)} + 2 M_{(1)} M_{(*)} M_{(1)};$$

this comes, upon substituting the M's, to $Z_{(1,*,1)}(x, y) = (x + 1) y^3 - y$.]

Fix a $t \in [0, \infty)$, and for every $n \in \mathbb{N}$ let $\varphi_n : \mathcal{A}_t^n \to \mathbb{C}$ be the n-th moment functional of $(\mathcal{A}_t, \varphi_t)$. Then one has

$$\varphi_n\big(u_t^{\omega(1)}, \ldots, u_t^{\omega(n)} \big) = M_\omega(t, e^{-t/2}), \quad \forall\, n \in \mathbb{N} \text{ and } \omega \in \{1, *\}^n. \tag{3.5}$$

This in turn implies that, for every $n \in \mathbb{N}$ and $\omega \in \{1, *\}^n$:

$$\kappa_n\big(u_t^{\omega(1)}, \ldots, u_t^{\omega(n)} \big) = \sum_{\pi \in NC(n)} \mathrm{Moeb}(\pi, 1_n) \cdot \Big(\prod_{V \in \pi} \varphi_{|V|}\big((u_t^{\omega(1)}, \ldots, u_t^{\omega(n)}) \mid V \big) \Big)$$

$$= \sum_{\pi \in NC(n)} \mathrm{Moeb}(\pi, 1_n) \cdot \big(\prod_{V \in \pi} M_{\omega|V}(t, e^{-t/2}) \big) = Z_\omega(t, e^{-t/2})$$

(where we first used the moment–cumulant formula (2.1), then we invoked Equations (3.5) and (3.4)). Thus Z_ω has the property stated in Equation (3.1).

The uniqueness of Z_ω with the property stated in Equation (3.1) follows from general considerations (a polynomial in $\mathbb{Q}[x, y]$ is determined by its values on pairs $(t, e^{-t/2})$, with $t \in [0, \infty)$).

Finally, let us also verify the specific form of Z_ω that was indicated in Equation (3.2). It suffices to show that: for every $\pi \in NC(n)$, the term indexed by π in the sum on the right-hand side of (3.4) is of the form $y^s \cdot T(x)$, where $s \in \{0, 1, \ldots, n\}$ has same parity as n, and where $T \in \mathbb{Q}[x]$. So fix a partition $\pi = \{V_1, \ldots, V_k\} \in NC(n)$, and for every $1 \le j \le k$ denote by p_j and by q_j the number of occurrences of "1" and respectively "$*$" in the restricted word $\omega|V_j$. The term indexed by π in the sum on the right-hand side of (3.4) is $\mathrm{Moeb}(\pi, 1_n) \cdot \prod_{j=1}^k Q_{|p_j - q_j|}(x) y^{|p_j - q_j|}$, where we set $Q_0 := 1$. This is indeed of the form $y^s \cdot T(x)$, with $s := \sum_{j=1}^k |p_j - q_j|$, and we are only left to check that $n - s$ is an even non-negative integer. But the latter fact follows from

$$n - s = \sum_{j=1}^k (p_j + q_j) - \sum_{j=1}^k |p_j - q_j| = \sum_{j=1}^k (p_j + q_j - |p_j - q_j|),$$

where every $p_j + q_j - |p_j - q_j|$ is an even non-negative integer. □

Example 3.2. Here are some concrete examples of polynomials Z_ω:

$$\begin{array}{lcl} Z_{(1,*)}(x, y) & = & -y^2 + 1, \\ Z_{(1,1,*)}(x, y) & = & (x + 1)y^3 - y, \\ Z_{(1,1,1,*)}(x, y) & = & -(\frac{3}{2}x^2 + 2x + 1)y^4 + (x + 1)y^2, \\ Z_{(1,1,*,*)}(x, y) & = & -(x^2 + 2x + 2)y^4 + 2y^2, \\ Z_{(1,*,1,*)}(x, y) & = & -(2x + 3)y^4 + 4y^2 - 1. \end{array}$$

If we add to this list the formula for a polynomial $Z_{(1,1,\ldots,1)}$ from Equation (1.7), and if we take into account some obvious invariance properties of the Z_ω's (as recorded in the next remark), then these examples cover all strings $\omega \in \{1, *\}^n$ for $n \le 4$.

Remark 3.3. The polynomials Z_ω have some invariance properties which follow directly from their definition.

1^o Let $\omega, \omega' \in \{1, *\}^n$ be such that ω' is obtained from ω via a cyclic permutation. The invariance of free cumulants under cyclic permutations of entries gives $Z_\omega(t, e^{-t/2}) = Z_{\omega'}(t, e^{-t/2})$, $t \in [0, \infty)$, which implies that the polynomials Z_ω and $Z_{\omega'}$ coincide.

2^o The same conclusion as in 1^o holds if we take ω' to be obtained from ω by reversing the order of its components, $\omega' = (\omega(n), \ldots, \omega(1))$. (This time we use the invariance property of free cumulants that was reviewed in Remark 2.5.3(b).)

3^o The moments of the free unitary Brownian motion u_t are real numbers (as reviewed at the beginning of the preceding proof). This has the consequence that u_t^* can also serve as free unitary Brownian motion at time t, which in turn implies that

$$\kappa_n\big(u_t^{\omega(1)}, \ldots , u_t^{\omega(n)} \big) = \kappa_n\big(u_t^{\omega'(1)}, \ldots , u_t^{\omega'(n)} \big), \ \ \forall\, t \in [0, \infty),$$

with ω' obtained out of $\omega \in \{1, *\}^n$ by swapping the roles of 1 and $*$ (every $\omega(i)$ which is a 1 is replaced by a $*$, and vice-versa). The uniqueness property of Z_ω thus implies that $Z_\omega = Z_{\omega'}$ in this situation, too.

We next put into evidence a very useful recursion satisfied by the polynomials Z_ω. This is done in Proposition 3.5; the essence of the argument is a calculation which holds for any unitary in a $*$-probability space, and is presented in the next lemma.

Lemma 3.4. *Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $u \in \mathcal{A}$ be a unitary element. Consider a string $\omega = (\omega(1), \ldots , \omega(n)) \in \{1, *\}^n$ with $n \geq 3$ and where $\omega(1) = 1$, $\omega(n) = *$. Then*

$$\kappa_n\big(u^{\omega(1)}, \ldots , u^{\omega(n)} \big) = -\sum_{m=1}^{n-1} \kappa_m\big(u^{\omega(1)}, \ldots , u^{\omega(m)} \big) \cdot \kappa_{n-m}\big(u^{\omega(m+1)}, \ldots , u^{\omega(n)} \big) \tag{3.6}$$

(where $\kappa_n, \kappa_m, \kappa_{n-m}$ denote free cumulant functionals for $(\mathcal{A}, \varphi)$).

Proof. We may assume (by replacing $\mathcal{A}$ with the $*$-algebra generated by u) that $(\mathcal{A}, \varphi)$ is tracial. In particular, we can write

$$\kappa_n\big(u^{\omega(1)}, \ldots , u^{\omega(n)} \big) = \kappa_n\big(u^{\omega(n)}, u^{\omega(1)}, \ldots , u^{\omega(n-1)} \big) = \kappa_n\big(u^*, u, u^{\omega(2)}, \ldots , u^{\omega(n-1)} \big).$$

Now, we know that $\kappa_{n-1}\big(u^*u, u^{\omega(2)}, \ldots , u^{\omega(n-1)} \big) = 0$ (a free cumulant of length ≥ 2 always vanishes when one of its entries is equal to $1_{\mathcal{A}}$). On the other hand, the formula (2.3) for cumulants with products as entries gives

$$\begin{aligned} &\kappa_{n-1}\big(u^*u, u^{\omega(2)}, \ldots , u^{\omega(n-1)} \big) \\ &= \sum_{\substack{\pi \in NC(n) \text{ such} \\ \text{that } \pi \vee \sigma = 1_n}} \prod_{V \in \pi} \kappa_{|V|}\big((u^*, u, u^{\omega(2)}, \ldots , u^{\omega(n-1)}) \mid V \big), \end{aligned} \tag{3.7}$$

where $\sigma \in NC(n)$ is the partition consisting of the 2-element block $\{1, 2\}$ and of $n - 2$ blocks with one element.

Let $\pi \in NC(n)$ be such that $\pi \vee \sigma = 1_n$, and let V' and V'' be the blocks of π which contain 1 and 2, respectively. We observe that $V' \cup V'' = \{1, \ldots , n\}$; indeed, in the opposite case we could consider the partition $\widetilde{\pi} \in NC(n)$ which is obtained from π by joining together the blocks V and V', and[1] this $\widetilde{\pi}$ would satisfy $\pi, \sigma \leq \widetilde{\pi} \neq 1_n$, in contradiction with the assumption that $\pi \vee \sigma = 1_n$. If $V' = V''$ then $\pi = 1_n$. If $V' \neq V''$ then either $V' = \{1\}$, $V'' = \{2, 3, \ldots , n\}$, or V'' is nested inside V'. In the latter case, denoting $|V''| =: m$, we find that $V'' = \{2, \ldots , m + 1\}$ and $V' = \{1\} \cup \{m + 2, \ldots , n\}$, where $1 \leq m \leq n - 2$.

The conclusion of the discussion in the preceding paragraph is that the sum on the right-hand side of (3.7) can be written explicitly as

$$\kappa_n\big(u^*, u, u^{\omega(2)}, \ldots , u^{\omega(n-1)} \big) + \sum_{m=1}^{n} \kappa_{\pi_m}\big(u^*, u, u^{\omega(2)}, \ldots , u^{\omega(n-1)} \big), \tag{3.8}$$

[1] The partition $\widetilde{\pi}$ thus consists of $V' \cup V''$ and of all the blocks $V \in \pi$ such that $V \neq V', V''$. The detail which prevents $\widetilde{\pi}$ from having crossings is that V' and V'' contain the adjacent points 1 and 2.

with $\pi_m = \{ \{2, \ldots, m+1\}, \{1, \ldots, n\} \setminus \{2, \ldots, m+1\} \}$, $1 \leq m \leq n-1$. It is immediately seen (by doing the suitable cyclic permutation of entries in $\kappa_{n-m}\big(u^*, u^{\omega(m+1)}, \ldots, u^{\omega(n-1)} \big)$) that for every $1 \leq m \leq n-1$ one has

$$\kappa_{\pi_m}\big(u^*, u, u^{\omega(2)}, \ldots, u^{\omega(n-1)} \big) = \kappa_m\big(u^{\omega(1)}, \ldots, u^{\omega(m)} \big) \cdot \kappa_{n-m}\big(u^{\omega(m+1)}, \ldots, u^{\omega(n)} \big).$$

But the sum in (3.8) is equal to 0 (since it started as an expansion for $\kappa_{n-1}(u^*u, \ldots) = 0$), and the statement of the lemma follows. □

Proposition 3.5. *Suppose that $n \geq 3$ and that $\omega = (\omega(1), \ldots, \omega(n)) \in \{1, *\}^n$ has $\omega(1) = 1$ and $\omega(n) = *$. Then it follows that*

$$Z_\omega = - \sum_{m=1}^{n-1} Z_{(\omega(1), \ldots, \omega(m))} \cdot Z_{(\omega(m+1), \ldots, \omega(n))} \tag{3.9}$$

(equality of polynomials in two variables).

Proof. Let $Z \in \mathbb{Q}[x, y]$ be the polynomial which appears on the right-hand side of (3.9). By using Lemma 3.4 one immediately sees that $Z(t, e^{-t/2}) = \kappa_n\big(u_t^{\omega(1)}, \ldots, u_t^{\omega(n)} \big)$, $t \in [0, \infty)$, and this implies $Z = Z_\omega$. □

As an application of Proposition 3.5 we show the following: the number of relevant terms (counting from the top) in the expansion given for Z_ω in Equation (3.2) is limited by how many times we switch between the symbols '1' and '$*$', upon going around the string ω. For instance if $\omega = (1, 1, \ldots, 1)$ then the expansion (3.2) amounts to just $Z_\omega(x, y) = Z_\omega^{(n)}(x) \cdot y^n$, if $\omega = (1, \ldots, 1, *, \ldots, *)$ then $Z_\omega(x, y) = Z_\omega^{(n)}(x) \cdot y^n + Z_\omega^{(n-2)}(x) \cdot y^{n-2}$, and so on. This fact is stated precisely in Theorem 3.8 below. In order to come to it, we first record the (natural) definition for what is the "number of switches between 1 and $*$" in a string ω.

Definition and Remark 3.6. For every $n \in \mathbb{N}$ and $\omega \in \{1, *\}^n$ we define the *switch-number* of ω to be

$$\text{Switch}(\omega) := \overline{\delta}_{\omega(n), \omega(1)} + \overline{\delta}_{\omega(1), \omega(2)} + \cdots + \overline{\delta}_{\omega(n-1), \omega(n)}, \tag{3.10}$$

where the $\overline{\delta}$'s on the right-hand side of the equation are assigned by putting

$$\overline{\delta}_{1,*} = \overline{\delta}_{*,1} = 1 \text{ and } \overline{\delta}_{1,1} = \overline{\delta}_{*,*} = 0.$$

It is easily seen that Switch(ω) is an even integer such that $0 \leq \text{Switch}(\omega) \leq n$. Another immediate observation is that $\text{Switch}(\omega) = \text{Switch}(\omega')$ whenever ω' is obtained out of ω via one of the transformations discussed in Remark 3.3.

Lemma 3.7. *Let $n \geq 3$ be an integer, and let $\omega = (\omega(1), \ldots, \omega(n)) \in \{1, *\}^n$ be such that $\omega(1) = 1$, $\omega(n) = *$. Let m be a number in $\{1, \ldots, n-1\}$, and consider the strings*

$$\omega' := (\omega(1), \ldots, \omega(m)) \in \{1, *\}^m, \quad \omega'' := (\omega(m+1), \ldots, \omega(n)) \in \{1, *\}^{n-m}.$$

Then we have

$$\text{Switch}(\omega') + \text{Switch}(\omega'') \leq \text{Switch}(\omega). \tag{3.11}$$

Proof. We will prove the required inequality by assuming that $1 < m < n-1$ (the cases when $m = 1$ or $m = n-1$ are analogous, and simpler). Look at the difference

$$\text{Switch}(\omega) - \Big(\text{Switch}(\omega') + \text{Switch}(\omega'') \Big).$$

By cancelling common terms in the expressions which define these three switch-numbers, we find that the above difference is equal to

$$\big(\overline{\delta}_{\omega(m),\omega(m+1)} + \overline{\delta}_{\omega(n),\omega(1)} \big) - \big(\overline{\delta}_{\omega(m),\omega(1)} + \overline{\delta}_{\omega(n),\omega(m+1)} \big).$$

Since $\overline{\delta}_{\omega(n),\omega(1)} = 1$ (while the other $\overline{\delta}$'s appearing above are 0 or 1), we get that

$$\text{Switch}(\omega) - \Big(\text{Switch}(\omega') + \text{Switch}(\omega'') \Big) \geq -1.$$

But switch-numbers are always even; so in the latter inequality we are actually forced to have "≥ 0", and (3.11) follows. □

Theorem 3.8. *Let n be a positive integer, and let ω be a string in $\{1,*\}^n$. Consider the polynomial $Z_\omega(x,y)$ and its expansion as sum of terms $Z_\omega^{(n-2j)}(x) \cdot y^{n-2j}$, with $0 \leq j \leq n/2$, which was obtained in Proposition 3.1. One has*

$$Z_\omega^{(n-2j)} = 0 \quad \textit{whenever } 2j > \text{Switch}(\omega). \tag{3.12}$$

Proof. We first observe that the statement of the theorem holds when ω is of the form $(1,1,\ldots,1)$ or $(*,*,\ldots,*)$. In this case we have $\text{Switch}(\omega) = 0$; so Equation (3.12) says that $Z_\omega^{(n-2j)} = 0$ for every $j \neq 0$, i.e. that $Z_\omega(x,y) = Z_\omega^{(n)}(x) \cdot y^n$. This is indeed true, as noticed in Equation (1.7) of the introduction.

We now prove by induction on n that the statement of the theorem holds for general strings $\omega \in \{1,*\}^n$. The case $n = 1$ is included in the preceding paragraph. Let us also verify the case $n = 2$. In this case, the strings $(1,1)$ and $(*,*)$ are covered by the preceding paragraph, while the strings $(1,*)$ and $(*,1)$ have switch-number equal to 2 — so for the latter two strings, Equation (3.12) is fulfilled vacuously (there is no j in the range $0 \leq j \leq n/2$ such that $2j > \text{Switch}(\omega)$).

In the remaining part of the proof we do the induction step: we fix an integer $n \geq 3$, we assume that the statement of the theorem holds for strings of length $\leq n-1$, and we will prove that it also holds for strings of length n.

So let us also fix an $\omega = (\omega(1),\ldots,\omega(n))$ in $\{1,*\}^n$, for which we will verify that (3.12) holds. We distinguish three cases.

Case 1. $\omega = (1,1,\ldots,1)$ or $\omega = (*,*,\ldots,*)$.
This case was verified in the first paragraph of the proof.

Case 2. ω is such that $\omega(1) = 1$ and $\omega(n) = *$.
Consider a $j \in \mathbb{N}$ such that $0 \leq j \leq n/2$ and such that $2j > \text{Switch}(\omega)$. (We assume that such j's exist, otherwise there is nothing to prove.) We are in a situation where we can invoke Proposition 3.5. By extracting the coefficient of y^{n-2j} on both sides of the recursion provided by that proposition, we find that

$$Z_\omega^{(n-2j)} = -\sum_{m=1}^{n-1} \sum_{\substack{0 \leq k \leq m/2,\ 0 \leq \ell \leq (n-m)/2 \\ \text{such that } k+\ell=j}} Z_{(\omega(1),\ldots,\omega(m))}^{(m-2k)} \cdot Z_{(\omega(m+1),\ldots,\omega(n))}^{(n-m-2\ell)} \tag{3.13}$$

(equality of polynomials in $\mathbb{Q}[x]$). We will show that $Z_\omega^{(n-2j)} = 0$ by verifying that every term in the double sum on the right-hand side of (3.13) is the zero polynomial. Indeed, let us pick such a term (indexed by an m, and then by a pair (k, ℓ)), and let us denote

$$\omega' := (\omega(1), \ldots, \omega(m)) \in \{1, *\}^m, \quad \omega'' := (\omega(m+1), \ldots, \omega(n)) \in \{1, *\}^{n-m}.$$

We have $2k + 2\ell = 2j > \text{Switch}(\omega) \geq \text{Switch}(\omega') + \text{Switch}(\omega'')$ (where at the second inequality we use Lemma 3.7). So either $2k > \text{Switch}(\omega')$ or $2\ell > \text{Switch}(\omega'')$, and the induction hypothesis gives us that either $Z^{(m-2k)}_{(\omega(1),\ldots,\omega(m))} = 0$ or $Z^{(n+m-2\ell)}_{(\omega(m+1),\ldots,\omega(n))} = 0$. Either way, the product of the latter two polynomials is 0, and this completes the verification of Case 2.

Case 3. ω does not fall in either Case 1 or Case 2 above.

Since ω is not in Case 1, both symbols 1 and $*$ must appear among its components. It is then easy to see that there exists a string ω' obtained from ω via a cyclic permutation of components, such that $\omega'(1) = 1$ and $\omega'(n) = *$. The string ω' has $Z_{\omega'} = Z_\omega$ (Remark 3.3.1), and has $\text{Switch}(\omega') = \text{Switch}(\omega)$ (Remark 3.6). For any j such that $2j > \text{Switch}(\omega) = \text{Switch}(\omega')$ we have $Z_{\omega'}^{(n-2j)} = 0$, because ω' falls in the Case 2 discussed above. It follows that $Z_\omega^{(n-2j)} = 0$ as well. This concludes the verification of the induction step, and the proof of the theorem. □

4. A special case of Z_ω's

In the present section we determine what is the polynomial Z_ω for a string of the form $\omega = (1, \ldots, 1, *, \ldots, *)$, having k occurrences of "1" followed by ℓ occurrences of "$*$", for some $k, \ell \geq 1$. In this case, Theorem 3.8 says that the expansion from Equation (1.6) is reduced to its top two terms:

$$Z_\omega(x, y) = Z_\omega^{(k+\ell)}(x)\, y^{k+\ell} + Z_\omega^{(k+\ell-2)}(x)\, y^{k+\ell-2}.$$

We will retrieve this fact, and we will moreover show how the polynomials $Z_\omega^{(k+\ell)}(x)$, $Z_\omega^{(k+\ell-2)}(x)$ can be written explicitly as some Laplace transform integrals. We start with a calculation (consequence of the above Lemma 3.4) which holds for any unitary in a $*$-probability space.

Lemma 4.1. *Let $(\mathcal{A}, \varphi)$ be a $*$-probability space and let $u \in \mathcal{A}$ be a unitary element. It makes sense to define an analytic function $F_u : \{(z, w) \in \mathbb{C}^2 \mid |z|, |w| < 1/16\} \to \mathbb{C}$ by putting*

$$F_u(z, w) := \sum_{k,\ell=1}^{\infty} \kappa_{k+\ell}(\underbrace{u, \ldots, u}_{k}, \underbrace{u^*, \ldots, u^*}_{\ell})\, z^k w^\ell. \tag{4.1}$$

Moreover, there exists $r \in (0, 1/16)$ such that for $|z|, |w| < r$ one has

$$F_u(z, w) = \frac{zw - R_u(z) R_{u^*}(w)}{1 + R_u(z) + R_{u^*}(w)}, \tag{4.2}$$

where $R_u, R_{u^*} : \{z \in \mathbb{C} \mid |z| < 1/16\} \to \mathbb{C}$ are R-transforms (as discussed in Remark 2.5.4).

Proof. The fact that $F_u(z,w)$ defined in Equation (4.1) is well-defined and analytic on $\{(z,w) \in \mathbb{C}^2 \mid |z|,|w| < 1/16\}$ follows from the bound

$$\kappa_{k+\ell}(\underbrace{u,\ldots,u}_{k},\underbrace{u^*,\ldots,u^*}_{\ell}) \le 16^{k+\ell}, \quad k,\ell \in \mathbb{N},$$

which was mentioned in Remark 2.5.4.

Let us next consider the analytic function defined on $\{(z,w) \in \mathbb{C}^2 \mid |z|,|w| < 1/16\}$ by

$$(z,w) \mapsto F_u(z,w)\,(1 + R_u(z) + R_{u^*}(w)) + R_u(z)R_{u^*}(w). \tag{4.3}$$

We will prove that this function is just $(z,w) \mapsto zw$; the formula claimed in the lemma for $F(z,w)$ will then clearly follow (with r picked e.g. such that $|R_u(z)| < 1/2$ for $|z| < r$).

It thus suffices to prove that the coefficient of $z^k w^\ell$ in the analytic function from (4.3) is equal to 1 for $k = \ell = 1$, and is equal to 0 for any $(k,\ell) \neq (1,1)$ in $\mathbb{N}^2$. In the special case when $k = \ell = 1$, the coefficient in question comes out as $\kappa_2(u,u^*) + \kappa_1(u)\,\kappa_1(u^*)$, which is equal to $\varphi(uu^*)$ by the moment-cumulant formula, and thus is indeed equal to 1. The cases when $(k,\ell) \neq (1,1)$ are covered by Lemma 3.4. Indeed, let us say for instance that both k and ℓ are ≥ 2 (if $k = 1 < \ell$ or if $\ell = 1 < k$ then the argument is analogous, and shorter). Direct inspection shows that the coefficient of $z^k w^\ell$ in the function from (4.3) is equal to

$$\kappa_{k+\ell}(\underbrace{u,\ldots,u}_{k},\underbrace{u^*,\ldots,u^*}_{\ell}) + \sum_{i=1}^{k-1} \kappa_i(u,\ldots,u)\cdot\kappa_{(k-i)+\ell}(\underbrace{u,\ldots,u}_{k-i},\underbrace{u^*,\ldots,u^*}_{\ell})$$
$$+\sum_{j=1}^{\ell-1} \kappa_{k+(\ell-j)}(\underbrace{u,\ldots,u}_{k},\underbrace{u^*,\ldots,u^*}_{\ell-j})\cdot\kappa_j(u^*,\ldots,u^*) + \kappa_k(u,\ldots,u)\cdot\kappa_\ell(u^*,\ldots,u^*).$$

But Lemma 3.4 (used for the string in $\{1,*\}^{k+\ell}$ which has k occurrences of 1 followed by ℓ occurrences of $*$) says precisely that the latter sum is equal to 0. □

We now turn to the case of interest, of the free unitary Brownian motion.

Lemma 4.2. *Let us fix $t \in (0,\infty)$. In the framework of Lemma 4.1 let us put $u = u_t$ (free unitary Brownian motion at time t), and let us consider the analytic function $F_{u_t}(z,w)$ defined as in Equation (4.1). Then for $|z|,|w|$ small enough we have that*

$$F_{u_t}(z,w) = \frac{1}{t}\cdot\frac{t^2 zw - W(te^{-t/2}z)W(te^{-t/2}w)}{t + W(te^{-t/2}z) + W(te^{-t/2}w)}, \tag{4.4}$$

or equivalently that

$$F_{u_t}(z,w) = tzw\int_0^1 e^{-ts}\,e^{-sW(te^{-t/2}z)}\,e^{-sW(te^{-t/2}w)}\,ds, \tag{4.5}$$

with W the Lambert function (viewed here as analytic function on $\{z \in \mathbb{C} \mid |z| < 1/e\}$).

Proof. As mentioned in the introduction, the rescaling $v_t = e^{t/2}u_t$ has R-transform $R_{v_t}(z) = t^{-1}W(tz)$. But $R_{u_t}(z) = R_{v_t}(e^{-t/2}z)$, so we find the R-transform of u_t to be

$$R_{u_t}(z) = \frac{1}{t}W(te^{-t/2}z). \tag{4.6}$$

The equality (4.6) holds when $|z|$ is small enough so that both sides are defined (one can e.g. use $|z| < 1/16$ on the left-hand side and $|z| < e^{t/2}/(et)$ on the right-hand side). The adjoint u_t^* has the same R-transform as u_t itself. We replace all this into the result of Lemma 4.1. Upon also requiring the condition that $|z|, |w|$ are small enough such that

$$|W(te^{-t/2}z)|,\ |W(te^{-t/2}w)| < t/2$$

(which ensures that the denominator $t + W(te^{-t/2}z) + W(te^{-t/2}w)$ does not vanish), we arrive to the formula for F_{u_t} that was stated in Equation (4.4).

In order to go from (4.4) to (4.5), let us fix $z, w \in \mathbb{C}$ such that $|z|, |w|$ satisfy the restrictions mentioned above, and let us denote

$$W(te^{-t/2}z) =: \alpha,\ W(te^{-t/2}w) =: \beta.$$

From the definition of the Lambert function it follows that $\alpha e^{\alpha} = te^{-t/2}z, \quad \beta e^{\beta} = te^{-t/2}w$, and multiplying together the latter equations gives

$$t^2 zw = \alpha\beta e^{t+\alpha+\beta}. \tag{4.7}$$

We write the right-hand side of (4.4) in terms of α and β (where $t^2 zw$ is substituted from Equation (4.7)), and we obtain

$$F_{u_t}(z, w) = \frac{1}{t} \cdot \frac{\alpha\beta e^{t+\alpha+\beta} - \alpha\beta}{t + \alpha + \beta} = \frac{\alpha\beta}{t} \int_0^1 e^{x(t+\alpha+\beta)}\, dx.$$

Finally, in the latter integral we make the substitution $s = 1 - x$, which leads to

$$F_{u_t}(z, w) = \frac{\alpha\beta}{t} \cdot e^{t+\alpha+\beta} \cdot \int_0^1 e^{-s(t+\alpha+\beta)}\, ds.$$

The constant $(\alpha\beta e^{t+\alpha+\beta})/t$ is (by (4.7)) equal to tzw, hence reverting back from α, β to z, w takes us precisely to the integral formula stated in Equation (4.5). $\square$

Proposition 4.3. *Let us fix $t \in (0, \infty)$ and let u_t be as above (free unitary Brownian motion at time t). For every $k, \ell \in \mathbb{N}$ we have*

$$\kappa_{k+\ell}(\underbrace{u_t, \ldots, u_t}_{k}, \underbrace{u_t^*, \ldots, u_t^*}_{\ell}) = \frac{(-1)^{k+\ell}}{(k-1)!(\ell-1)!} t^{k+\ell-1} \, (e^{-t/2})^{k+\ell-2} \cdot I_{k,\ell}(t), \tag{4.8}$$

where

$$I_{k,\ell}(t) := \int_0^1 e^{-ts}\, s^2\, (s + k - 1)^{k-2}\, (s + \ell - 1)^{\ell-2}\, ds. \tag{4.9}$$

Proof. It is known (see e.g. [5]) that for any $s \in [0, 1]$ and $y \in \mathbb{C}$ with $|y| < 1/e$ one has the series expansion

$$e^{-sW(y)} = 1 - sy + \frac{s(s+2)}{2!} y^2 - \frac{s(s+3)^2}{3!} y^3 + \cdots + (-1)^n \frac{s(s+n)^{n-1}}{n!} y^n + \cdots ,$$

which we will find convenient to write concisely as

$$e^{-sW(y)} = \sum_{n=0}^{\infty} \frac{s(s+n)^{n-1}}{n!} (-y)^n. \tag{4.10}$$

Let us then pick some z, w with $|z|, |w|$ small enough (in the sense discussed in Lemma 4.2) and such that moreover z, w are real negative numbers. By using the expansion (4.10) in the Equation (4.5) of Lemma 4.2 we infer that

$$F_{u_t}(z, w) = tzw \int_0^1 e^{-ts} \cdot \sum_{m=0}^{\infty} \frac{s(s+m)^{m-1}}{m!} (-te^{-t/2}z)^m \cdot \sum_{n=0}^{\infty} \frac{s(s+n)^{n-1}}{n!} (-te^{-t/2}w)^n\, ds$$

$$= \int_0^1 \Big(\sum_{m,n=0}^{\infty} tzw \cdot e^{-ts} \cdot \frac{s(s+m)^{m-1}}{m!} (-te^{-t/2}z)^m \cdot \frac{s(s+n)^{n-1}}{n!} (-te^{-t/2}w)^n \Big) \, ds.$$

The terms of the infinite double-sum are non-negative, hence the monotone convergence theorem allows us to interchange the double-sum with the integral. When we do that, and we move the powers of $-z, -w, t, e^{-t/2}$ outside the integral, we come to the fact that (for z, w picked as above) we have

$$F_{u_t}(z,w) = \sum_{m,n=0}^{\infty} (-z)^{m+1}(-w)^{n+1} \cdot \frac{t^{m+n+1}(e^{-t/2})^{m+n}}{m!n!} \cdot \Big(\int_0^1 s(s+m)^{m-1} \, s(s+n)^{n-1} \, ds \Big).$$

It is convenient to also make here the shift of indices $m+1 = k$, $n+1 = \ell$, and conclude that

$$F_{u_t}(z,w) = \sum_{k,\ell=1}^{\infty} (-z)^k (-w)^\ell \cdot \frac{t^{k+\ell-1}(e^{-t/2})^{k+\ell-2}}{(k-1)!(\ell-1)!} \cdot I_{k,\ell}(t), \tag{4.11}$$

with $I_{k,\ell}(t)$ defined in (4.9).

Now, it is easy to see that if we put

$$\lambda_{k,\ell}(t) := \frac{(-1)^{k+\ell}}{(k-1)!(\ell-1)!} t^{k+\ell-1} \, (e^{-t/2})^{k+\ell-2} \cdot I_{k,\ell}(t), \quad k, \ell \in \mathbb{N},$$

then the formula

$$G_t(z,w) := \sum_{k,\ell=1}^{\infty} \lambda_{k,\ell}(t) z^k w^\ell$$

gives an analytic function defined for $|z|, |w|$ small enough. Indeed, one can simply bound the integrand in $I_{k,\ell}(t)$ by $k^{k-2}\ell^{\ell-2}$ to conclude that

$$0 \le I_{k,\ell}(t) \le k^{k-2}\ell^{\ell-2} \le \gamma \cdot e^k (k-1)! \cdot e^\ell (\ell-1)!,$$

where $\gamma > 0$ is an absolute constant (not depending on k, ℓ) — the second inequality displayed above follows from Stirling's formula. This implies in turn the bound

$$|\lambda_{k,\ell}(t)| \le (\gamma e^t / t) \cdot (ete^{-t/2})^{k+\ell}, \quad \forall \, k, \ell \in \mathbb{N},$$

and gives the claim about the existence of $G_t(z,w)$.

Finally, Equation (4.11) can be read as saying that $F_{u_t}(z,w) = G_t(z,w)$ for z, w real negative numbers of small enough absolute value. This implies that F_{u_t} and G_t must have the same series expansion around $(0,0)$, which is exactly the statement that had to be proved. □

The formula for cumulants found in Proposition 4.3 can be re-phrased as a formula for the corresponding polynomials Z_ω, as follows.

Theorem 4.4. *Let k, ℓ be positive integers. There exist polynomials $U_{k,\ell}, V_{k,\ell} \in \mathbb{Z}[x]$, uniquely determined, such that*

$$\begin{cases} U_{k,\ell}(x) = -x^{k+\ell-1} \int_0^\infty e^{-xs} \big((s+1)^2 (s+k)^{k-2} (s+\ell)^{\ell-2} \big) ds, \\ V_{k,\ell}(x) = x^{k+\ell-1} \int_0^\infty e^{-xs} \big(s^2 (s+k-1)^{k-2} (s+\ell-1)^{\ell-2} \big) ds, \end{cases} \quad x \in [0, \infty). \tag{4.12}$$

One has

$$Z_{(\underbrace{1,\ldots,1}_{k}\underbrace{*,\ldots,*}_{\ell})}(x,y) = \frac{(-1)^{k+\ell}}{(k-1)!(\ell-1)!}\Big(U_{k,\ell}(x)y^{k+\ell} + V_{k,\ell}(x)y^{k+\ell-2}\Big). \tag{4.13}$$

Proof. In order to verify that the function $U_{k,\ell}(x)$ defined by the first integral in (4.12) is indeed a polynomial, we expand the product $(s+1)^2(k+s)^{k-2}(\ell+s)^{\ell-2}$ in powers of s, then use the fact that

$$x^{k+\ell-1}\int_0^\infty e^{-xs}s^m ds = m!\,x^{(k+\ell-1)-(m+1)},\quad 0\le m\le k+\ell-2.$$

A similar calculation shows that $V_{k,\ell}(x)$ is a polynomial as well.

In order to prove that (4.13) holds, it suffices to fix a $t\in[0,\infty)$ and to verify the following fact: when evaluated at $(t,e^{-t/2})$, the polynomial in (x,y) from the right-hand side of (4.13) yields the free cumulant $\kappa_{k+\ell}(u_t,\ldots,u_t,u_t^*,\ldots,u_t^*)$ (with k entries of u_t and ℓ entries of u_t^*). By comparing this fact against the result of Proposition 4.3, and by doing some obvious simplifications, we see that it is actually sufficient to check that

$$t^{k+\ell-1}\,I_{k,\ell}(t) = U_{k,\ell}(t)\,e^{-t} + V_{k,\ell}(t),$$

where $I_{k,\ell}(t)$ is the integral defined in Equation (4.9). But the latter verification is immediately obtained when one writes the integral "$\int_0^1$" which defines $I_{k,\ell}(t)$ as a difference "$\int_0^\infty - \int_1^\infty$" (by using the same integrand). Indeed, the very definition of $V_{k,\ell}$ says that

$$t^{k+\ell-1}\int_0^\infty e^{-ts}\,s^2\,(s+k-1)^{k-2}\,(s+\ell-1)^{\ell-2}\,ds = V_{k,\ell}(t),$$

while on the other hand the change of variable $\widetilde{s}=s-1$ gives

$$\begin{aligned} &t^{k+\ell-1}\int_1^\infty e^{-ts}\,s^2\,(s+k-1)^{k-2}\,(s+\ell-1)^{\ell-2}\,ds \\ &= t^{k+\ell-1}\int_0^\infty e^{-t(\widetilde{s}+1}\,(\widetilde{s}+1)^2\,(\widetilde{s}+k)^{k-2}\,(\widetilde{s}+\ell)^{\ell-2}\,d\widetilde{s}, \end{aligned}$$

which is $-e^{-t}U_{k,\ell}(t)$. □

Remark 4.5. Let us illustrate the explicit writing of the polynomials $U_{k,\ell}$ and $V_{k,\ell}$ in the special case $\ell=1$ (this gives, in some sense, the simplest possible example of free cumulants of u_t and u_t^* that are truly "joint"). The formulas defining $U_{k,\ell}$ and $V_{k,\ell}$ become here

$$U_{k,1}(x) = -x^k\int_0^\infty e^{-xs}(s+1)(s+k)^{k-2}ds,\;\; V_{k,1}(x) = x^k\int_0^\infty e^{-xs}s(s+k-1)^{k-2}ds,\;\; k\in\mathbb{N}.$$

We note the special relation

$$U_{k,1} = -\frac{1}{k}V_{k+1,1},\quad \forall\, k\ge 1,$$

which is easily derived by writing $V_{k+1,1}(x) = -x^k\int_0^\infty (e^{-xs})'\cdot s(s+k-1)^{k-2}ds$, and by doing an integration by parts. We thus only need to write explicitly the $V_{k,1}$'s; this is done in the way shown at the beginning of the preceding proof, which gives $V_{1,1}(x)=V_{2,1}(x)=1$ and

$$V_{k,1}(x) = \sum_{j=0}^{k-2}\binom{k-2}{j}\cdot(k-1-j)!\cdot(k-1)^j\;t^j,\;\; k\ge 3. \tag{4.14}$$

In terms of the actual $*$-cumulants of u_t, the above considerations say that for every $k \in \mathbb{N}$ we have:

$$\begin{aligned}\kappa_{k+1}(\underbrace{u_t,\ldots,u_t}_{k},u_t^*) &= Z_{(\underbrace{1,\ldots,1}_{k},*)}(t,e^{-t/2}) \\ &= \frac{(-1)^{k+1}}{(k-1)!}\Big(U_{k,1}(t)(e^{-t/2})^{k+1} + V_{k,1}(t)(e^{-t/2})^{k-1}\Big) \\ &= \frac{(-e^{-t/2})^{k-1}}{(k-1)!}\Big(-\frac{1}{k}V_{k+1,1}(t)e^{-t} + V_{k,1}(t)\Big).\end{aligned}$$

Thus, if we consider the sequence of polynomials

$$V_k := V_{k,1}/(k-1)!, \quad k \in \mathbb{N},$$

(with $V_{k,1}$ taken from Equation (4.14)), the conclusion is that for every $k \in \mathbb{N}$ and $t \in [0,\infty)$ we have

$$\kappa_{k+1}(\underbrace{u_t,\ldots,u_t}_{k},u_t^*) = (-e^{-t/2})^{k-1}\big(V_k(t) - e^{-t}V_{k+1}(t)\big). \tag{4.15}$$

So for instance for $k \leq 4$ we have

$$\begin{cases} \kappa_2(u_t,u_t^*) & = & (-e^{-t/2})^0\,(1-e^{-t}), \\ \kappa_3(u_t,u_t,u_t^*) & = & (-e^{-t/2})^1\,\big(1-e^{-t}(t+1)\big), \\ \kappa_4(u_t,u_t,u_t,u_t^*) & = & (-e^{-t/2})^2\,\big((t+1)-e^{-t}(\frac{3}{2}t^2+2t+1)\big), \\ \kappa_5(u_t,u_t,u_t,u_t,u_t^*) & = & (-e^{-t/2})^3\,\big((\frac{3}{2}t^2+2t+1)-e^{-t}(\frac{8}{3}t^3+4t^2+3t+1)\big). \end{cases}$$

5. Another special case — alternating ω's

The special form of free joint cumulants for a Haar unitary and its adjoint (reviewed in Equation (1.4) of the introduction) suggests that in our discussion of the polynomials Z_ω we should consider the case when ω is an alternating string of even length. The polynomial Z_ω associated to the alternating string $(1,*,\ldots,1,*) \in \{1,*\}^{2k}$ is of the form

$$(-1)^{k-1}\Big(C_{k-1} - T_1^{(k)}(x)\,y^2 + T_2^{(k)}(x)\,y^4 - \cdots + (-1)^k T_k^{(k)}(x)\,y^{2k}\Big), \tag{5.1}$$

where C_{k-1} is the $(k-1)$-th Catalan number, and every $T_j^{(k)}$ $(1 \leq j \leq k)$ is a polynomial of degree $j-1$ with strictly positive rational coefficients. Examples for small k:

$$\begin{cases} Z_{(1,*)}(x,y) & = & 1-y^2, \\ Z_{(1,*,1,*)}(x,y) & = & -1+4y^2-(2x+3)y^4, \\ Z_{(1,*,1,*,1,*)}(x,y) & = & 2-15y^2+(12x+30)y^4-(6x^2+18x+17)y^6, \\ Z_{(1,*,1,*,1,*,1,*)}(x,y) & = & -5+56y^2-28(2x+7)y^4+8(6x^2+26x+33)y^6 \\ & & -\big(\frac{64}{3}x^3+96x^2+172x+119\big)\,y^8. \end{cases}$$

The inductive verification that the pattern (5.1) holds for general k is not hard (based on the recursion from Proposition 3.5), and is left as exercise to the reader. In this section we do not focus on coefficients of Z_ω's, but we find it more interesting to look at the actual cumulants

$$\xi_n(t) := \kappa_{2n}(u_t, u_t^*, \ldots, u_t, u_t^*) = Z_{(\underbrace{1,*,\ldots,1,*}_{2n})}(t, e^{-t/2}),\ n \geq 1, \tag{5.2}$$

where u_t is the free unitary Brownian motion at time t. The notation introduced in (5.2) emphasizes the dependence on t. This is of relevance because the main point of the section is to put into evidence a recursion for the derivative of ξ_n with respect to t, as shown next.

Theorem 5.1. *Let $\xi_n(t)$ be as above. Then for every $n \geq 2$ one has*

$$-\frac{1}{n}\frac{d\xi_n}{dt}(t) = \xi_n(t) + \sum_{m=1}^{n-1} \xi_m(t)\xi_{n-m}(t), \quad t \in [0, \infty). \tag{5.3}$$

Proof. For convenience of notation, throughout this proof we will fix a tracial $*$-probability space $(\mathcal{A}, \varphi)$ which is large enough to contain all the unitaries u_t for $t \in [0, \infty)$. By enlarging $(\mathcal{A}, \varphi)$ a bit[2] further, we will moreover assume that $\mathcal{A}$ contains two families of elements $\{p_\theta \mid 0 < \theta < 1/2\}$ and $\{q_\theta \mid 0 < \theta < 1/2\}$ such that

(i) $p_\theta^2 = p_\theta^* = p_\theta,\ q_\theta^2 = q_\theta^* = q_\theta$ and $p_\theta q_\theta = q_\theta p_\theta = 0, \quad \forall\, \theta \in (0, 1/2)$;
(ii) $\varphi(p_\theta) = \varphi(q_\theta) = \theta, \quad \forall\, \theta \in (0, 1/2)$;
(iii) $\{p_\theta, q_\theta\}$ is free from $\{u_t, u_t^*\}$, for all $\theta \in (0, 1/2)$ and $t \in [0, \infty)$.

We consider the rescaled elements $v_t = e^{t/2} u_t$. Following [2], for every $n \in \mathbb{N}$ we define a function $f_{2n} : [0, \infty) \times (0, 1/2) \to \mathbb{R}$ by

$$f_{2n}(t, \theta) := \varphi\big(\, (p_\theta\, v_t\, q_\theta\, v_t^*)^n \,\big), \quad \forall\, t \geq 0 \text{ and } 0 < \theta < 1/2. \tag{5.4}$$

[For instance for $n = 1$ we have $f_2(t, \theta) := \varphi(p_\theta\, v_t\, q_\theta\, v_t^*)$, and an immediate application of formula (2.2) for alternating moments yields $f_2(t, \theta) = \theta^2(e^t - 1)$.]

Claim. For every $n \in \mathbb{N}$, the function f_{2n} is of the form

$$f_{2n}(t, \theta) = \sum_{j=1}^{2n} g_{n,j}(t)\, \theta^j, \tag{5.5}$$

where the $g_{n,j}$'s are quasi-polynomials, and where (for $j = 2n$) we have

$$g_{n,2n}(t) = e^{nt} \xi_n(t). \tag{5.6}$$

Verification of Claim. Fix $n \in \mathbb{N}$ for which we will verify that (5.5) and (5.6) hold. We write $f_{2n}(t, \theta)$ as $\varphi(\, v_t\, q_\theta\, v_t^*\, p_\theta \cdots v_t\, q_\theta\, v_t^*\, p_\theta\,)$, and we expand this alternating moment of order $4n$ in the way indicated in Remark 2.5.1, in terms of moments of p_θ, q_θ and of free cumulants of v_t, v_t^*. In this way we obtain the formula

$$f_{2n}(t, \theta) = \sum_{\sigma \in NC(2n)} g_\sigma(t) \cdot h_\sigma(\theta), \tag{5.7}$$

where for $\sigma \in NC(2n)$ we put

$$\begin{cases} g_\sigma(t) &= \prod_{V \in \sigma} \kappa_{|V|}\big(\, (v_t, v_t^*, \ldots, v_t, v_t^*) \mid V \,\big), \\ h_\sigma(\theta) &= \prod_{W \in \mathrm{Kr}_{2n}(\sigma)} \varphi_{|W|}\big(\, (q_\theta, p_\theta, \ldots, q_\theta, p_\theta) \mid W \,\big). \end{cases}$$

[2] For instance we can replace $(\mathcal{A}, \varphi)$ by the free product $(\mathcal{A}, \varphi) * (L^\infty[0,1], dt)$, and take $p_\theta, q_\theta \in L^\infty[0,1], dt)$ to be the indicator functions of the intervals $[0, \theta]$ and $[1 - \theta, 1]$, respectively.

Note that every g_σ can be written as

$$g_\sigma(t) = e^{nt} \cdot \prod_{V \in \sigma} \kappa_{|V|}\big((u_t, u_t^*, \ldots, u_t, u_t^*) \mid V \,) \big),$$

and is thus a quasi-polynomial by Proposition 3.1.

Let us next observe that for every non-empty set $W \subseteq \{1, \ldots, 2n\}$, the moment $\varphi_{|W|}\big((q_\theta, p_\theta, \ldots, q_\theta, p_\theta) \mid W \,) \big)$ is equal to either 0 or θ. Indeed, if W contains both odd and even numbers, then the moment in discussion vanishes due to the hypothesis that $p_\theta q_\theta = q_\theta p_\theta = 0$. In the opposite case, we are looking either at $\varphi(p_\theta^{|W|})$ or at $\varphi(q_\theta^{|W|})$, and both these moments are equal to θ.

The observation from the preceding paragraph implies that, for every $\sigma \in NC(2n)$, the value of $h_\sigma(\theta)$ is either 0 or $\theta^{j(\sigma)}$, with $j(\sigma) :=|\ \mathrm{Kr}_{2n}(\sigma)\ | = \ (2n+1) - |\sigma|$, where $|\sigma|$ denotes the number of blocks of σ. Moreover, the case $h_\sigma(\theta) = \theta^{j(\sigma)}$ occurs if and only if every block W of $\mathrm{Kr}_{2n}(\sigma)$ either is contained in $\{1, 3, \ldots, 2n-1\}$ or is contained in $\{2, 4, \ldots, 2n\}$. The latter condition on $\mathrm{Kr}_{2n}(\sigma)$ is easily seen to be equivalent to the fact that every block of σ has even cardinality (cf. [9], Exercise 9.42(1) on p. 154). We thus come to the conclusion that we can re-write Equation (5.7) in the form

$$f_{2n}(t, \theta) = \sum_{j=1}^{2n} g_{n,j}(t) \cdot \theta^j,$$

where for $1 \leq j \leq 2n$ we define the quasi-polynomial $g_{n,j}$ to be

$$g_{n,j} := \sum_{\substack{\sigma \in NC(2n),\ |\sigma| = 2n+1-j \\ and\ |V|\ even\ for\ all\ V \in \sigma}} g_\sigma(t). \tag{5.8}$$

In the special case $j = 2n$, the only partition involved in the sum from (5.8) is $\sigma = 1_{2n}$, which has $g_{1_{2n}}(t) = \kappa_{2n}(v_t, v_t^*, \ldots, v_t, v_t^*) = e^{nt} \cdot \kappa_{2n}(u_t, u_t^*, \ldots, u_t, u_t^*)$, and (5.6) also follows. *[End of Verif. of Claim]*

Besides the $f_2, f_4, \ldots, f_{2n}, \ldots$ introduced in (5.5) we consider, also following [2], the function $f_0 : [0, \infty) \times (0, 1/2) \to \mathbb{R}$ defined by

$$f_0(t, \theta) := \theta, \quad \forall\, t \geq 0 \text{ and } 0 < \theta < 1/2.$$

(Note that the definition of f_0 is not obtained by extending the range of n from $\mathbb{N}$ to $\mathbb{N} \cup \{0\}$ in Equation (5.4)!) Theorem 3.4 in [2] gives us that for every $n \geq 1$, the partial derivative $\partial_t f_{2n}$ satisfies the following recursion:

$$\partial_t f_{2n}(t, \theta) = - \sum_{\substack{1 \leq k < \ell \leq 2n \\ k = \ell \ mod\ 2}} f_{2n-(\ell-k)}(t, \theta)\, f_{\ell-k}(t, \theta)$$

$$+ e^t \sum_{\substack{1 \leq k < \ell \leq 2n \\ k \neq \ell \ mod\ 2}} f_{2n-(\ell-k)-1}(t, \theta)\, f_{\ell-k-1}(t, \theta). \tag{5.9}$$

[For illustration we record that the special cases $n = 1$ and $n = 2$ of (5.9) come to $\partial_t f_2 = e^t f_0^2$, and respectively to $\partial_t f_4 = -2 f_2^2 + e^t \cdot 4 f_0 f_2$.]

For a fixed $t \in [0, \infty)$, both sides of Equation (5.9) are polynomials of degree $2n$ in θ; so it makes sense to extract the coefficient of θ^{2n} in this equation. On the left-hand side, the coefficient of θ^{2n} is equal to the derivative of $g_{n,2n}(t)$, thus to

$$e^{nt} \cdot \big(n\xi_n(t) + \frac{d\xi_n}{dt}(t) \big). \tag{5.10}$$

On the right-hand side of (5.9) only the terms from the first of the two sums contribute to θ^{2n}, giving a coefficient equal to

$$- \sum_{\substack{1 \le k < \ell \le 2n \\ k = \ell \ mod\ 2}} \big(e^{(n-(\ell-k)/2)t} \xi_{n-(\ell-k)/2}(t) \big) \cdot \big(e^{((\ell-k)/2)t} \xi_{(\ell-k)/2}(t) \big) = -e^{nt} \cdot n \sum_{m=1}^{n-1} \xi_m(t) \xi_{n-m}(t).$$

When we equate the latter quantity with the one in (5.10), formula (5.3) follows. □

Corollary 5.2. *Consider the function*

$$H(t,z) := \frac{1}{2} + \sum_{n=1}^{\infty} \xi_n(t) z^n \tag{5.11}$$

defined on $\{(t,z) \mid t \in [0,\infty),\ z \in \mathbb{C},\ |z| < 1/16^2\}$. *Then* H *satisfies the partial differential equation*

$$\partial_t H + 2z\, H\, \partial_z H = z, \tag{5.12}$$

with initial condition $H(0,z) = 1/2$.

Proof. The domain of H is considered by taking into account the bounds for $\xi_n(t)$'s that follow from Remark 2.5.4. In order to obtain (5.12), we square both sides of (5.11) and then we take partial derivative ∂_z, to find that

$$z \cdot \partial_z H^2(t,z) = \xi_1(t) z + 2\Big(\xi_2(t) + \xi_1^2(t) \Big) z^2 + \cdots + n \Big(\xi_n(t) + \sum_{m=1}^{n-1} \xi_m(t) \xi_{n-m}(t) \Big) z^n + \cdots$$

The latter equation can be written as

$$\begin{aligned} z \cdot \partial_z H^2(t,z) &= \xi_1(t) z - \xi_2'(t) z^2 - \cdots - \xi_n'(t) z^n - \cdots && \text{(by Theorem 5.1)} \\ &= (1 - \xi_1'(t)) z - \xi_2'(t) z^2 - \cdots - \xi_n'(t) z^n - \cdots && \text{(because } \xi_1(t) = 1 - e^{-t}) \\ &= z - \partial_t H(t,z), \end{aligned}$$

and (5.12) follows. The condition on $H(0,z)$ is also clear, since $\xi_n(0) = 0$ for all $n \in \mathbb{N}$. □

Remark 5.3. 1^o Starting from $\xi_1(t) = 1 - e^{-t}$ and the initial condition $\xi_n(0) = 0,\ \forall\, n \ge 2$, one can use Theorem 5.1 to calculate all the ξ_n's, getting $\xi_2(t) = -1 + 4e^{-t} - (2t+3)e^{-2t}$, then $\xi_3(t) = 2 - 15e^{-t} + 6(2t+5)e^{-2t} - (6t^2 + 18t + 17)e^{-3t}$, and so on.

2^o It stands to reason that one should also look for a description of the functions $\xi_n(t)$ that is done by plain algebra (without resorting to the derivative $\frac{d}{dt}$), for a given value of t. That is, we are interested in an algebraic description for the function

$$H_t : \{ z \in \mathbb{C} \mid \ |z| < 1/16^2 \} \to \mathbb{C}, \qquad H_t(z) := H(t,z) = \frac{1}{2} + \sum_{n=1}^{\infty} \xi_n(t) z^n. \tag{5.13}$$

We will achieve this by examining the characteristic curves of the p.d.e. found in Corollary 5.2.

In order to state precisely what is the algebraic description obtained for H_t, we introduce an auxiliary complex parameter c and we consider, for every $t \in [0, \infty)$, the function

$$\chi_t : \Omega_t \to \mathbb{C}, \quad \chi_t(c) := \frac{c^2(1-c^2)e^{ct}}{(\,(1+c)-(1-c)e^{ct}\,)^2}, \tag{5.14}$$

where Ω_t is the open set $\{c \in \mathbb{C} \mid 1+c \neq (1-c)e^{ct}\}$. One has $\Omega_t \ni 1$, with $\chi_t(1) = 0$ and $\chi_t'(1) = e^{-t/2} \neq 0$. The inverse function theorem thus gives a $\delta_t > 0$ such that an analytic inverse for χ_t can be defined on $\{z \in \mathbb{C} \mid \ |z| < \delta_t\}$, sending 0 back to 1. We denote this compositional inverse as

$$\chi_t^{\langle -1 \rangle} : \{z \in \mathbb{C} \mid \ |z| < \delta_t\} \to \mathbb{C}. \tag{5.15}$$

$\chi_t^{\langle -1 \rangle}$ is injective and its range-set $\mathrm{Ran}(\,\chi_t^{\langle -1 \rangle}\,)$ is an open subset of Ω_t.

Without loss of generality, we may assume that in (5.15) we have $\delta_t < 1/16^2$, so that $H_t(z)$ from Equation (5.13) is sure to be defined, too, for $|z| < \delta_t$.

Theorem 5.4. *Let $t \in [0, \infty)$ be fixed, and consider the analytic functions H_t and $\chi_t^{\langle -1 \rangle}$ defined in Remark 5.3.2. Then one has*

$$[\,H_t(z)\,]^2 = z + \frac{1}{4}[\,\chi_t^{\langle -1 \rangle}(z)\,]^2, \quad |z| < \delta_t. \tag{5.16}$$

Proof. Our strategy will be to prove the following fact.

$$\text{(5.17)} \quad \left[\begin{array}{l} \text{There exists } \varepsilon_t > 0 \text{ such that for every } c \in (1-\varepsilon_t, 1+\varepsilon_t) \subseteq \mathbb{R} \text{ one has:} \\ \quad \to \ c \in \Omega_t \text{ and } |\chi_t(c)| < 1/16^2 \text{ (hence } H_t(\,\chi_t(c)\,) \text{ is defined);} \\ \quad \to \ [H_t(\,\chi_t(c)\,)]^2 = \chi_t(c) + \frac{c^2}{4}. \end{array}\right.$$

This fact implies the statement of the theorem. Indeed, let us assume that (5.17) holds. Take a strictly decreasing sequence $(c_n)_{n=1}^{\infty}$ in $(1-\varepsilon_t, 1+\varepsilon_t) \cap \mathrm{Ran}(\,\chi_t^{\langle -1 \rangle}\,)$, with $\lim_{n\to\infty} c_n = 1$, and put $z_n := \chi_t(c_n)$, $n \in \mathbb{N}$. Then $(z_n)_{n=1}^{\infty}$ are distinct points in $\{z \in \mathbb{C} \mid \ |z| < \delta_t\}$, with $\lim_{n\to\infty} z_n = 0$, and by applying the last line of (5.17) to the c_n we get

$$[H_t(z_n)]^2 = z_n + \frac{1}{4}[\chi_t^{\langle -1 \rangle}(z_n)]^2, \quad \forall\, n \in \mathbb{N}.$$

Hence the analytic functions appearing on the two sides of (5.16) coincide on a subset of $\{z \in \mathbb{C} \mid \ |z| < \delta_t\}$ which has 0 as accumulation point, and (5.16) follows.

We now start towards the proof of the fact stated in (5.17). We consider the rectangular strip

$$R := [0, \infty) \times (-1/16^2, +1/16^2) \subseteq \mathbb{R}^2,$$

and we consider the restriction of H (from its domain stated in Corollary 5.2) to R. This restriction will still be denoted as H, and [3] we put

$$\Gamma := \{(s, x, u) \mid (s, x) \in R, \ u = H(s, x) \in \mathbb{R}\} \quad \text{(graph of restricted } H).$$

We also consider the vector field $V : R \times \mathbb{R} \to \mathbb{R}^3$ defined by

$$V(s, x, u) := (1, 2xu, x), \quad \text{for } (s, x) \in R,\ u \in \mathbb{R}. \tag{5.18}$$

[3] Since "t" is here a specific time that was fixed in the statement of the theorem, we will use the generic letter "s" for the first component of a point in R.

The partial differential equation (5.11) says that for every $(s,x) \in R$, the vector $V\big(s,x,H(s,x)\big)$ is orthogonal to the normal direction $\big((\partial_t H)(s,x),(\partial_x H)(s,x),-1\big)$ to Γ at the point $\big(s,x,H(s,x)\big)$. It follows that $V\big(s,x,H(s,x)\big)$ gives a tangent direction to the graph Γ, at the point $\big(s,x,H(s,x)\big)$.

We next pick an $a \in (-1/16^2, +1/16^2)$ and we consider a path (a.k.a. characteristic curve) $L_a : [0,\beta(a)) \to \mathbb{R}^3$ which has

$$L_a(0) = (0,a,1/2) \in \Gamma \tag{5.19}$$

and follows the vector field V:

$$L_a'(s) = V(\,L_a(s)\,), \quad \forall\, s \in [0,\beta(a)). \tag{5.20}$$

When we write L_a componentwise,

$$L_a(s) = (p_a(s), q_a(s), r_a(s)), \quad 0 \le s < \beta(a),$$

the Equation (5.20) becomes a system of ordinary differential equations, for which (5.19) gives an initial condition:

$$\begin{cases} p_a'(s) = 1,\ q_a'(s) = 2q_a(s)\,r_a(s),\ r_a'(s) = q_a(s), \\ \\ \quad \text{with } p_a(0) = 0,\ q_a(0) = a,\ r_a(0) = 1/2. \end{cases} \tag{5.21}$$

Luckily, the Cauchy problem from (5.21) can be solved explicitly. More precisely: considering the auxiliary[4] constants

$$c = \sqrt{1-4a}, \quad \alpha = \frac{1-c}{1+c} = \frac{4a}{(1+c)^2}, \tag{5.22}$$

we get

$$p_a(s) = s,\ q_a(s) = \frac{c^2\alpha e^{cs}}{(1-\alpha e^{cs})^2},\ r_a(s) = \frac{c}{2}\cdot\frac{1+\alpha e^{cs}}{1-\alpha e^{cs}}, \tag{5.23}$$

for $0 \le s < \beta(a)$. A significant detail which comes up while solving (5.21) (and can, of course, be checked directly on (5.23)) is that one has

$$q_a(s) - \big(\,r_a(s)\,\big)^2 = a - \frac{1}{4}, \quad \forall\, s \in [0,\beta(a)). \tag{5.24}$$

It is quite useful if at this point we take a moment to assess what we want to have for "$\beta(a)$" in the discussion from the preceding paragraph. Clearly, $\beta(a)$ must be in any case picked such that

$$\text{(i) } 1 - \alpha e^{cs} > 0,\ \ \forall\, s \in [0,\beta(a)), \qquad \text{and} \qquad \text{(ii) } \Big|\, \tfrac{c^2\alpha e^{cs}}{(1-\alpha e^{cs})^2} \,\Big| < \tfrac{1}{16^2},\ \forall\, s \in [0,\beta(a)).$$

The condition (i) ensures that the formulas (5.23) give indeed a well-defined path $L_a(s) = (p_a(s), q_a(s), r_a(s))$, $0 \le s < \beta(a)$; then (ii) ensures that $L_a(s) \in R \times \mathbb{R}$ (hence that "$V(L_a(s))$" makes sense) for every $s \in [0,\beta(a))$. We will moreover insist that

(iii) $\beta(a)$ is a continuous function of $a \in (-1/16^2, +1/16^2)$,

and

(iv) $\beta(0) > t$ (= the time fixed in the statement of the theorem).

We leave it as a routine (though tedious) exercise to the reader to check that all the conditions (i)–(iv) are fulfilled if we go with

$$\beta(a) = \min\big(t+1, \beta_{\text{(i)}}(a), \beta_{\text{(ii)}}(a)\big) \ \text{ for } |a| < 1/16^2,$$

[4] It is useful to keep in mind that c runs in a neighbourhood of 1 (it satisfies $\sqrt{63}/8 < c < \sqrt{65}/8$), while α runs in a neighbourhood of 0 (has $\text{sign}(\alpha) = \text{sign}(a)$ and $|\alpha| < 4|a| < 1/64$).

where $\beta_{\text{(i)}}, \beta_{\text{(ii)}} : (-1/16^2, +1/16^2) \to [0, \infty]$ are continuous functions describing the natural bounds up to which an $s \in [0, \infty)$ fulfills the conditions (i) and (ii), respectively. For instance $\beta_{\text{(i)}}(a)$ comes out as

$$\beta_{\text{(i)}}(a) = \begin{cases} \frac{1}{c}\ln\frac{1}{\alpha}, & \text{if } \alpha > 0, \\ \\ \infty, & \text{if } \alpha \leq 0, \end{cases}$$

with $c = c(a)$ and $\alpha = \alpha(a)$ as defined in Equations (5.22).

We now invoke a basic result from the theory of quasi-linear partial differential equations, which states that: since it starts at a point $L_a(0) \in \Gamma$, the characteristic curve L_a cannot leave the graph Γ of H. (See e.g. the theorem on page 10 of [6].) In other words, one has

$$H(p_a(s), q_a(s)) = r_a(s), \quad \forall\, a \in (-1/16^2, +1/16^2) \text{ and } s \in [0, \beta(a)). \tag{5.25}$$

If we square both sides of (5.25) and take into account the formula (5.24) (also the fact that $p_a(s) = s$), we arrive to

$$[\,H(s, q_a(s))\,]^2 = q_a(s) + \big(\frac{1}{4} - a\big), \quad \forall\, a \in (-1/16^2, +1/16^2) \text{ and } s \in [0, \beta(a)). \tag{5.26}$$

Finally, let us return to the time $t \in [0, \infty)$ that was fixed in the statement of the theorem. Since β is continuous and has $\beta(0) > t$, we can find $0 < \lambda_t < 1/16^2$ such that $\beta(a) > t$ for all $a \in (-\lambda_t, \lambda_t)$. For $|a| < \lambda_t$ we can thus put $s = t$ in (5.26), to obtain that

$$[\,H(t, q_a(t))\,]^2 = q_a(t) + \big(\frac{1}{4} - a\big). \tag{5.27}$$

On the other hand, we make the following claim.

Claim. If $|a| < \lambda_t$, then $c := \sqrt{1 - 4a}$ belongs to the domain Ω_t of the function χ_t, and one has $q_a(t) = \chi_t(c)$.

Verification of Claim. We have $t < \beta(a) \leq \beta_{\text{(i)}}(a)$, hence (from how $\beta_{\text{(i)}}(a)$ is defined) we get $1 - \alpha e^{ct} > 0$, where $\alpha = (1-c)/(1+c)$. This implies $(1+c) - (1-c)e^{ct} > 0$, and it follows that $c \in \Omega_t$. The equality $q_a(t) = \chi_t(c)$ is then immediately obtained by comparing the formulas which describe $q_a(t)$ and $\chi_t(c)$ (cf. Equation (5.14) and the case $s = t$ of (5.23)).

[End of Verif. of Claim]

By using the above claim, we convert Equation (5.27) into

$$\begin{cases} [H_t(\,\chi_t(c)\,)\,]^2 = \chi_t(c) + \frac{c^2}{4}, \quad \text{with } c = \sqrt{1-4a}, \\ \\ \quad \text{for every } a \in (-\lambda_t, \lambda_t). \end{cases}$$

But when a runs in $(-\lambda_t, \lambda_t)$, the quantity $c = \sqrt{1-4a}$ covers $(\sqrt{1 - 4\lambda_t}, \sqrt{1 + 4\lambda_t}\,)$, which contains an open interval centered at 1. This implies the fact stated in (5.17), and concludes the proof. □

Remark 5.5. The formula (5.16) from the preceding theorem can be used to calculate the alternating cumulants $\xi_n(t)$ without doing a derivative $\frac{d}{dt}$, but rather by starting from the Taylor expansion around 1 of the function $\chi_t(c)$ defined in Remark 5.3.2:

$$\chi_t(c) = (c-1)\,\chi_t'(1) + \frac{(c-1)^2}{2}\,\chi_t''(1) + \cdots$$

$$= (-\frac{1}{2}e^t)\,(c-1) + \big(\frac{1}{2}e^{2t} - (\frac{3}{4} + \frac{t}{2})e^t\big)\,(c-1)^2 + \cdots$$

Indeed, considering the expansion $\chi_t^{\langle -1\rangle}(z) = 1 + \lambda_1 z + \lambda_2 z^2 + \cdots$ of $\chi_t^{\langle -1\rangle}$ around 0, one can then calculate recursively the λ_n by writing that

$$z = \chi_t\big(\chi_t^{\langle -1\rangle}(z)\big) = (-\frac{1}{2}e^t)\cdot\big(\chi_t^{\langle -1\rangle}(z)-1\big)+\big(\frac{1}{2}e^{2t}-(\frac{3}{4}+\frac{t}{2})e^t\big)\cdot\big(\chi_t^{\langle -1\rangle}(z)-1\big)^2+\cdots \tag{5.28}$$
$$= (-\frac{1}{2}e^t)(\lambda_1 z + \lambda_2 z^2 + \cdots) + \big(\frac{1}{2}e^{2t} - (\frac{3}{4} + \frac{t}{2})e^t\big)\,(\lambda_1 z + \lambda_2 z^2 + \cdots)^2 + \cdots,$$

and by identifying coefficients. The λ_n come out as quasi-polynomials in $-t$ (for instance, as immediately seen from the few terms recorded in (5.28), one gets $\lambda_1 = -2e^{-t}$ and $\lambda_2 = 4e^{-t} - (4t+6)e^{-2t}$).

Finally, Equation (5.16) says that

$$\frac{1}{4} + \xi_1(t)z + (\xi_2(t) + \xi_1^2(t))z^2 + (\xi_3(t) + 2\xi_1(t)\xi_2(t))z^3 + \cdots$$
$$= z + \frac{1}{4}\Big(1 + 2\lambda_1 z + (2\lambda_2 + \lambda_1^2)z^2 + (2\lambda_3 + 2\lambda_1\lambda_2)z^3 + \cdots\Big),$$

which allows the recursive calculation of the $\xi_n(t)$ (e.g. $\xi_1(t) = 1 + \frac{\lambda_1}{2} = 1 - e^{-t}$, then

$$\xi_2(t) = \frac{1}{4}(2\lambda_2 + \lambda_1^2) - \xi_1(t)^2$$
$$= (2e^{-t} - (2t+2)e^{-2t}) - (1-e^{-t})^2 = -1 + 4e^{-t} - (2t+3)e^{-2t},$$

which agrees, of course, with the formulas stated in Remark 5.3.1).

Remark 5.6. The proof presented above for Theorem 5.4 is a standard application of the method of characteristics, and has in its favour the fact that the relevant function χ_t from Equation (5.14) is "discovered" as we move through the argument. The referee to the paper pointed out to us how an alternative, shorter proof of the theorem can be made by starting from the observation that, for fixed c, the function $t \mapsto \chi_t(c)$ satisfies the differential equation

$$\partial_t \chi_t(c) = 2\chi_t(c)\,\Big(\,\chi_t(c) + \frac{c^2}{4}\,\Big)^{1/2}. \tag{5.29}$$

The present remark gives a sketch of this alternative argument.

For the convenience of having all our functions defined around the origin, let us consider the shifted sets $\widetilde{\Omega_t} := \{c-1 \mid c \in \Omega_t\}$, $t \geq 0$, and let us define

$$\widetilde{\chi}(t,z) = \widetilde{\chi}_t(z) := \chi_t(z+1), \quad t \geq 0,\ z \in \widetilde{\Omega}_t. \tag{5.30}$$

Formula (5.29) then gives a family of ordinary differential equations satisfied by the functions $t \mapsto \widetilde{\chi}(t,z)$ (with the parameter z running in a neighbourhood of 0), namely

$$\begin{cases} \partial_t \widetilde{\chi}(t,z) = 2\widetilde{\chi}(t,z)\,\Big(\,\widetilde{\chi}(t,z) + \frac{(z+1)^2}{4}\,\Big)^{1/2}, \\ \text{with initial condition } \widetilde{\chi}(0,z) = (1-(z+1)^2)/4. \end{cases} \tag{5.31}$$

Now, Theorem 5.4 can be recast as the statement that a certain function constructed out of H is equal to $\widetilde{\chi}$. Indeed, the conclusion of the theorem can be rewritten as

$$\big[4(H_t(z)^2 - z)\big]^{1/2} - 1 = \chi_t^{\langle -1\rangle}(z) - 1 = \widetilde{\chi}_t^{\langle -1\rangle}(z). \tag{5.32}$$

So then let us denote

$$G(t,z) = G_t(z) := \big[4(H_t(z)^2 - z)\big]^{1/2} - 1. \tag{5.33}$$

By starting from the explicit series expansion of $H_t(z)$ in (5.13) and by following through the algebra, one finds an explicit series expansion for $G_t(z)$,

$$G_t(z) = 2(\xi_1(t) - 1)z + 2(\xi_2(t) + 2\xi_1(t) - 1)z^2 + \cdots$$

What interests us here is that the expansion of $G_t(z)$ has no constant term, and has linear term $2(\xi_1(t) - 1) = 2\big((1 - e^{-t}) - 1\big) = -2e^{-t} \neq 0$; this implies that G_t is invertible under composition. We can therefore define a function $K(t, z) = K_t(z)$ via the requirement that

$$K_t = G_t^{\langle -1 \rangle} \quad \text{(compositional inverse)}, \quad \forall\, t \in [0, \infty).$$

With these notations and in view of the calculation from (5.32), the statement of Theorem 5.4 amounts to checking that

$$K(t, z) = \widetilde{\chi}(t, z) \tag{5.34}$$

(for enough pairs (t, z), with $t \in [0, \infty)$ and z running in a neighbourhood of 0).

The final step of this line of proof is to verify that the function $t \mapsto K(t, z)$ satisfies the same differential equation as found for $t \mapsto \widetilde{\chi}(t, z)$ in Equation (5.31). This is obtained by invoking the partial differential equation known for H from Corollary 5.2, which expresses $\partial_t H$ as $z(1 - 2H \cdot \partial_z H)$. Indeed, upon working out the ∂_t and ∂_z in the definition (5.33) of G, one finds the said p.d.e. for H to have the nice consequence that

$$\partial_t G(t, z) = -2z\, H(t, z)\, \partial_z G(t, z). \tag{5.35}$$

So then if we take ∂_t in the identity $G(t, K(t, z)) = z$, we get

$$\begin{aligned} 0 &= \partial_t G(t, K(t, z)) + \partial_z G(t, K(t, z))\, \partial_t K(t, z) \\ &= \partial_z G(t, K(t, z)) \, \Big(-2K(t, z)\, H(t, K(t, z) + \partial_t K(t, z) \Big), \end{aligned}$$

where at the second equality sign we made use of (5.35). In the resulting product we are sure that $\partial_z G(t, K(t, z)) \neq 0$ (because taking ∂_z in the identity $G(t, K(t, z)) = z$ gives $\partial_z G(t, K(t, z)) \cdot \partial_z K(t, z) = 1$); so we can divide it out, and conclude that

$$\partial_t K(t, z) = 2K(t, z)\, H(t, K(t, z)). \tag{5.36}$$

We are left to observe that

$$\begin{aligned} 4\big(H(t, K(t, z))^2 - K(t, z) \big) &= \big(G(t, K(t, z)) + 1 \big)^2 \quad \text{(by def. of } G) \\ &= (z + 1)^2; \end{aligned}$$

this implies $H(t, K(t, z)) = \big(K(t, z) + (z+1)^2/4 \big)^{1/2}$, hence (5.36) is precisely the differential equation which had been sought for $t \mapsto K(t, z)$. The required initial condition $K(0, z) = (1 - (z + 1)^2)/4$ is also easily verified, by writing $K_0 = G_0^{\langle -1 \rangle}$ with $G_0(z) = (1 - 4z)^{1/2} - 1$.

6. Behaviour when $t \to \infty$

In this section we look at the behaviour of a joint cumulant $\kappa_n(u_t^{\omega(1)}, \ldots, u_t^{\omega(n)})$, for general $\omega \in \{1, *\}^n$, when $t \to \infty$. Specifically, we discuss how the limit and derivative at ∞ relate to the corresponding polynomial Z_ω.

When it comes to just taking a plain limit $t \to \infty$, things are straightforward: we have

$$\lim_{t \to \infty} \kappa_n\big(u_t^{\omega(1)}, \ldots, u_t^{\omega(n)} \big) = \kappa_n\big(u^{\omega(1)}, \ldots, u^{\omega(n)} \big), \tag{6.1}$$

where u is a Haar unitary; and the $*$-cumulants of a Haar unitary have a very nice form, first found in [10], which puts the spotlight on alternating strings of even length. For later perusal throughout the section, it is convenient to include the latter concept into the following definition.

Definition 6.1. 1^o Let n be an even positive integer. A string $\omega \in \{1,*\}^n$ is said to be *alternating* if it is equal either to $(1,*,1,*,\ldots,1,*)$ or to $(*,1,*,1,\ldots,*,1)$.

2^o Let n be an odd positive integer. A string $\omega \in \{1,*\}^n$ is said to be *alternating* if it is obtained by a cyclic permutation of components from either $(1,*,1,\ldots,*,1)$ or $(*,1,*\ldots,1,*)$.

[So note that we only have 2 alternating strings of length n when n is even, but we have $2n$ alternating strings of length n when n is odd. A concrete example:

$$(1,1,*,1,*),\ (*,1,1,*,1),\ (1,*,1,1,*),\ (*,1,*,1,1),\ (1,*,1,*,1)$$

are 5 of the 10 alternating strings of length 5, and the remaining alternating strings of length 5 are obtained by swapping the roles of '1' and '$*$' in the above list.]

Proposition 6.2. *Let $\omega = (\omega(1),\ldots,\omega(n))$ be in $\{1,*\}^n$, for $n \in \mathbb{N}$.*

1^o *We have*

$$\lim_{t\to\infty} \kappa_n\big(u_t^{\omega(1)},\ldots,u_t^{\omega(n)}\big) = \begin{cases} (-1)^{k-1}C_{k-1}, & \text{if } n \text{ is even, } n=2k, \text{ and} \\ & \omega \text{ is an alternating string,} \\ 0, & \text{otherwise,} \end{cases} \tag{6.2}$$

with C_{k-1} the $(k-1)$-th Catalan number (same as in Equation (1.4) of the introduction).

2^o *Suppose that n is even, $n=2k$. Consider the polynomial Z_ω and as written in Proposition 3.1,*

$$Z_\omega(x,y) = Z_\omega^{(2k)}(x)\,y^{2k} + Z_\omega^{(2k-2)}(x)\,y^{2k-2} + \cdots + Z_\omega^{(2)}(x)\,y^2 + Z_\omega^{(0)}(x).$$

Then $Z_\omega^{(0)}(x)$ is a constant polynomial, where the constant is given by the right-hand side of Equation (6.2).

Proof. 1^o This is the limit from (6.1), where we also invoke the explicit formula for the $*$-cumulants of a Haar unitary that was found in [10]. (See Section 3.4 of [10], or Proposition 15.1 in the monograph [9].)

2^o In view of how Z_ω is defined, from 1^o it follows that $\lim_{t\to\infty} Z_\omega(t, e^{-t/2})$ exists. Since $\lim_{t\to\infty} Z_\omega^{(n-2j)}(t)\cdot(e^{-t/2})^{n-2j} = 0$ for $j < n/2$, we then infer that $\lim_{t\to\infty} Z_\omega^{(0)}(t)$ exists as well. But this can only happen if $Z_\omega^{(0)}$ is a constant (and the constant in question must be the one appearing on the right-hand side of (6.2)). ☐

For the present paper it is important that upon looking at strings of odd length, we get the following analogue of Proposition 6.2.2.

Theorem 6.3. *Let* $\omega = (\omega(1), \ldots, \omega(n))$ *be in* $\{1,*\}^n$*. Suppose that* n *is odd,* $n = 2k-1$*, and consider the polynomial* Z_ω*, written in the form*

$$Z_\omega(x,y) = Z_\omega^{(2k-1)}(x)\, y^{2k-1} + Z_\omega^{(2k-3)}(x)\, y^{2k-3} + \cdots + Z_\omega^{(3)}(x)\, y^3 + Z_\omega^{(1)}(x)\, y.$$

Then $Z_\omega^{(1)}(x)$ *is a constant polynomial, and more precisely:*

$$Z_\omega^{(1)}(x) = \begin{cases} (-1)^{k-1} C_{k-1}, & \text{if } \omega \text{ is alternating,} \\ \\ 0, & \text{otherwise.} \end{cases} \tag{6.3}$$

Proof. It is easily seen that every $\omega \in \{1,*\}^n$ has $\mathrm{Switch}(\omega) \leq n-1$, with equality holding if and only if ω is alternating. Thus if ω is not alternating, then Theorem 3.8 can be applied with $j = (n-1)/2$, and gives that $Z_\omega^{(1)}(x)$ is constantly equal to 0.

We are left to prove the following:

$$\begin{cases} \text{if } n = 2k-1 \text{ and if } \omega \in \{1,*\}^n \text{ is alternating,} \\ \text{then } Z_\omega^{(1)}(x) \text{ is constantly equal to } (-1)^{k-1} C_{k-1}. \end{cases} \tag{6.4}$$

We will prove this statement by induction on k. The case $k = 1$ is clear, since $Z_{(1)}(x,y) = Z_{(*)}(x,y) = y$ (corresponding to the fact that u_t has expectation $e^{-t/2}$, $\forall\, t \in [0,\infty)$). The remaining part of the proof is devoted to the induction step: we fix $k \geq 2$, we assume that (6.4) holds for alternating strings of length $1, 3, \ldots, 2k-3$, and we prove that it also holds for alternating strings of length $2k-1$.

Since any two alternating strings of length $2k-1$ can be obtained from each other by operations which do not affect Z_ω's, it will suffice to verify that, for the k that was fixed, we have

$$Z^{(1)}_{(\underbrace{1,1,*,\ldots,1,*}_{2k-1})}(x) = (-1)^{k-1} C_{k-1}. \tag{6.5}$$

In order to verify (6.5), we invoke the recursion from Proposition 3.5 (used for the string $\omega = (1,1,*,\ldots,1,*) \in \{1,*\}^{2k-1}$), and we extract the coefficient of y on both sides of that recursion. On the right-hand side of the resulting equation we get a sum which (same as in Equation (3.9) of Proposition 3.5) is indexed by m, with $1 \leq m \leq n-1 = 2k-2$. By grouping the terms of the sum according to the parity of m, we obtain that

$$Z^{(1)}_{(\underbrace{1,1,*,\ldots,1,*}_{2k-1})} = -\big(\, \Sigma_{\mathrm{odd}} + \Sigma_{\mathrm{even}} \,\big), \tag{6.6}$$

where

$$\Sigma_{\mathrm{odd}} = Z^{(1)}_{(1)}\, Z^{(0)}_{(\underbrace{1,*,\ldots,1,*}_{2k-2})} + Z^{(1)}_{(1,1,*)}\, Z^{(0)}_{(\underbrace{1,*,\ldots,1,*}_{2k-4})} + \cdots + Z^{(0)}_{(\underbrace{1,1,*,\ldots,1,*}_{2k-3})}\, Z^{(0)}_{(1,*)}$$

and

$$\Sigma_{\mathrm{even}} = Z^{(0)}_{(1,1)}\, Z^{(1)}_{(\underbrace{*,1,*,\ldots,1,*}_{2k-3})} + Z^{(0)}_{(1,1,*,1)}\, Z^{(1)}_{(\underbrace{*,1,*,\ldots,1,*}_{2k-5})} + \cdots + Z^{(0)}_{(\underbrace{1,1,*,1\ldots,*,1}_{2k-2})}\, Z^{(1)}_{(*)}.$$

The sum Σ_{even} is equal to 0, because of

$$Z^{(0)}_{(1,1)} = Z^{(0)}_{(1,1,*,1)} = \cdots = Z^{(0)}_{(\underbrace{1,1,*,1\ldots,*,1}_{2k-2})} = 0$$

(cf. Proposition 6.2.2, case of non-alternating strings). On the other hand, the induction hypothesis and the case of alternating strings in Proposition 6.2.2 give us that

$$\Sigma_{\text{odd}} = (-1)^0 C_0 \cdot (-1)^{k-2} C_{k-2} + (-1)^1 C_1 \cdot (-1)^{k-3} C_{k-3} + \cdots + (-1)^{k-2} C_{k-2} \cdot (-1)^0 C_0.$$

Thus Equation (6.6) comes, after all, to

$$Z^{(1)}_{(\underbrace{1,1,*,\ldots,1,*}_{2k-1})}(x) = (-1)^{k-1} \sum_{j=0}^{k-2} C_j \cdot C_{k-2-j}.$$

A basic recursion for Catalan numbers says that the sum on the right-hand side of the latter equality is just C_{k-1}, and the required formula (6.5) follows. □

We can now follow the same kind of connection as in Proposition 6.2 (but going in reverse) in order to obtain the "derivative at $t = \infty$" for $*$-cumulants of the u_t's.

Corollary 6.4. *Let n be a positive integer and let $\omega = (\omega(1), \ldots, \omega(n))$ be a string in $\{1, *\}^n$. We consider the limit*

$$\lim_{t\to\infty} \frac{\kappa_n\big(u_t^{\omega(1)}, \ldots, u_t^{\omega(n)}\big) - \kappa_n\big(u^{\omega(1)}, \ldots, u^{\omega(n)}\big)}{e^{-t/2}} \tag{6.7}$$

where u_t is the free unitary Brownian motion at time t, and u is a Haar unitary. This limit exists, and is equal to

$$\begin{cases} (-1)^{k-1} C_{k-1}, & \text{if } n \text{ is odd, } n = 2k-1, \text{ and} \\ & \omega \text{ is an alternating string,} \\ 0, & \text{otherwise.} \end{cases} \tag{6.8}$$

Proof. If n is even, $n = 2k$, then the difference on the numerator of the fraction in (6.7) is

$$Z^{(2k)}_\omega(t) \cdot (e^{-t/2})^{2k} + Z^{(2k-2)}_\omega(t) \cdot (e^{-t/2})^{2k-2} + \cdots + Z^{(2)}_\omega(t) \cdot (e^{-t/2})^2,$$

and when divided by $e^{-t/2}$ this is sure to go to 0 as $t \to \infty$.

If n is odd, $n = 2k-1$, then the difference on the numerator of the fraction in (6.7) is

$$Z^{(2k-1)}_\omega(t) \cdot (e^{-t/2})^{2k-1} + Z^{(2k-3)}_\omega(t) \cdot (e^{-t/2})^{2k-3} + \cdots + Z^{(3)}_\omega(t) \cdot (e^{-t/2})^3 + Z^{(1)}_\omega(t) \cdot (e^{-t/2}).$$

When divided by $e^{-t/2}$ this converges to the constant $Z^{(1)}_\omega(t)$ described in Theorem 6.3, and the result follows. □

Remark 6.5. The limit from Corollary 6.4 points towards an "infinitesimal structure" which accompanies the $*$-distribution of a Haar unitary, in the sense of the paper of Belinschi and Shlyakhtenko [1]. In order to relate to the framework of [1], one has to do a change of variable: consider the noncommutative probability spaces $(\mathcal{B}_s, \psi_s)$, defined for $s \in [0,1]$, where for $s \neq 0$ we put $\mathcal{B}_s = \mathcal{A}_{-2\log s}$, $\psi_s = \varphi_{-2\log s}$, while for $s = 0$ we take $(\mathcal{B}_0, \psi_0)$ to be the space where the Haar unitary lives. With this change of variable, the limit from (6.7) becomes a derivative at 0, as prescribed in [1].

Remark 6.6. As reviewed in Remark 2.6, the Haar unitary is a basic example of R-diagonal element, with determining sequence consisting of signed Catalan numbers. For a Haar unitary u, Corollary 6.4 brings into the picture an additional "infinitesimal determining sequence", which happens to also consist of signed Catalan numbers, and which determines the derivatives at ∞ of all joint cumulants of u_t and u_t^* (with u_t seen as an approximation of u). It is natural to extend this concept of infinitesimal determining sequence to the case of an R-diagonal element $a = uq$ as appearing in Remark 2.6. Indeed, we can approximate such an a with elements of the form $a_t := u_t q$ (where q is now assumed to also be free from $\{u_t, u_t^*\}$), and we can consider the same kind of limits as in Equation (6.7) of Corollary 6.4, but now in connection to a and a_t. This leads to the next proposition, which provides a nice infinitesimal analogue for the facts reviewed in Remark 2.6.

Proposition 6.7. *Let a and $\{a_t \mid t \in [0,\infty)\}$ be as in the preceding remark. There exists a sequence $(\beta_k)_{k=1}^{\infty}$ such that for $\omega = (\omega(1), \ldots, \omega(n)) \in \{1,*\}^n$ one has:*

$$\lim_{t\to\infty} \frac{\kappa_n\big(a_t^{\omega(1)}, \ldots, a_t^{\omega(n)}\big) - \kappa_n\big(a^{\omega(1)}, \ldots, a^{\omega(n)}\big)}{e^{-t/2}} = \begin{cases} \beta_{(n+1)/2}, & \text{if } n \text{ is odd and } \omega \text{ is alternating,} \\ 0, & \text{otherwise.} \end{cases} \tag{6.9}$$

The β_k's can be written in terms of the free cumulants of q and q^2 via a formula similar to Equation (2.6), as follows:

$$\beta_k = \sum_{\pi \in NC(k)} \Big(\mathrm{Moeb}(\pi, 1_k) \cdot \prod_{V\in\pi} \kappa_{|V|}\big(\, (q^2, \ldots, q^2, q) \mid V \, \big) \Big), \quad k \in \mathbb{N}. \tag{6.10}$$

[A concrete example: for $\omega = (1,*,1)$ the above proposition says that

$$\lim_{t\to\infty} \frac{\kappa_3(u_t q, q u_t^*, u_t q) - \kappa_3(uq, qu^*, uq)}{e^{-t/2}} = \beta_2, \text{ with } \beta_2 = -\kappa_1(q^2)\kappa_1(q) + \kappa_2(q^2, q).]$$

The remaining part of this section is devoted to discussing the proof of Proposition 6.7. The arguments revolve around a certain set of non-crossing partitions $NC_\omega(2n) \subseteq NC(2n)$ that is associated to an $\omega \in \{1,*\}^n$. The sets $NC_\omega(2n)$ are introduced next, and their relevance for the limits on the left-hand side of Equation (6.9) is explained in Lemma 6.10 below. The conclusion of Proposition 6.7 will then be derived via a calculation which relies on the structure of these $NC_\omega(2n)$'s. In order to not make the discussion excessively long, we will merely state (in Lemmas 6.11 and 6.13) the relevant facts we need about $NC_\omega(2n)$, and we will leave the proofs of these purely combinatorial facts as an exercise to the interested reader.

Definition 6.8. Consider a string $\omega = (\omega(1), \ldots, \omega(n)) \in \{1,*\}^n$.

1^o We denote $U_\omega := \{2i-1 \mid 1 \le i \le n, \ \omega(i) = 1\} \cup \{2i \mid 1 \le i \le n, \ \omega(i) = *\}$, and $Q_\omega := \{1, 2, \ldots, 2n\} \setminus U_\omega = \{2i \mid 1 \le i \le n, \ \omega(i) = 1\} \cup \{2i-1 \mid 1 \le i \le n, \ \omega(i) = *\}$. (Thus $U_\omega, Q_\omega \subseteq \{1, \ldots, 2n\}$, and they have n elements each.)

2^o We will use the notation $NC_\omega(2n)$ for the set of partitions $\tau \in NC(2n)$ which fulfill all of the following conditions (i)–(v).

(i) $\tau \vee \{ \{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\} \} = 1_{2n}$.

(ii) For every $V \in \tau$ we have that either $V \subseteq Q_\omega$ or $V \subseteq U_\omega$.

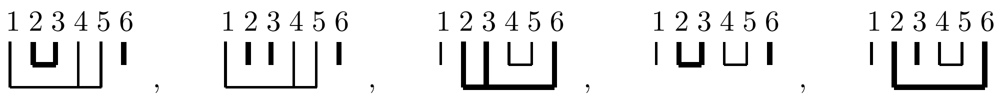
(iii) There exists precisely one block V_o of τ such that $V_o \subseteq U_\omega$ and $|V_o|$ is odd.

(iv) If $V_o = \{i_1, \ldots, i_p\}$ (with $i_1 < \cdots < i_p$) is as in (iii) then, modulo a cyclic permutation, the numbers $i_1, \ldots, i_p$ have alternating parities.

(v) If $V = \{j_1, \ldots, j_r\}$ (with $j_1 < \cdots < j_r$) is a block of τ such that $V \subseteq U_\omega$ and r is even, then $j_1, \ldots, j_r$ have alternating parities.

[Note: the meaning of (iv) is that if in $(i_1, \ldots, i_p)$ we replace every i_h which is odd by a "1" and every i_h which is even by a "$*$", then we get an alternating string in $\{1,*\}^p$, in the sense of Definition 6.1.2. The same happens for (v), but there we don't need to mention the possibility of a cyclic permutation.]

Example 6.9. To illustrate the above terminology, let us pursue the case when $n = 3$ and $\omega = (1, *, 1)$. Then $U_\omega = \{1, 4, 5\}$ and $Q_\omega = \{2, 3, 6\}$. Direct inspection shows that for a partition $\tau \in NC_\omega(6)$, the restriction $\tau \mid U_\omega$ has to be one of $\{\, \{1, 4, 5\}\,\}$ or $\{\, \{1\}, \{4, 5\}\,\}$. (If we try to make $\tau \mid U_\omega = \{\, \{1, 4\}, \{5\}\,\}$ then condition (i) of Definition 6.8.2 cannot be satisfied. Likewise, trying to make $\tau \mid U_\omega$ be one of $\{\, \{1, 5\}, \{4\}\,\}$ or $\{\, \{1\}, \{4\}, \{5\}\,\}$ violates condition (v), respectively (iii).) We find in this way that $NC_\omega(6)$ consists of 5 partitions, depicted as follows.



Lemma 6.10. *Let a and $\{a_t \mid t \in [0, \infty)\}$ be as in Proposition 6.7, and let ω be a string in $\{1, *\}^n$. The limit considered on the left-hand side of Equation (6.9) exists, and is equal to*

$$\sum_{\tau \in NC_\omega(2n)} \mathrm{term}_\tau^{(U)} \cdot \mathrm{term}_\tau^{(Q)}, \tag{6.11}$$

where for every $\tau \in NC_\omega(2n)$ the numbers $\mathrm{term}_\tau^{(U)}$ and $\mathrm{term}_\tau^{(Q)}$ are defined as follows:
• *Let V_o be the unique block of τ such that $V_o \subseteq U_\omega$ and $|V_o|$ is odd, and let $V_1, \ldots, V_k$ be the other blocks of τ which are contained in U_ω. Then*

$$\mathrm{term}_\tau^{(U)} = (-1)^{(|V_o|-1)/2} C_{(|V_o|-1)/2} \cdot \prod_{i=1}^{k} (-1)^{(|V_i|-2)/2} C_{(|V_i|-2)/2}. \tag{6.12}$$

• *Let $W_1, \ldots, W_\ell$ be the blocks of τ which are contained in Q_ω. Then*

$$\mathrm{term}_\tau^{(Q)} = \prod_{j=1}^{\ell} \kappa_{|W_j|}(q, \ldots, q). \tag{6.13}$$

In the case (which may occur) when $NC_\omega(2n) = \emptyset$, the quantity (6.11) should be read as 0.

Proof. In the cumulant $\kappa_n(a_t^{(\omega(1))}, \ldots, a_t^{(\omega(n))})$ we replace every a_t by $u_t q$ and every a_t^* by $q u_t^*$, and we invoke the formula for a cumulant with products of entries which was reviewed in Remark 2.5.2. This gives

$$\kappa_n(a_t^{(\omega(1))}, \ldots, a_t^{(\omega(n))}) = \sum_{\substack{\tau \in NC(2n) \text{ with} \\ \tau \vee \{\{1,2\},\{3,4\},\ldots\} = 1_{2n}}} \mathrm{Term}_\tau, \tag{6.14}$$

where every Term_τ is a product of cumulants with entries from $\{q, u_t, u_t^*\}$. But q is free from $\{u_t, u_t^*\}$; hence free cumulants which mix q with $\{u_t, u_t^*\}$ vanish, and this implies that on the right-hand side of (6.14) we can restrict the sum to the smaller set

$$\mathcal{T}_\omega := \{\tau \in NC(2n) \mid \tau \text{ fulfills conditions (i) and (ii) in Definition 6.8.2}\}.$$

For every $\tau \in \mathcal{T}_\omega$, the quantity Term_τ appearing in (6.14) is a product where some factors are joint cumulants of u_t and u_t^*, while some other factors are cumulants of q. The dependence on t is coming exclusively from the factors involving u_t and u_t^*, which are quasi-polynomials in $t/2$, in the way found earlier in the paper. If we are interested in the limit on the left-hand side of Equation (6.9), then what we have to do is pick the coefficient of $e^{-t/2}$ in every Term_τ. (Note that the coefficient of $(e^{-t/2})^0$ in Term_τ, if existing, will be removed by the subtraction of $\kappa_n(a^{(\omega(1))}, \ldots, a^{(\omega(n))})$ in the numerator of (6.9).) By invoking Proposition 6.2.2 and Theorem 6.3, one easily sees that a partition $\tau \in \mathcal{T}_\omega$ can include a contribution of order $e^{-t/2}$ in Term_τ only if τ also satisfies the conditions (iii)–(v) listed in Definition 6.8.2, i.e. only if $\tau \in NC_\omega(2n)$. (If $NC_\omega(2n) = \emptyset$, then we see at this point that the limit in (6.9) is equal to 0.) Finally, for every $\tau \in NC_\omega(2n)$ one has $\mathrm{Term}_\tau = \mathrm{term}_\tau^{(U)} \cdot \mathrm{term}_\tau^{(Q)}$, with $\mathrm{term}_\tau^{(U)}$ and $\mathrm{term}_\tau^{(Q)}$ described as in Equations (6.12), (6.13); this verification is immediate (with the signed Catalan numbers in (6.12) coming from Proposition 6.2.2 and Theorem 6.3), and is left as exercise to the reader. □

Now, there are many strings $\omega \in \{1, *\}^n$ with $NC_\omega(2n) = \emptyset$. An obvious necessary condition for $NC_\omega(2n)$ being non-empty is that $|\ell - \ell'| = 1$, where

$$\ell := |\{1 \le i \le n \mid \omega(i) = 1\}| \text{ and } \ell' := |\{1 \le i \le n \mid \omega(i) = *\}|;$$

indeed, if it is not true that $|\ell - \ell'| = 1$, then no partition $\tau \in NC(2n)$ can satisfy conditions (iii)–(v) of Definition 6.8.2. But even when $|\ell - \ell'| = 1$, it still turns out that $NC_\omega(2n) = \emptyset$ unless ω is alternating. This is caused by the condition (i) of Definition 6.8.2. The next lemma records the precise statement that we will need later on; the proof of the lemma (which goes in the same spirit as those of Propositions 11.25 or 15.1 in [9]) is left as exercise.

Lemma 6.11. *Let $\omega = (\omega(1), \ldots, \omega(n))$ be a string in $\{1, *\}^n$ for some $n \ge 2$, such that $\omega(1) = \omega(n) = 1$. If $NC_\omega(2n) \ne \emptyset$, then n is odd and ω is the alternating string $(1, *, 1, \ldots, *, 1)$.* □

We are thus prompted to focus on alternating strings of odd length. In order to describe what is going on in this case, we introduce some additional bits of notation.

Remark and Notation 6.12. Let k be a positive integer, and consider the alternating string $\omega_k := (1, *, 1, \ldots, *, 1) \in \{1, *\}^{2k-1}$. Note that the sets $U_{\omega_k}, Q_{\omega_k} \subseteq \{1, \ldots, 4k-2\}$ associated to ω_k in Definition 6.8.2 are

$$U_{\omega_k} = \{1, 4, 5, 8, 9, \ldots, 4k-4, 4k-3\} \text{ and } Q_{\omega_k} = \{2, 3, 6, 7, \ldots, 4k-6, 4k-5, 4k-2\}.$$

1^o For every partition $\pi \in NC(k)$ we denote by $\pi^{(\text{u-points})}$ the partition of U_{ω_k} which is defined by "converting" the points $1, 2, \ldots, k$ into the groups of points $\{1\}$, then $\{4, 5\}$, then $\{8, 9\}, \ldots$, then $\{4k-4, 4k-3\}$ of U_{ω_k}. That is, $\pi^{(\text{u-points})}$ has blocks of the form

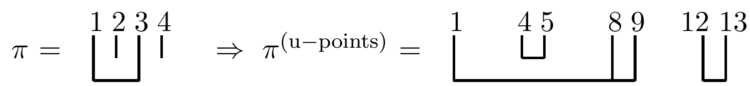
$$\widetilde{V} := \bigcup_{i \in V} \{4i-4, 4i-3\}, \text{ where } V \text{ is a block of } \pi \text{ such that } 1 \notin V,$$

and also has a block

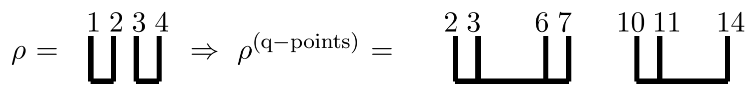
$$\widetilde{V_o} := \{1\} \cup \Big(\bigcup_{\substack{i \in V_o, \\ i \neq 1}} \{4i-4, 4i-3\} \Big),$$

where V_o is the block of π such that $1 \in V_o$.

Likewise, for every $\rho \in NC(k)$ we denote by $\rho^{(\text{q}-\text{points})}$ the partition of Q_{ω_k} which is defined by converting the points $1, 2, \ldots, k$ into the groups of points $\{2, 3\}$, then $\{6, 7\}, \ldots$, then $\{4k-6, 4k-5\}$, then $\{4k-2\}$ of Q_{ω_k}. For example, for $k = 4$ we have:



and



2^o There exists an analogy between the notation for $\pi^{(\text{u}-\text{points})}$ and $\rho^{(\text{q}-\text{points})}$ that was just introduced, and the notation for $\pi^{(\text{odd})}$ and $\rho^{(\text{even})}$ used in the description of the Kreweras complementation map, in Notation 2.1.3. This is due to the fact that if we think of the sets

$$\{1\},\ \{2,3\}, \{4,5\}, \ldots, \{4k-4, 4k-3\},\ \{4k-2\}$$

as of a sequence of $2k$ consecutive "fat points", then U_{ω_k} covers the odd positions and Q_{ω_k} covers the even positions among these fat points.

As a consequence of the above, it is easily seen that if we start with a partition $\pi \in NC(k)$ and we draw the combined pictures of $\pi^{(\text{u}-\text{points})}$ and $(\text{Kr}(\pi))^{(\text{q}-\text{points})}$, then we obtain a partition in $NC(4k-2)$. Moreover, $(\text{Kr}(\pi))^{(\text{q}-\text{points})}$ can be characterized as the largest (with respected to reverse refinement order) partition σ of the set Q_{ω_k} which has the property that $\pi^{(\text{u}-\text{points})} \sqcup \sigma \in NC(4k-2)$.

3^o Let ρ be in $NC(k)$, and consider the partition $\rho^{(\text{q}-\text{points})}$ of Q_{ω_k}. We will need to work with a special way of breaking the blocks of $\rho^{(\text{q}-\text{points})}$ into pairs (plus a singleton), which is described as follows. Let $W = \{j_1, \ldots, j_p\}$ be a block of ρ, where $j_1 < \cdots < j_p$. We distinguish two cases.

Case 1. $j_p \neq k$. In this case, the block of $\rho^{(\text{q}-\text{points})}$ that corresponds to W is

$$\widetilde{W} := \{4j_1 - 2, 4j_1 - 1, 4j_2 - 2, 4j_2 - 1, \ldots, 4j_p - 2, 4j_p - 1\},$$

and we break it into p pairs by going cyclically as follows:

$$\{4j_1 - 1, 4j_2 - 2\},\ \{4j_2 - 1, 4j_3 - 2\}, \ldots, \{4j_{p-1} - 1, 4j_p - 2\},\ \{4j_1 - 2, 4j_p - 1\}. \tag{6.15}$$

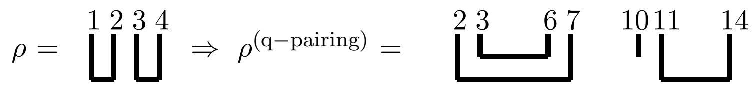
Case 2. $j_p = k$. In this case, the block of $\rho^{(\text{q}-\text{points})}$ that corresponds to W is

$$\widetilde{W} := \{4j_1 - 2, 4j_1 - 1, 4j_2 - 2, 4j_2 - 1, \ldots, 4j_{p-1} - 2, 4j_{p-1} - 1, 4j_p - 2\},$$

and we break it into $p - 1$ pairs and a singleton as follows:

$$\{4j_1 - 2\},\ \{4j_1 - 1, 4j_2 - 2\},\ \{4j_2 - 1, 4j_3 - 2\}, \ldots, \{4j_{p-1} - 1, 4j_p - 2\}. \tag{6.16}$$

The partial pairing (with one singleton block) of Q_{ω_k} which results upon doing all the breaking described in (6.15) and (6.16) will be denoted as $\rho^{(\text{q-pairing})}$. In the example with $k = 4$ depicted in part 1^o of this notation, we get



Based on the notation introduced above we describe, in the next lemma, the structure of a general partition in $NC_{\omega_k}(4k-2)$. (The statement of the lemma also uses the lattice structure of $NC(Q_{\omega_k})$, and invokes the "$\sqcup$" operation — this is analogous to the note recorded in Remark 2.2.)

Lemma 6.13. *Let k be a positive integer, let ω_k be the alternating string $(1, *, 1 \ldots, *, 1) \in \{1, *\}^{2k-1}$, and consider the terminology introduced by Notation 6.12 in connection to ω_k.*

1^o *Let τ be a partition in $NC_{\omega_k}(4k-2)$. Then the restricted partition $\tau \mid U_{\omega_k}$ is of the form $\pi^{(\text{u-points})}$ for a (uniquely determined) $\pi \in NC(k)$. The quantity $\text{term}_\tau^{(U)}$ appearing in (6.12) of Lemma 6.10 is precisely equal to $\text{Moeb}(0_k, \pi)$ for this $\pi \in NC(k)$.*

2^o *Let τ be a partition in $NC_{\omega_k}(4k-2)$, let $\pi \in NC(k)$ be such that $\tau \mid U_{\omega_k} = \pi^{(\text{u-points})}$, and let us denote $\text{Kr}(\pi) =: \rho \in NC(k)$. Then the partition $\sigma := \tau \mid Q_{\omega_k} \in NC(Q_{\omega_k})$ has the properties that $\sigma \leq \rho^{(\text{q-points})}$ and $\sigma \vee \rho^{(\text{q-pairing})} = \rho^{(\text{q-points})}$.*

3^o *Conversely: let π be in $NC(k)$, put $\rho := \text{Kr}(\pi)$, and let σ be a partition in $NC(Q_{\omega_k})$ with the properties that $\sigma \leq \rho^{(\text{q-points})}$ and $\sigma \vee \rho^{(\text{q-pairing})} = \rho^{(\text{q-points})}$. Then the partition $\tau := \pi^{(\text{u-points})} \sqcup \sigma$ of $\{1, \ldots, 4k-2\}$ is in $NC_{\omega_k}(4k-2)$.* □

Proof of Proposition 6.7. If n is even, then it is immediate that $NC_\omega(2n) = \emptyset$ for every $\omega \in \{1, *\}^n$ (cf. discussion preceding Lemma 6.11). Thus in this case the limit on the left-hand side of Equation (6.9) is indeed equal to 0, as required. For the rest of the proof we will assume that n is odd.

We next note a couple of reductions that can be done on ω.

(a) Observe that if the conclusion of the proposition holds for a string $\omega = (\omega(1), \ldots, \omega(n))$, then it also holds for the "adjoint" string ω^* with entries

$$\omega^*(i) = \left\{ \begin{array}{ll} 1, & \text{if } \omega(n+1-i) = * \\ *, & \text{if } \omega(n+1-i) = 1 \end{array} \right\}, \quad 1 \leq i \leq n.$$

This follows from the left-right symmetry of free cumulants that was noted in part (c) of Remark 2.5.3, where we also take into account the fact that $\{u_t^* \mid t \geq 0\}$ form a free unitary Brownian motion, and that (due to the hypothesis $q = q^*$) all the limits postulated on the right-hand side of Equation (6.9) are real numbers.

(b) Observe that if the conclusion of the proposition holds for a string ω, then it also holds for an ω' obtained by cyclically permuting the entries of ω. This is immediate from the invariance property of free cumulants noted in part (a) of Remark 2.5.3.

As a result of the above observation (a), we see that it suffices to handle strings $\omega \in \{1, *\}^n$ that have $|\{1 \leq i \leq n \mid \omega(i) = 1\}| > |\{1 \leq i \leq n \mid \omega(i) = *\}|$. By doing a suitable cyclic permutation of entries for such an ω and by using observation (b), we then see that it suffices to prove the proposition under the additional assumption that $\omega(1) = \omega(n) = 1$.

If $\omega(1) = \omega(n) = 1$ but ω is not alternating, then Lemma 6.11 says that $NC_\omega(2n) = \emptyset$. The limit on the left-hand side of (6.9) is therefore equal to 0, as required.

We are left to discuss the case when ω is alternating of odd length, with $\omega(1) = \omega(n) = 1$. This is exactly the case when $\omega = \omega_k = (1, *, 1, \ldots, *, 1) \in \{1, *\}^{2k-1}$ for some $k \in \mathbb{N}$, as discussed in Notation 6.12. In the remaining part of the proof we fix k, and we will prove that for $\omega = \omega_k$, the limit on the left-hand side of (6.9) is equal to the β_k described in Equation (6.10). In view of Lemma 6.10, it will suffice to verify the equality between the summation formulas that appear in (6.11) and on the right-hand side of Equation (6.10). The sum from (6.11) takes here the form

$$\sum_{\tau \in NC_{\omega_k}(4k-2)} \mathrm{term}_\tau^{(U)} \cdot \mathrm{term}_\tau^{(Q)}. \tag{6.17}$$

But a partition $\tau \in NC_{\omega_k}(4k-2)$ is parametrized in Lemma 6.13 in terms of a pair (π, σ) with $\pi \in NC(k)$ and $\sigma \in NC(Q_{\omega_k})$; recalling the specifics of how that goes (which includes the writing of $\mathrm{term}_\tau^{(U)}$ as $\mathrm{Moeb}(0_k, \pi)$ and the writing of $\mathrm{term}_\tau^{(Q)}$ in terms of σ), we see that (6.17) can be rewritten in the form

$$\sum_{\substack{\pi \in NC(k) \\ \textit{with } \mathrm{Kr}(\pi) =: \rho}} \mathrm{Moeb}(0_k, \pi) \Big(\sum_{\substack{\sigma \in NC(Q_{\omega_k}), \\ \sigma \vee \rho^{(\mathrm{q-pairing})} = \rho^{(\mathrm{q-points})}}} \prod_{W \in \sigma} \kappa_{|W|}(q, q, \ldots, q) \Big).$$

In the above expression it is convenient that in the first sum we do the change of variable $\rho = \mathrm{Kr}(\pi)$ (where we also substitute $\mathrm{Moeb}(0_k, \pi)$ as $\mathrm{Moeb}(\rho, 1_k)$); the quantity in (6.17) takes then the form

$$\sum_{\rho \in NC(k)} \mathrm{Moeb}(\rho, 1_k) \Big(\sum_{\substack{\sigma \in NC(Q_{\omega_k}), \\ \sigma \vee \rho^{(\mathrm{q-pairing})} = \rho^{(\mathrm{q-points})}}} \prod_{W \in \sigma} \kappa_{|W|}(q, q, \ldots, q) \Big). \tag{6.18}$$

Now let us fix for the moment a partition $\rho = \{B_1, \ldots, B_\ell\} \in NC(k)$ and, for this particular ρ, let us examine the summation over σ which appears in (6.18). Let $W_1, \ldots, W_\ell$ be the blocks of $\rho^{(\mathrm{q-points})}$, where W_j corresponds to B_j in the natural way (cf. discussion in Notation 6.12.3), and where let us assume that $k \in B_\ell$, hence $4k-2 \in W_\ell$. It is easy to see that the summation over σ in (6.18) amounts in fact to doing ℓ independent summations, over partitions $\sigma_1 \in NC(W_1), \ldots, \sigma_\ell \in NC(W_\ell)$. Moreover, the formula (2.3) for cumulants with products as entries applies to each of these ℓ summations, leading to the conclusion that the result of the j-th summation is

$$\begin{cases} \kappa_{|W_j|}(q^2, \ldots, q^2), & \text{if } j < \ell, \\ \kappa_{|W_j|}(q, \underbrace{q^2, \ldots, q^2}_{|W_j|-1}) & \text{if } j = \ell. \end{cases}$$

Since $|W_j| = |B_j|$ for every $1 \le j \le \ell$, and since $\kappa_{|B_\ell|}(q, q^2, \ldots, q^2) = \kappa_{|B_\ell|}(q^2, \ldots, q^2, q)$, we obtain that for our fixed ρ we have:

$$\sum_{\substack{\sigma \in NC(Q_{\omega_k}), \\ \sigma \vee \rho^{(\mathrm{q-pairing})} = \rho^{(\mathrm{q-points})}}} \prod_{W \in \sigma} \kappa_{|W|}(q, q, \ldots, q) = \prod_{B \in \rho} \kappa_{|B|}\big((q^2, \ldots, q^2, q) \mid B \big). \tag{6.19}$$

Finally, we let ρ run in $NC(k)$, we substitute Equation (6.19) into the second summation from (6.18), and we arrive to the right-hand side of Equation (6.10), as we wanted. □

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