COMPLETION OF CONTINUITY SPACES WITH UNIFORMLY VANISHING ASYMMETRY

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Abstract. The classical Cauchy completion of a metric space (by means of Cauchy sequences) as well as the completion of a uniform space (by means of Cauchy filters) are well-known to rely on the symmetry of the metric space or uniform space in question. For qausi-metric spaces and quasi-uniform spaces various non-equivalent completions exist, often defined on a certain subcategory of spaces that satisfy a key property required for the particular completion to exist. The classical filter completion of a uniform space can be adapted to yield a filter completion of a metric space. We show that this completion by filters generalizes to continuity spaces that satisfy a form of symmetry which we call uniformly vanishing asymmetry.

1. Introduction

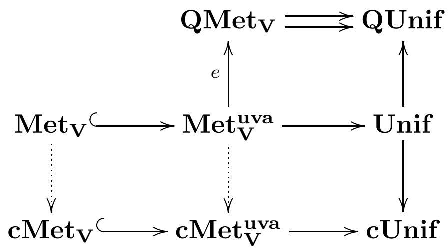
The theories of the completion of metric spaces and the completion of uniform spaces are well-known and understood. There is little to no doubt as to what completion should mean in these cases and there are several (equivalent of course) constructions of the completions. The situation is different when considering quasi-metric spaces and quasi-uniform spaces. The lack of symmetry (see [8] for a detailed account of symmetry and completions in the context of quantaloid enriched categories) sabotages the standard completion constructions that work in the symmetric case and the theory bifurcates with several different notions of complete objects and different completion processes existing in the literature (see e.g., [1, 3–6, 9–15]).

Consider the category **QMet** of quasi-metric spaces and uniformly continuous functions, and the category **QUnif** of quasi-uniform spaces and their morphisms. With a given quasi-metric space (X, d) one can associate two quasi-uniform structures, one generated by the entourages $U_\varepsilon = \{(x, y) \in X \times X \mid d(x, y) < \varepsilon\}$, the other generated by the entourages $U^\varepsilon = \{(x, y) \in X \times X \mid d(y, x) < \varepsilon\}$, giving rise to the two parallel functors in the diagram

$$\mathbf{Met^{uva}} \longrightarrow \mathbf{QMet} \underset{U(-^{op})}{\overset{U(-)}{\rightrightarrows}} \mathbf{QUnif}$$

and it is natural to consider their equalizer. From the universal property of the equalizer it follows that $\mathbf{Met^{uva}}$ extends **Met**, the full subcategory of **QMet** spanned by the ordinary metric spaces, but it is strictly larger.

A quasi-metric space is the same thing as a V-*continuity space* or a V-*space,* a concept introduced by Flagg in [7], where V is the value quantale $[0, \infty]$, viewed as a complete lattice, with ordinary addition. Everything above can be repeated with **Met** and **QMet** replaced, respectively, by $\mathbf{Met_V}$ (the category of symmetric V-spaces and uniformly continuous functions) and $\mathbf{QMet_V}$ (the category of V-spaces and uniformly continuous functions) for any value quantale V. In more detail, the aim of this work is the construction of the dotted functors in the commutative diagram



where the three lower vertical arrows are completion functors, thus showing that the classical completion extends to the equalizer. In more detail, in the diagram, the lower right vertical functor is the standard construction of the completion of a uniform space via minimal Cauchy filters. We show that this construction extends to separated V-spaces in the equalizer. The construction is a metric re-incarnation of that giving rise to the lower right functor. Most of the existing notions of completions of V-spaces, when restricted to symmetric spaces, yield (essentially) the same completion but these constructions bifurcate to non-isometric completions for general V-spaces. It is quite straightforward to manually check for most completions in the literature that when restricted to $\mathbf{Met_V^{uva}}$, the results are isometric. We thus expand the domain of the definition of the standard completion to what appears to be the maximum possible.

In the context of the completion of metric spaces, one of the striking differences between the completion by means of Cauchy sequences and Cauchy filters is that the former requires a quotient construction, identifying sequences of distance 0, while the latter enjoys a canonical choice of representative, namely the round filter $\mathcal{F}_\succ$ generated by fattening the elements in $\mathcal{F}$. In the construction of the completion we give below we also treat round filters in the full generality of V-spaces and we exhibit the 'roundification' process as a left adjoint on appropriately constructed categories.

The plan of the article is as follows. Section 2 recounts some basic facts on value quantales and V-spaces following [7]. Section 3 introduces the concept of uniformly vanishing asymmetry, the notion of symmetry required for the completion, which is then presented in Section 4.

2. Value quantales and V-spaces

Recall that a *complete lattice* L is a poset which possesses, for all $S \subseteq L$, a meet $\bigwedge S$ and a join $\bigvee S$. The top and bottom elements are denoted, respectively, by ∞ and 0. The *well-above* relation $\succ$ is derived from the poset structure in L (or any poset) as follows. For $a, b \in L$, a is said to be well-above b, denoted by $a \succ b$, or $b \prec a$, if, given any $S \subseteq L$ such that $b \geq \bigwedge S$, there exists $s \in S$ with $a \geq s$.

A *value quantale*, as introduced in [7] by Flagg, is a pair $(V, +)$ where V is a complete lattice and $+$ is an associative and commutative binary operation on V such that

- $x + 0 = x$
- $x = \bigwedge \{y \in V \mid y \succ x\}$
- $x + \bigwedge S = \bigwedge (x + S)$
- $a \wedge b \succ 0$

for all $x \in V$, $S \subseteq V$, and $a, b \in V$ with $a \succ 0$ and $b \succ 0$ ($x+S$ means $\{x+s \mid s \in S\}$). When the ambient value quantale V is clear from the context, we will write $\varepsilon \succ 0$ as shorthand for the claim that $\varepsilon \in V$ and that $\varepsilon \succ 0$ holds in V.

Remark 2.1. More commonly, a quantale is defined by the duals of the axioms above, but in the context of this work we adhere to Flagg's original notation.

2.1. **Value quantale fundamentals.** We list those properties of value quantales that are needed for the proofs that follow. We provide no arguments for the claims we make in this section since the proofs are either immediate or are found in [7]. Let V be a value quantale.

- For all $x, y, z \in V$,
  (1) if $x \succ y$, then $x \geq y$
  (2) if $x \succ y$ and $y \geq z$, then $x \succ z$
  (3) if $x \geq y$ and $y \succ z$, then $x \succ z$.
- For all $x, y, a, b \in V$, if $x \leq y$ and $a \leq b$, then $x + a \leq y + b$.
- For all $\varepsilon \succ 0$, there exists $\delta \succ 0$ such that $\delta + \delta \leq \varepsilon$. More generally, for all $a \in V$ and $n \geq 1$ there exists $\delta \succ 0$ such that $n \cdot \delta \leq \varepsilon$, where $n \cdot \delta$ denotes the n-fold addition of δ with itself.
- For all $x, z \in V$, if $x \prec z$, then there exists $y \in V$ with $x \prec y \prec z$ (this result is known as the *interpolation property*).
- Fix $b \in V$. Since $b + \square : V \to V$ preserves meets, it has a left adjoint denoted by $\square - b : V \to V$, characterized by the property that $a - b \leq c \iff a \leq b + c$, for all $a, c \in V$. Among the numerous properties of this notion of subtraction, the one we will use is $(a - b) - c = a - (b + c)$.
- For all $a \in V$, we have $a = \bigwedge\{a + \varepsilon \mid \varepsilon \succ 0\}$. Consequently, for all $a, b \in V$, $a \leq b$ if, and only if, $a \leq b + \varepsilon$ for all $\varepsilon \succ 0$.

Value quantales are the objects of a 2-category $\mathbb{V}$ as follows. A morphism $\alpha : V \to W$ in $\mathbb{V}$ is a monotone function of the underlying lattices such that

- $\alpha(0) = 0$, and
- $\alpha(a + b) \leq \alpha(a) + \alpha(b)$

for all $a, b \in V$. Each hom set $\mathbb{V}(V, W)$ is given a poset structure by declaring, for $\alpha, \beta : V \to W$, that $a \leq \beta$ precisely when $\beta(a) \leq \alpha(a)$ for all $a \in V$. Interpreting the poset $\mathbb{V}(V, W)$ as a category thus describes the 2-cells in the 2-category $\mathbb{V}$.

2.2. **V-spaces.** Flagg introduced value quantales to replace the traditional non-negative extended real numbers and act as generalized codomains for distance functions $d : X \times X \to V$. Thus, a *V-space* (called a *V-continuity space* in [7]), is a triple (X, d, V) where V is a value quantale, X is a set, and $d : X \times X \to V$ is a function satisfying

- $d(x, x) = 0$; and
- $d(x, z) \leq d(x, y) + d(y, z)$

for all $x, y, z \in X$. A V-space (X, d, V) is *symmetric* if $d(x, y) = d(y, x)$ for all $x, y \in X$ and it is called *separated* if, for all $x, y \in X$, the equalities $d(x, y) = 0 = d(y, x)$ imply $x = y$.

For a given value quantale V, the category $\mathbf{Met_V}$ consists of all symmetric V-spaces as objects and all *uniformly continuous* mappings as morphisms $f : X \to Y$ (i.e., those functions satisfying that for all $\varepsilon \succ 0$ there exists $\delta \succ 0$ such that $d(f(x_1), f(x_2)) \leq \varepsilon$ whenever $d(x_1, x_2) \leq \delta$). Ignoring size issues, the assignment

$V \mapsto \mathbf{Met_V}$ extends to a 2-functor $\mathbf{Met}_{\_} : \mathbb{V} \to \mathbf{Cat}$ into the 2-category of categories. In more detail, if $\alpha : V \to W$ is a morphism of value quantales, then $\mathbf{Met}_\alpha : \mathbf{Met_V} \to \mathbf{Met_W}$ sends a V-space (X, d) to $(X, \alpha_* d)$ where $\alpha_* d(x, y) = \alpha(d(x, y))$, which is easily seen to be a W-space. If now $\beta : V \to W$ is another morphism such that $\alpha \le \beta$, then it is immediately verified that there is a natural transformation $\mathbf{Met}_\alpha \to \mathbf{Met}_\beta$, where the component at X is the identity function on the underlying sets.

Similarly, one defines the categories $\mathbf{QMet_V}$ of V-spaces and again one obtains a similar 2-functor $\mathbf{QMet}_{\_} : \mathbb{V} \to \mathbf{Cat}$. The full subcategory $\mathbf{QMet_V^0}$ of $\mathbf{QMet_V}$ spanned by the separated V-spaces is reflective, with the reflector $-_0 : \mathbf{QMet_V} \to \mathbf{QMet_V^0}$ mapping X to $X_0 = X/\sim$, where $x \sim y$ precisely when $d(x, y) = d(y, x) = 0$. Similarly, the full subcategory $\mathbf{Met_V^0}$ of $\mathbf{Met_V}$ spanned by the separated symmetric V-spaces is reflective. It is immediate that if (X, d) is a V-space, then so is the *dual space* $X^{op} = (X, d^{op})$, where $d^{op}(x, y) = d(y, x)$.

Remark 2.2. V-spaces are in fact enriched V-categories. However, notice that then enriched functors correspond to non-expanding functions rather than the uniformly continuous ones we consider.

V-spaces are general enough to capture all topological spaces in the sense that for every topological space X, there is a value quantale V such that X is V-metrizable (Theorem 4.15 in [7]). Further, the category $\mathbf{Top}$ is equivalent to the category $\mathbf{QMet}_T$ whose objects are all pairs (V, X) where V is a value quantale and X is a V-space, and a morphism $(V, X) \to (W, Y)$ is a continuous function $f : X \to Y$ (see [16] for more details).

3. Uniformly vanishing asymmetry

We introduce now a class of spaces with a sufficient amount of symmetry to allow for the classical completion via Cauchy filters to carry through.

For a V-space X, a point $x \in X$, and $\varepsilon \succ 0$ let $\mathbf{B}_\varepsilon(x) = \{y \in X \mid d(x, y) \le \varepsilon\}$ and similarly let $\mathbf{B}^\varepsilon(x) = \{y \in X \mid d(y, x) \le \varepsilon\}$. We extend the notation $\mathbf{B}_\varepsilon(x)$ to subsets $S \subseteq X$ by defining $\mathbf{B}_\varepsilon(S) = \bigcup_{s \in S} \mathbf{B}_\varepsilon(s)$, with $\mathbf{B}^\varepsilon(S)$ defined similarly. Notice that $\mathbf{B}_\varepsilon(x)$ in X^{op} is precisely $\mathbf{B}^\varepsilon(x)$ in X. The set $\mathbf{B}_\varepsilon(x)$ is a closed set in the topology generated by the sets of the form $\{y \in X \mid d(x, y) \prec \varepsilon\}$, where x varies over X and $\varepsilon \succ 0$ varies in V (see Theorem 4.4 in [7]). That topology is denoted by $\mathcal{O}(X)$ and one obtains the functor $\mathcal{O} : \mathbf{QMet_V} \to \mathbf{Top}$. A straightforward verification shows that a V-space X gives rise to a quasi-uniform space $U(X)$, where the entourages are generated by $\{(x, y) \in X \times X \mid d(x, y) \le \varepsilon\}$, where $\varepsilon \succ 0$ varies in V, giving rise to a fully faithful functor $U : \mathbf{QMet_V} \to \mathbf{QUnif}$.

For a V-space X, the conditions

- for all $x \in X$ and for all $\varepsilon \succ 0$ there exists $\delta \succ 0$ such that $\mathbf{B}^\delta(x) \subseteq \mathbf{B}_\varepsilon(x)$ and such that $\mathbf{B}_\delta(x) \subseteq \mathbf{B}^\varepsilon(x)$ (any such δ will be called a *modulus of symmetry* for ε);
- the identity function $X^{op} \to X$ is a homeomorphism;
- $\mathcal{O}(X) = \mathcal{O}(X^{op})$

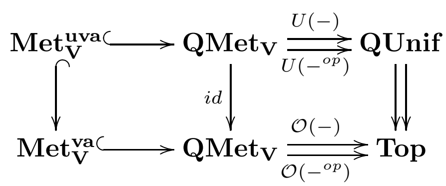
are equivalent. If X satisfies these conditions, then X is said to have *vanishing asymmetry.* Similarly, the conditions

- for all $\varepsilon \succ 0$ there exists $\delta \succ 0$ such that if $d(y, x) \le \delta$, then $d(x, y) \le \varepsilon$ (any such δ will be called a *uniform modulus of symmetry* for ε);

- the identity functions $X^{op} \to X$ and $X \to X^{op}$ are uniformly continuous;
- $U(X) = U(X^{op})$

are equivalent. If X satisfies these conditions, then X is said to have *uniformly vanishing asymmetry*. Clearly, if X has uniformly vanishing asymmetry, then X has vanishing asymmetry.

Let $\mathbf{Met}_{\mathbf{V}}^{\mathbf{va}}$ and $\mathbf{Met}_{\mathbf{V}}^{\mathbf{uva}}$ be the full subcategories of $\mathbf{QMet}_{\mathbf{V}}$ spanned by the spaces with vanishing asymmetry and the spaces with uniformly vanishing asymmetry, respectively. Consider the diagram



where the square on the left consists of inclusions, and the right vertical arrows are the standard constructions of the topology associated to a quasi-uniform space. The diagram commutes as long as one does not incorrectly mix different functors in the square on the right, and we note that from the definition of (uniformly) vanishing asymmetry, the top and bottom parts of the diagram are equalizers.

4. Completion

From this point onwards, we fix a value quantale V and a V-space X. We develop the relevant ingredients for constructing a completion of X as the set of all minimal Cauchy filters on X.

Recall that a *filter* on a set X is a non-empty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ such that $A \in \mathcal{F} \implies B \in \mathcal{F}$ for all $A \subseteq B \subseteq X$, and $A \cap B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$ (we do not require that $\emptyset \notin \mathcal{F}$, so in particular the power-set $\mathcal{P}(X)$ is a filter, the unique filter containing the empty set, referred to as an *improper filter*). A *filter base* is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ such that for all $A, B \in \mathcal{B}$ there exists $C \in \mathcal{B}$ with $C \subseteq A \cap B$. It follows immediately that a filter base $\mathcal{B}$ gives rise to a filter $\mathcal{F}$, the least filter containing $\mathcal{B}$, given explicitly by $\mathcal{F} = \{D \subseteq X \mid \exists C \in \mathcal{B}, C \subseteq D\}$. By a filter (resp. filter base) on a V-space is meant a filter (resp. filter base) on the underlying set.

A filter $\mathcal{F}$ is said to *converge* to $x \in X$, written $\mathcal{F} \to x$, if $\mathbf{B}_\varepsilon(x) \in \mathcal{F}$ for all $\varepsilon \succ 0$. Convergence interpreted in X^{op} is referred to as op-convergence, thus $\mathcal{F}$ *op-converges* to x, denoted by $\mathcal{F} \to^{op} x$, when $\mathbf{B}^\varepsilon(x) \in \mathcal{F}$ for all $\varepsilon \succ 0$.

Definition 4.1. A filter $\mathcal{F}$ on X is said to be a *Cauchy filter* if for all $\varepsilon \succ 0$ there exists $x \in X$ such that $\mathbf{B}_\varepsilon(x) \in \mathcal{F}$. If, moreover, $\mathcal{F}$ does not contain any proper Cauchy subfilter, then $\mathcal{F}$ is called a *minimal Cauchy filter*.

X is said to be *Cauchy complete* if every proper Cauchy filter on X converges. The dual notion of a Cauchy filter is that of an *op-Cauchy filter*, namely when for all $\varepsilon \succ 0$ there exists an $x \in X$ with $\mathbf{B}^\varepsilon(x) \in \mathcal{F}$, that is $\mathcal{F}$ is op-Cauchy in X precisely when $\mathcal{F}$ is Cauchy in X^{op}. The V-space X is *op-Cauchy complete* if every proper op-Cauchy filter on X op-converges. For spaces with uniformly vanishing asymmetry introduced above, the dual concepts of Cauchy completeness and op-Cauchy completeness coincide. A *completion* of X is a Cauchy complete V-space $\hat{X}$ together with an isometry $X \to \hat{X}$ with dense image in $\hat{X}$.

Definition 4.2. A filter $\mathcal{F}$ in X is said to be a *round filter* if for all $F \in \mathcal{F}$ there exists $\varepsilon \succ 0$ such that $\mathbf{B}_\varepsilon(x) \in \mathcal{F}$ implies $\mathbf{B}_\varepsilon(x) \subseteq F$, for all $x \in X$.

The omitted proof of the following result is completely formal.

Proposition 4.3. *If $\mathcal{F}$ is Cauchy and round, then $\mathcal{F}$ is minimal Cauchy.*

Let $\mathcal{F}$ be a filter. Consider the collection $\{\mathbf{B}_\varepsilon(F) \mid F \in \mathcal{F},\ \varepsilon \succ 0\}$, which is a filter base since $\mathbf{B}_{\varepsilon\wedge\delta}(F \cap F') \subseteq \mathbf{B}_\varepsilon(F) \cap \mathbf{B}_\delta(F')$ (recalling that $\varepsilon \wedge \delta \succ 0$). The generated filter is denoted by $\mathcal{F}_\succ$.

Proposition 4.4. *If $\mathcal{F}$ is Cauchy, then $\mathcal{F}_\succ$ is Cauchy.*

Proof. Let $\varepsilon \succ 0$ and $\delta \succ 0$ with $2 \cdot \delta \le \varepsilon$. Let $x \in X$ with $\mathbf{B}_\delta(x) \in \mathcal{F}$, and so $\mathbf{B}_\delta(\mathbf{B}_\delta(x)) \in \mathcal{F}_\succ$. Then $\mathbf{B}_\varepsilon(x) \in \mathcal{F}_\succ$ follows by $\mathbf{B}_\delta(\mathbf{B}_\delta(x)) \subseteq \mathbf{B}_{2\cdot\delta}(x) \subseteq \mathbf{B}_\varepsilon(x)$. □

Lemma 4.5. *If X has uniformly vanishing asymmetry and $\mathcal{F} \neq \mathcal{P}(X)$ is a filter on X, then $\mathcal{F}_\succ$ is round.*

Proof. It suffices to show that for a given basis element $\mathbf{B}_\varepsilon(F) \in \mathcal{F}_\succ$ there exists a $\delta \succ 0$ such that if $\mathbf{B}_\delta(y) \in \mathcal{F}_\succ$, then $\mathbf{B}_\delta(y) \subseteq \mathbf{B}_\varepsilon(F)$. Let $\delta_1 \succ 0$ with $2 \cdot \delta_1 \le \varepsilon$ and let $\delta_2 \succ 0$ be a uniform modulus of symmetry for δ_1. Set $\delta = \delta_1 \wedge \delta_2$. Suppose that $\mathbf{B}_\delta(y) \in \mathcal{F}_\succ$ for some $y \in X$. Then $\mathbf{B}_\delta(y) \supseteq \mathbf{B}_{\varepsilon'}(F') \supseteq F'$ for some $F' \in \mathcal{F}$ and $\varepsilon' \succ 0$. Choose some $x \in F \cap F'$. Then, $x \in \mathbf{B}_\delta(y)$ and so $d(y, x) \le \delta$. To show now that $\mathbf{B}_\delta(y) \subseteq \mathbf{B}_\varepsilon(F)$, notice that if $z \in \mathbf{B}_\delta(y)$, then $d(y, z) \le \delta$, and so $d(x, z) \le d(x, y) + d(y, z) \le \delta_1 + \delta_1 \le \varepsilon$, and thus $z \in \mathbf{B}_\varepsilon(F)$. □

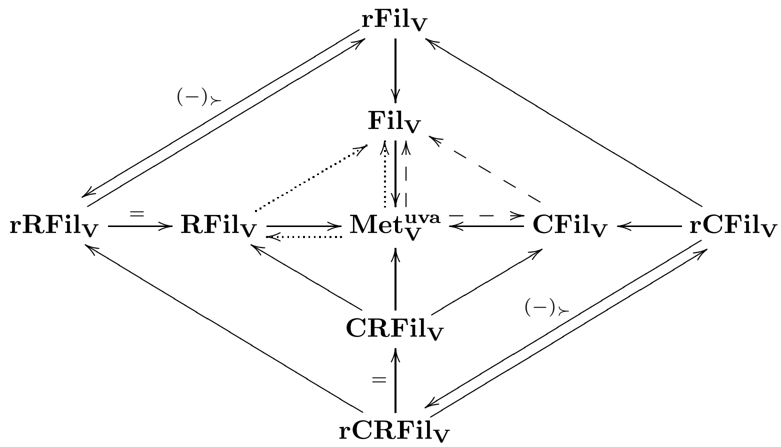
Corollary 4.6. *If X has uniformly vanishing asymmetry and $\mathcal{F} \neq \mathcal{P}(X)$ is a Cauchy filter on X, then $\mathcal{F}_\succ$ is a minimal Cauchy filter.*

Corollary 4.7. *If X has uniformly vanishing asymmetry, then a filter $\mathcal{F}$ is minimal Cauchy if, and only if, $\mathcal{F}$ is Cauchy and round.*

Proof. One direction is Proposition 4.3. For the other direction, if $\mathcal{F}$ is minimal Cauchy, then $\mathcal{F} = \mathcal{F}_\succ$, which is round. □

The results above translate to interesting categorical relations between Cauchy and round filters, as we now show. Let $\mathbf{Fil_V}$ be the category whose objects are all pairs $(X, \mathcal{F})$ where X is a V-space with uniformly vanishing asymmetry and $\mathcal{F}$ is a filter on X. The morphisms $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ are uniformly continuous functions $f : X \to Y$ with the property that $f(\mathcal{F}) \supseteq \mathcal{G}$, where $f(\mathcal{F}) = \{S \subseteq Y \mid f^{-1}(S) \in \mathcal{F}\}$ (which is easily seen to be a filter). Let $\mathbf{RFil_V}$ and $\mathbf{CFil_V}$ be the full subcategories of $\mathbf{Fil_V}$ spanned by round filters and by Cauchy filters, respectively. Let $\mathbf{CRFil_V} = \mathbf{CFil_V} \cap \mathbf{RFil_V}$. Finally, let $\mathbf{rFil_V}$ be the full subcategory of $\mathbf{Fil_V}$ spanned by the *restricted* objects, i.e., objects $(X, \mathcal{F})$ where $\mathcal{F} \neq \mathcal{P}(X)$ if $X \neq \emptyset$.

Similarly one defines the other restricted full subcategories. Consider the diagram



where the upwards directed arrows in both diamonds are inclusion functors and all of the arrows in the smaller diamond pointing towards the centre are the obvious forgetful functors. The other arrows (which are detailed below), with the exception of the pair on the upper left side of the outer diamond, are all adjunctions, with the left adjoint depicted on top or to the left of its right adjoint. The left adjoint $\mathbf{Met_V^{uva}} \to \mathbf{Fil_V}$ sends X to $(X, \mathcal{P}(X))$ while the right adjoint sends X to $(X, \{X\})$. For these functors the dotted and the dashed triangles commute. We note that in the degenerate case $V = \{0 = \infty\}$, the inner diamond reduces to identity functors, $\mathbf{Met_V^{uva}} \cong \mathbf{Set}$, and $\mathbf{Fil_V}$ is the category of filters introduced in [2].

Remark 4.8. Regarding the forgetful functor $p : \mathbf{CRFil_V} \to \mathbf{Met_V^{uva}}$, recall that the fiber over an object X is the category consisting of all of the objects in $\mathbf{CRFil_V}$ that project to X and all morphisms that project to the identity on X. This category is essentially a set and is precisely the completion of X we construct below.

Proposition 4.9. *The construction $(X, \mathcal{F}) \mapsto (X, \mathcal{F}_\succ)$ is the object part of a functor $(-)_\succ : \mathbf{Fil_V} \to \mathbf{Fil_V}$ which further sends $f : \mathcal{F} \to \mathcal{G}$ to $f : \mathcal{F}_\succ \to \mathcal{G}_\succ$. The restriction of this functor to $\mathbf{rFil_V}$ gives rise to the functor at the top left of the diagram above.*

Proof. Note that uniform continuity of $f : X \to Y$ implies that for all $\varepsilon \succ 0$ there exists $\delta \succ 0$ such that $f^{-1}(\mathbf{B}_\varepsilon(S)) \supseteq \mathbf{B}_\delta(f^{-1}(S))$, for all $S \subseteq Y$. Now, to show that $(-)_\succ$ is functorial, suppose that $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is a morphism, i.e., that $f(\mathcal{F}) \supseteq \mathcal{G}$, and we need to show that $f : (X, \mathcal{F}_\succ) \to (Y, \mathcal{G}_\succ)$ is a morphism, i.e., that $f(\mathcal{F}_\succ) \supseteq \mathcal{G}_\succ$. Indeed, if $G \in \mathcal{G}_\succ$, then $G \supseteq \mathbf{B}_\varepsilon(G')$ for some $G' \in \mathcal{G}$ and $\varepsilon \succ 0$. It thus follows that $f^{-1}(G) \supseteq f^{-1}(\mathbf{B}_\varepsilon(G')) \supseteq \mathbf{B}_\delta(f^{-1}(G'))$ for a suitable $\delta \succ 0$. Since $f^{-1}(G') \in \mathcal{F}$ we conclude that $f^{-1}(G) \in \mathcal{F}_\succ$. The claim about the image of the functor is Corollary 4.6. □

Note that generally speaking $\mathcal{G} \supseteq \mathcal{G}_\succ$ but strict inclusion may hold even if $\mathcal{G}$ is already round. The fact that for a round filter that is also Cauchy $\mathcal{G}_\succ = \mathcal{G}$ (by minimality) is crucial in the following proof.

Proposition 4.10. *The functor $(-)_\succ : \mathbf{rFil_V} \to \mathbf{rRFil_V}$ restricts to a functor $(-)_\succ : \mathbf{rCFil_V} \to \mathbf{rCRFil_V}$. This functor is left adjoint to the inclusion functor $\mathbf{rCRFil_V} \to \mathbf{rCFil_V}$.*

Proof. The claim about the restriction landing in Cauchy filters is Proposition 4.4. To establish that $(-)_{\succ}$ is left adjoint to the inclusion, we need to show for a Cauchy and round filter $\mathcal{G}$ on Y and an arbitrary Cauchy filter $\mathcal{F}$ on X, that $f:(X,\mathcal{F}_{\succ})\to(Y,\mathcal{G})$ is a morphism, i.e., that $f(\mathcal{F}_{\succ})\supseteq\mathcal{G}$, if, and only if, $f:(X,\mathcal{F})\to(Y,\mathcal{G})$ is a morphism, i.e., $f(\mathcal{F})\supseteq\mathcal{G}$. Since $\mathcal{F}\supseteq\mathcal{F}_{\succ}$, it follows that $f(\mathcal{F})\supseteq f(\mathcal{F}_{\succ})$, and thus one of the implications is trivial. Assume now that $f(\mathcal{F})\supseteq\mathcal{G}$, and we need to show that $f(\mathcal{F}_{\succ})\supseteq\mathcal{G}$. Let $G\in\mathcal{G}$. As $\mathcal{G}$ is Cauchy and round, thus minimal Cauchy, we have that $\mathcal{G}_{\succ}=\mathcal{G}$, and so there exists some $G'\in\mathcal{G}$ and $\varepsilon\succ 0$ with $G\supseteq\mathbf{B}_{\varepsilon}(G')$. Then $f^{-1}(G)\supseteq f^{-1}(\mathbf{B}_{\varepsilon}(G'))\supseteq\mathbf{B}_{\delta}(f^{-1}(G'))$ for some $\delta\succ 0$, and since $f^{-1}(G')\in\mathcal{F}$, we conclude that $f^{-1}(G)\in\mathcal{F}_{\succ}$. □

This concludes the description of the functors in the diagram above. We now turn to the details of the completion construction. For $x\in X$ let $\mathcal{F}_x$ be the filter generated by the filter base $\mathcal{B}_x=\{\mathbf{B}_{\varepsilon}(x)\mid\varepsilon\succ 0\}$, which is clearly Cauchy. The dual construction is the filter $\mathcal{F}^x$ generated by the filter base $\mathcal{B}^x=\{\mathbf{B}^{\varepsilon}(x)\mid\varepsilon\succ 0\}$. For general V-spaces, a filter may be Cauchy without being op-Cauchy and $\mathcal{F}_x=\mathcal{F}^x$ need not hold.

Proposition 4.11. *If X has vanishing asymmetry, then $\mathcal{F}_x=\mathcal{F}^x$ for all $x\in X$. If X has uniformly vanishing asymmetry, then a filter $\mathcal{F}$ is Cauchy if, and only if, it is op-Cauchy. Consequently, X is Cauchy complete if, and only if, it is op-Cauchy complete.*

Proof. To show that $\mathcal{F}_x=\mathcal{F}^x$ it suffices to argue on basis elements. If X has vanishing asymmetry, then given $\mathbf{B}_{\varepsilon}(x)\in\mathcal{B}_x$ let $\delta\succ 0$ be such that $\mathbf{B}^{\delta}(x)\subseteq\mathbf{B}_{\varepsilon}(x)$, which thus shows that $\mathbf{B}_{\varepsilon}(x)\in\mathcal{F}^x$, and so $\mathcal{F}_x\subseteq\mathcal{F}^x$. The reverse inequality follows similarly. Suppose now that X has uniformly vanishing asymmetry and that $\mathcal{F}$ is Cauchy. Given $\varepsilon\succ 0$, let $\delta\succ 0$ be a corresponding modulus of uniform symmetry. There is then $x\in X$ with $\mathbf{B}_{\delta}(x)\in\mathcal{F}$, and since $\mathbf{B}_{\delta}(x)\subseteq\mathbf{B}^{\varepsilon}(x)$, it follows that $\mathbf{B}^{\varepsilon}(x)\in\mathcal{F}$, and so $\mathcal{F}$ is op-Cauchy. The reverse implication is similar. The last assertion in the proposition follows since $\mathcal{F}\to x$ is equivalent to $\mathcal{F}_x\subseteq\mathcal{F}$, and $\mathcal{F}\to^{op}x$ is equivalent to $\mathcal{F}^x\subseteq\mathcal{F}$. □

Proposition 4.12. *If X has uniformly vanishing asymmetry, then $\mathcal{F}_x$ is round.*

Proof. Given $\mathbf{B}_{\varepsilon}(x)\in\mathcal{F}_x$, let $\delta_1\succ 0$ with $2\cdot\delta_1\leq\varepsilon$ and let $\delta_2\succ 0$ be a uniform modulus of symmetry for δ_1. Set $\delta=\delta_1\wedge\delta_2$ and suppose $\mathbf{B}_{\delta}(y)\in\mathcal{F}_x$. Then clearly $x\in\mathbf{B}_{\delta}(y)$, thus $d(y,x)\leq\delta$, implying that $d(x,y)\leq\delta_1$. Now, to show that $\mathbf{B}_{\delta}(y)\subseteq\mathbf{B}_{\varepsilon}(x)$, notice that if $z\in\mathbf{B}_{\delta}(y)$, then $d(x,z)\leq d(x,y)+d(y,z)\leq\delta_1+\delta_1\leq\varepsilon$, and so $z\in\mathbf{B}_{\varepsilon}(x)$. □

For subsets $S,T\subseteq X$, let $d(S,T)=\bigwedge_{s\in S,t\in T}d(s,t)$, and for collections $\mathcal{E}_1,\mathcal{E}_2\subseteq\mathcal{P}(X)$, let $d(\mathcal{E}_1,\mathcal{E}_2)=\bigvee_{S\in\mathcal{E}_1,T\in\mathcal{E}_2}d(S,T)$, giving rise to a function $d:\mathcal{P}(\mathcal{P}(X))\times\mathcal{P}(\mathcal{P}(X))\to V$. Let $\tilde{X}\subseteq\mathcal{P}(\mathcal{P}(X))$ be the set of all proper (i.e., $\mathcal{P}(X)$ is excluded) Cauchy filters on X and let $\hat{X}\subseteq\tilde{X}$ be the set of all minimal Cauchy filters. It is easy to see that if $\mathcal{B}_{\mathcal{F}}$ and $\mathcal{B}_{\mathcal{G}}$ are filter bases for $\mathcal{F}$ and $\mathcal{G}$ respectively, then $d(\mathcal{F},\mathcal{G})=d(\mathcal{B}_{\mathcal{F}},\mathcal{B}_{\mathcal{G}})$. (Alternatively, notice that $\mathcal{F}_x$ is $(\mathcal{P}_x)_{\succ}$, where $\mathcal{P}_x$ is the principal filter on x.)

The following computation is convenient to record for the proofs below.

Proposition 4.13. *Suppose that $S\subseteq\mathbf{B}^{\delta}(x)$ and $T\subseteq\mathbf{B}_{\varepsilon}(y)$ and $S,T\neq\emptyset$. Then $d(S,T)\leq d(x,y)+\delta+\varepsilon$ and $d(y,x)\leq d(T,S)+\delta+\varepsilon$.*

Proof. Let $s \in S$ and $t \in T$ be arbitrary. Then $d(S,T) \le d(s,t) \le d(s,x)+d(x,y)+d(y,t) \le \delta + d(x,y) + \varepsilon$, which is the first inequality. By the distributivity law in V, the second inequality will follow by showing that $d(y,x) \le d(t,s) + \delta + \varepsilon$ for all $s \in S$ and $t \in T$. Indeed, $d(y,x) \le d(y,t) + d(t,s) + d(s,x) \le \varepsilon + d(t,s) + \delta$. $\square$

Lemma 4.14. *If X has uniformly vanishing asymmetry, then $(\tilde{X}, d)$ is a V-space, which itself has uniformly vanishing asymmetry.*

Proof. $d(\mathcal{F},\mathcal{F}) = 0$ since for all $F, F' \in \mathcal{F}$, $F \cap F' \neq \emptyset$. To establish that $d(\mathcal{F},\mathcal{H}) \le d(\mathcal{F},\mathcal{G}) + d(\mathcal{G},\mathcal{H})$ it suffices to show, for fixed $\varepsilon \succ 0$, $F \in \mathcal{F}$, and $H \in \mathcal{H}$, that there exists $G \in \mathcal{G}$ such that $d(F,H) \le d(F,G) + d(G,H) + \varepsilon$. Let $\delta \succ 0$ be such that $2 \cdot \delta \le \varepsilon$ and let $\delta' \succ 0$ be a uniform modulus of symmetry for δ, and set $\eta = \delta \wedge \delta'$. As $\mathcal{G}$ is Cauchy, there is $x \in X$ such that $G = \mathbf{B}_\eta(x) \in \mathcal{G}$. And then

$$\begin{aligned} d(F,G) + d(G,H) + \varepsilon &\ge \bigwedge_{f \in F, y,z \in G, h \in H} d(f,y) + d(z,h) + 2 \cdot \delta \\ &\ge \bigwedge_{f \in F, y,z \in G, h \in H} d(f,y) + d(y,x) + d(x,z) + d(z,h) \\ &\ge \bigwedge_{f \in F, h \in H} d(f,h) = d(F,H) \end{aligned}$$

as required for showing that $\tilde{X}$ is a V-space.

To show that $\tilde{X}$ has uniformly vanishing asymmetry, let $\varepsilon \succ 0$ be given and let $\eta \succ 0$ with $2 \cdot \eta \le \varepsilon$. Let $\delta_1 \succ 0$ be a uniform modulus of symmetry for η, and $\delta \succ 0$ with $2 \cdot \delta \le \delta_1$. Suppose that $d(\mathcal{G},\mathcal{F}) \le \delta$, which means that $d(G,F) \le \delta$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$. To show that $d(\mathcal{F},\mathcal{G}) \le \varepsilon$ it suffices to show that $d(F_0, G_0) \le \varepsilon$ for fixed $F_0 \in \mathcal{F}$ and $G_0 \in \mathcal{G}$. Since $\mathcal{F}$ is Cauchy (and thus op-Cauchy) and since $\mathcal{G}$ is Cauchy, there exist $x, y \in X$ with $\mathbf{B}^{\delta'}(x) \in \mathcal{F}$ and $\mathbf{B}_{\delta'}(y) \in \mathcal{G}$, where $\delta' \succ 0$ satisfies $2 \cdot \delta' \le \delta$. Let $S = F_0 \cap \mathbf{B}^{\delta'}(x)$ and $T = G_0 \cap \mathbf{B}_{\delta'}(y)$. Then, using Proposition 4.13 (here and in the following computation), $d(y,x) \le d(T,S) + \delta \le 2 \cdot \delta \le \delta_1$ and thus $d(x,y) \le \eta$. Finally, $d(F_0, G_0) \le d(S,T) \le d(x,y) + \eta \le 2 \cdot \eta \le \varepsilon$, as required. $\square$

For any V-space, setting $x \sim y$ whenever $d(x,y) = d(y,x) = 0$ is an equivalence relation, and X_0, the set of equivalence classes becomes a separated V-space where the distance function is given by $d([x],[y]) = d(x,y)$. We note that if X has vanishing asymmetry, then $d(x,y) = 0$ implies $d(y,x) = 0$ and if X is also separated, then $\mathcal{O}(X)$ is Hausdorff. In particular, the following result (whose proof is immediate and thus omitted), implies that if X has vanishing asymmetry, then X_0 is Hausdorff.

Proposition 4.15. *If X has (uniformly) vanishing asymmetry, then so does X_0.*

Theorem 4.16. *If X has uniformly vanishing asymmetry, then $\hat{X}$ is isometric to $\tilde{X}_0$.*

Proof. For any two Cauchy filters $\mathcal{F}, \mathcal{G}$ on X, their intersection is again a filter but it need not be Cauchy. However, if $d(\mathcal{F},\mathcal{G}) = 0$, then $\mathcal{F} \cap \mathcal{G}$ is Cauchy. Indeed, let $\varepsilon \succ 0$ and let $\delta_1 \succ 0$ with $2 \cdot \delta_1 \le \varepsilon$. Let $\delta_2 \succ 0$ be a uniform modulus of symmetry for δ_1, and let $\delta_3 \succ 0$ satisfy $2 \cdot \delta_3 \le \delta_2$. Set $\delta = \delta_2 \wedge \delta_3$. There exists $x \in X$ with $\mathbf{B}_\delta(x) \in \mathcal{F}$ and $y \in X$ with $\mathbf{B}^\delta(y) \in \mathcal{G}$, and since $d(\mathcal{F},\mathcal{G}) = 0$ it follows that $d(\mathbf{B}_\delta(x), \mathbf{B}^\delta(y)) = 0$, and thus that $d(x,y) \le 2 \cdot \delta \le \delta_1$. Now, $d(y,s) \le \delta_1$ for all

$s \in \mathbf{B}^\delta(y)$ and so $d(x,s) \le d(x,y) + d(y,s) \le 2\cdot\delta_1 \le \varepsilon$, leading to $\mathbf{B}^\delta(y) \subseteq \mathbf{B}_\varepsilon(x)$. It thus follows that $\mathbf{B}_\varepsilon(x) \in \mathcal{G}$, which establishes that $\mathcal{F}\cap\mathcal{G}$ is Cauchy.

It now follows that each equivalence class $[\mathcal{F}]$ contains a unique minimal Cauchy representative. Indeed, it is easily seen that $d(\mathcal{F}, \mathcal{F}_\succ) = 0$ so that $\mathcal{F}_\succ \in [\mathcal{F}]$. If $\mathcal{F}_1, \mathcal{F}_2$ are two minimal Cauchy filters with $\mathcal{F}_1 \sim \mathcal{F}_2$, then $\mathcal{F}_1 \cap \mathcal{F}_2$ is Cauchy so that minimality forces $\mathcal{F}_1 = \mathcal{F}_2$. The bijective isometry $\tilde{X}_0 \to \hat{X}$ is thus given by $[\mathcal{F}] \mapsto \mathcal{F}_\succ$. □

Corollary 4.17. *If X has uniformly vanishing asymmetry, then $\hat{X}$ is a separated V-space with uniformly vanishing asymmetry.*

Recall that when X has uniformly vanishing asymmetry every the filters $\mathcal{F}_x$ are round (and clearly Cauchy). We then obtain the function $\iota : X \to \hat{X}$, given by $\iota(x) = \mathcal{F}_x$, called the *canonical embedding* (even though it is injective if, and only if, X is separated).

Lemma 4.18. *If X has uniformly vanishing asymmetry, then the canonical embedding $\iota : X \to \hat{X}$ is an isometry.*

Proof. Clearly, $d(\mathbf{B}_\varepsilon(x), \mathbf{B}_\delta(y)) \le d(x,y)$, thus $d(\mathcal{B}_x, \mathcal{B}_y) \le d(x,y)$, and therefore $d(\mathcal{F}_x, \mathcal{F}_y) \le d(x,y)$. For the other direction, we will use the fact that $\mathcal{F}_y = \mathcal{F}^y$ (cf. Proposition 4.11), so it suffices to show that $d(x,y) \le d(\mathcal{B}_x, \mathcal{B}^y)$. To that end, let $\rho \succ 0$, and $\eta \succ 0$ with $2\cdot\eta \le \rho$. Since in general $d(x,y) - \varepsilon - \delta \le d(\mathbf{B}_\varepsilon(x), \mathbf{B}^\delta(y))$ we have $d(\mathcal{B}_x, \mathcal{B}^y) \ge d(\mathbf{B}_\eta(x), \mathbf{B}^\eta(y)) \ge (d(x,y) - \eta) - \eta = d(x,y) - (\eta+\eta) \ge d(x,y) - \rho$. Thus, $d(x,y) \le d(\mathcal{B}_x, \mathcal{B}^y) + \rho$, and as $\rho \succ 0$ is arbitrary, the desired inequality follows. □

Corollary 4.19. *If X is separated and has uniformly vanishing asymmetry, then the canonical embedding $\iota : X \to \hat{X}$ is injective.*

Lemma 4.20. *If X has uniformly vanishing asymmetry, then the image $\iota(X)$ is dense in $\hat{X}$.*

Proof. Fix $\mathcal{G} \in \hat{X}$ and $\varepsilon \succ 0$. Let $\delta \succ 0$ be a uniform modulus of symmetry for ε, and since $\mathcal{G}$ is Cauchy we may find $x \in X$ with $\mathbf{B}_\delta(x) \in \mathcal{G}$. To show that $d(\mathcal{G}, \mathcal{F}_x) \le \varepsilon$ it suffices to show that $d(G, \mathbf{B}_\eta(x)) \le \varepsilon$ for all $\eta \succ 0$ and $G \in \mathcal{G}$. Let $y \in G \cap \mathbf{B}_\delta(x)$. Then $d(x,y) \le \delta$ implies $d(G, \mathbf{B}_\eta(x)) \le d(G \cap \mathbf{B}_\delta(x), x) \le d(y,x) \le \varepsilon$. □

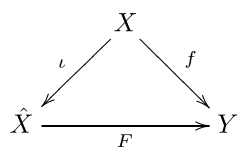
Theorem 4.21. *If X has uniformly vanishing asymmetry, then $\hat{X}$ is Cauchy complete.*

Proof. It suffices to show that every proper Cauchy filter on $\hat{X}$ converges to a minimal Cauchy filter on X. Let $\mathbb{A}$ be a Cauchy filter on $\hat{X}$ and $\varepsilon \succ 0$. Then there is a minimal Cauchy filter $\mathcal{L} \in \hat{X}$ such that $\mathbf{B}_\varepsilon(\mathcal{L}) = \{\mathcal{G} \in \hat{X} \mid d(\mathcal{L}, \mathcal{G}) \le \varepsilon\}$ is in $\mathbb{A}$. It is straightforward to verify that $\mathcal{F} = \{F \subseteq X \mid \exists A \in \mathbb{A}, F \in \bigcap A\}$ is a filter. Next, to show that $\mathcal{F}$ is Cauchy, let $\varepsilon \succ 0$ and $\delta_1 \succ 0$ with $4\cdot\delta_1 \le \varepsilon$. Further, let $\delta_2 \succ 0$ be a uniform modulus of symmetry for δ_1. Note $\delta_3 = \delta_1 \wedge \delta_2 \succ 0$. Fix $\mathcal{G} \in \mathbf{B}_\varepsilon(\mathcal{L})$. Since $\mathcal{G}$ is Cauchy, there is $y \in X$ such that $\mathbf{B}_{\delta_3}(y) \in \mathcal{G}$. For $a \in \mathbf{B}_{\delta_3}(y)$, we have $d(x,a) \le d(x,y) + d(y,a) \le d(\mathbf{B}_{\delta_1}(x), \mathbf{B}_{\delta_3}(y)) + 2\cdot\delta_1 + \delta_3 \le 4\cdot\delta_1 \le \varepsilon$, thus $\mathbf{B}_{\delta_3}(y) \subseteq \mathbf{B}_\varepsilon(x)$ which implies $\mathbf{B}_\varepsilon(x) \in \mathcal{G}$. Since $\mathcal{G} \in \mathbf{B}_\varepsilon(\mathcal{L})$ is arbitrary, $\mathbf{B}_\varepsilon(x) \in \bigcap \mathbf{B}_\varepsilon(\mathcal{L})$, thus $\mathbf{B}_\varepsilon(x) \in \mathcal{F}$.

Finally, we show that $\mathbb{A}$ converges to the minimal Cauchy filter $\mathcal{F}_\succ$. Let $\varepsilon \succ 0$ and $\delta_1 \succ 0$ with $2\cdot\delta_1 \le \varepsilon$, and further let $\delta_2 \succ 0$ be a uniform modulus of symmetry for δ_1. Note that $\delta = \delta_1 \wedge \delta_2 \succ 0$. There is $\mathcal{L} \in \hat{X}$ such that $\mathbf{B}_\delta(\mathcal{L}) \in \mathbb{A}$. Then it suffices to show that $\mathbf{B}_\delta(\mathcal{L}) \subseteq \mathbf{B}_\varepsilon(\mathcal{F}_\succ)$. Let $\mathcal{M} \in \mathbf{B}_\delta(\mathcal{L})$, $L_0 \in \mathcal{L}$ and $F_0 \in \mathcal{F}$. This means that there is $A \in \mathbb{A}$ such that $F_0 \in \bigcap A$. Since $\mathbb{A}$ is a proper filter, $\mathbf{B}_\delta(\mathcal{L}) \cap A \neq \emptyset$ and $d(\mathcal{L}, \mathcal{G}) \leq \delta \leq \delta_2$ implies $d(\mathcal{G}, \mathcal{L}) \leq \delta_1$ for every $\mathcal{G} \in \mathbf{B}_\delta(\mathcal{L}) \cap A$. Since F_0 is also in $\mathcal{G}$, $d(F_0, L_0) \leq \bigvee_{G \in \mathcal{G}, L \in \mathcal{L}} d(G, L) = d(\mathcal{G}, \mathcal{L}) \leq \delta_1$, thus $d(F_0, L_0) \leq \delta_1$. Since $F_0 \in \mathcal{F}$ and $L_0 \in \mathcal{L}$ are arbitrary, we obtain $d(\mathcal{F}, \mathcal{L}) \leq \delta_1$. Then $d(\mathcal{F}_\succ, \mathcal{L}) \leq d(\mathcal{F}_\succ, \mathcal{F}) + d(\mathcal{F}, \mathcal{L}) \leq 0 + \delta_1$ which implies that $d(\mathcal{F}_\succ, \mathcal{M}) \leq d(\mathcal{F}_\succ, \mathcal{L}) + d(\mathcal{L}, \mathcal{M}) \leq \delta_1 + \delta \leq \delta_1 + \delta_1 \leq \varepsilon$, thus $\mathcal{M} \in \mathbf{B}_\varepsilon(\mathcal{F}_\succ)$. Since $\mathcal{M} \in \mathbf{B}_\delta(\mathcal{L})$ is arbitrary, $\mathbf{B}_\delta(\mathcal{L}) \subseteq \mathbf{B}_\varepsilon(\mathcal{F}_\succ)$ and since $\varepsilon \succ 0$ is arbitrary, it follows that $\mathbb{A}$ converges to $\mathcal{F}_\succ$. □

Obviously, the construction $X \mapsto \hat{X}$ is functorial. The following two corollaries follow by standard arguments from Lemma 4.18, Lemma 4.20, and Theorem 4.21:

Corollary 4.22. *The universal property*



stating that for any Cauchy complete V-space Y with uniformly vanishing asymmetry and any uniformly continuous function f there exists a unique uniformly continuous extension F, holds for all separated V-spaces X with uniformly vanishing asymmetry.

Corollary 4.23. *Every separated V-space X with uniformly vanishing asymmetry has a completion, unique up to a unique isomorphism.*

Relating back to the categorical point-of-view, i.e., to the 2-functor $\mathbf{QMet}_{-} : \mathbb{V} \to \mathbf{Cat}$ from Section 2.2, the constructions above may be summarized as follows. Consider the obvious 2-functors $\mathbf{sMet}^{\mathbf{uva}}_{-}, \mathbf{scMet}^{\mathbf{uva}}_{-} : \mathbb{V} \to \mathbf{Cat}$ mapping V to $\mathbf{sMet}^{\mathbf{uva}}_{\mathbf{V}}$ and to $s\mathbf{cMet}^{\mathbf{uva}}_{\mathbf{V}}$ (the categories of separated V-spaces with uniformly vanishing asymmetry and of complete separated V-spaces with uniformly vanishing asymmetry), respectively. The completion functor $\mathbf{sMet}^{\mathbf{uva}}_{\mathbf{V}} \to \mathbf{scMet}^{\mathbf{uva}}_{\mathbf{V}}$ defines a 2-natural transformation $\hat{-} : \mathbf{sMet}^{\mathbf{uva}}_{-} \to \mathbf{scMet}^{\mathbf{uva}}_{-}$.

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