

MAXIMALLY FROBENIUS-DESTABILIZED VECTOR BUNDLES OVER SMOOTH ALGEBRAIC CURVES

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ABSTRACT. Vector bundles in positive characteristics have a tendency to be destabilized after pulling back by the Frobenius morphism. In this paper, we closely examine vector bundles over curves that are, in an appropriate sense, maximally destabilized by the Frobenius morphism. Then we prove that such bundles of rank 2 exist over any curve in characteristic 3, and are unique up to twisting by a line bundle. We also give an application of such bundles to the study of ample vector bundles, which is valid in all characteristics.

0. INTRODUCTION

Given a normal projective variety X over an algebraically closed field k of characteristic $p \neq 0$, with a fixed ample divisor, it can happen that pulling back by an inseparable morphism $f : Y \rightarrow X$ destroys semistability of vector bundles over X . In the simplest case, where X is a smooth curve and $f = F_{\text{rel}}$ is the (relative) Frobenius morphism, semistable vector bundles whose pullback under f fail to be semistable are called *Frobenius-destabilized* vector bundles. They are closely related to the study of the generalized Verschiebung.

In this paper, we are primarily interested in rank- r vector bundles E over a curve X of genus $g \geq 2$ with the following property:

(*) the Harder-Narasimhan filtration of $F_{\text{rel}}^* E$ admits line bundle quotients L_1, \dots, L_r , and $\deg(L_i) = \deg(L_{i+1}) + 2g - 2$ for all $i = 1, \dots, r - 1$.

If themselves semistable, such bundles are the “most destabilized” ones possible. We will call a semistable vector bundle E with property (*) a *maximally Frobenius-destabilized* vector bundle (cf. Def. 2.11). These vector bundles appear in the works of many authors. Notably, over a genus-2 curve, all rank-2 Frobenius-destabilized vector bundles are maximally Frobenius-destabilized (cf. [8, Prop. 3.3]); the work of S. Mochizuki [17] gave a precise formula counting the number of such bundles with trivial determinant over a general curve in arbitrary characteristic, and B. Osserman [18] counted them over an *arbitrary* curve in small characteristics. In another direction, K. Joshi *et al.* [9] gave a relation between certain Frobenius-destabilized bundles and pre-opers, a concept originated from the geometric Langlands program; their observation was later used by X. Sun [20] to prove that stability is preserved under Frobenius-pushforward.

We prove that vector bundles with property (*) exhibit an interesting trichotomy:

Theorem. *Let E be a rank- r vector bundle over a curve X of genus $g \geq 2$ with property (*). Then*

- (i) *if $r < p$ and p does not divide $g - 1$, then E is stable (Prop. 2.6);*
- (ii) *if $r > p$, then E is not semistable (Prop. 2.8); and*
- (iii) *if $r = p$, then E is stable if and only if E is the Frobenius-pushforward of a line bundle (Prop. 2.10; this is already discovered by L.-G. Li, H. Lange and C. Pauly).*

The result (i) reflects a general observation of K. Joshi *et al.* that high instability of the Frobenius-pullback $F_{\text{rel}}^* E$ implies high stability of E . The main technical ingredient in the proof of (i) is an improvement of an inequality due to N. Shepherd-Barron [19, Cor. 2^p]: $L_{\max}(E) - L_{\min}(E) \leq (r - 1)(2g - 2)/p$. Here L_{\max} and L_{\min} are measures of maximal slope of subbundles and minimal slope of quotient bundles respectively, taken over all finite pullbacks. We split Shepherd-Barron’s inequality into

$$L_{\max}(E) - \mu(E), \quad \mu(E) - L_{\min}(E) \leq (r - 1)(g - 1)/p \tag{0.1}$$

as well as giving an equality criterion for both. As we will see, the improved inequalities (0.1) have many independent applications as well. The result (ii) shows that maximally Frobenius-destabilized vector

bundles do not exist with rank $r > p$. For the critical value $r = p$, property (*) is intimately related to pre-operators; the result (iii) is known (cf. Li [15, Lem. 3.1] and Lange and Pauly [13, Prop. 1.2]) and a more general version is given by C. Liu and M. Zhou [16].

In characteristic $p = 3$, B. Osserman [18] has constructed rank-2 maximally Frobenius-destabilized vector bundles with trivial determinant over an arbitrary curve of genus 2, and showed that there are exactly $\#\text{Pic}(X)[2] = 16$ of them. We derive from the previous theorem that, for *arbitrary* genus, such bundles are precisely the rank-2 vector bundles E satisfying

$$S^2(E) \cong (F_{\text{rel}})_*T \quad (0.2)$$

where $S^2(E)$ is the second symmetric power of E , and T is the tangent sheaf. Then we turn (0.2) around, and construct vector bundle E whose second symmetric power is isomorphic to $(F_{\text{rel}})_*T$. Our main result in this direction is

Theorem (Thm. 3.10). *Over any curve X of genus $g \geq 2$ in characteristic 3, there exist precisely $\#\text{Pic}(X)[2] = 2^{2g}$ rank-2 maximally Frobenius-destabilized vector bundles of trivial determinant, and they are related to one another via twisting by an order-2 line bundle.*

Y. Wakabayashi [21, Thm. A] recently gave a formula counting the number of rank-2 Frobenius-destabilized bundles on a general curve of genus g , subject to the condition $2(g-1) < p$. In contrast, our result holds for $p = 3$ and arbitrary genus.

In characteristic zero, a vector bundle E over a curve is ample if and only if the minimal slope of vector bundle quotients of E is positive (cf. [6, Thm. 3.2.7]). In positive characteristics, a numerical criterion for ample vector bundles over curves is given by H. Brenner [3, Thm. 2.3] and I. Biswas [2]. We generalize their result to higher dimensions:

Theorem (Thm. 4.3). *Let X be a smooth projective variety over an algebraically closed field k , and E be a vector bundle over X . Then E is ample if and only if there exists some real number $\varepsilon > 0$, such that for any integral, closed curve C in X , there holds $L_{\min}(E|_C) \geq \varepsilon\|C\|$.*

Here, the norm $\|\cdot\|$ on $A_1(X)$ is fixed with respect to a chosen basis.

The improved inequalities (0.1) together with the above ample criterion show that semistable vector bundles with degree greater than $r(r-1)(g-1)/p$ over nonrational curves are ample. On the other hand, maximally Frobenius-destabilized vector bundles with degree less than $r(r-1)(g-1)/p$ are *not* ample (Cor. 4.5). We use this last observation to construct certain non-ample semistable vector bundles of positive degree.

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1. PRELIMINARIES

We first fix some notations. Let X be a normal projective variety over an algebraically closed field k , and let E be a torsion-free coherent sheaf over X . Fix an ample line bundle H over X . With respect to H , we set $\mu_{\max}(E)$ to be the maximal slope of a coherent subsheaf of E , and $\mu_{\min}(E)$ to be the minimal slope of a torsion-free coherent quotient sheaf of E . If $0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$ is the Harder-Narasimhan filtration of E with successive quotients $Q_i = E_i/E_{i-1}$, then

$$\mu_{\max}(E) = \mu_1(E) > \mu_2(E) > \cdots > \mu_l(E) = \mu_{\min}(E)$$

where $\mu_i(E) = \mu(Q_i)$ is the slope of Q_i . With respect to the ample line bundles pulled back from H , we also set

$$L_{\max}(E) = \sup_{f:Y \rightarrow X} \frac{\mu_{\max}(f^*E)}{\deg(f)}, \quad \text{and} \quad L_{\min}(E) = \inf_{f:Y \rightarrow X} \frac{\mu_{\min}(f^*E)}{\deg(f)} \quad (1.1)$$

where both the supremum and the infimum are taken over all finite morphisms $f : Y \rightarrow X$ of normal projective varieties over k . We note the following standard

Lemma 1.1. *Let $f : Y \rightarrow X$ be a finite, separable morphism of normal projective varieties. Let E be a torsion-free coherent sheaf over X with Harder-Narasimhan filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$. Then $0 = f^*E_0 \subset f^*E_1 \subset \cdots \subset f^*E_l = f^*E$ is the Harder-Narasimhan filtration of f^*E .*

Sketch of proof. Using an argument similar to that of [6, Lem. 3.2.2], one first proves that if E_1 is the maximal destabilizing subsheaf of E , then f^*E_1 is the maximal destabilizing subsheaf of f^*E . Apply this fact with E replaced by E/E_i for all $i = 1, \dots, l-1$, and the result follows. \square

Corollary 1.2. *If the ground field k has characteristic zero, then $L_{\max}(E) = \mu_{\max}(E)$, and $L_{\min}(E) = \mu_{\min}(E)$.*

Proof. In this case, every finite morphism $f : Y \rightarrow X$ of normal projective varieties is separable. Hence the lemma shows that

$$\frac{\mu_{\max}(f^*E)}{\deg(f)} = \mu_{\max}(E), \quad \text{and} \quad \frac{\mu_{\min}(f^*E)}{\deg(f)} = \mu_{\min}(E)$$

The corollary then follows from the definitions of $L_{\max}(E)$ and $L_{\min}(E)$. \square

If the ground field k has characteristic $p \neq 0$, we may let $F_{\text{abs}} : X \rightarrow X$ be the absolute Frobenius morphism, which factors through the relative Frobenius morphism $F_{\text{rel}} : X \rightarrow X^{(1)}$ as in the following commutative diagram:

$$\begin{array}{ccccc} X & & & & \\ & \searrow^{F_{\text{rel}}} & & \searrow^{F_{\text{abs}}} & \\ & & X^{(1)} & \xrightarrow{\quad} & X \\ & \searrow & \downarrow & & \downarrow \\ & & \text{Spec}(k) & \xrightarrow{a \rightarrow a^p} & \text{Spec}(k) \end{array}$$

Here, the commutative square is a pullback diagram.

Corollary 1.3. *If the ground field k has characteristic $p \neq 0$, then*

$$L_{\max}(E) = \lim_{k \rightarrow \infty} \frac{\mu_{\max}((F_{\text{rel}}^k)^*E)}{p^k}, \quad \text{and} \quad L_{\min}(E) = \lim_{k \rightarrow \infty} \frac{\mu_{\min}((F_{\text{rel}}^k)^*E)}{p^k} \quad (1.2)$$

Observe that we may replace F_{rel} by F_{abs} in (1.2) since X and $X^{(1)}$ are isomorphic as schemes. This corollary shows that our definitions of $L_{\max}(E)$ and $L_{\min}(E)$ agree with those in [14].

Proof. Note that the sequence $\mu_{\max}((F_{\text{rel}}^k)^*E)/p^k$ is nondecreasing. We only need to show for every finite morphism $f : Y \rightarrow X$ of normal projective varieties, there exists some $k \in \mathbb{N}$ such that

$$\frac{\mu_{\max}(f^*E)}{\deg(f)} \leq \frac{\mu_{\max}((F_{\text{rel}}^k)^*E)}{p^k}$$

Indeed, given such a morphism $f : Y \rightarrow X$, upon further pulling back, we may assume the induced field extension $K(X) \subset K(Y)$ is normal. Thus we may factor f into the composition of a separable morphism $g : Y \rightarrow Y'$ of normal projective varieties, and a purely inseparable morphism $h : Y' \rightarrow X$; moreover, h is “dominated” by a composition of F_{rel} as illustrated in the following commutative diagram:

$$\begin{array}{ccccc} Y & \xrightarrow{g} & Y' & \longleftarrow & X^{(-k)} \\ & \searrow f & \downarrow h & & \swarrow F_{\text{rel}}^k \\ & & X & & \end{array}$$

Applying Lem. 1.1 to morphism g , we obtain

$$\frac{\mu_{\max}(f^*E)}{\deg(f)} = \frac{\mu_{\max}(h^*E)}{\deg(h)} \leq \frac{\mu_{\max}((F_{\text{rel}}^k)^*E)}{p^k}$$

as desired. The proof for the second identity is similar. \square

One of the most important properties of $L_{\max}(E)$ and $L_{\min}(E)$ is that they are determined after pulling back by F_{rel} finitely many times. This result is due to A. Langer:

Theorem 1.4 (A. Langer). *Let X be a smooth projective variety over an algebraically closed field of positive characteristic p , and let E be a torsion-free coherent sheaf over X . Then there exists a natural number k_0 such that if*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = (F_{\text{rel}}^{k_0})^* E$$

is the Harder-Narasimhan filtration of $(F_{\text{rel}}^{k_0})^ E$, then for every $k \geq k_0$,*

$$0 = (F_{\text{rel}}^{k-k_0})^* E_0 \subset (F_{\text{rel}}^{k-k_0})^* E_1 \subset \cdots \subset (F_{\text{rel}}^{k-k_0})^* E_l = (F_{\text{rel}}^k)^* E$$

is the Harder-Narasimhan filtration of $(F_{\text{rel}}^k)^ E$. In other words, the successive quotients $Q_i = E_i/E_{i-1}$ are strongly semistable.*

Proof. This is [14, Thm. 2.7]. □

We now assume the ground field k has characteristic $p \neq 0$. Given a coherent sheaf E over X , and a connection $\nabla : E \rightarrow E \otimes \Omega_X$, the p -curvature of ∇ is the p -linear sheaf morphism $\text{Der}_k(\mathcal{O}_X) \rightarrow \mathcal{E}nd_{\mathcal{O}_X}(E)$ given by

$$\theta \mapsto (\nabla_{\theta})^p - \nabla_{\theta^p}$$

There is a canonical connection $\nabla_{\text{can}} : F_{\text{rel}}^* E \rightarrow F_{\text{rel}}^* E \otimes \Omega_{X^{(-1)}}$, defined locally in a straightforward manner. A theorem of Cartier (see, for example, [11, Thm. 5.1]) shows the following equivalence of categories:

$$\left(\begin{array}{c} \text{coherent sheaves} \\ \text{over } X \end{array} \right) \xrightarrow{E \mapsto (F_{\text{rel}}^* E, \nabla_{\text{can}})} \left(\begin{array}{c} \text{coherent sheaves on } X^{(-1)} \text{ with integrable} \\ \text{connections of vanishing } p\text{-curvature} \end{array} \right)$$

An immediate consequence of Cartier's theorem is the following

Lemma 1.5. *Let X be a smooth projective variety, and E be a torsion-free coherent sheaf over X . Then for every coherent subsheaf $F \subset F_{\text{rel}}^* E$, we have*

- (i) *F is the pullback of some subsheaf of E along F_{rel} if and only if $\nabla_{\text{can}}(F) \subset F \otimes \Omega_{X^{(-1)}}$.*
- (ii) *If F is not the pullback of any subsheaf of E along F_{rel} , then the induced map $\nabla_{\text{can}} : F \rightarrow (F_{\text{rel}}^* E/F) \otimes \Omega_{X^{(-1)}}$ is a nonzero $\mathcal{O}_{X^{(-1)}}$ -linear morphism.*

Proof. For (i), the ‘‘only if’’ direction is clear. For the ‘‘if’’ direction, observe that ∇_{can} , when restricted to F , gives an integrable connection of vanishing p -curvature. By Cartier's theorem, the inclusion $F \rightarrow F_{\text{rel}}^* E$ is the pullback of some morphism $E' \rightarrow E$. On the other hand, F_{rel} is flat when X is smooth, so this morphism $E' \rightarrow E$ has to be an inclusion. (ii) follows directly from (i), except that the $\mathcal{O}_{X^{(-1)}}$ -linearity has to be checked locally. □

2. MAXIMALLY FROBENIUS-DESTABILIZED VECTOR BUNDLES

In this entire section, X will denote a smooth projective curve over an algebraically closed field k of characteristic $p \neq 0$, with genus $g \geq 1$. For notational convenience, we will denote $X^{(-1)}$ by Y , thus writing the relative Frobenius morphism $F_{\text{rel}} : Y \rightarrow X$.

We start by refining the following inequality discovered by I. N. Shepherd-Barron [19, Cor. 2^p] and generalized to higher dimensions by A. Langer [14, Cor. 6.2]:

$$L_{\max}(E) - L_{\min}(E) \leq \frac{(r-1)(2g-2)}{p} \tag{2.1}$$

for any rank- r semistable vector bundle E over X .

Proposition 2.1. *Let E be a semistable vector bundle over X of rank r . Then*

$$L_{\max}(E) - \mu(E) \leq \frac{(r-1)(g-1)}{p} \quad \text{and} \quad \mu(E) - L_{\min}(E) \leq \frac{(r-1)(g-1)}{p} \tag{2.2}$$

Furthermore, the following are equivalent, provided $g \geq 2$:

- (i) $L_{\max}(E) - \mu(E) = (r-1)(g-1)/p$;

- (ii) $\mu(E) - L_{\min}(E) = (r-1)(g-1)/p$;
 (iii) F_{rel}^*E satisfies the following condition:

$$\begin{aligned} & \text{Its Harder-Narasimhan filtration consists entirely of rank-one quotients} \\ & L_1, \dots, L_r \text{ and } \mu(L_i) = \mu(L_{i+1}) + 2g - 2 \text{ for all } i = 1, \dots, r. \end{aligned} \quad (2.3)$$

We will see later (Prop. 2.8) that (iii) cannot be satisfied for semistable vector bundles E of rank $r > p$. Hence the inequalities (2.2) are not sharp for $r > p$. In that case, we will supply a better inequality (Prop. 2.16).

Proof. We start with a technique used by Langer in the proof of [14, Cor. 6.2]. By Thm. 1.4, we may fix some sufficiently large $k \in \mathbb{N}$ such that

$$L_{\max}(E) = \frac{\mu_{\max}((F_{\text{rel}}^k)^*E)}{p^k} \quad \text{and} \quad L_{\min}(E) = \frac{\mu_{\min}((F_{\text{rel}}^k)^*E)}{p^k}$$

There is a connection

$$\eta : (F_{\text{rel}}^k)^*E \rightarrow (F_{\text{rel}}^k)^*E \otimes (\Omega_{X^{(-k)}} \oplus \dots \oplus (F_{\text{rel}}^{k-1})^*\Omega_Y)$$

given by $\eta = (\nabla_{\text{can}}, \dots, (F_{\text{rel}}^{k-1})^*\nabla_{\text{can}})$, where ∇_{can} is the canonical connection induced by the Frobenius morphism. Let $0 = E_0 \subset \dots \subset E_l = (F_{\text{rel}}^k)^*E$ be the Harder-Narasimhan filtration of $(F_{\text{rel}}^k)^*E$. Since E is semistable, none of the E_i 's are pullbacks of subbundles of E by F_{rel}^k . By Lem. 1.5, there is a nonzero $\mathcal{O}_{X^{(-k)}}$ -linear morphism for each E_i :

$$E_i \rightarrow ((F_{\text{rel}}^k)^*E/E_i) \otimes (\Omega_{X^{(-k)}} \oplus \dots \oplus (F_{\text{rel}}^{k-1})^*\Omega_Y)$$

i.e. given each i , there is a nonzero morphism $E_i \rightarrow ((F_{\text{rel}}^k)^*E/E_i) \otimes (F_{\text{rel}}^j)^*\Omega_{X^{(j-k)}}$ for some $j = 0, \dots, k-1$. Thus

$$\mu_i((F_{\text{rel}}^k)^*E) \leq \mu_{i+1}((F_{\text{rel}}^k)^*E) + p^j(2g-2) \leq \mu_{i+1}((F_{\text{rel}}^k)^*E) + p^{k-1}(2g-2)$$

where the second equality is attained if and only if E_i is the pullback of some subbundle of F_{rel}^*E by F_{rel}^{k-1} .

Claim 2.2 (cf. [9], Lem. 4.2.4). *Fix $\mu, \eta \in \mathbb{R}$ and $r \in \mathbb{N}$; assume $\eta \geq 0$. For any sequence of real numbers $\mu_1 > \mu_2 > \dots > \mu_l$ with $\mu_i \leq \mu_{i+1} + \eta$, and any sequence of natural numbers r_1, \dots, r_l of indeterminate length such that $r_1 + \dots + r_l = r$ and $r_1\mu_1 + \dots + r_l\mu_l = r\mu$, there hold*

$$\mu_1 \leq \mu + \frac{(r-1)\eta}{2} \quad \text{and} \quad \mu_l \geq \mu - \frac{(r-1)\eta}{2} \quad (2.4)$$

Furthermore, the following are equivalent if we assume $\eta > 0$:

- (i) $\mu_1 = \mu + (r-1)\eta/2$;
 (ii) $\mu_l = \mu - (r-1)\eta/2$;
 (iii) $l = r$, $r_i = 1$ for all i , and $\mu_i = \mu_{i+1} + \eta$.

Assuming the claim and applying it to $\mu = \mu((F_{\text{rel}}^k)^*E)$, $\eta = p^{k-1}(2g-2)$, $\mu_i = \mu_i((F_{\text{rel}}^k)^*E)$, and $r_i = \text{rank}(E_i)$, the inequalities (2.2) readily follow. Furthermore, in the $g \geq 2$ case, both equalities are attained when each E_i is pullback of a subbundle of F_{rel}^*E by F_{rel}^{k-1} , and the Harder-Narasimhan filtration of F_{rel}^*E must have rank-one quotients L_1, \dots, L_r , with $\mu(L_i) = \mu(L_{i+1}) + 2(g-1)$.

We now prove the elementary claim. First, if $\eta = 0$, then $l = 1$ and $\mu = \mu_1 = \mu_l$; the inequalities (2.4) are trivially satisfied. We now assume $\eta > 0$. Indeed, using $\mu_i \geq \mu_1 - (i-1)\eta$, we find

$$\mu = \frac{1}{r}(r_1\mu_1 + \dots + r_l\mu_l) \geq \mu_1 - \frac{\eta}{r}(r_2 + 2r_3 + \dots + (l-1)r_l)$$

where equality is attained if and only if $\mu_i = \mu_{i+1} + \eta$ for each i . If $l = 1$, we obtain strict inequalities in (2.4) thank to the $\eta > 0$ assumption. Now, for any $l = 2, \dots, r$, the maximum of $r_2 + 2r_3 + \dots + (l-1)r_l$, subject to the condition $r_1 + r_2 + \dots + r_l = r$, is obtained when $r_1 = \dots = r_{l-1} = 1$ and $r_l = r - l + 1$. Altogether,

$$r_2 + 2r_3 + \dots + (l-1)r_l \leq -\frac{1}{2}l^2 + \left(r + \frac{1}{2}\right)l - r \leq \frac{1}{2}r(r-1)$$

where the last equality is attained when $l = r$. Summarizing these inequalities, we get

$$\mu_1 \leq \mu + \frac{\eta}{r} \cdot \frac{1}{2} r(r-1) = \mu + \frac{(r-1)\eta}{2}$$

and the equivalence of (i) and (iii). The other inequality in (2.4) and the equivalence of (ii) and (iii) can be deduced in a similar manner. \square

Remark 2.3. In the last condition (iii) in Prop. 2.1, the numerical equation $\mu(L_i) = \mu(L_{i+1}) + 2g - 2$ is equivalent to $L_i \cong L_{i+1} \otimes \Omega_Y$ (provided that E is semistable). This can be seen as a consequence of the following easy

Lemma 2.4. *Suppose $g \geq 2$, and $F_{\text{rel}}^* E$ satisfies (2.3), and none of the subbundles E_i ($1 \leq i \leq r-1$) in the Harder-Narasimhan filtration of $F_{\text{rel}}^* E$ occurs as pullback of a subbundle of E along F_{rel} . Then ∇_{can} maps E_i to $E_{i+1} \otimes \Omega_Y$, and induces isomorphism $L_i \xrightarrow{\sim} L_{i+1} \otimes \Omega_Y$ for all rank-one quotients $L_i = E_i/E_{i-1}$.*

Proof. By hypothesis and Lem. 1.5, we have a nonzero morphism $E_i \rightarrow (F_{\text{rel}}^* E/E_i) \otimes \Omega_Y$. Let j be the largest integer such that the image of E_i is not contained in $(E_j/E_i) \otimes \Omega_Y$; in particular, $i \leq j \leq r$. Then we have an induced nonzero morphism

$$E_i \rightarrow (E_{j+1}/E_j) \otimes \Omega_Y$$

and it follows that $\mu(L_i) \leq \mu(L_{j+1}) + 2g - 2$. Given $\mu(L_i) = \mu(L_{i+1}) + 2g - 2$ for all i , we must have $i = j$. Consequently, ∇_{can} maps E_i to $E_{i+1} \otimes \Omega_Y$, and induces a nonzero morphism $E_i \rightarrow L_{i+1} \otimes \Omega_Y$. Since ∇_{can} also maps E_{i-1} to $E_i \otimes \Omega_Y$, the above morphism $E_i \rightarrow L_{i+1} \otimes \Omega_Y$ vanishes on E_{i-1} . Thus we obtain a nonzero morphism $L_i \rightarrow L_{i+1} \otimes \Omega_Y$ of line bundles with the same degree, and as such, it must be an isomorphism. \square

Vector bundles E whose pullbacks $F_{\text{rel}}^* E$ satisfy (2.3) display an interesting trichotomy corresponding to the relative size of their rank r and the characteristic p . The following lemma is necessary for our discussion of the $r < p$ case.

Lemma 2.5. *Let E be a vector bundle over X , whose pullback $F_{\text{rel}}^* E$ has Harder-Narasimhan filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_l = F_{\text{rel}}^* E$. If the canonical connection ∇_{can} induces isomorphisms $E_i/E_{i-1} \xrightarrow{\sim} E_{i+1}/E_i \otimes \Omega_Y$ for all i , then $F_{\text{rel}}^* E' \not\subset E_{l-1}$ for every subbundle $E' \subset E$.*

Proof. Suppose the converse that $F_{\text{rel}}^* E' \subset E_i$ for some $i \leq l-1$, and $F_{\text{rel}}^* E' \not\subset E_{i-1}$. Then we have a commutative square

$$\begin{array}{ccc} F_{\text{rel}}^* E' & \xrightarrow{\text{nonzero}} & E_i/E_{i-1} \otimes \Omega_Y \\ \nabla_{\text{can}} \downarrow & & \parallel \nabla_{\text{can}} \\ F_{\text{rel}}^* E' \otimes \Omega_X & \xrightarrow{\text{zero}} & E_{i+1}/E_i \otimes \Omega_Y \end{array}$$

which gives a contradiction since the upper composition is nonzero, but the lower one is zero. \square

Proposition 2.6 ($r < p$ case). *Suppose $g \geq 2$ and $g-1$ is not divisible by p . Let E be a vector bundle over X of rank $r < p$ such that $F_{\text{rel}}^* E$ satisfies (2.3), then E is stable.*

Proof. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_r = F_{\text{rel}}^* E$ be the Harder-Narasimhan filtration of $F_{\text{rel}}^* E$, with rank-one quotients L_1, \dots, L_r . We first claim that under the hypothesis, none of the subbundles E_i , for $i = 1, \dots, r-1$, is a pullback along F_{rel} . Indeed, since $\deg(L_i) = \deg(L_{i+1}) + 2g - 2$, we find

$$\deg(E_i) = \sum_{j=1}^i \deg(L_j) = i \deg(L_1) - i(i-1)(g-1) \quad (2.5)$$

In particular, for $i = r$, we obtain $\deg(L_1) = \mu(F_{\text{rel}}^* E) + (r-1)(g-1)$. Substitute this equation back into (2.5) and simplify, we find

$$\deg(E_i) = i\mu(F_{\text{rel}}^* E) + i(r-i)(g-1) = \frac{ip \deg(E)}{r} + i(r-i)(g-1)$$

Now, suppose to the contrary of the claim that $E_i = F^*E'$ for some subbundle E' of E . Then

$$\deg(E') = \frac{i \deg(E)}{r} + \frac{i(r-i)(g-1)}{p} \quad (2.6)$$

is an integer. However, by the hypothesis, p does not divide $i(r-i)(g-1)$ for all $i = 1, \dots, r-1$, and r is coprime to p . Hence (2.6) cannot possibly be an integer, and the claim is proved.

Now, let E' be any *stable* proper subbundle of E . It suffices to show that $\mu(E') < \mu(E)$. Lem. 2.4 shows that ∇_{can} induces isomorphisms $L_i \xrightarrow{\sim} L_{i+1} \otimes \Omega_Y$ for all i . Therefore, by Lem. 2.5, $F_{\text{rel}}^*E' \not\subset E_{r-1}$, and we obtain a nonzero morphism $F_{\text{rel}}^*E' \rightarrow L_r$. Hence

$$L_{\min}(E') \leq \frac{\mu_{\min}(F_{\text{rel}}^*E')}{p} \leq \frac{\deg(L_r)}{p} \quad (2.7)$$

A computation similar to (2.5) shows that

$$p \deg(E) = \deg(F_{\text{rel}}^*E) = r \deg(L_r) + r(r-1)(g-1) \quad (2.8)$$

Let r' be the rank of E' . Then the second inequality in Prop. 2.1, applied to the stable bundle E' , shows that

$$\mu(E') - \frac{(r'-1)(g-1)}{p} \leq L_{\min}(E') \leq \frac{\deg(L_r)}{p} \leq \frac{\deg(E)}{r} - \frac{(r-1)(g-1)}{p}$$

where in the second and third inequalities, we used (2.7) and (2.8), respectively. Simplify this inequality, and we obtain

$$\mu(E') \leq \mu(E) - \frac{(r-r')(g-1)}{p} < \mu(E)$$

by the hypotheses that $g \geq 2$ and $r > r'$. \square

The condition that $g-1$ is not divisible by p is necessary in Prop. 2.6. Indeed, one can construct counterexamples to Prop. 2.6 when this condition is dropped.

Remark 2.7. Suppose $g \geq 2$. Let E be a vector bundle over X such that F_{rel}^*E satisfies (2.3). Let $0 = E_0 \subset E_1 \subset \dots \subset E_r = F_{\text{rel}}^*E$ be its Harder-Narasimhan filtration. The proof of Prop. 2.6 shows that if none of the E_i 's ($1 \leq i \leq r-1$) comes from pulling back along F_{rel} , then E is stable.

Suppose now that $E_{i_j} = F_{\text{rel}}^*G_j$ are those subbundles coming from pullbacks. Applying the argument to $E_{i_{j+1}}/E_{i_j} = F^*(G_{j+1}/G_j)$ shows that G_{j+1}/G_j is stable. Furthermore, the inequality

$$\mu(G_j/G_{j-1}) = \frac{\mu(E_{i_j}/E_{i_{j-1}})}{p} > \frac{\mu(E_{i_{j+1}}/E_{i_j})}{p} = \mu(G_{j+1}/G_j)$$

proves that $0 = G_0 \subset G_1 \subset \dots \subset G_j \subset \dots \subset E$ is in fact the Harder-Narasimhan filtration of E , and has stable quotients.

Proposition 2.8 ($r > p$ case). *Suppose $g \geq 2$. Let E be a vector bundle over X of rank $r > p$ such that F_{rel}^*E satisfies (2.3), then E is not semistable.*

Proof. Again let $0 = E_0 \subset E_1 \subset \dots \subset E_r = F_{\text{rel}}^*E$ be the Harder-Narasimhan filtration of F_{rel}^*E , with rank-one quotients L_1, \dots, L_r . Since $\mu(E_i) > \mu(F_{\text{rel}}^*E)$ for each $i = 1, \dots, r-1$, we see that if at least one such E_i is a pullback along F_{rel} , then E is not semistable.

Hence, we may suppose to the contrary that none of the E_i 's ($1 \leq i \leq r-1$) is a pullback along F_{rel} . We pick a sufficiently small open affine $\text{Spec}(A) \subset X$, whose preimage under F_{rel} is $\text{Spec}(B)$, where $B \cong A[x]/(x^p - a)$ for some $a \in A$, and the inclusion $A \rightarrow A[x]/(x^p - a)$ corresponds to F_{rel} . We use M_i to denote the free B -module corresponding to the vector bundle E_i , and M for E . By shrinking $\text{Spec}(A)$ if necessary, we may assume $\Omega_{B/k} \cong Bdx$ is free as well. Let $e_1, \dots, e_r \in M \otimes_A B$ be such that for each i , the elements e_1, \dots, e_i form a B -basis for M_i . Write ∇ for ∇_{can} , and by Lem. 2.4, we may suppose

$$\nabla(e_i) = \sum_{j=1}^{i+1} \omega_{ij} e_j$$

where $\omega_{ij} = b_{ij}dx \in \Omega_{B/k}^1$ form a matrix of 1-forms, and $\omega_{i(i+1)} \neq 0$ for all i . On the other hand, let $\theta = \partial/\partial x$, so that

$$\nabla_{(\theta^p)}(e_i) = \sum_{j=1}^{i+1} \omega_{ij}(\theta^p)e_j = \sum_{j=1}^{i+1} b_{ij}\theta^p(x)e_j = 0$$

We now evaluate the p -curvature of ∇ on θ and e_1 :

$$(\nabla_\theta)^p(e_1) - \nabla_{(\theta^p)}(e_1) = (\nabla_\theta)^p(e_1)$$

Note that the coefficient of $(\nabla_\theta)^p(e_1)$ in front of e_{p+1} is given by the expression $\omega_{12}(\theta)\omega_{23}(\theta)\cdots\omega_{p(p+1)}(\theta)$. Since $\omega_{i(i+1)} \neq 0$ for all i and the p -curvature vanishes, we must have $e_{p+1} = 0$. This is absurd given $r > p$. \square

The critical value $r = p$ is a more intricate case. However, using results of K. Joshi, S. Ramanan, E. Xia, and J.-K. Yu [9, §5.3], we can classify all semistable vector bundles of rank $r = p$ that are maximally destabilized by the Frobenius morphism.

Lemma 2.9. *Let L be a line bundle over Y , then the pushforward $(F_{\text{rel}})_*L$ is stable if $g \geq 2$, and semistable if $g = 1$.*

We point out that this lemma is neither new, nor the most general to date. It is proved by Lange and Pauly [13, Prop. 1.2] using the classical inequality (2.1) and relative duality, and later generalized by Sun [20, Thm. 2.2] to the case where L is replaced by any stable vector bundle. We provide a simple proof using the improved inequality (2.2).

Proof. Let E be a rank- r stable subbundle of $(F_{\text{rel}})_*L$. Then the inclusion $E \subset (F_{\text{rel}})_*L$ gives, by adjunction, a nonzero morphism $F_{\text{rel}}^*E \rightarrow L$. Hence $\mu_{\min}(F_{\text{rel}}^*E) \leq \deg(L)$. The Riemann-Roch formula implies that $\deg(L) = \deg((F_{\text{rel}})_*L) - (p-1)(g-1)$. Apply Prop. 2.1 to the first inequality in the following chain:

$$\begin{aligned} \mu(E) - \frac{(r-1)(g-1)}{p} &\leq L_{\min}(E) \leq \frac{\mu_{\min}(F_{\text{rel}}^*E)}{p} \leq \frac{\deg(L)}{p} \\ &= \mu((F_{\text{rel}})_*L) - \frac{(p-1)(g-1)}{p} \end{aligned}$$

Therefore $\mu(E) \leq \mu((F_{\text{rel}})_*L) - (p-r)(g-1)/p$. If $g = 1$, then $\mu(E) \leq \mu((F_{\text{rel}})_*L)$, and $(F_{\text{rel}})_*L$ is semistable. If $g \geq 2$, then $\mu(E) < \mu((F_{\text{rel}})_*L)$ for the proper subbundle E , and $(F_{\text{rel}})_*L$ is stable. \square

Proposition 2.10 ($r = p$ case). *Suppose $g \geq 2$. Let E be a vector bundle over X of rank $r = p$. Then the following are equivalent:*

- (i) E is stable, and F_{rel}^*E satisfies (2.3);
- (ii) E is semistable, and F_{rel}^*E satisfies (2.3);
- (iii) $E = (F_{\text{rel}})_*L$ for some line bundle L .

Proof. (i) trivially implies (ii). To show that (ii) implies (iii), let L denote the last rank-one quotient in the Harder-Narasimhan filtration of F_{rel}^*E . Then we have a nonzero morphism $F_{\text{rel}}^*E \rightarrow L$, which by adjunction, gives rise to a nonzero morphism $E \rightarrow (F_{\text{rel}})_*L$. Under the assumption $r = p$, the computation (2.8) shows that

$$\deg(E) = \deg(L) + (p-1)(g-1)$$

On the other hand, we obtain from the Riemann-Roch theorem that $\deg(L) = \deg((F_{\text{rel}})_*L) - (p-1)(g-1)$. Hence

$$\deg((F_{\text{rel}})_*L) = \deg(L) + (p-1)(g-1) = \deg(E)$$

and consequently $\mu(E) = \mu((F_{\text{rel}})_*L)$. Note that $(F_{\text{rel}})_*L$ is stable by Lem. 2.9. Because E is semistable and $\mu(E) = \mu((F_{\text{rel}})_*L)$, the morphism $E \rightarrow (F_{\text{rel}})_*L$ is necessarily surjective. On the other hand, $\text{rank}(E) = r = p = \text{rank}((F_{\text{rel}})_*L)$, so this morphism is an isomorphism.

It remains to show that (iii) implies (i). Indeed, E is stable by Lem. 2.9. On the other hand, K. Joshi *et al.* (cf. [9, §5], although a detailed proof of this result is given in [20, Lem. 2.1]) proved that the subbundles $0 = E_0 \subset E_1 \subset \cdots \subset E_l = F_{\text{rel}}^*E$ defined by

- (i) $E_l = F^*E$ and $E_{l-1} = \ker(E_l \rightarrow L)$, where the morphism $E_l \rightarrow L$ is the counit of the adjunction, as $E_l = F_{\text{rel}}^*(F_{\text{rel}})_*L$.
- (ii) $E_i = \ker(E_{i+1} \xrightarrow{\nabla_{\text{can}}} F_{\text{rel}}^*E \otimes \Omega_Y \rightarrow (F_{\text{rel}}^*E/E_{i+1}) \otimes \Omega_Y)$ for $0 \leq i \leq p-2$.

is a pre-oper, i.e. $\nabla_{\text{can}}(E_i) \subset E_{i+1} \otimes \Omega_Y$ for $0 \leq i \leq l-1$, and the induced \mathcal{O}_Y -linear morphism $E_i/E_{i-1} \rightarrow (E_{i+1}/E_i) \otimes \Omega_Y$ is an isomorphism for $1 \leq i \leq l-1$. In particular, the quotient E_l/E_{l-1} is line bundle L , and inductively, each quotient $E_i/E_{i-1} \cong (E_{i+1}/E_i) \otimes \Omega_Y$ is a line bundle L_i , for $i = 1, \dots, l-1$. So $l = r$, the subbundles $0 = E_0 \subset E_1 \subset \dots \subset E_r = F_{\text{rel}}^*E$ is the Harder-Narasimhan filtration of F^*E with line bundle quotients $L_1, \dots, L_{r-1}, L_r = L$, and $\deg(L_i) = \deg(L_{i+1}) + 2g - 2$. \square

Having concluded the discussion on vector bundles whose pullback satisfies (2.3), we make the following definition. As before, X denotes a smooth projective curve over an algebraically closed field k of characteristic $p \neq 0$, with genus $g \geq 1$.

Definition 2.11. A vector bundle E over X is *maximally Frobenius-destabilized* if E is stable, the Harder-Narasimhan filtration of F_{rel}^*E consists entirely of rank-one quotients L_1, \dots, L_r , and $\mu(L_i) = \mu(L_{i+1}) + 2g - 2$ for all $i = 1, \dots, r$ (cf. condition (2.3)).

When $g = 1$, there is no maximally Frobenius-destabilized vector bundles for trivial reasons. Prop. 2.8 shows that maximally Frobenius-destabilized bundles necessarily have rank $r \leq p$. Prop. 2.10 guarantees the existence of rank- p maximally Frobenius-destabilized bundles over an arbitrary curve of genus $g \geq 2$. In general, we cannot expect such bundles with rank $r < p$ to exist over an *arbitrary* curve. However, for most g , there exists a curve X of genus g and a maximally Frobenius-destabilized bundle E over X of arbitrary rank r with $2 \leq r < p$. We construct such bundles in the following way:

Lemma 2.12 (S. Mochizuki, B. Osserman). *Suppose $p > 2$. Then there exists a rank-2 maximally Frobenius-destabilized vector bundle E over a general curve X of genus 2.*

Proof. Osserman's main theorem (cf. [18, Thm. 1.2], although the part we are using here is originally due to Mochizuki [17]) states that, over a general curve of genus 2, the number of rank-2 semistable vector bundles with trivial determinant and whose pullback under F is not semistable equals $2(p^3 - p)/3 > 0$.

We argue that every such vector bundle E is maximally Frobenius-destabilized (this is [8, Prop. 3.3]). Indeed, since F_{rel}^*E is not semistable and of trivial determinant, there exists a line bundle L of positive degree admitting an exact sequence

$$0 \rightarrow L \rightarrow F_{\text{rel}}^*E \rightarrow L^{-1} \rightarrow 0$$

Furthermore, Lem. 1.5 gives a nonzero morphism $L \rightarrow L^{-1} \otimes \Omega_Y$, so $2 \deg(L) \leq \deg(\Omega_X) = 2$. Thus we must have $\deg(L) = 1$, and $\deg(L) = \deg(L^{-1}) + 2$ as desired. \square

Lemma 2.13. *If a rank-2 vector bundle E over a curve of genus $g \geq 2$ satisfies condition (2.3), then so do its symmetric and exterior powers $S^t(E)$ and $\Lambda^t(E)$ for all $t \in \mathbb{N}$.*

We prove the lemma for $S^t(E)$. The proof for the exterior power case is completely analogous.

Proof. Indeed, a rank-2 vector bundle E satisfies (2.3) if and only if there exist line bundles L_1, L_2 , and an exact sequence

$$0 \rightarrow L_1 \rightarrow E \rightarrow L_2 \rightarrow 0$$

such that $\deg(L_1) = \deg(L_2) + 2g - 2$. Now $S^t(E)$ has a filtration $0 = E_0 \subset E_1 \subset \dots \subset E_{t+1} = S^t(E)$ with rank-one quotients

$$E_{i+1}/E_i \cong S^{t-i}(L_1) \otimes S^i(L_2) \cong L_1^{t-i} \otimes L_2^i$$

so $\deg(E_{i+1}/E_i) = t \deg(L_1) - i(2g - 2)$. Thus this filtration is the Harder-Narasimhan filtration of $S^t(E)$, and its successive quotients satisfy $\deg(E_i/E_{i-1}) = \deg(E_{i+1}/E_i) + 2g - 2$. \square

Proposition 2.14. *Suppose natural number p, g , and r are all at least 2, and either*

- (i) $r = p$, or
- (ii) $r < p$ and p does not divide $g - 1$,

Then there exists a rank- r maximally Frobenius-destabilized vector bundle E over some genus- g curve X .

Proof. Existence in case (i) follows directly from (2.10), and holds over an *arbitrary* curve of genus g .

For case (ii), we first consider a general curve Z of genus 2. By Lem. 2.12, there exists a rank-2 vector bundle G over Z which is maximally Frobenius-destabilized.

Fix $n = g - 1$. Then n is not divisible by p , so finite étale coverings of Z of degree n are parametrized by the n -torsion elements of the Picard group $\text{Pic}(Y)[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$. In particular, there exists such a covering $f : X \rightarrow Z$ of degree n . Riemann-Hurwitz formula shows that the genus g_X of X satisfies $g_X = n + 1 = g$. Furthermore, $f^*\Omega_Z = \Omega_X$, so the exact sequence (cf. Rem. 2.3)

$$0 \rightarrow L \otimes \Omega_{Z^{(-1)}} \rightarrow F_{\text{rel}}^* G \rightarrow L \rightarrow 0$$

pulls back to an exact sequence under the flat morphism f :

$$0 \rightarrow f^* L \otimes \Omega_Y \rightarrow F_{\text{rel}}^* f^* G \rightarrow f^* L \rightarrow 0 \quad (2.9)$$

In particular, $f^* G$ is a rank-2 vector bundle over X whose pullback along F_{rel} satisfies condition (2.3). Taking $E = S^{r-1}(f^* G)$ and applying Lem. 2.13 and Prop. 2.6 completes the construction of the rank- r vector bundle E . \square

We also note the following

Corollary 2.15. *Suppose $p > 2$, $g \geq 2$ and $g - 1$ is not divisible by p . Let E be a rank-2 vector bundle over a genus- g curve such that $F_{\text{rel}}^* E$ satisfies (2.3). Then $S^{p-1}(E) = (F_{\text{rel}})_* L$ for some line bundle L .*

Proof. $F_{\text{rel}}^* S^{p-1}(E)$ also satisfies (2.3), so we know from Prop. 2.10 that it suffices to prove the stability of $S^{p-1}(E)$. Let $0 = E_0 \subset E_1 \subset \dots \subset E_p = F_{\text{rel}}^* S^{p-1}(E)$ be the Harder-Narasimhan filtration of $F_{\text{rel}}^* S^{p-1}(E)$. We only need to show that none of the E_i 's ($1 \leq i \leq p - 1$) comes from pulling back along F_{rel} (cf. Rem. 2.7). If some $E_i = F_{\text{rel}}^* E'$ for some subbundle E' of $S^{p-1}(E)$, then the computation (2.6) shows that

$$\deg(E') = \frac{i \deg(S^{p-1}(E))}{p} + \frac{i(p-i)(g-1)}{p} \quad (2.10)$$

On the other hand, it follows from a Chern class computation that

$$\deg(S^{p-1}(E)) = \frac{p(p-1)}{2} \deg(E)$$

Therefore, under the hypothesis that $g - 1$ is not divisible by p , the expression in (2.10) cannot be an integer for $1 \leq i \leq p - 1$. \square

and a partial improvement of the bound in Prop. 2.1 using a theorem of X. Sun:

Proposition 2.16. *Suppose $g \geq 2$. Let E be a semistable vector bundle over a genus- g curve X . Then*

$$L_{\max}(E) - \mu(E) < g - 1 \quad \text{and} \quad \mu(E) - L_{\min}(E) < g - 1$$

Let r denote the rank of E . When $r > p$, these inequalities are sharper than Prop. 2.1.

Proof. We first prove the second inequality. Using Thm. 1.4, we may fix some sufficiently large k such that $L_{\min}(E) = \mu_{\min}((F_{\text{rel}}^k)^* E)/p^k$. Consider the last quotient Q_l in the Harder-Narasimhan filtration of $(F_{\text{rel}}^k)^* E$, and let Q be the last quotient in the Jordan-Hölder filtration of Q_l . Then

$$\mu(Q) = \mu(Q_l) = \mu_{\min}((F_{\text{rel}}^k)^* E)$$

and there is a surjection $(F_{\text{rel}}^k)^* E \rightarrow Q$. By adjunction, we obtain a nonzero morphism $E \rightarrow (F_{\text{rel}}^k)_* Q$. Sun's theorem [20, Thm. 2.2] shows $(F_{\text{rel}}^k)_* Q$ is stable. Hence $\mu(E) \leq \mu((F_{\text{rel}}^k)_* Q)$. Let q denote the rank of Q . Expressing $\mu((F_{\text{rel}}^k)_* Q)$ in terms of $\mu(Q)$ by the Riemann-Roch formula, we compute

$$\mu(E) \leq \mu((F_{\text{rel}}^k)_* Q) = \frac{\mu(Q)}{p^k} + \left(1 - \frac{1}{p^k}\right)(g-1) = L_{\min}(E) + \left(1 - \frac{1}{p^k}\right)(g-1) \quad (2.11)$$

and the desired inequality follows.

For the first inequality, consider the maximal destabilizing subsheaf E_1 of $(F_{\text{rel}}^k)^* E$, and let S be the first nontrivial subbundle in the Jordan-Hölder filtration of E_1 . Then there is a surjection $(F_{\text{rel}}^k)^* E^\vee \rightarrow S^\vee$, where S^\vee is stable with $\mu(S^\vee) = -\mu(S) = -\mu_{\max}((F_{\text{rel}}^k)^* E)$. The above computation (2.11) then shows that $\mu(E^\vee) = -\mu(E) \leq -L_{\max}(E) + (g-1)$, which gives the first inequality. \square

3. THE CHARACTERISTIC-3 CASE

In this section, X will denote a smooth projective curve over an algebraically closed field k of characteristic $p = 3$, with genus $g \geq 2$. We will again denote $X^{(-1)}$ by Y , thus writing the relative Frobenius morphism $F_{\text{rel}} : Y \rightarrow X$.

We try to turn Cor. 2.15 around by taking a line bundle L , and asking whether $(F_{\text{rel}})_*L$ can be expressed as the symmetric power of a rank-2 vector bundle. The problem can be reduced to vector bundles of trivial determinant.

Lemma 3.1. *Suppose E is a rank-2 vector bundle, and F_{rel}^*E satisfies condition (2.3), i.e. there exists a line bundle L over Y and an exact sequence*

$$0 \rightarrow L \otimes \Omega_Y \rightarrow F_{\text{rel}}^*E \rightarrow L \rightarrow 0 \quad (3.1)$$

Then after twisting by a line bundle, E has trivial determinant.

The lemma actually holds in all positive characteristic $p \neq 2$.

Proof. Pulling back further by the isomorphism $X \xrightarrow{\sim} Y$ of abstract schemes, we see that (3.1) gives rise to an exact sequence

$$0 \rightarrow L' \otimes \Omega_X \rightarrow F_{\text{abs}}^*E \rightarrow L' \rightarrow 0$$

over X , for some line bundle L' . Let Θ be a theta characteristic over X (which exists for $p \neq 2$). Then $\Theta^2 \cong \Omega_X$. On the other hand, the above exact sequence shows that

$$3c_1(E) = c_1(F_{\text{abs}}^*E) = c_1(L' \otimes \Omega_X) + c_1(L') = 2c_1(L' \otimes \Theta)$$

Let $M = \det(E) \otimes (L' \otimes \Theta)^{-1}$. Then

$$c_1(E \otimes M) = c_1(E) + 2c_1(M) = 3c_1(E) - 2c_1(L' \otimes \Theta) = 0$$

and therefore $\det(E \otimes M) \cong \mathcal{O}_X$. □

Proposition 3.2. *Suppose E is a rank-2 vector bundle over X . Then the following are equivalent:*

- (i) E is maximally Frobenius-destabilized;
- (ii) $S^2(E)$ is maximally Frobenius-destabilized;
- (iii) $S^2(E) = (F_{\text{rel}})_*L$ for some line bundle L ;

Furthermore, if E is assumed to have trivial determinant, then we can replace L by T_Y in (iii).

Proof. The equivalence of (ii) and (iii) is already proved in Prop. 2.10.

For (i) \implies (ii), note that by Lem. 2.13 and the equivalence (i) \iff (ii) in Prop. 2.10, we only need to show that $S^2(E)$ is semistable. By twisting E with a line bundle, we may assume $\deg(E) = 0$. On the other hand, there is an exact sequence

$$0 \rightarrow N \rightarrow E^{\otimes 2} \rightarrow S^2(E) \rightarrow 0$$

where N is a line bundle. Since E has degree zero by assumption, all three vector bundles in this sequence have vanishing slope. The theorem of S. Ilangovan, V. B. Mehta, and A. J. Parameswaran [7] shows that the tensor product of semistable vector bundles whose ranks add up to less than $p + 2$ is again semistable. Therefore, $E^{\otimes 2}$ is semistable, so all quotients of $E^{\otimes 2}$ are of nonnegative degree. Since all quotients of $S^2(E)$ are quotients of $E^{\otimes 2}$, the same holds for $S^2(E)$.

For (ii) \implies (i), to prove the stability of E , let M be any quotient line bundle of E . Therefore, we have a surjection $S^2(E) \rightarrow M^2$. But $S^2(E)$ is stable by assumption, so $\mu(S^2E) < 2 \deg(M)$. On the other hand, a Chern class computation shows that $\deg(S^2(E)) = 3 \deg(E)$. Thus $2\mu(E) = \mu(S^2(E)) < 2 \deg(M)$, and E is stable. Again, because $S^2(F_{\text{rel}}^*E) \cong F_{\text{rel}}^*S^2(E)$ is not semistable, the Ilangovan-Mehta-Parameswaran theorem shows that F_{rel}^*E is not semistable. Thus F_{rel}^*E fits into an exact sequence

$$0 \rightarrow N \rightarrow F_{\text{rel}}^*E \rightarrow R \rightarrow 0 \quad (3.2)$$

where $\deg(N) > \mu(F_{\text{rel}}^*E) > \deg(R)$. Therefore, $S^2(F_{\text{rel}}^*E)$ admits a filtration with successive quotients $N^{\otimes 2}$, $N \otimes R$, and R^2 . The fact that they are stable, and $\deg(N^2) > \deg(N \otimes R) > \deg(R^2)$ shows that this is the Harder-Narasimhan filtration of $S^2(F_{\text{rel}}^*E)$. On the other hand, $S^2(F_{\text{rel}}^*E)$ satisfies condition

(2.3), so $N \cong R \otimes \Omega_Y$. Looking back at the exact sequence (3.2), we see that E is maximally Frobenius-destabilized.

For the second claim, note that the assumption $\det(E) \cong \mathcal{O}_X$ implies $N \cong R^{-1}$ in (3.2). Hence $R^2 \cong T_Y$. We also observe (cf. proof of “(iii) \implies (i)” in Prop. 2.10) that the Harder-Narasimhan filtration of $F_{\text{rel}}^*(F_{\text{rel}})_*L$ has successive quotients $L \otimes \Omega_Y^2$, $L \otimes \Omega_Y$, and L . It follows from the uniqueness of Harder-Narasimhan filtration that if $(F_{\text{rel}})_*L \cong S^2(E)$, then $L \cong R^2 \cong T_Y$. \square

Restricting ourselves to the trivial-determinant case, we seek to express $(F_{\text{rel}})_*T_Y$ as a second symmetric power. We do so using the following rather general machinery:

Lemma 3.3. *There is a bijection of sets:*

$$\Phi : \left(\begin{array}{c} \text{isomorphism classes of rank-2} \\ \text{vector bundles } E \text{ over } X \end{array} \right) / \text{twisting by line bundles} \xrightarrow{\sim} \left(\begin{array}{c} \text{pairs } (F, q) \text{ where } F \text{ is a rank-3 vector bundle with trivial} \\ \text{determinant and } q \in H^0(X, S^2(F)) \text{ is a nondegenerate quadric} \end{array} \right) / \text{equivalences}$$

where the underlying rank-3 bundle of $\Phi(E)$ is $S^2(E) \otimes \det(E)^{-1}$.

We do not go into details in defining equivalences of pairs (F, q) , although they will be clear from the proof. For our application, we will be looking at a specific vector bundle F , for which the choice of q is unique up to scaling.

Proof. A pair (F, q) of rank-3 vector bundle F with trivial determinant and $q \in H^0(X, S^2(F))$ a nondegenerate quadric corresponds to an SO_3 -principal bundle, via the standard representation of SO_3 . On the other hand, SO_3 equipped with the standard representation is equivalent to PGL_2 equipped with the adjoint representation \mathfrak{g} . Consider the exact sequence of groups

$$0 \rightarrow \mathbb{G}_m \rightarrow \text{GL}_2 \xrightarrow{p} \text{PGL}_2 \rightarrow 0$$

Let V be the standard representation of GL_2 . Then the precomposition of \mathfrak{g} by p is isomorphic to the representation $S^2(V) \otimes \det(V)^{-1}$ of GL_2 . Taking cohomology, we obtain

$$H_{\text{ét}}^1(X, \mathbb{G}_m) \rightarrow H_{\text{ét}}^1(X, \text{GL}_2) \rightarrow H_{\text{ét}}^1(X, \text{PGL}_2) \rightarrow H_{\text{ét}}^2(X, \mathbb{G}_m)$$

where $H_{\text{ét}}^2(X, \mathbb{G}_m)$ vanishes by Tsen’s theorem. The bijection Φ then follows from the isomorphism $H_{\text{ét}}^1(X, \mathbb{G}_m) \cong \text{Pic}(X)$. \square

For later computations in Prop. 3.8, we first work out an explicit description of relative duality for the morphism F_{rel} . This material is standard. Around any point $x \in X$, pick an affine neighborhood $\text{Spec}(A) \subset X$, whose preimage under F_{rel} is given by $\text{Spec}(B) \subset Y$, with $B \cong A[x]/(x^3 - a)$ for some $a \in A$. The morphism F_{rel} corresponds to the embedding $A \rightarrow B \cong A[x]/(x^3 - a)$. By further shrinking, we may assume $\Omega_{A/k} \cong A da$ and $\Omega_{B/k} \cong B dx$ are both free; we abbreviate $\Omega_{A/k}$ and $\Omega_{B/k}$ by Ω_A and Ω_B . There is a trace map (see, for example, [5, Tag 0ADY]):

$$\text{tr} : (F_{\text{rel}})_*\Omega_Y \rightarrow \Omega_X \tag{3.3}$$

defined locally by a map $\text{tr}_{x,d} : (\Omega_B)_A \rightarrow \Omega_A$ of A -modules:

$$\text{tr}_{x,d}(dx) = 0, \quad \text{tr}_{x,d}(xdx) = 0, \quad \text{and} \quad \text{tr}_{x,d}(x^2dx) = da$$

By the adjunction $((F_{\text{rel}})_*, F_{\text{rel}}^!)$, we have a nonzero morphism $\Omega_Y \rightarrow F_{\text{rel}}^!\Omega_X$. Locally, this morphism is defined by a map $\Omega_B \rightarrow \text{Hom}_A(B, \Omega_A)$, sending dx to the A -linear map:

$$1 \mapsto 0, \quad x \mapsto 0, \quad \text{and} \quad x^2 \mapsto da$$

We see from the local description that this morphism is injective and surjective. In summary,

Lemma 3.4. *There is an isomorphism $\Omega_Y \rightarrow F_{\text{rel}}^!\Omega_X$ which, by adjunction of $((F_{\text{rel}})_*, F_{\text{rel}}^!)$, gives the trace map (3.3).* \square

We also note that the isomorphism $F_{\text{rel}}^*\Omega_X \xrightarrow{\sim} \Omega_Y^3$, which is locally given by a map $\Omega_A \otimes_A B \rightarrow \Omega_B^3$, sends $da \otimes 1$ to $(dx)^3$. The following lemma proves itself.

Lemma 3.5. *For every line bundle L over Y , there is a chain of isomorphisms*

$$\begin{aligned} (F_{\text{rel}})_*L &\cong (F_{\text{rel}})_*\mathcal{H}\text{om}(L^{-1} \otimes \Omega_Y, \Omega_Y) \cong (F_{\text{rel}})_*\mathcal{H}\text{om}(L^{-1} \otimes \Omega_Y, F_{\text{rel}}^1\Omega_X) \\ &\cong \mathcal{H}\text{om}((F_{\text{rel}})_*(L^{-1} \otimes \Omega_Y), \Omega_X) \cong \mathcal{H}\text{om}((F_{\text{rel}})_*(L^{-1} \otimes T_Y^2 \otimes \Omega_Y^3), \Omega_X) \\ &\cong \mathcal{H}\text{om}((F_{\text{rel}})_*(L^{-1} \otimes T_Y^2) \otimes \Omega_X, \Omega_X) \cong ((F_{\text{rel}})_*(L^{-1} \otimes T_Y^2))^\vee \end{aligned}$$

where Lem. 3.4 is used in the second isomorphism, the adjunction $((F_{\text{rel}})_*, F_{\text{rel}}^1)$ in the third, and the projection formula in the fifth. \square

In particular, $(F_{\text{rel}})_*T_Y$ is self-dual and $(F_{\text{rel}})_*\mathcal{O}_Y$ is dual to $(F_{\text{rel}})_*(T_Y^2)$.

For notational simplicity, we will now use F to denote $(F_{\text{rel}})_*T_Y$. Observe that there is a canonical surjection

$$S^2(F) \rightarrow (F_{\text{rel}})_*(T_Y^2) \rightarrow 0 \quad (3.4)$$

Since F is self-dual (cf. §3), it has trivial determinant. Given any vector bundle E over X , the nondegenerate pairing on $E^{\otimes 2}$ and $(E^\vee)^{\otimes 2}$ induces, using $2 < 3$, a nondegenerate pairing on $S^2(E)$ and $S^2(E^\vee)$. Thus, the dual of (3.4) gives a canonical injection

$$0 \rightarrow (F_{\text{rel}})_*\mathcal{O}_Y \rightarrow S^2(F^\vee) \cong S^2(F) \quad (3.5)$$

In fact, the above two morphisms combine into an exact sequence:

Proposition 3.6. *There is a canonically defined exact sequence*

$$0 \rightarrow (F_{\text{rel}})_*\mathcal{O}_Y \rightarrow S^2(F) \rightarrow (F_{\text{rel}})_*(T_Y^2) \rightarrow 0 \quad (3.6)$$

Proof. Indeed, we have a composition $(F_{\text{rel}})_*\mathcal{O}_Y \rightarrow S^2(F) \rightarrow (F_{\text{rel}})_*(T_Y^2)$ from (3.4) and (3.5). This is a zero morphism, since

$$\deg((F_{\text{rel}})_*\mathcal{O}_Y) = 2g - 2 > -(2g - 2) = \deg((F_{\text{rel}})_*(T_Y^2))$$

by a computation using the Riemann-Roch formula, and the two vector bundles are stable by Lem. 2.9. Therefore the injective morphism $(F_{\text{rel}})_*\mathcal{O}_Y \rightarrow S^2(F)$ factors through the kernel of $S^2(F) \rightarrow (F_{\text{rel}})_*(T_Y^2)$. By rank considerations, $(F_{\text{rel}})_*\mathcal{O}_Y$ is identified with this kernel. \square

Corollary 3.7. *$S^2(F)$ has a unique global section up to scaling, and it is given by the image of $1 \in H^0(X, (F_{\text{rel}})_*\mathcal{O}_Y)$ under the canonical injection (3.5).*

Proof. (3.6) gives rise to an exact sequence

$$0 \rightarrow H^0(X, (F_{\text{rel}})_*\mathcal{O}_Y) \rightarrow H^0(X, S^2(F)) \rightarrow H^0(X, (F_{\text{rel}})_*(T_Y^2))$$

Since $(F_{\text{rel}})_*(T_Y^2)$ is stable of negative degree $-(2g - 2)$, there is no nonzero morphism $\mathcal{O}_X \rightarrow (F_{\text{rel}})_*(T_Y^2)$. Therefore $H^0(X, (F_{\text{rel}})_*(T_Y^2)) = 0$, so $H^0(X, S^2(F)) = H^0(X, (F_{\text{rel}})_*\mathcal{O}_Y)$ has a unique section up to scaling. \square

Proposition 3.8. *The image of $1 \in H^0(X, (F_{\text{rel}})_*\mathcal{O}_Y)$ under the canonical injection (3.5) is a nondegenerate quadric on every fiber of $S^2(F)$.*

Proof. We compute the image of $1 \in H^0(X, (F_{\text{rel}})_*\mathcal{O}_Y)$ in $H^0(X, S^2(F))$ explicitly. In order to do so, we have to understand the isomorphism $(F_{\text{rel}})_*\mathcal{O}_Y \cong ((F_{\text{rel}})_*(T_Y^2))^\vee$ given by relative duality, find out the image of 1 , and then dualize the canonical surjection $S^2(F) \rightarrow (F_{\text{rel}})_*(T_Y^2)$.

Around any point $x \in X$, pick a sufficiently small affine neighborhood, and use the notations in §3 for the relevant local expressions. The chain of isomorphisms in Lem. 3.5 can be expressed locally as

$$\begin{array}{ccc} & \text{Hom}_B(\Omega_B, \text{Hom}_A(B, \Omega_A))_A = \text{Hom}_A((\Omega_B)_A, \Omega_A) & \\ & \nearrow \text{identity} \rightarrow \text{tr}_{x,d} & \searrow \text{contraction} \\ \text{Hom}_B(\Omega_B, \Omega_B)_A & & \text{Hom}_A((T_B^2 \otimes \Omega_B^3)_A, \Omega_A) \\ \parallel \text{contraction} & & \parallel \text{projection formula} \\ B_A & \xrightarrow{\quad ? \quad} & ((T_B^2)_A)^\vee \xrightarrow{\text{contraction}} \text{Hom}_A((T_B^2)_A \otimes \Omega_A, \Omega_A) \end{array}$$

where $1 \in B_A$ is sent to the identity morphism in $\text{Hom}_B(\Omega_B, \Omega_B)_A$, and consequently the map $\text{tr}_{x,d}$ in $\text{Hom}_A((\Omega_B)_A, \Omega_A)$, which is then the morphism

$$\left(\frac{\partial}{\partial x}\right)^{\otimes 2} \otimes da \mapsto 0, \quad x \left(\frac{\partial}{\partial x}\right)^{\otimes 2} \otimes da \mapsto 0, \quad \text{and} \quad x^2 \left(\frac{\partial}{\partial x}\right)^{\otimes 2} \otimes da \mapsto da$$

in $\text{Hom}_A((T_B^2)_A \otimes \Omega_A, \Omega_A)$ after contraction and applying the projection formula. This is then finally the element

$$\left(\frac{\partial}{\partial x}\right)^{\otimes 2} \mapsto 0, \quad x \left(\frac{\partial}{\partial x}\right)^{\otimes 2} \mapsto 0, \quad \text{and} \quad x^2 \left(\frac{\partial}{\partial x}\right)^{\otimes 2} \mapsto 1$$

in $((T_B^2)_A)^\vee$. Let $X_1 = \partial/\partial x$, $X_2 = x(\partial/\partial x)$, and $X_3 = x^2(\partial/\partial x)$ be an A -basis for $(T_B)_A$, and $\{X_i X_j\}$ a basis for $S^2((T_B)_A)$. The composition $B_A \xrightarrow{\sim} ((T_B^2)_A)^\vee \rightarrow S^2((T_B)_A)^\vee$ sends $1 \in B_A$ to the sum $(X_1 X_3)^\vee + (X_2^2)^\vee$, where $(X_1 X_3)^\vee$ denotes the dual basis for $X_1 X_3$, and similarly for $(X_2^2)^\vee$. Since the isomorphism for symmetric powers $S^2((T_B)_A)^\vee \cong S^2(((T_B)_A)^\vee)$ is induced from that for the tensor powers, the element in $S^2(((T_B)_A)^\vee)$ corresponding to $(X_1 X_3)^\vee + (X_2^2)^\vee$ is $X_1^\vee X_3^\vee + (X_2^\vee)^2$. This expression is a nondegenerate quadric. Since nondegeneracy is preserved under linear equivalences, the corresponding expression in $S^2((T_B)_A)$ via self-duality of $(T_B)_A$ is still nondegenerate. The proof is complete, as the choice of $x \in X$ is arbitrary. \square

Corollary 3.9. *There exists a rank-2 vector bundle E over X with trivial determinant, such that $S^2(E) \cong F$. Furthermore, the choice of such a vector bundle E is unique up to twisting by a line bundle in $\text{Pic}(X)[2]$.*

Proof. Prop. 3.8 and Cor. 3.7 together show that there exists a unique nondegenerate quadric $q \in H^0(X, S^2(F))$ up to scaling. Therefore, Lem. 3.3 shows that all rank-2 vector bundles E with $S^2(E) \otimes \det(E)^{-1} \cong F$ occur, by uniqueness of q , as preimage of the pair (F, q) under Φ . Thus they all differ by line bundle twists. Given such a vector bundle E , since $S^2(E) \cong F \otimes \det(E)$ is maximally Frobenius-destabilized, the same holds for E itself (Prop. 3.2). It follows from Lem. 3.1 that after twisting by a line bundle, E has trivial determinant, and thus $S^2(E) \cong F$. Finally, note that given two rank-2 vector bundles E, E' with trivial determinant such that $E' = E \otimes L$ for some line bundle L , then $L \in \text{Pic}(X)[2]$ because $\det(E') \cong \det(E) \otimes L^2$. \square

We summarize the results of this section.

Theorem 3.10. *In characteristic 3, maximally Frobenius-destabilized vector bundles of rank-2 with trivial determinant exist over an arbitrary smooth projective curve of genus $g \geq 2$, and are unique up to twisting by an arbitrary line bundle in $\text{Pic}(X)[2]$. Furthermore, such vector bundles are precisely the rank-2 vector bundles E satisfying $S^2(E) = (F_{\text{rel}})_* T_Y$.*

In particular, there are $\#\text{Pic}(X)[2]$ number of such vector bundles.

Proof. We already know from Prop. 3.2 that rank-2 maximally Frobenius-destabilized vector bundles with trivial determinant are precisely the rank-2 vector bundles E satisfying $S^2(E) \cong (F_{\text{rel}})_* T_Y$. On the other hand, by Cor. 3.9, bundles with this property exist and are unique up to twisting by any line bundle in $\text{Pic}(X)[2]$. \square

Even more generally,

Theorem 3.11. *In characteristic 3, maximally Frobenius-destabilized vector bundles of rank-2 exist over an arbitrary smooth projective curve of genus $g \geq 2$, and are unique up to twisting by line bundles. Furthermore, such vector bundles are precisely the rank-2 vector bundles E satisfying $S^2(E) \cong (F_{\text{rel}})_* L$ for some line bundle L over Y .*

Proof. The second claim again follows from Prop. 3.2. Up to twisting by a line bundle, every rank-2 maximally Frobenius-destabilized vector bundle E has trivial determinant, by Lem. 3.1. The result then reduces to thm. 3.10. \square

Remark 3.12. In [18, Thm. 1.2], B. Osserman proves that over an arbitrary genus-2 curve in characteristic 3, there are exactly $\#\text{Pic}(X)[2]$ number of rank-2 Frobenius-destabilized vector bundles. Since over genus-2 curves, Frobenius-destabilized bundles are precisely the maximally Frobenius-destabilized ones (cf. [9, Prop. 3.3], or look in the proof of our Lem. 2.12), Thm. 3.10 specializes to Osserman’s counting formula in the genus-2 case.

4. AN APPLICATION: AMPLE VECTOR BUNDLES

We first give a new criterion for ample vector bundles, which is valid over an arbitrary smooth projective variety over an algebraically closed field k (of arbitrary characteristic). C. Barton [1] proved a numerical criterion for ample vector bundles in terms of their pullbacks. His result will form the basis of ours, so we briefly review it. Given a projective variety X over an algebraically closed field k , we let $N_1(X)$ denote the group of integral 1-cycles of X modulo numerical equivalence, and let $A_1(X) = N_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$. It is a finite-dimensional vector space (cf. [12, §IV, Prop. 4]). Thus we may fix a basis z_1, \dots, z_q for $A_1(X)$, and let $\|\cdot\|$ be the norm on $A_1(X)$ for which the basis z_1, \dots, z_q is orthonormal.

Theorem 4.1 (C. Barton). *A vector bundle E over X is ample if and only if there exists a real number ε such that for any smooth projective curve Y , any finite morphism $f : Y \rightarrow X$, and any line bundle quotient $f^*E \rightarrow L$, there holds $\deg(L)/\deg(f) \geq \varepsilon\|f(Y)\|$.*

Proof. This is [1, Thm. 2.1]. The main ingredient of the proof is a criterion of amplitude due to S. Kleiman [12, IV-2, Prop. 2]. \square

We also need a result about vector bundles over curves proved by S.-W. Zhang in the arithmetic setting (though it is possibly older), and we choose the more geometric presentation in [4]:

Theorem 4.2 (S.-W. Zhang). *Given a vector bundle E over a smooth projective curve C and any real number $\varepsilon > 0$, there exists a smooth projective curve C' , a finite morphism $f : C' \rightarrow C$, and a line bundle quotient $f^*E \rightarrow L$ such that $\deg(L)/\deg(f) < \mu(E) + \varepsilon$.*

Proof. See [4, Thm. 5.7]. Although the authors prove that there exists a finite, flat cover of C , their proof actually shows that C' can be chosen to be smooth. \square

Our criterion generalizes [3, Thm. 2.3] and [2, Thm. 1.1] to arbitrary dimensions, and is essentially a consequence of [12, IV-2, Prop. 2]:

Theorem 4.3. *Let X be a smooth projective variety over an algebraically closed field k , and E be a vector bundle over X . Then E is ample if and only if there exists some real number $\varepsilon > 0$, such that for any integral, closed curve C in X , there holds $L_{\min}(E|_C) \geq \varepsilon\|C\|$.*

As before, the norm $\|\cdot\|$ on $A_1(X)$ is fixed with respect to a chosen basis. If C is a non-smooth curve, the notation $L_{\min}(E|_C)$ is used to denote $L_{\min}(\nu^*E)$ where $\nu : \tilde{C} \rightarrow C$ is the normalization map. The proof in characteristic zero is a simplified version of the proof in positive characteristic, so we omit the former.

Proof in characteristic $p \neq 0$. First suppose that E is ample. Thm. 4.1 provides some $\varepsilon' > 0$ such that for any smooth projective curve Y , any finite morphism $f : Y \rightarrow X$, and any line bundle quotient $f^*E \rightarrow L$, Barton’s inequality $\deg(L)/\deg(f) \geq \varepsilon'\|f(Y)\|$ holds.

Now consider any integral, closed curve C in X , and let $\nu : \tilde{C} \rightarrow C$ be the normalization map (with $\tilde{C} \cong C$ if C is already smooth). Let $\tilde{E} = \nu^*(E|_C)$. By Thm. 1.4, we may fix a sufficiently large k , such that

$$L_{\min}(E|_C) = L_{\min}(\tilde{E}) = \frac{\mu_{\min}((F_{\text{rel}}^k)^*\tilde{E})}{p^k}$$

Let Q be the last quotient in the Harder-Narasimhan filtration of $(F_{\text{rel}}^k)^*\tilde{E}$. Then $\mu_{\min}((F_{\text{rel}}^k)^*\tilde{E}) = \mu(Q)$. Furthermore, Thm. 4.2 allows us to find a smooth projective curve Y , a finite morphism $g : Y \rightarrow \tilde{C}^{(-k)}$, and a line bundle quotient $g^*Q \rightarrow L$ such that

$$\frac{\deg(L)}{\deg(g)} < \mu(Q) + \frac{p^k \varepsilon'}{2} \cdot \|C\|$$

Divide by p^k , and let $f = \nu \circ F_{\text{rel}}^k \circ g$, then

$$\frac{\deg(L)}{\deg(f)} < \frac{\mu_{\min}((F_{\text{rel}}^k)^* \tilde{E})}{p^k} + \frac{\varepsilon'}{2} \cdot \|C\| \quad (4.1)$$

using $\deg(f) = p^k \deg(g)$, and $\mu_{\min}((F_{\text{rel}}^k)^* \tilde{E}) = \mu(Q)$. Since $f(Y) = C$, Barton's inequality $\deg(L)/\deg(f) \geq \varepsilon' \|C\|$ and (4.1) together imply

$$\frac{\mu_{\min}((F_{\text{rel}}^k)^* \tilde{E})}{p^k} > \frac{\varepsilon'}{2} \cdot \|C\|$$

Therefore, by letting $\varepsilon = \varepsilon'/2$, we obtain $L_{\min}(E|_C) \geq \varepsilon \|C\|$.

Conversely, suppose we have some real number $\varepsilon > 0$, and inequality $L_{\min}(E|_C) \geq \varepsilon \|C\|$ for any integral, closed curve C in X . We will check that E is ample using Thm. 4.1. Let ν , \tilde{C} , and \tilde{E} be as before. Indeed, given any finite morphism $f : Y \rightarrow X$ where Y is a smooth projective curve, and any line bundle quotient $f^*E \rightarrow L$, there is a factorization

$$\begin{array}{ccc} Y & \xrightarrow{g} & \tilde{C}^{(-k)} \\ & \searrow f & \downarrow \nu \circ F_{\text{rel}}^k \\ & & X \end{array}$$

where g is a finite, separable morphism of smooth projective curves. Applying Lem. 1.1 to the morphism g , we obtain

$$\frac{\deg(L)}{\deg(f)} \geq \frac{\mu_{\min}(f^*E)}{\deg(f)} = \frac{\mu_{\min}((F_{\text{rel}}^k)^* \tilde{E})}{p^k} \geq L_{\min}(\tilde{E}) = L_{\min}(E|_C) \geq \varepsilon \|C\|$$

and the result follows. \square

Corollary 4.4 (H. Brenner [3] and I. Biswas [2]). *Let X be a smooth projective curve over an algebraically closed field k , and E be a vector bundle over X . Then E is ample if and only if $L_{\min}(E) > 0$.*

Proof. This is Thm. 4.3 in dimension one. \square

Combining our criterion for amplitude with earlier results, we find

Corollary 4.5. *Let X be a smooth projective curve over an algebraically closed field k of characteristic $p > 0$. Let E be a rank- r semistable vector bundle over X . Suppose $r \leq p$, then*

- (i) $\deg(E) > r(r-1)(g-1)/p$ implies that E is ample;
- (ii) if E is maximally Frobenius-destabilized, then $\deg(E) \leq r(r-1)(g-1)/p$ implies that E is not ample.

Suppose $r > p$. Then $\deg(E) \geq r(g-1)$ implies that E is ample.

Proof. In the $r \leq p$ case, Prop. 2.1 shows that $\mu(E) - L_{\min}(E) \leq (r-1)(g-1)/p$, and equality is attained if and only if condition (2.3) holds. Since E is ample if and only if $L_{\min}(E) > 0$, by Cor. 4.4, the statements (i) and (ii) readily follow. In the $r > p$ case, we use the sharper inequality $\mu(E) - L_{\min}(E) < g-1$ from Prop. 2.16. \square

We now use Cor. 4.5 to construct non-ample semistable vector bundles.

Corollary 4.6 ($r = p$). *Suppose $g \geq 2$. For every integer $d \leq (p-1)(g-1)$, there exists a rank- p vector bundle E over any smooth projective curve X of genus g such that*

- (i) $\deg(E) = d$, and
- (ii) E is semistable but not ample.

Proof. Choose a line bundle L over $X^{(-1)}$ of degree $d - (p-1)(g-1)$. Then $E := (F_{\text{rel}})_* L$ is a maximally Frobenius-destabilized vector bundle (Prop. 2.10) of rank p and degree d . In particular, E is semistable, and Cor. 4.5 shows that E is not ample. \square

Corollary 4.7 ($r < p$). *Suppose $g \geq 2$ and p does not divide $g - 1$; fix a natural number $r < p$. For every integer $d \leq r(r - 1)(g - 1)/p$ which is divisible by r , there exist a smooth projective curve X of genus g and a rank- r vector bundle E over X such that*

- (i) $\deg(E) = d$, and
- (ii) E is semistable but not ample.

Proof. It follows from Prop. 2.14 that there exists a genus- g curve X , and a rank- r maximally Frobenius-destabilized vector bundle E_0 over X . The construction there shows that $\deg(E_0) = 0$. For some line bundle L over X , the twist $E := E_0 \otimes L$ is of degree d . Since E is still maximally Frobenius-destabilized, Cor. 4.5 shows that E is not ample. \square

Remark 4.8. In the $r < p$ case, our technique can only be used to construct non-ample semistable bundles whose degree is divisible by r . Indeed, the computation (2.8) shows that the degree of *any* maximally Frobenius-destabilized vector bundle (with rank $r < p$) is divisible by r .

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