

DEGENERATIONS OF CALABI-YAU THREEFOLDS AND BCOV INVARIANTS

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ABSTRACT. In [1], [2], by expressing the physical quantity F_1 in two distinct ways, Bershadsky-Cecotti-Ooguri-Vafa discovered a remarkable equivalence between Ray-Singer analytic torsion and elliptic instanton numbers for Calabi-Yau threefolds. After their discovery, in [7], a holomorphic torsion invariant for Calabi-Yau threefolds corresponding to F_1 , called BCOV invariant, was constructed. In this article, we study the asymptotic behavior of BCOV invariants for algebraic one-parameter degenerations of Calabi-Yau threefolds. We prove the rationality of the coefficient of logarithmic divergence and give its geometric expression by using a semi-stable reduction of the given family.

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INTRODUCTION

In [1], [2], by expressing the physical quantity F_1 in two distinct ways, Bershadsky-Cecotti-Ooguri-Vafa discovered a remarkable equivalence between Ray-Singer analytic torsion and elliptic instanton numbers for Calabi-Yau threefolds. After their discovery, in [7], a holomorphic torsion invariant for Calabi-Yau threefolds corresponding to F_1 , called *BCOV invariant*, was constructed. Because of its invariance property, BCOV invariant gives rise to a function τ_{BCOV} on the moduli space of Calabi-Yau threefolds. In physics literatures, $-\log \tau_{\text{BCOV}}$ is denoted by F_1 . The prediction of Bershadsky-Cecotti-Ooguri-Vafa concerning the equivalence of holomorphic torsion and elliptic instanton numbers for Calabi-Yau threefolds can be stated as follows: τ_{BCOV} admits an explicit infinite product expression of Borchers type near the large complex structure limit point of the compactified moduli space of Calabi-Yau threefolds, and the exponents of the infinite product are given by explicit linear combinations of rational and elliptic instanton numbers of the mirror Calabi-Yau threefold corresponding to the large complex structure limit point.

As was done in [1], [2], a possible first step towards the conjecture of Bershadsky-Cecotti-Ooguri-Vafa is to determine the section of certain holomorphic line bundle

on the moduli space corresponding to τ_{BCOV} . Thanks to the curvature theorem of Bismut-Gillet-Soulé [4], the complex Hessian $dd^c \log \tau_{\text{BCOV}}$ is expressed as an explicit linear combination of the Weil-Petersson form and its Ricci-form on the moduli space [1], [2], [6], [7]. However, since the moduli space of Calabi-Yau threefolds are non-compact in general, the complex Hessian $dd^c \log \tau_{\text{BCOV}}$ does not determine uniquely its potential and hence the corresponding holomorphic section. To determine its potential up to a constant, $dd^c \log \tau_{\text{BCOV}}$ must be determined as a current on some compactified moduli space. In this way, we are led to the following two problems: one is to understand the behaviors of Weil-Petersson and its Ricci forms as well as their potentials near the boundary locus of the compactified moduli space; the other is to understand the behavior of τ_{BCOV} near the boundary locus of the compactified moduli space. We refer to [10], [11], [7], [12] for the first problem. In this article, we focus on the second problem.

In this direction, in [7], the following results were obtained as an application of the theory of Quillen metrics [5], [3], [18]: $\log \tau_{\text{BCOV}}$ always has logarithmic singularity for arbitrary algebraic one-parameter degenerations of Calabi-Yau threefolds and the logarithmic singularity of $\log \tau_{\text{BCOV}}$ is determined for smoothings of Calabi-Yau varieties with at most one ordinary double point under an additional assumption of the dimension of moduli space. These results, together with the formula for $dd^c \log \tau_{\text{BCOV}}$ and the known boundary behaviors of Weil-Petersson and its Ricci forms, are sufficient to determine τ_{BCOV} for quintic mirror threefolds [7]. However, the results in [7] concerning the singularity of τ_{BCOV} are not sufficient to determine an explicit formula for τ_{BCOV} for wider classes of Calabi-Yau threefolds, e.g. Calabi-Yau threefolds of Borcea-Voisin. For this reason, it is strongly desired to improve the above results in [7]. The purpose of the present article is to give such improvements. Let us explain our main results.

Let $f: \mathcal{X} \rightarrow C$ be a surjective morphism from an irreducible projective fourfold \mathcal{X} to a compact Riemann surface C . Assume that there exists a finite subset $\Delta_f \subset C$ such that $f|_{C \setminus \Delta_f}: \mathcal{X}|_{C \setminus \Delta_f} \rightarrow C \setminus \Delta_f$ is a smooth morphism and such that $X_t = f^{-1}(t)$ is a Calabi-Yau threefold for all $t \in C \setminus \Delta_f$.

Theorem 0.1. *For every $0 \in \Delta_f$, there exists $\alpha = \alpha_0 \in \mathbf{Q}$ such that*

$$\log \tau_{\text{BCOV}}(X_t) = \alpha \log |t|^2 + O(\log(-\log |t|)) \quad (t \rightarrow 0),$$

where t is a local parameter of C centered at $0 \in \Delta_f$.

We remark that in the corresponding theorem in [7], the rationality of α was missing. After Theorem 0.1, a natural question is how the coefficient α is determined by the family $f: \mathcal{X} \rightarrow C$. For this, following [7], we consider its semi-stable reduction [13]. Let $g: (\mathcal{Y}, Y_0) \rightarrow (B, 0)$ be a semi-stable reduction of $f: (\mathcal{X}, X_0) \rightarrow (C, 0)$. By definition, \mathcal{Y} is a smooth projective fourfold, $(B, 0)$ is a pointed compact Riemann surface and there is a surjective morphism of pointed compact Riemann surfaces $\phi: (B, 0) \rightarrow (C, 0)$ such that $Y_0 = g^{-1}(0)$ is a *reduced* normal crossing divisor of \mathcal{Y} and such that $\mathcal{Y} \setminus Y_0 \cong (\mathcal{X} \setminus X_0) \times_{C \setminus \{0\}} (B \setminus \{0\})$.

By choosing a small neighborhood V of 0 in B , $g^{-1}(V)$ carries a canonical form whose zero divisor is contained in Y_0 . The zero divisor of any canonical form on $g^{-1}(V)$ with this property is independent of the choice of such canonical form and is denoted by $\mathfrak{K}_{(\mathcal{Y}, Y_0)}$. We call $\mathfrak{K}_{(\mathcal{Y}, Y_0)}$ the normalized canonical divisor.

Let $\Omega_{\mathcal{Y}/B}^1$ be the sheaf of relative Kähler differentials on \mathcal{Y} and let $\Omega_{\mathcal{Y}/B}^1(\log)$ be its logarithmic version. Set $\mathcal{Q} = \Omega_{\mathcal{Y}/B}^1(\log)/\Omega_{\mathcal{Y}/B}^1$. Every direct image $R^q g_* \mathcal{Q}|_V$ is

a finitely generated torsion sheaf on V supported at $0 \in B$. We set $\chi(Rg_*\mathcal{Q}|_V) = \sum_{q \geq 0} (-1)^q \dim_{\mathbf{C}}(R^q g_* \mathcal{Q})_0 \in \mathbf{Z}$.

Let Σ_g be the critical locus of g . Let $\mathbf{P}(T\mathcal{Y})^\vee$ be the projective bundle over \mathcal{Y} whose fiber $\mathbf{P}(T\mathcal{Y})_y^\vee$ is the projective space of hyperplanes of $T_y\mathcal{Y}$. Then the Gauss map $\mu: \mathcal{Y} \setminus \Sigma_g \ni y \rightarrow [T_y Y_{g(y)}] \in \mathbf{P}(T\mathcal{Y})^\vee$ extends to a meromorphic map from \mathcal{Y} to $\mathbf{P}(T\mathcal{Y})^\vee$. Namely, there exists a blowing-up $\sigma: \tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ inducing an isomorphism $\tilde{\mathcal{Y}} \setminus \sigma^{-1}(\Sigma_g) \cong \mathcal{Y} \setminus \Sigma_g$ such that the composite $\tilde{\mu} = \mu \circ \sigma$ extends to a holomorphic map from $\tilde{\mathcal{Y}}$ to $\mathbf{P}(T\mathcal{Y})^\vee$. Set $\tilde{g} = g \circ \sigma$. We have a new family of Calabi-Yau threefolds $\tilde{g}: \tilde{\mathcal{Y}} \rightarrow B$, whose critical locus $\Sigma_{\tilde{g}}|_V$ defines a divisor of $\tilde{\mathcal{Y}}$.

Let U be the universal hyperplane bundle over $\mathbf{P}(T\mathcal{Y})^\vee$ and let H be the universal quotient line bundle over $\mathbf{P}(T\mathcal{Y})^\vee$. Following [7], set

$$a_p(g, \Sigma_g) = \sum_{j=0}^p (-1)^{p-j} \int_{\text{Exc}(\sigma)} \tilde{\mu}^* \left\{ \text{Td}(U) \frac{\text{Td}(c_1(H)) - e^{-(p-j)c_1(H)}}{c_1(H)} \right\} \sigma^* \text{ch}(\Omega_{\mathcal{Y}}^j),$$

where $\text{Exc}(\sigma)$ is the exceptional divisor of $\sigma: \tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$, $\text{Td}(\cdot)$ is the Todd genus, and $\Omega_{\mathcal{Y}}^j$ is the holomorphic vector bundle of holomorphic j -forms on \mathcal{Y} . Define

$$\rho(g, \Sigma_g) = -3a_0(g, \Sigma_g) + 2a_1(g, \Sigma_g) - \chi(Rg_*\mathcal{Q}|_V) + \frac{1}{12} \int_{\Sigma_{\tilde{g}}|_V} \tilde{\mu}^* c_3(U) \in \mathbf{Q},$$

$$\kappa(g, \Sigma_g, \mathfrak{K}_{(\mathcal{Y}, Y_0)}) = \int_{\sigma^* \mathfrak{K}_{(\mathcal{Y}, Y_0)}} \tilde{\mu}^* c_3(U) \in \mathbf{Z}.$$

Theorem 0.2. *The rational number α in Theorem 0.1 is given by*

$$\alpha = \frac{1}{\deg\{\phi: (B, 0) \rightarrow (C, 0)\}} \left\{ \rho(g, \Sigma_g) - \frac{1}{12} \kappa(g, \Sigma_g, \mathfrak{K}_{(\mathcal{Y}, Y_0)}) \right\}.$$

Since every algebraic one-parameter degeneration of Calabi-Yau threefolds admits a semi-stable reduction [13], in principle, one can compute the singularity of τ_{BCOV} for those degenerations by Theorems 0.1 and 0.2, once one knows their semi-stable reductions. In this sense, the problem of understanding the singularity of τ_{BCOV} is reduced to the algebro-geometric problem of classifying possible semi-stable degenerations of Calabi-Yau threefolds.

As an application of Theorems 0.1 and 0.2, we shall prove certain locality of the singularity of τ_{BCOV} . Namely, under some additional assumptions about the family $f: \mathcal{X} \rightarrow C$ (cf. Section 4 for the required conditions), the coefficient α in Theorem 0.1 depends only on the function germ of f around the critical locus Σ_f . (See Theorem 4.1 for the precise statement.) In some cases, this locality is quite powerful, because we have only to compute one particular example to determine the singularity of τ_{BCOV} . This locality result plays a crucial role to determine the BCOV invariant for Borcea-Voisin threefolds [21].

The strategy to the proof of Theorems 0.1 and 0.2 is quite parallel to that of [7, Th. 9.1]. In [7], it was proved that the L^2 -metric on the line bundle $\det R^q g_* \Omega_{\mathcal{Y}/B}^p(\log)$ has at most an algebraic singularity at the discriminant locus when $p + q = 3$. In this article, we shall improve this estimate. Namely, under the assumption of semi-stability, the L^2 -metric on $\det R^q g_* \Omega_{\mathcal{Y}/B}^p(\log)$ has at most a logarithmic singularity, which enables us to determine various inexplicit constants in [7, §9] and hence α in Theorem 0.1.

This article is organized as follows. In Section 1, we recall the construction of BCOV invariants. In Section 2, we study the asymptotic behavior of the L^2 -metric on $\det R^q g_* \Omega_{Y/B}^p(\log)$ and prove the key fact that it has at most a mild singularity when $p + q = 3$. In Section 3, we prove Theorems 0.1 and 0.2. In Section 4, we prove the locality of the singularity of τ_{BCOV} . In Section 5, we determine the singularity of τ_{BCOV} for general one-parameter smoothings of Calabi-Yau varieties with at most ordinary double points.

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1. BCOV INVARIANTS

1.1. Analytic torsion and Quillen metrics. Let (M, g) be a compact Kähler manifold of dimension d with Kähler form ω . Let $\square_{p,q} = (\bar{\partial} + \bar{\partial}^*)^2$ be the Hodge-Kodaira Laplacian acting on C^∞ (p, q) -forms on M or equivalently $(0, q)$ -forms on M with values in Ω_M^p , where Ω_M^1 is the holomorphic cotangent bundle of M and $\Omega_M^p := \Lambda^p \Omega_M^1$. Let $\sigma(\square_{p,q}) \subset \mathbf{R}_{\geq 0}$ be the set of eigenvalues of $\square_{p,q}$. The spectral zeta function of $\square_{p,q}$ is defined as

$$\zeta_{p,q}(s) := \sum_{\lambda \in \sigma(\square_{p,q}) \setminus \{0\}} \lambda^{-s} \dim E(\lambda, \square_{p,q}),$$

where $E(\lambda, \square_{p,q})$ is the eigenspace of $\square_{p,q}$ corresponding to the eigenvalue λ . Then $\zeta_{p,q}(s)$ converges on the half-plane $\{s \in \mathbf{C}; \Re s > \dim M\}$, extends to a meromorphic function on \mathbf{C} , and is holomorphic at $s = 0$. By Ray-Singer [15], the *analytic torsion* of (M, Ω_M^p) is the real number defined as

$$\tau(M, \Omega_M^p) := \exp\left\{-\sum_{q \geq 0} (-1)^q q \zeta'_{p,q}(0)\right\}.$$

Obviously, $\tau(M, \Omega_M^p)$ depends not only on the complex structure of M but also on the metric g . When we emphasize the dependence of analytic torsion on the metric, we write $\tau(M, \Omega_M^p, g)$.

In [2], Bershadsky-Cecotti-Ooguri-Vafa introduced the following combination of analytic torsions.

Definition 1.1. The *BCOV torsion* of (M, g) is the real number defined as

$$T_{\text{BCOV}}(M, g) := \prod_{q \geq 0} \tau(M, \Omega_M^p)^{(-1)^p} = \exp\left\{-\sum_{p,q \geq 0} (-1)^{p+q} pq \zeta'_{p,q}(0)\right\}.$$

If γ is the Kähler form of g , then we often write $T_{\text{BCOV}}(M, \gamma)$ for $T_{\text{BCOV}}(M, g)$. In general, $T_{\text{BCOV}}(M, g)$ does depend on the choice of Kähler metric g and hence is not a holomorphic invariant of M . When M is a Calabi-Yau threefold, it is possible to construct a holomorphic invariant of M from $T_{\text{BCOV}}(M, g)$ by multiplying a correction factor. Following [7], let us recall the construction of this invariant.

1.2. Calabi-Yau threefolds and BCOV invariants. A compact connected Kähler manifold X is *Calabi-Yau* if $h^{0,q}(X) = 0$ for $0 < q < \dim X$ and $K_X \cong \mathcal{O}_X$, where K_X is the canonical line bundle of X . Our particular interest is the case where X is a threefold. Let X be a Calabi-Yau threefold. Let $g = \sum_{i,j} g_{i\bar{j}} dz_i \otimes d\bar{z}_j$ be a

Kähler metric on X and let $\gamma = \gamma_g := \sqrt{-1} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ be the corresponding Kähler form. Following the convention in Arakelov geometry, we define

$$\text{Vol}(X, \gamma) := \frac{1}{(2\pi)^3} \int_X \frac{\gamma^3}{3!}.$$

The covolume of $H^2(X, \mathbf{Z})_{\text{free}} := H^2(X, \mathbf{Z})/\text{Torsion}$ with respect to $[\gamma]$ is defined as

$$\text{Vol}_{L^2}(H^2(X, \mathbf{Z}), [\gamma]) := \det(\langle \mathbf{e}_i, \mathbf{e}_j \rangle_{L^2, [\gamma]})_{1 \leq i, j \leq b_2(X)}.$$

Here $\{\mathbf{e}_1, \dots, \mathbf{e}_{b_2(X)}\}$ is a basis of $H^2(X, \mathbf{Z})_{\text{free}} = \text{Im}\{H^2(X, \mathbf{Z}) \rightarrow H^2(X, \mathbf{R})\}$ over \mathbf{Z} and $\langle \cdot, \cdot \rangle_{L^2, [\gamma]}$ is the inner product on $H^2(X, \mathbf{R})$ induced by integration of harmonic forms. Namely, if $\mathcal{H}\mathbf{e}_i$ denotes the harmonic representative of $\mathbf{e}_i \in H^2(X, \mathbf{R})$ with respect to γ and if $*$ denotes the Hodge star operator, then

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle_{L^2, [\gamma]} := \frac{1}{(2\pi)^3} \int_X \mathcal{H}\mathbf{e}_i \wedge *(\mathcal{H}\mathbf{e}_j).$$

The covolume $\text{Vol}_{L^2}(H^2(X, \mathbf{Z}), [\gamma])$ is the volume of real torus $H^2(X, \mathbf{R})/H^2(X, \mathbf{Z})_{\text{free}}$ with respect to the L^2 -metric $\langle \cdot, \cdot \rangle_{L^2, [\gamma]}$ on $H^2(X, \mathbf{R})$.

As the correction term to the BCOV torsion $T_{\text{BCOV}}(X, \gamma)$, we introduce a Bott-Chern term.

Definition 1.2. For a Calabi-Yau threefold X equipped with a Kähler form, define

$$A(X, \gamma) := \exp \left[-\frac{1}{12} \int_X \log \left(\sqrt{-1} \frac{\eta \wedge \bar{\eta}}{\gamma^3/3!} \frac{\text{Vol}(X, \gamma)}{\|\eta\|_{L^2}^2} \right) c_3(X, \gamma) \right],$$

where $c_3(X, \gamma)$ is the top Chern form of (X, γ) , $\eta \in H^0(X, K_X) \setminus \{0\}$ is a nowhere vanishing canonical form on X and $\|\eta\|_{L^2}$ is its L^2 -norm, i.e.,

$$\|\eta\|_{L^2} := \frac{1}{(2\pi)^3} \int_X \sqrt{-1} \eta \wedge \bar{\eta}.$$

Obviously, $A(X, \gamma)$ is independent of the choice of $\eta \in H^0(X, K_X) \setminus \{0\}$. We remark that our definition of $A(X, \gamma)$ differs from the one in [7, Def. 4.1] by the factor $\text{Vol}(X, \gamma)^{\chi(X)/12}$, where $\chi(X)$ denotes the topological Euler number of X . Notice that $A(X, \gamma) = 1$ if γ is Ricci-flat.

Definition 1.3. The *BCOV invariant* of X is the real number defined as

$$\tau_{\text{BCOV}}(X) := \text{Vol}(X, \gamma)^{-3 + \frac{\chi(X)}{12}} \text{Vol}_{L^2}(H^2(X, \mathbf{Z}), [\gamma])^{-1} T_{\text{BCOV}}(X, \gamma) A(X, \gamma).$$

As an application of the curvature formula for Quillen metrics [4, Th. 0.1], we get the invariance property of $\tau_{\text{BCOV}}(X)$ in [7].

Theorem 1.4. For a Calabi-Yau threefold X , $\tau_{\text{BCOV}}(X)$ is independent of the choice of a Kähler form on X .

Proof. See [7, Th. 4.16]. □

After Theorem 1.4, we regard τ_{BCOV} as a function on the moduli space of Calabi-Yau threefolds. In this article, we study the behavior of τ_{BCOV} for algebraic one-parameter families of Calabi-Yau threefolds and improve some results in [7, §9].

2. ASYMPTOTIC BEHAVIOR OF THE L^2 -METRIC ON HODGE BUNDLE

Let $\Delta := \{z \in \mathbf{C}; |z| < 1\}$ be the unit disc and let $\Delta^* := \Delta \setminus \{0\}$ be the unit punctured disc. Let $\mathfrak{H} := \{z \in \mathbf{C}; \Im z > 0\}$ be the complex upper half-plane. We regard \mathfrak{H} as the universal covering of Δ^* by the map $\varpi: \mathfrak{H} \ni z \rightarrow \exp(2\pi iz) \in \Delta^*$.

Let $f: \mathcal{Z} \rightarrow \Delta$ be a proper surjective holomorphic map from a smooth complex manifold of dimension $n+1$. We set $Z_t := f^{-1}(t)$ for $t \in \Delta$. If Z_t is smooth for all $t \in \Delta^*$ and if Z_0 is a reduced normal crossing divisor of \mathcal{Z} , then the family $f: \mathcal{Z} \rightarrow \Delta$ is called a *semi-stable degeneration* of relative dimension n .

Let $f: \mathcal{Z} \rightarrow \Delta$ be a semi-stable degeneration of relative dimension n . Set $f^\circ := f|_{\Delta^*}$ and $\mathcal{Z}^\circ := \mathcal{Z} \setminus Z_0$. Let \mathcal{L} be an ample line bundle on \mathcal{Z} . We consider the cohomology of middle degree and set $\ell := \dim H^n(Z_t, \mathbf{C})$ for $t \neq 0$. Assume that

$H^n(Z_t, \mathbf{C})$ consists of primitive cohomology classes with respect to $c_1(\mathcal{L})|_{Z_t}$.

By the primitivity, each component $H^{p,q}(Z_t)$, $p+q = n$, carries the L^2 -inner product

$$(2.1) \quad (u, v)_{L^2, t} := (\sqrt{-1})^{p-q} (-1)^{\frac{n(n-1)}{2}} \int_{Z_t} u \wedge \bar{v}.$$

In particular, $H^n(Z_t, \mathbf{C})$ is endowed with the L^2 -Hermitian structure, which is independent of the choice of polarization.

2.1. L^2 -length of flat section. The holomorphic vector bundle $R^n f_* \mathbf{C} \otimes \mathcal{O}_{\Delta^*}$ is endowed with the Gauss-Manin connection. Fix a reference point $t_0 \in \Delta$ and set $V := H^n(Z_{t_0}, \mathbf{C})$. Fix a basis $\{v_1, \dots, v_\ell\}$ of V , which is unitary with respect to the L^2 -inner product at $t = t_0$. Since \mathfrak{H} is simply connected, $\varpi^*(R^n f_* \mathbf{C})$ is a trivial local system of rank ℓ over \mathfrak{H} . By fixing a point $z_0 \in \mathfrak{H}$ with $t_0 = \varpi(z_0)$, each $v_i \in \varpi^*(R^n f_* \mathbf{C})|_{z_0}$ extends uniquely to a flat section \mathbf{v}_i of $\varpi^*(R^n f_* \mathbf{C}) = V \times \mathfrak{H}$ with respect to the Gauss-Manin connection such that $\mathbf{v}_i(z_0) = v_i$. We regard $\mathbf{v}_i(z)$ as a V -valued holomorphic function on \mathfrak{H} . Namely, $\mathbf{v}_i(z) \in \mathcal{O}(\mathfrak{H}) \otimes_{\mathbf{C}} V$.

On $\varpi^*(R^n f_* \mathbf{C}) = V \times \mathfrak{H}$, the generator of $\pi_1(\Delta^*) = \mathbf{Z}$ acts as the Picard-Lefschetz transformation: There exists $T \in \text{Aut}(V)$ such that for all $z \in \mathfrak{H}$,

$$\mathbf{v}_i(z+1) = T\mathbf{v}_i(z) \quad (i = 1, \dots, \ell).$$

Since $\mathbf{v}_1(z), \dots, \mathbf{v}_\ell(z)$ are flat with respect to the Gauss-Manin connection, there exist constants t_{ij} , $1 \leq i, j \leq \ell$, such that $T\mathbf{v}_i(z) = \sum_j t_{ij} \mathbf{v}_j(z)$. Since $f: \mathcal{Z} \rightarrow \Delta$ is a semi-stable degeneration, $T = (t_{ij})$ is unipotent.

Let $(\cdot, \cdot)_{L^2, z}$ be the L^2 -inner product on $H^n(Z_{\varpi(z)}, \mathbf{C})$. Under the identification of $H^n(Z_{\varpi(z)}, \mathbf{C})$ with $V = H^n(Z_{\varpi(z_0)}, \mathbf{C})$ via the Gauss-Manin connection, $(\cdot, \cdot)_{L^2, z}$ is regarded as a family of Hermitian structures on V such that

$$(2.2) \quad (\mathbf{v}(z+1), \mathbf{v}'(z+1))_{L^2, z+1} = (T\mathbf{v}(z), T\mathbf{v}'(z))_{L^2, z}$$

for any flat sections $\mathbf{v}(z), \mathbf{v}'(z) \in \mathcal{O}(\mathfrak{H}) \otimes_{\mathbf{C}} V$ with respect to the Gauss-Manin connection.

Let $a < b$ be real numbers with $0 < b - a < 1$. By [8, p.46 Prop. 25], there exist positive constants $C_1 = C_1(a, b)$, $C_2 = C_2(a, b)$, $k = k(a, b) > 0$ such that for all $z \in \mathfrak{H}$ with $a < \Re z < b$, $\Im z \gg 0$ and for all $\mathbf{c} = (c_1, \dots, c_\ell) \in \mathbf{C}^\ell$, one has

$$(2.3) \quad C_1 (\Im z)^{-k} \|\mathbf{c}\|^2 \leq \left\| \sum_{i=1}^{\ell} c_i \mathbf{v}_i(z) \right\|_{L^2, z}^2 \leq C_2 (\Im z)^k \|\mathbf{c}\|^2,$$

where $\|\mathbf{c}\|^2 = \sum_{i=1}^{\ell} |c_i|^2 = \left\| \sum_{i=1}^{\ell} c_i \mathbf{v}_i(z) \right\|_{L^2, z_0}^2$.

Remark 2.1. By the SL_2 -orbit theorem of Schmid [16, Th. 6.6 and its proof] (in particular, the last equality of [16, p.253]), there is an integer $\nu(\mathbf{c}) \in \mathbf{Z}_{\geq 0}$ depending on \mathbf{c} such that the ratio $\|\sum_{i=1}^{\ell} c_i \mathbf{v}_i(z)\|_{L^2, z}^2 / (\Im z)^{\nu(\mathbf{c})}$ is bounded from below and above by positive constants when $a < \Re z < b$, $\Im z \gg 0$. Hence we indeed have the following better estimate

$$(2.4) \quad C_1 \|\mathbf{c}\|^2 \leq \left\| \sum_{i=1}^{\ell} c_i \mathbf{v}_i(z) \right\|_{L^2, z}^2 \leq C_2 (\Im z)^k \|\mathbf{c}\|^2,$$

where $k \in \mathbf{Z}_{\geq 0}$. For our later purpose, the weaker estimate (2.3), whose proof is much easier than that of the SL_2 -orbit theorem, is sufficient.

Let $H(z)$ be the positive-definite $\ell \times \ell$ -Hermitian matrix defined as

$$H(z) := \begin{pmatrix} (\mathbf{v}_1(z), \mathbf{v}_1(z))_{L^2, z} & \cdots & (\mathbf{v}_1(z), \mathbf{v}_\ell(z))_{L^2, z} \\ \vdots & \ddots & \vdots \\ (\mathbf{v}_\ell(z), \mathbf{v}_1(z))_{L^2, z} & \cdots & (\mathbf{v}_\ell(z), \mathbf{v}_\ell(z))_{L^2, z} \end{pmatrix}.$$

Let $\lambda_{\min}(z)$ (resp. $\lambda_{\max}(z)$) be the smallest (resp. largest) eigenvalue of the positive-definite Hermitian matrix $H(z)$. By (2.3), we get

$$(2.5) \quad C_1 (\Im z)^{-k} \leq \lambda_{\min}(z) \leq \lambda_{\max}(z) \leq C_2 (\Im z)^k.$$

For a multi-index $I = \{i_1 < i_2 < \cdots < i_r\}$ with $|I| := r \leq \ell$, we define

$$\mathbf{v}_I(z) := \mathbf{v}_{i_1}(z) \wedge \mathbf{v}_{i_2}(z) \wedge \cdots \wedge \mathbf{v}_{i_r}(z) \in \Lambda^r V.$$

For $1 \leq r \leq \ell$, let $\Lambda^r H(z)$ be the positive-definite $\binom{\ell}{r} \times \binom{\ell}{r}$ Hermitian matrix defined as

$$\Lambda^r H(z) := ((\mathbf{v}_I(z), \mathbf{v}_J(z))_{L^2, z})_{|I|=|J|=r}.$$

Here $\Lambda^r V$ is equipped with the Hermitian structure induced by $(\cdot, \cdot)_{L^2, z}$, which is again denoted by the same symbol. By (2.5), we have the following inequality of positive-definite Hermitian endomorphisms on $\Lambda^r V$

$$(2.6) \quad C_1^r (\Im z)^{-kr} I_{\Lambda^r V} \leq \lambda_{\min}(z)^r I_{\Lambda^r V} \leq \Lambda^r H(z) \leq \lambda_{\max}(z)^r I_{\Lambda^r V} \leq C_2^r (\Im z)^{kr} I_{\Lambda^r V}.$$

By (2.6), for all $z \in \mathfrak{H}$ with $a < \Re z < b$ and $\xi = (\xi_I)_{|I|=r} \in \mathbf{C}^{\binom{\ell}{r}}$, we get

$$(2.7) \quad C_1^r (\Im z)^{-kr} \|\xi\|^2 \leq \left\| \sum_{|I|=r} \xi_I \mathbf{v}_I(z) \right\|_{L^2, z}^2 \leq C_2^r (\Im z)^{kr} \|\xi\|^2.$$

2.2. L^2 -length of the canonical section associated to flat section. From the flat sections $\mathbf{v}_1(z), \dots, \mathbf{v}_\ell(z)$, one can construct nowhere vanishing $\pi_1(\Delta^*)$ -invariant holomorphic sections $\mathbf{s}_1, \dots, \mathbf{s}_\ell$ as follows. Let $N \in \mathrm{End}(V)$ be the logarithm of the Picard-Lefschetz transformation T . Since T is unipotent by our assumption, we have $N = \sum_{l \geq 1} (-1)^{l+1} (T - 1_V)^l / l$. Since the entries of $T \in \mathrm{Aut}(V)$ with respect to the basis $\{\mathbf{v}_1(z), \dots, \mathbf{v}_\ell(z)\}$ are constant, so are the entries of $N \in \mathrm{End}(V)$. We define

$$(2.8) \quad \mathbf{s}_i(\exp(2\pi\sqrt{-1}z)) := e^{-zN} \mathbf{v}_i(z) = \sum_{k \geq 0} \frac{(-1)^k}{k!} z^k N^k \mathbf{v}_i(z) \in \mathcal{O}(\mathfrak{H}) \otimes_{\mathbf{C}} V.$$

Since $\mathbf{v}_i(z+1) = e^N \mathbf{v}_i(z)$, \mathbf{s}_i is $\pi_1(\Delta^*)$ -invariant and descends to a nowhere vanishing holomorphic section of $R^n f_*^{\circ} \mathbf{C} \otimes \mathcal{O}_{\Delta^*}$. Since the inner product $(\mathbf{s}_i, \mathbf{s}_j)_{L^2, z}$

is $\pi_1(\Delta^*)$ -invariant by (2.2), it is denoted by $(\mathbf{s}_i, \mathbf{s}_j)_{L^2, t}$, where $t = \exp(2\pi\sqrt{-1}z)$. After Schmid [16, p.235], the *canonical extension* of $R^n f_* \mathbf{C} \otimes \mathcal{O}_{\Delta^*}$ to Δ , denoted by \mathcal{H}^n , is defined as the holomorphic vector bundle of rank ℓ over Δ generated by the frame fields $\{\mathbf{s}_1, \dots, \mathbf{s}_\ell\}$ (see [17], [22] for algebro-geometric construction):

$$\mathcal{H}^n := \mathcal{O}_{\Delta} \mathbf{s}_1 \oplus \dots \oplus \mathcal{O}_{\Delta} \mathbf{s}_\ell.$$

Since N is nilpotent and has constant entries with respect to $\{\mathbf{v}_1(z), \dots, \mathbf{v}_\ell(z)\}$, there exists by (2.8) polynomials $P_{ij}(z) \in \mathbf{C}[z]$ such that for all $z \in \mathfrak{H}$

$$\mathbf{s}_i(e^{2\pi\sqrt{-1}z}) = \sum_{j=1}^{\ell} P_{ij}(z) \mathbf{v}_j(z) \quad (i = 1, \dots, \ell).$$

Since $\{\mathbf{s}_1(e^{2\pi\sqrt{-1}z}), \dots, \mathbf{s}_\ell(e^{2\pi\sqrt{-1}z})\}$ is a basis of $H^n(Z_{\overline{\sigma}(\exp(2\pi\sqrt{-1}z))}, \mathbf{C})$, we get

$$\det(P_{ij}(z)) \neq 0 \quad (\forall z \in \mathfrak{H}).$$

As before, for a multi-index $I = \{i_1 < i_2 < \dots < i_r\}$, we set

$$\mathbf{s}_I(\exp(2\pi\sqrt{-1}z)) := \mathbf{s}_{i_1}(\exp(2\pi\sqrt{-1}z)) \wedge \dots \wedge \mathbf{s}_{i_r}(\exp(2\pi\sqrt{-1}z)).$$

Then we have

$$\sum_{|I|=r} \xi_I \mathbf{s}_I(e^{2\pi\sqrt{-1}z}) = \sum_{|J|=r} \left(\sum_{|I|=r} \xi_I \begin{vmatrix} P_{i_1 j_1}(z) & \dots & P_{i_1 j_r}(z) \\ \vdots & \ddots & \vdots \\ P_{i_r j_1}(z) & \dots & P_{i_r j_r}(z) \end{vmatrix} \right) \mathbf{v}_J(z),$$

where $I = \{i_1 < \dots < i_r\}$, $J = \{j_1 < \dots < j_r\}$. We deduce from (2.7) that for all $z \in \mathfrak{H}$ with $a < \Re z < b$ and $\xi = (\xi_I)_{|I|=r} \in \mathbf{C}^{\binom{\ell}{r}}$,

$$(2.9) \quad C_1^r(\Im z)^{-kr} \leq \frac{\|\sum_{|I|=r} \xi_I \mathbf{s}_I(e^{2\pi\sqrt{-1}z})\|_{L^2, z}^2}{\sqrt{\sum_{|J|=r} \left| \sum_{|I|=r} \xi_I \begin{vmatrix} P_{i_1 j_1}(z) & \dots & P_{i_1 j_r}(z) \\ \vdots & \ddots & \vdots \\ P_{i_r j_1}(z) & \dots & P_{i_r j_r}(z) \end{vmatrix} \right|^2}} \leq C_2^r(\Im z)^{kr}.$$

We define an invertible $\binom{\ell}{r} \times \binom{\ell}{r}$ -matrix $\Lambda^r P(z)$ by

$$\Lambda^r P(z) := \left(\begin{vmatrix} P_{i_1 j_1}(z) & \dots & P_{i_1 j_r}(z) \\ \vdots & \ddots & \vdots \\ P_{i_r j_1}(z) & \dots & P_{i_r j_r}(z) \end{vmatrix} \right)_{|I|=|J|=r} \in GL\left(\mathbf{C}^{\binom{\ell}{r}}\right).$$

Let $\mu_{\min}(z)$ (resp. $\mu_{\max}(z)$) be the smallest (resp. largest) eigenvalue of the positive-definite $\binom{\ell}{r} \times \binom{\ell}{r}$ -matrix $G(z) := {}^t(\Lambda^r P(z)) \overline{\Lambda^r P(z)}$. Then we have

$$(2.10) \quad \mu_{\min}(z)^{-1} \leq \text{Tr}\{G(z)^{-1}\}, \quad \mu_{\max}(z) \leq \text{Tr} G(z).$$

For multi-indices I, J with $|I| = |J| = r$, let $r_{IJ}(z) \in \mathbf{C}(z)$ be the (I, J) -entry of the $\binom{\ell}{r} \times \binom{\ell}{r}$ -matrix $(\Lambda^r P(z))^{-1}$. By the definition of $G(z)$, we get

$$\begin{aligned} \text{Tr}\{G(z)^{-1}\} &= \sum_{|I|=|J|=r} |r_{IJ}(z)|^2, \\ \text{Tr} G(z) &= \sum_{|I|=|J|=r} \left| \det \begin{pmatrix} P_{i_1 j_1}(z) & \dots & P_{i_1 j_r}(z) \\ \vdots & \ddots & \vdots \\ P_{i_r j_1}(z) & \dots & P_{i_r j_r}(z) \end{pmatrix} \right|^2. \end{aligned}$$

Since $r_{IJ}(z)$ is a rational function in the variable z with $\det(r_{IJ}(z)) \neq 0$ in $\mathbf{C}(z)$ and since $P_{ij}(z)$ is a polynomial in the variable z with $\det(P_{IJ}(z)) \neq 0$ in $\mathbf{C}[z]$, there exist $\mu \in \mathbf{Z}$, $\nu \in \mathbf{Z}_{\geq 0}$ and constants $C_3, C_4 > 0$ such that

$$(2.11) \quad \mathrm{Tr}\{G(z)^{-1}\} \leq C_3|z|^{2\mu}, \quad \mathrm{Tr} G(z) \leq C_4|z|^{2\nu}$$

for all $z \in \mathfrak{H}$ with $|z| \gg 1$. Since

$$\sum_{|J|=r} \left| \sum_{|I|=r} \xi_I \begin{vmatrix} P_{i_1 j_1}(z) & \cdots & P_{i_1 j_r}(z) \\ \vdots & \ddots & \vdots \\ P_{i_r j_1}(z) & \cdots & P_{i_r j_r}(z) \end{vmatrix} \right|^2 = \|\Lambda^r P(z)\xi\|^2 = (G(z)\xi, \xi),$$

we deduce from (2.10), (2.11) that for $z \in \mathfrak{H}$ with $|z| \gg 1$ and $\xi = (\xi_I)_{|I|=r} \in \mathbf{C}^{\binom{\ell}{r}}$

$$(2.12) \quad C_3^{-1} \|\xi\|^2 |z|^{-2\mu} \leq \sum_{|J|=r} \left| \sum_{|I|=r} \xi_I \begin{vmatrix} P_{i_1 j_1}(z) & \cdots & P_{i_1 j_r}(z) \\ \vdots & \ddots & \vdots \\ P_{i_r j_1}(z) & \cdots & P_{i_r j_r}(z) \end{vmatrix} \right|^2 \leq C_4 \|\xi\|^2 |z|^{2\nu}.$$

By (2.9), (2.12), we get for all $z \in \mathfrak{H}$ with $a < \Re z < b$, $|z| \gg 1$ and $\xi \in \mathbf{C}^{\binom{\ell}{r}}$

$$(2.13) \quad C_5 \|\xi\|^2 (\Im z)^{-(kr+2\mu)} \leq \left\| \sum_{|I|=r} \xi_I \mathbf{s}_I(e^{2\pi\sqrt{-1}z}) \right\|_{L^2, z}^2 \leq C_6 \|\xi\|^2 (\Im z)^{(kr+2\nu)},$$

where C_5, C_6 are constants. Here we used the fact that $|z|/\Im z$ is bounded from below and above by positive constants on the domain $\{z \in \mathfrak{H}; a < \Re z < b, |z| \gg 1\}$. Since $\|\mathbf{s}_I(e^{2\pi\sqrt{-1}z})\|_{L^2, z}$ is \mathbf{Z} -invariant, it follows from (2.13) that for $t \in \Delta^*$ with $0 < |t| \ll 1$,

$$(2.14) \quad C_5 \|\xi\|^2 (-\log |t|)^{-(kr+2\mu)} \leq \left\| \sum_{|I|=r} \xi_I \mathbf{s}_I(t) \right\|_{L^2, t}^2 \leq C_6 \|\xi\|^2 (-\log |t|)^{(kr+2\nu)}.$$

2.3. L^2 -length of a nowhere vanishing section of $\det \mathcal{F}^p$. On \mathcal{H}^n , we have the Hodge filtration

$$0 \subset \mathcal{F}^n \subset \mathcal{F}^{n-1} \subset \cdots \subset \mathcal{F}^1 \subset \mathcal{F}^0 = \mathcal{H}^n,$$

where \mathcal{F}^p is a holomorphic subbundle of \mathcal{H}^n satisfying $\mathcal{F}_t^p = \bigoplus_{k \geq p} H^{k, n-k}(Z_t)$ for $t \in \Delta^*$ and

$$\mathcal{F}^p / \mathcal{F}^{p+1} \cong R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log Z_0).$$

Here $\Omega_{\mathcal{Z}/\Delta}^p(\log Z_0) := \Lambda^p \Omega_{\mathcal{Z}/\Delta}^1(\log Z_0)$ and $\Omega_{\mathcal{Z}/\Delta}^1(\log Z_0) := \Omega_{\mathcal{Z}}^1(\log Z_0) / \mathcal{O}_{\mathcal{Z}} f^*(dt/t)$.

On the open subset of \mathcal{Z} on which $f(z) = z_1 \cdots z_k$, $\Omega_{\mathcal{Z}}^1(\log Z_0)$ is given by

$$\Omega_{\mathcal{Z}}^1(\log Z_0) = \mathcal{O}_{\mathcal{Z}}(dz_1/z_1) + \cdots + \mathcal{O}_{\mathcal{Z}}(dz_k/z_k) + \mathcal{O}_{\mathcal{Z}} dz_{k+1} + \cdots + \mathcal{O}_{\mathcal{Z}} dz_n.$$

See [17], [22] for algebro-geometric account of the Hodge filtrations. In what follows, we often write $R^q f_* \Omega_{\mathcal{Z}/\Delta}^p(\log)$ for $R^q f_* \Omega_{\mathcal{Z}/\Delta}^p(\log Z_0)$.

Set $\ell_p := \mathrm{rk} \mathcal{F}^p$. There exist holomorphic sections $\varphi_1, \dots, \varphi_{\ell} \in \Gamma(\Delta, \mathcal{H}^n)$ with

$$\mathcal{F}^p = \mathcal{O}_{\Delta} \varphi_1 \oplus \cdots \oplus \mathcal{O}_{\Delta} \varphi_{\ell_p} \quad (p = n, n-1, \dots, 1, 0).$$

Since $\{\mathbf{s}_1, \dots, \mathbf{s}_{\ell}\}$ is a basis of \mathcal{H}^n as an \mathcal{O}_{Δ} -module, there exist holomorphic functions $a_{\alpha i}(t) \in \mathcal{O}(\Delta)$, $1 \leq i, \alpha \leq \ell$ such that

$$\varphi_{\alpha}(t) = \sum_{i=1}^{\ell} a_{\alpha i}(t) \mathbf{s}_i(t) \quad (\alpha = 1, \dots, \ell), \quad \det(a_{\alpha i}(0))_{1 \leq \alpha, i \leq \ell} \neq 0.$$

Proposition 2.2. *There exist constants $C \in \mathbf{R}_{\geq 0}$ and $C' \in \mathbf{R}_{> 0}$ such that for all $t \in \Delta^*$ with $|t| \ll 1$ and $1 \leq m \leq \ell$,*

$$\left| \log \|\varphi_1(t) \wedge \cdots \wedge \varphi_m(t)\|_{L^2, t} \right| \leq C' + C \log(-\log |t|).$$

Proof. Since

$$\varphi_1(t) \wedge \cdots \wedge \varphi_m(t) = \sum_{|J|=m} \begin{vmatrix} a_{1j_1}(t) & \cdots & a_{1j_m}(t) \\ \vdots & \ddots & \vdots \\ a_{mj_1}(t) & \cdots & a_{mj_m}(t) \end{vmatrix} \mathbf{s}_J(t),$$

there exist by (2.14) constants $k_1, k_2 \geq 0$ such that for all $t \in \Delta^*$ with $0 < |t| \ll 1$ (2.15)

$$C_5 (-\log |t|)^{-k_1} \leq \frac{\|\varphi_1(t) \wedge \cdots \wedge \varphi_m(t)\|_{L^2, t}^2}{\sum_{|J|=m} \left| \det \begin{pmatrix} a_{1j_1}(t) & \cdots & a_{1j_m}(t) \\ \vdots & \ddots & \vdots \\ a_{mj_1}(t) & \cdots & a_{mj_m}(t) \end{pmatrix} \right|^2} \leq C_6 (-\log |t|)^{k_2}.$$

By (2.15), it suffices to prove that

$$(2.16) \quad \sum_{|J|=m} \left| \det \begin{pmatrix} a_{1j_1}(0) & \cdots & a_{1j_m}(0) \\ \vdots & \ddots & \vdots \\ a_{mj_1}(0) & \cdots & a_{mj_m}(0) \end{pmatrix} \right|^2 \neq 0.$$

Set

$$A := \begin{pmatrix} a_{11}(0) & \cdots & a_{1\ell}(0) \\ \vdots & \ddots & \vdots \\ a_{\ell 1}(0) & \cdots & a_{\ell\ell}(0) \end{pmatrix} \in GL(\mathbf{C}^\ell).$$

Then the endomorphism on $\Lambda^m \mathbf{C}^\ell$ induced by A is given by the $\binom{\ell}{m} \times \binom{\ell}{m}$ -matrix

$$\Lambda^m A := \left(\begin{vmatrix} a_{i_1 j_1}(0) & \cdots & a_{i_1 j_m}(0) \\ \vdots & \ddots & \vdots \\ a_{i_m j_1}(0) & \cdots & a_{i_m j_m}(0) \end{vmatrix} \right)_{|I|=|J|=m}.$$

Since $\det A \neq 0$, $\Lambda^m A \in \text{End}(\Lambda^m \mathbf{C}^\ell)$ is invertible. In particular, the row vector of $\Lambda^m A$ corresponding to the multi-index $I = \{1, 2, \dots, m\}$ is non-zero, which implies (2.16). This proves the result. \square

Since $R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log)|_{\Delta^*} = R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p|_{\Delta^*}$ is identified with the holomorphic vector bundle over Δ^* with fiber $H^{n-p}(Z_t, \Omega_{Z_t}^p)$ over $t \in \Delta^*$, $R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log)|_{\Delta^*}$ is equipped with the L^2 -Hermitian metric by identifying $H^{n-p}(Z_t, \Omega_{Z_t}^p)$ with the corresponding vector space of harmonic forms of bidegree $(p, n-p)$.

Under the canonical identification $R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log) = \mathcal{F}^p / \mathcal{F}^{p+1}$, the L^2 -metric on $R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log)|_{\Delta^*}$ is identified with the quotient metric on $\mathcal{F}^p / \mathcal{F}^{p+1}$ induced by the L^2 -metric on $\mathcal{H}^p|_{\Delta^*}$. Hence we have an isometry of holomorphic line bundles equipped with singular Hermitian metrics

$$(2.17) \quad \left(\det R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log), \|\cdot\|_{L^2} \right) \cong \left(\det \mathcal{F}^p, \|\cdot\|_{L^2} \right) \otimes \left(\det \mathcal{F}^{p+1}, \|\cdot\|_{L^2} \right)^\vee.$$

Corollary 2.3. *Let $e_1(t), \dots, e_{n_p}(t) \in \Gamma(\Delta, R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log))$ be a basis of the free \mathcal{O}_Δ -module $R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log)$, where $n_p := h^{p, n-p}(Z_t)$, $t \neq 0$. Then there exist constants $C \geq 0$ and $C' > 0$ such that for all $t \in \Delta^*$ with $|t| \ll 1$*

$$\left| \log \|e_1(t) \wedge \dots \wedge e_{n_p}(t)\|_{L^2, t} \right| \leq C' + C \log(-\log |t|).$$

Proof. There exist nowhere vanishing holomorphic sections $\mathbf{f}_p(t) \in \Gamma(\Delta, \det \mathcal{F}^p)$ and $\mathbf{f}_{p+1}(t) \in \Gamma(\Delta, \det \mathcal{F}^{p+1})$ such that

$$e_1(t) \wedge \dots \wedge e_{n_p}(t) = \mathbf{f}_p(t) \otimes \mathbf{f}_{p+1}(t)^{-1}$$

under the identification (2.17). Since

$$\|e_1(t) \wedge \dots \wedge e_{n_p}(t)\|_{L^2, t} = \|\mathbf{f}_p(t)\|_{L^2, t} \cdot \|\mathbf{f}_{p+1}(t)\|_{L^2, t}^{-1}$$

the result follows from Proposition 2.2 applied to the nowhere vanishing sections $\mathbf{f}_p(t)$ and $\mathbf{f}_{p+1}(t)$. \square

3. SINGULARITY OF BCOV INVARIANTS FOR SEMI-STABLE DEGENERATIONS

Set up. Throughout this section except Section 3.9, we assume the following:

- (1) There exist a smooth projective fourfold \mathcal{X} , a compact Riemann surface C , an embedding $\Delta \subset C$, and a surjective holomorphic map $f: \mathcal{X} \rightarrow C$ such that $f: \mathcal{X}|_\Delta \rightarrow \Delta$ is a semi-stable degeneration. In particular, $f^{-1}(0)$ is a reduced normal crossing divisor of \mathcal{X} .
- (2) The regular fibers of $f: \mathcal{X} \rightarrow C$ are Calabi-Yau threefolds.

Set $X_t := f^{-1}(t)$ for $t \in C$. Then X_t is a Calabi-Yau threefold for all $t \in \Delta^*$. In this section, we determine the asymptotic behavior of the function on Δ^*

$$t \mapsto \log \tau_{\text{BCOV}}(X_t) \quad (t \rightarrow 0).$$

Let \mathcal{L} be an ample line bundle on \mathcal{X} . For all $t \in \Delta^*$, $c_1(\mathcal{L}_t)$ is a Kähler class on X_t , so that every $H^{p,q}(X_t, \mathbf{C})$ is endowed with the L^2 -Hermitian structure with respect to the Kähler class $c_1(\mathcal{L}_t)$. When $p+q = 3$, $H^{p,q}(X_t, \mathbf{C})$ consists of primitive cohomology classes and the L^2 -Hermitian structure on $H^{p,q}(X_t, \mathbf{C})$ with respect to $c_1(\mathcal{L}_t)$ is given by (2.1).

Let $\Sigma_f := \{x \in \mathcal{X}; df_x = 0\}$ be the critical locus of f and let $\Delta_f := f(\Sigma_f) \subset C$ be the discriminant locus of $f: \mathcal{X} \rightarrow C$. Then $\Omega_{\mathcal{X}/C}^p$ is a holomorphic vector bundle over $\mathcal{X} \setminus \Sigma_f$, where $\Omega_{\mathcal{X}/C}^p := \Lambda^p \Omega_{\mathcal{X}/C}^1$ and $\Omega_{\mathcal{X}/C}^1 := \Omega_{\mathcal{X}}^1 / f^* \Omega_C^1$. Since possible extension of $\Omega_{\mathcal{X}/C}^p$ to a coherent sheaf on \mathcal{X} is not unique in general, we regard $\Omega_{\mathcal{X}/C}^p$ as a locally free sheaf on $\mathcal{X} \setminus \Sigma_f$ rather than a coherent sheaf on \mathcal{X} .

Recall that the determinant of the cohomologies of $\Omega_{\mathcal{X}/C}^p$ is the holomorphic line bundle on $C \setminus \Delta_f$ defined as

$$\lambda(\Omega_{\mathcal{X}/C}^p) := \bigotimes_{q \geq 0} (\det R^q f_* \Omega_{\mathcal{X}/C}^p)^{(-1)^q}.$$

Since X_t , $t \in C \setminus \Delta_f$, is equipped with the Kähler class $c_1(\mathcal{L}_t)$, $H^q(X_t, \Omega_{X_t}^p)$ is equipped with the L^2 Hermitian metric by identifying it with the corresponding vector space of harmonic forms. In this way, for all $p, q \geq 0$, $R^q f_* \Omega_{\mathcal{X}/C}^p$ is a holomorphic vector bundle over $C \setminus \Delta_f$ equipped with the L^2 Hermitian metric. (This Hermitian metric coincides with the one considered in Section 2.3 when $p+q =$

3.) Hence $\lambda(\Omega_{\mathcal{X}/C}^p)$ is a holomorphic Hermitian line bundle on $C \setminus \Delta_f$ equipped with the L^2 metric $\|\cdot\|_{L^2}$.

We fix a Kähler metric $g^{\mathcal{X}}$ on \mathcal{X} with Kähler class $c_1(\mathcal{L})$. For $t \in C \setminus \Delta_f$, we set $g_t := g^{\mathcal{X}}|_{X_t}$. Then $\{(X_t, g_t)\}_{t \in C \setminus \Delta_f}$ is a family of compact Kähler manifolds with constant Kähler class $c_1(\mathcal{L}_t) = c_1(\mathcal{L})|_{X_t}$. As in Section 1.1, we have analytic torsion $\tau(X_t, \Omega_{X_t}^p)$ for all $t \in C \setminus \Delta_f$. By Quillen [14] and Bismut-Gillet-Soulé [4], the *Quillen metric* on $\lambda(\Omega_{\mathcal{X}/C}^p)$ is defined as

$$\|\cdot\|_{\mathcal{Q}}^2(t) := \tau(X_t, \Omega_{X_t}^p) \cdot \|\cdot\|_{L^2, t}^2, \quad t \in C \setminus \Delta_f.$$

The curvature and anomaly formulae for Quillen metrics were obtained by Bismut-Gillet-Soulé [4]. As an application of the Bismut-Lebeau embedding formula [5], the singularity of Quillen metric as $t \rightarrow 0 \in \Delta_f$ was determined in [7], which we shall recall in Section 3.4.

3.1. Asymptotic behavior of the L^2 -metric on $\lambda(\mathcal{O}_{\mathcal{X}})|_{\Delta}$.

Proposition 3.1. *Let $\varsigma_0 \in \Gamma(\Delta, \lambda(\mathcal{O}_{\mathcal{X}})|_{\Delta})$ be a nowhere vanishing holomorphic section. Then the following holds as $t \rightarrow 0$:*

$$\log \|\varsigma_0(t)\|_{L^2}^2 = O(\log(-\log|t|)).$$

Proof. Since X_t is a Calabi-Yau threefold for all $t \in \Delta^*$, we have $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_C \cdot 1$ and $R^3 f_*\mathcal{O}_{\mathcal{X}}|_{\Delta} = \mathcal{O}_{\Delta} \cdot e$, where $e(t) \in \Gamma(\Delta, R^3 f_*\mathcal{O}_{\mathcal{X}})$ is a nowhere vanishing section. By the definition of the L^2 -metric on $H^0(X_t, \mathbf{C})$, we see that $\|1\|_{L^2}^2 = \deg(L_t)/(2\pi)^3$ is a constant function on Δ . By Corollary 2.3 in the case $n = 3$, $q = 0$, we have

$$\log \|e(t)\|_{L^2}^2 = O(\log(-\log|t|)) \quad (t \rightarrow 0).$$

Since $\sigma_0 = 1 \otimes e^{\vee}$, we get the result. \square

3.2. Asymptotic behavior of the L^2 -metric on $\lambda(\Omega_{\mathcal{X}/C}^1)|_{\Delta}$. We define

$$\Omega_{\mathcal{X}/C}^1 := \Omega_{\mathcal{X}}^1 / f^*\Omega_C^1.$$

Since $\Omega_{\mathcal{X}}^1 \subset \Omega_{\mathcal{X}}^1(\log)$ and $\Omega_C^1 \subset \Omega_C^1(\log)$, we have the natural inclusion of sheaves

$$\Omega_{\mathcal{X}/C}^1 \subset \Omega_{\mathcal{X}/C}^1(\log)$$

and we set

$$\mathcal{Q} := \Omega_{\mathcal{X}/C}^1(\log) / \Omega_{\mathcal{X}/C}^1.$$

Then $\mathcal{Q}|_{f^{-1}(\Delta)}$ is a coherent sheaf on $\mathcal{X}|_{\Delta}$ supported on $\text{Sing } X_0$. The short exact sequence of coherent sheaves on \mathcal{X}

$$0 \longrightarrow \Omega_{\mathcal{X}/C}^1 \longrightarrow \Omega_{\mathcal{X}/C}^1(\log) \longrightarrow \mathcal{Q} \longrightarrow 0$$

induces the long exact sequence of direct image sheaves on C

$$(3.1) \quad \begin{aligned} &\longrightarrow R^{q-1} f_* \Omega_{\mathcal{X}/C}^1(\log) \longrightarrow R^{q-1} f_* \mathcal{Q} \longrightarrow R^q f_* \Omega_{\mathcal{X}/C}^1 \longrightarrow R^q f_* \Omega_{\mathcal{X}/C}^1(\log) \longrightarrow R^q f_* \mathcal{Q} \longrightarrow \\ &\longrightarrow R^{q+1} f_* \Omega_{\mathcal{X}/C}^1(\log) \longrightarrow R^{q+1} f_* \mathcal{Q} \longrightarrow R^{q+2} f_* \Omega_{\mathcal{X}/C}^1 \longrightarrow R^{q+2} f_* \Omega_{\mathcal{X}/C}^1(\log) \longrightarrow R^{q+2} f_* \mathcal{Q} \longrightarrow \end{aligned}$$

Following [7, proof of Prop. 9.5], set

$$M_q := (R^q f_* \Omega_{\mathcal{X}/C}^1)_{\text{tors}}|_{\Delta}, \quad N_q := R^q f_* \Omega_{\mathcal{X}/C}^1(\log) / R^q f_* \Omega_{\mathcal{X}/C}^1|_{\Delta}.$$

Since $R^q f_* \Omega_{\mathcal{X}/C}^1(\log)$ is a locally free sheaf on C , we deduce from (3.1) the isomorphism of \mathcal{O}_{Δ} -modules

$$(3.2) \quad R^q f_* \mathcal{Q}|_{\Delta} \cong N_q \oplus M_{q+1}.$$

Set

$$\chi(Rf_*\mathcal{Q}|_\Delta) := \sum_q (-1)^q \dim_{\mathbf{C}}(R^q f_*\mathcal{Q})_0.$$

By (3.2), we get

$$(3.3) \quad \chi(Rf_*\mathcal{Q}|_\Delta) = \sum_q (-1)^q (\dim_{\mathbf{C}}(N_q)_0 - \dim_{\mathbf{C}}(M_q)_0).$$

Since $\mathcal{Q}|_{f^{-1}(\Delta)}$ depends only on the function germ of f near Σ_f , so is $\chi(Rf_*\mathcal{Q}|_\Delta)$.

Proposition 3.2. *Let $\varsigma_1 \in \Gamma(\Delta, \lambda(\Omega_{\mathcal{X}/C}^1))$ be a nowhere vanishing holomorphic section. Then the following holds as $t \rightarrow 0$:*

$$\log \|\varsigma_1(t)\|_{\lambda(\Omega_{\mathcal{X}/C}^1), L^2}^2 = \chi(Rf_*\mathcal{Q}|_\Delta) \log |t|^2 + O(\log(-\log |t|)).$$

Proof. Let $e_1(t), \dots, e_{h^{1,q}}(t) \in \Gamma(\Delta, R^q f_*\Omega_{\mathcal{X}/C}^1(\log))$ be a basis of $R^q f_*\Omega_{\mathcal{X}/C}^1(\log)|_\Delta$ as a free \mathcal{O}_Δ -module. By [7, Prop. 9.4], there exists $\delta_q \in \mathbf{R}$ such that

$$\log \|e_1(t) \wedge \dots \wedge e_{h^{1,q}}(t)\|_{L^2}^2 = \delta_q \log |t|^2 + O(\log(-\log |t|)) \quad (t \rightarrow 0).$$

It follows from [7, Eq. (9.15)] and (3.3) that as $t \rightarrow 0$

$$(3.4) \quad \log \|\varsigma_1(t)\|_{\lambda(\Omega_{\mathcal{X}/C}^1), L^2}^2 = \{\chi(Rf_*\mathcal{Q}|_\Delta) + \sum_q (-1)^q \delta_q\} \log |t|^2 + O(\log(-\log |t|)).$$

By (3.4), it suffices to prove $\delta_q = 0$ for all $q \geq 0$. The vanishing $\delta_0 = \delta_1 = \delta_3 = 0$ was already proved in [7, proof of Prop. 9.4]. The vanishing $\delta_2 = 0$ follows from Corollary 2.3. This completes the proof. \square

3.3. The Kähler extension of $\lambda(\Omega_{\mathcal{X}/C}^p)|_{\Delta^*}$. Following [7], we recall an extension of $\lambda(\Omega_{\mathcal{X}/C}^p)$ from Δ^* to Δ , which we call the Kähler extension and which is distinct from $\lambda(\Omega_{\mathcal{X}/C}^p(\log))$ in general. For $p \geq 0$, set

$$\Omega_{\mathcal{X}/C}^p := \Lambda^p \Omega_{\mathcal{X}/C}^1.$$

Then $\Omega_{\mathcal{X}/C}^p$ is a coherent sheaf on \mathcal{X} , which is locally free on $\mathcal{X} \setminus \Sigma_f$. Following [7, Sect. 5], we recall the Kähler extension of $\lambda(\Omega_{\mathcal{X}/C}^p)|_{\Delta^*}$. On $\mathcal{X}|_{\Delta^*}$, we have the following exact sequence of holomorphic vector bundles:

$$0 \longrightarrow f^*\Omega_C^1 \longrightarrow \Omega_{\mathcal{X}}^1 \longrightarrow \Omega_{\mathcal{X}/C}^1 \longrightarrow 0.$$

Since $\text{rk } f^*\Omega_C^1 = 1$, this short exact sequence induces the following exact sequence of holomorphic vector bundles on $\mathcal{X}|_{\Delta^*}$:

$$(3.5) \quad 0 \longrightarrow \mathcal{E}_{\mathcal{X}/C}^p \longrightarrow \Omega_{\mathcal{X}/C}^p \longrightarrow 0,$$

where $\mathcal{E}_{\mathcal{X}/C}^p$ is the complex of holomorphic vector bundles over \mathcal{X} given by

$$\mathcal{E}_{\mathcal{X}/C}^p: (f^*\Omega_C^1)^{\otimes p} \rightarrow \Omega_{\mathcal{X}}^1 \otimes (f^*\Omega_C^1)^{\otimes p-1} \rightarrow \dots \rightarrow \Omega_{\mathcal{X}}^{p-1} \otimes f^*\Omega_C^1 \rightarrow \Omega_{\mathcal{X}}^p$$

and the map $\Omega_{\mathcal{X}}^p \rightarrow \Omega_{\mathcal{X}/C}^p$ is given by the canonical quotient map. Here, if θ is a local generator of Ω_C^1 , then the map $\Omega_{\mathcal{X}}^i \otimes (f^*\Omega_C^1)^{\otimes (p-i)} \rightarrow \Omega_{\mathcal{X}}^{i+1} \otimes (f^*\Omega_C^1)^{\otimes (p-i-1)}$ is given by $\omega \otimes (f^*\theta)^{\otimes (p-i)} \mapsto (\omega \wedge f^*\theta) \otimes (f^*\theta)^{\otimes (p-i-1)}$ for $\omega \in \Omega_{\mathcal{X}}^{p-i}$.

Definition 3.3. The *Kähler extension* of $\lambda(\Omega_{\mathcal{X}/C}^p)$ is the holomorphic line bundle over C defined as

$$\lambda(\mathcal{E}_{\mathcal{X}/C}^p) := \bigotimes_{i=0}^p \lambda \left(\Omega_{\mathcal{X}}^{p-i} \otimes (f^* \Omega_C^1)^{\otimes i} \right)^{(-1)^i}.$$

By the exactness of (3.5) on $\mathcal{X}|_{\Delta^*}$, we have the canonical isomorphism of holomorphic line bundles over Δ^* :

$$(3.6) \quad \lambda(\Omega_{\mathcal{X}/C}^p)|_{\Delta^*} \cong \lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_{\Delta^*}.$$

When $p = 0, 1$, the canonical isomorphism (3.6) extends to an isomorphism of holomorphic line bundles over C :

$$(3.7) \quad \lambda(\mathcal{O}_{\mathcal{X}}) = \lambda(\mathcal{E}_{\mathcal{X}/C}^0), \quad \lambda(\Omega_{\mathcal{X}/C}^1) \cong \lambda(\mathcal{E}_{\mathcal{X}/C}^1).$$

Via the canonical isomorphism (3.6), the L^2 -metric on $\lambda(\Omega_{\mathcal{X}/C}^p)|_{\Delta^*}$ induces a Hermitian metric on $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_{\Delta^*}$, which is denoted by $\|\cdot\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), L^2}$. Notice that $\|\cdot\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), L^2}$ does *not* coincide with the Hermitian metric on $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)$ defined as the product of the L^2 -metrics on $\lambda(\Omega_{\mathcal{X}}^{p-i} \otimes (f^* \Omega_C^1)^{\otimes i})$.

3.4. Asymptotic behavior of the Quillen metric on $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_{\Delta}$. Via the canonical isomorphism (3.6), the line bundle $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_{\Delta^*}$ is endowed with the Quillen metric $\|\cdot\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), Q}$. Following [7, Sect. 5], we recall the singularity of $\|\cdot\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), Q}$ at $t = 0$. For this, we introduce some tautological vector bundles over the projective bundle $\mathbf{P}(T\mathcal{X})^\vee$.

Let $\mathbf{P}(T\mathcal{X})^\vee$ be the projective bundle over \mathcal{X} with projection $\Pi: \mathbf{P}(T\mathcal{X})^\vee \rightarrow \mathcal{X}$ such that $\Pi^{-1}(x) = \mathbf{P}(T_x\mathcal{X})^\vee$. Here, for a complex vector space V , $\mathbf{P}(V)^\vee$ denotes the projective space of hyperplanes of V passing through the origin. Then we have the canonical isomorphism $\mathbf{P}(T\mathcal{X})^\vee \cong \mathbf{P}(\Omega_{\mathcal{X}}^1)$. We define the *Gauss map* $\mu: \mathcal{X} \setminus \Sigma_f \rightarrow \mathbf{P}(T\mathcal{X})^\vee$ by

$$\mu(x) := [T_x X_{f(x)}],$$

where $[T_x X_{f(x)}] \in \mathbf{P}(T_x\mathcal{X})^\vee$ is the point corresponding to the hyperplane $T_x X_{f(x)} \subset T_x\mathcal{X}$. Then μ is a meromorphic map from \mathcal{X} to $\mathbf{P}(T\mathcal{X})^\vee$. Let

$$\sigma: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$$

be a resolution of the indeterminacy of μ . Namely, there exists a birational holomorphic map $\sigma: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ inducing an isomorphism between $\tilde{\mathcal{X}} \setminus \sigma^{-1}(\Sigma_f)$ and $\mathcal{X} \setminus \Sigma_f$ such that the composite morphism

$$\tilde{\mu} := \mu \circ \sigma$$

extends to a holomorphic map from $\tilde{\mathcal{X}}$ to $\mathbf{P}(T\mathcal{X})^\vee$. We set

$$\text{Exc}(\sigma) := \sigma^{-1}(\Sigma_f).$$

Without loss of generality, we may and will assume that $\text{Exc}(\sigma)$ is a normal crossing divisor of $\tilde{\mathcal{X}}$.

Remark 3.4. Since $f: \mathcal{X}|_{\Delta} \rightarrow \Delta$ is a semi-stable degeneration, for any $p \in X_0$, there is a system of local coordinates (z_0, z_1, z_2, z_3) of \mathcal{X} centered at p such that

$$f(z) = z_0 \cdots z_k \quad (k \leq 3).$$

Near p , the Gauss map μ is expressed as the following explicit meromorphic map

$$\mu(z) = \left(\frac{1}{z_0} : \cdots : \frac{1}{z_k} : 0 : \cdots : 0 \right).$$

One can resolve the indeterminacy of μ in the canonical way as follows.

Let $X_0 = E_1 + \cdots + E_m$ be the irreducible decomposition. For $i_1 < \cdots < i_k$, we set $E_{i_1 \dots i_k} := E_{i_1} \cap \cdots \cap E_{i_k}$. Then $E_{i_1 \dots i_k}$ is a (possibly disconnected) submanifold of \mathcal{X} and $E_{i_1 \dots i_k} = \emptyset$ for $k \geq 5$. The indeterminacy locus of μ is given by $\bigcup_{i < j} E_{ij}$. Let $\sigma^{(1)}: \mathcal{X}^{(1)} \rightarrow \mathcal{X}$ be the blowing-up of $\bigcup_{i < j < k < l} E_{ijkl}$ and set $\mu^{(1)} := \mu \circ \sigma^{(1)}$. Let $E_i^{(1)}$ be the proper transform of E_i and set $E_{i_1 \dots i_k}^{(1)} := E_{i_1}^{(1)} \cap \cdots \cap E_{i_k}^{(1)}$ for $i_1 < \cdots < i_k$. Then $E_{i_1 \dots i_k}^{(1)} = \emptyset$ for $k \geq 4$ and $E_{i_1 \dots i_k}^{(1)}$ is the proper transform of $E_{i_1 \dots i_k}$ for $k \leq 3$. The indeterminacy locus of $\mu^{(1)}$ is given by $\bigcup_{i < j} E_{ij}^{(1)}$. Let $\sigma^{(2)}: \mathcal{X}^{(2)} \rightarrow \mathcal{X}^{(1)}$ be the blowing-up of $\bigcup_{i < j < k} E_{ijk}^{(1)}$ and set $\mu^{(2)} := \mu^{(1)} \circ \sigma^{(2)}$. Let $E_i^{(2)}$ be the proper transform of $E_i^{(1)}$ and set $E_{i_1 \dots i_k}^{(2)} := E_{i_1}^{(2)} \cap \cdots \cap E_{i_k}^{(2)}$ for $i_1 < \cdots < i_k$. Then $E_{i_1 \dots i_k}^{(2)} = \emptyset$ for $k \geq 3$ and $E_{i_1 \dots i_k}^{(2)}$ is the proper transform of $E_{i_1 \dots i_k}^{(1)}$ for $k \leq 2$. The indeterminacy locus of $\mu^{(2)}$ is given by $\bigcup_{i < j} E_{ij}^{(2)}$. Finally, let $\sigma^{(3)}: \mathcal{X}^{(3)} \rightarrow \mathcal{X}^{(2)}$ be the blowing-up of $\bigcup_{i < j} E_{ij}^{(2)}$ and set $\mu^{(3)} := \mu^{(2)} \circ \sigma^{(3)}$. Then $\mu^{(3)}: \mathcal{X}^{(3)} \rightarrow \mathbf{P}(T\mathcal{X})^\vee$ is regular. Setting $\tilde{\mathcal{X}} := \mathcal{X}^{(3)}$, $\sigma := \sigma^{(1)} \circ \sigma^{(2)} \circ \sigma^{(3)}$ and $\tilde{\mu} := \mu \circ \sigma$, we get a resolution of the indeterminacy of $\mu: \mathcal{X} \dashrightarrow \mathbf{P}(T\mathcal{X})^\vee$.

Let U be the universal hyperplane bundle over $\mathbf{P}(T\mathcal{X})^\vee$ and let H be the universal quotient line bundle over $\mathbf{P}(T\mathcal{X})^\vee$. Then we have the following exact sequence of holomorphic vector bundles over $\mathbf{P}(T\mathcal{X})^\vee$:

$$0 \rightarrow U \rightarrow \Pi^* T\mathcal{X} \rightarrow H \rightarrow 0.$$

After [7, Th. 5.4], we introduce the rational number $a_p(f, \Sigma_f) \in \mathbf{Q}$ by

$$a_p(f, \Sigma_f) := \sum_{j=0}^p (-1)^{p-j} \int_{\text{Exc}(\tilde{\sigma})} \tilde{\mu}^* \left\{ \text{Td}(U) \frac{\text{Td}(c_1(H)) - e^{-(p-j)c_1(H)}}{c_1(H)} \right\} \sigma^* \text{ch}(\Omega_{\mathcal{X}}^j).$$

Then $a_p(f, \Sigma_f)$ is determined by the function germ of $f: \mathcal{X}|_\Delta \rightarrow \Delta$ near Σ_f .

Theorem 3.5. *Let $0 \leq p \leq 3$ and let ς_p be a nowhere vanishing holomorphic section of $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_\Delta$. Then the following holds as $t \rightarrow 0$:*

$$\log \|\varsigma_p(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), \mathbf{Q}}^2 = a_p(f, \Sigma_f) \log |t|^2 + O(1).$$

Proof. See [7, Th. 5.4]. □

3.5. Asymptotic behavior of the L^2 -metric on $\lambda(\Omega_{\mathcal{X}/C}^p)|_{\Delta^*}$: the cases $p = 2, 3$. Following [7, Th. 8.1 and Prop. 9.6], we determine the singularity of the L^2 -metric on $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_\Delta$ for the remaining cases $p = 2, 3$.

Proposition 3.6. *Let ς_p be a nowhere vanishing holomorphic section of $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_\Delta$.*

(1) *When $p = 2$, the following holds as $t \rightarrow 0$:*

$$\begin{aligned} \log \|\varsigma_2(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^2), L^2}^2 &= \{a_2(f, \Sigma_f) - a_1(f, \Sigma_f) + \chi(Rf_* \mathcal{Q}|_\Delta)\} \log |t|^2 \\ &\quad + O(\log(-\log |t|)). \end{aligned}$$

(2) When $p = 3$, the following holds as $t \rightarrow 0$:

$$\log \|\varsigma_3(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^3), L^2}^2 = \{a_3(f, \Sigma_f) - a_0(f, \Sigma_f)\} \log |t|^2 + O(\log(-\log |t|)).$$

Proof. Let $0 \leq p \leq 3$. Then we have

$$\begin{aligned} & \log \|\varsigma_p(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), L^2}^2 - \log \|\varsigma_{3-p}(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^{3-p}), L^2}^2 \\ &= \log \|\varsigma_p(t) \otimes \varsigma_{3-p}(t)^\vee\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p) \otimes \lambda(\mathcal{E}_{\mathcal{X}/C}^{3-p})^\vee, L^2}^2 \\ (3.8) \quad &= \log \|\varsigma_p(t) \otimes \varsigma_{3-p}(t)^\vee\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p) \otimes \lambda(\mathcal{E}_{\mathcal{X}/C}^{3-p})^\vee, Q}^2 \\ &= \log \|\varsigma_p(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), Q}^2 - \log \|\varsigma_{3-p}(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^{3-p}), Q}^2 \\ &= \{a_p(f, \Sigma_f) - a_{3-p}(f, \Sigma_f)\} \log |t|^2 + O(1), \end{aligned}$$

where the second equality follows from [7, Eq. (8.4)] and the last equality follows from Theorem 3.5. The result for $p = 2$ follows from (3.8) and Proposition 3.2. The result for $p = 3$ follows from (3.8) and Proposition 3.1. \square

3.6. Asymptotic behavior of BCOV torsion. Following [7, Th. 8.2 and Sect. 9.2], we determine the singularity of $T_{\text{BCOV}}(X_t, g_t)$ as $t \rightarrow 0$.

Theorem 3.7. *The following holds as $t \rightarrow 0$:*

$$\begin{aligned} \log T_{\text{BCOV}}(X_t, g_t) &= \{-3a_0(f, \Sigma_f) + 2a_1(f, \Sigma_f) - \chi(Rf_* \mathcal{Q}|_\Delta)\} \log |t|^2 \\ &\quad + O(\log(-\log |t|)). \end{aligned}$$

Proof. For simplicity, write a_p for $a_p(f, \Sigma_f)$. Let ς_p be a nowhere vanishing holomorphic section of $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_\Delta$. By Theorem 3.5, we get

$$(3.9) \quad \sum_{p=0}^3 (-1)^p p \log \|\varsigma_p(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), Q}^2 = \left(\sum_{p=0}^3 (-1)^p p a_p \right) \log |t|^2 + O(1).$$

By Propositions 3.1, 3.2, 3.6, we get

$$\begin{aligned} (3.10) \quad & \sum_{p=0}^3 (-1)^p p \log \|\varsigma_p(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), L^2}^2 = \{3a_0 - 2a_1 + 2a_2 - 3a_3 + \chi(Rf_* \mathcal{Q}|_\Delta)\} \log |t|^2 \\ & \quad + O(\log(-\log |t|)). \end{aligned}$$

By the definition of Quillen metrics, we have

$$(3.11) \quad \log T_{\text{BCOV}}(X_t, g_t) = \sum_{p=0}^3 (-1)^p p \left\{ \log \|\varsigma_p(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), Q}^2 - \log \|\varsigma_p(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), L^2}^2 \right\}.$$

Substituting (3.9), (3.10) into (3.11), we get the result. \square

3.7. Asymptotic behavior of the Bott-Chern term. Following [7, Prop. 7.9], we determine the singularity of $A(X_t, g_t)$ as $t \rightarrow 0$.

Let $\Xi \in \Gamma(\mathcal{X}|_\Delta, K_{\mathcal{X}})$ be a canonical form on $\mathcal{X}|_\Delta$ such that (cf. [7, Lemma 7.7])

$$(3.12) \quad \text{div}(\Xi) \subset X_0.$$

Let ω_{X_t} be the dualizing sheaf of X_t . Then $\omega_{X_t} \cong K_{\mathcal{X}/C}|_{X_t}$ for all $t \in \Delta$. We define $\eta_t \in H^0(X_t, \omega_{X_t})$ as the canonical form on X_t such that

$$\Xi|_{X_t} = \eta_t \wedge df.$$

Let $\eta_{\mathcal{X}/\Delta} \in \Gamma(\Delta, f_*K_{\mathcal{X}/C})$ be the section defined by $\eta(t) = \eta_t$ for all $t \in \Delta$. Since the family $f: \mathcal{X}/\Delta \rightarrow \Delta$ is a semi-stable degeneration, $\eta_{\mathcal{X}/\Delta}$ is regarded as a holomorphic section of the Hodge bundle $\mathcal{F}^3 \subset \mathcal{H}^3$. If Ξ vanishes identically on X_0 , then there exists $\nu \in \mathbf{Z}_{>0}$ such that $t^{-\nu}\eta_{\mathcal{X}/\Delta}$ is a nowhere vanishing holomorphic section of \mathcal{F}^3 . Replacing Ξ by $f^*t^{-\nu} \cdot \Xi$ in this case, we may and will assume that

$$(3.13) \quad \Xi|_{X_0} \in H^0(X_0, K_{\mathcal{X}}|_{X_0}) \setminus \{0\}.$$

Namely, there is at least one irreducible component of X_0 , on which Ξ does not vanish. Then $\eta_{\mathcal{X}/\Delta}$ is a nowhere vanishing holomorphic section of \mathcal{F}^3 and hence

$$(3.14) \quad \log \|\eta_{\mathcal{X}/\Delta}(t)\|_{L^2} = O(\log(-\log|t|)) \quad (t \rightarrow 0)$$

by Proposition 2.2.

Lemma 3.8. *The divisor $\text{div}(\Xi)$ is independent of the choice of $\Xi \in \Gamma(\mathcal{X}/\Delta, K_{\mathcal{X}/C})$ satisfying (3.12), (3.13).*

Proof. Let $\Xi \otimes (f^*dt)^{-1}$ and $\Xi' \otimes (f^*dt)^{-1}$ be holomorphic 4-forms on \mathcal{X} satisfying (3.12), (3.13). Then the ratio Ξ/Ξ' descends to a nowhere vanishing holomorphic function on Δ^* . Since both Ξ and Ξ' correspond to nowhere vanishing holomorphic section of the *line* bundle \mathcal{F}^3 , we conclude that Ξ/Ξ' is a nowhere vanishing holomorphic function on Δ . Hence $\text{div}(\Xi) = \text{div}(\Xi')$. \square

After Lemma 3.8, the following definition makes sense.

Definition 3.9. The *normalized canonical divisor* $\mathfrak{K}_{(\mathcal{X}, X_0)}$ of (\mathcal{X}, X_0) is defined as

$$\mathfrak{K}_{(\mathcal{X}, X_0)} := \text{div}(\Xi), \quad \text{Supp}(\mathfrak{K}_{(\mathcal{X}, X_0)}) \subsetneq \text{Supp}(X_0),$$

where Ξ satisfies (3.12), (3.13).

To describe the asymptotic behavior of $A(X_t, g_t)$ as $t \rightarrow 0$, we use the notation in Section 3.4. Recall that $\sigma: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ is a resolution of the indeterminacy of the Gauss map $\mu: \mathcal{X} \setminus \Sigma_f \rightarrow \mathbf{P}(T\mathcal{X})^\vee$ as in Remark 3.4, that $\tilde{\mu}: \tilde{\mathcal{X}} \rightarrow \mathbf{P}(T\mathcal{X})^\vee$ is the resolved Gauss map, and that $U \rightarrow \mathbf{P}(T\mathcal{X})^\vee$ is the universal hyperplane bundle.

We set

$$\tilde{f} := f \circ \sigma$$

and we get a new family $\tilde{f}: \tilde{\mathcal{X}} \rightarrow C$, whose central fiber $\tilde{X}_0 := \tilde{f}^{-1}(0)$ is a possibly *non-reduced* normal crossing divisor. Hence $\tilde{f}: \tilde{\mathcal{X}} \rightarrow C$ is not necessarily a semi-stable degeneration. Let $\Sigma_{\tilde{f}}$ be the divisor of $\tilde{\mathcal{X}}$ defined as the critical locus of \tilde{f} : If $\tilde{X}_0 = \sum_{i=1}^k m_i E_i$ with E_i being an irreducible divisor of $\tilde{\mathcal{X}}$ and $m_i \in \mathbf{Z}_{>0}$, then

$$\Sigma_{\tilde{f}} := \text{div}(d\tilde{f}) = \sum_{i=1}^k (m_i - 1)E_i.$$

Since the resolution $\sigma: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ is canonically defined, $\Sigma_{\tilde{f}}$ is determined by the function germ of f near Σ_f .

Proposition 3.10. *The following holds as $t \rightarrow 0$:*

$$\log A(X_t, g_t) = -\frac{1}{12} \left(\int_{\sigma^* \mathfrak{R}(\mathcal{X}, X_0) - \Sigma_{\tilde{f}}} \tilde{\mu}^* c_3(U) \right) \log |t|^2 + O(\log(-\log |t|)).$$

Proof. Let $\chi(X_{\text{gen}})$ be the topological Euler number of a general fiber of $f: \mathcal{X} \rightarrow C$. Let g_U be the Hermitian metric on U induced from the Hermitian metric $\Pi^* g^{\mathcal{X}}$ on $\Pi^* T\mathcal{X}$ via the inclusion $U \subset \Pi^* T\mathcal{X}$ and let $c_3(U)$ be the top Chern form of (U, g_U) . Define the function $A(\mathcal{X}/\Delta)$ on Δ^* by $A(\mathcal{X}/\Delta)(t) := A(X_t, g_t)$. By [7, Eq. (7.12)], we have

$$\begin{aligned} \log A(\mathcal{X}/\Delta) &= -\frac{1}{12} \tilde{f}_* \left[\log \sigma^* \left(\frac{\|\Xi\|^2}{\|df\|^2} \right) \tilde{\mu}^* c_3(U, g_U) \right] + \frac{\chi(X_{\text{gen}})}{12} \log \|\eta_{\mathcal{X}/\Delta}\|_{L^2}^2 \\ &= -\frac{1}{12} \tilde{f}_* \left[\log \left(\frac{\|\sigma^* \Xi\|^2}{\|df\|^2} \right) \tilde{\mu}^* c_3(U, g_U) \right] + O(\log(-\log |t|)) \\ &= -\frac{1}{12} \left(\int_{\sigma^* \mathfrak{R}(\mathcal{X}, X_0) - \Sigma_{\tilde{f}}} \tilde{\mu}^* c_3(U, g_U) \right) \log |t|^2 + O(\log(-\log |t|)), \end{aligned}$$

where the second equality follows from (3.14) and the third equality follows from [18, Cor. 4.6] and the equalities of divisors $\text{div}(\sigma^* \Xi) = \sigma^* \mathfrak{R}(\mathcal{X}, X_0)$, $\Sigma_{\tilde{f}} = \text{div}(d\tilde{f})$ on $\tilde{\mathcal{X}}$. This completes the proof. \square

3.8. Asymptotic behavior of BCOV invariants for semi-stable degenerations. Define

$$\rho(f, \Sigma_f) := -3a_0(f, \Sigma_f) + 2a_1(f, \Sigma_f) - \chi(Rf_* \mathcal{Q}|_{\Delta}) + \frac{1}{12} \int_{\Sigma_{\tilde{f}}} \tilde{\mu}^* c_3(U) \in \mathbf{Q},$$

$$\kappa(f, \Sigma_f, \mathfrak{R}(\mathcal{X}, X_0)) := \int_{\pi^* \mathfrak{R}(\mathcal{X}, X_0)} \tilde{\mu}^* c_3(U) \in \mathbf{Z}.$$

Since there is a canonical way of resolving the indeterminacy of the Gauss map μ for the semi-stable degeneration $f: \mathcal{X}|_{\Delta} \rightarrow \Delta$ as explained in Remark 3.4, $\rho(f, \Sigma_f)$ is determined by the function germ of f near Σ_f , whereas $\kappa(f, \Sigma_f, \mathfrak{R}(\mathcal{X}, X_0))$ is determined by the function germ of f near Σ_f and the normalized canonical divisor $\mathfrak{R}(\mathcal{X}, X_0)$.

Theorem 3.11. *The following holds as $t \rightarrow 0$:*

$$\log \tau_{\text{BCOV}}(X_t) = \left\{ \rho(f, \Sigma_f) - \frac{1}{12} \kappa(f, \Sigma_f, \mathfrak{R}(\mathcal{X}, X_0)) \right\} \log |t|^2 + O(\log(-\log |t|)).$$

Proof. Since the Kähler metric g_t is induced from the Kähler metric $g^{\mathcal{X}}$ on \mathcal{X} , the functions on Δ^*

$$t \mapsto \text{Vol}(X_t, g_t), \quad t \mapsto \text{Vol}_{L^2}(H^2(X_t, \mathbf{Z}), [c_1(\mathcal{L}_t)])$$

are constant by [7, Lemma 4.12]. Hence there is a constant $C > 0$ such that

$$(3.15) \quad \log \tau_{\text{BCOV}}(X_t) = \log T_{\text{BCOV}}(X_t, g_t) + \log A(X_t, g_t) + C$$

for all $t \in \Delta^*$. Substituting the formulae in Theorem 3.7 and Proposition 3.10 into (3.15), we get the result. \square

3.9. Asymptotic behavior of BCOV invariants for general degenerations.

By Theorem 3.11, we get the rationality of the coefficient of the logarithmic divergence of $\log \tau_{\text{BCOV}}$ for general one-parameter degenerations. In this subsection, we do not assume that $f: \mathcal{X}|_{\Delta} \rightarrow \Delta$ is a semi-stable degeneration.

Theorem 3.12. *Let $f: \mathcal{X} \rightarrow C$ be a surjective morphism from an irreducible projective fourfold \mathcal{X} to a compact Riemann surface C . If there is a finite subset $\Delta_f \subset C$ such that $f|_{C \setminus \Delta_f}: \mathcal{X}|_{C \setminus \Delta_f} \rightarrow C \setminus \Delta_f$ is a smooth morphism and such that $X_t := f^{-1}(t)$ is a Calabi-Yau threefold for all $t \in C \setminus \Delta_f$, then for every $0 \in \Delta_f$, there exists a rational number $\alpha \in \mathbf{Q}$ such that*

$$\log \tau_{\text{BCOV}}(X_t) = \alpha \log |t|^2 + O(\log(-\log |t|)) \quad (t \rightarrow 0),$$

where t is a local parameter of C centered at 0. Let $g: (\mathcal{Y}, Y_0) \rightarrow (B, 0)$ be a semi-stable reduction of $f: (\mathcal{X}, X_0) \rightarrow (C, 0)$:

$$\begin{array}{ccc} (\mathcal{Y}, Y_0) & \xrightarrow{\Phi} & (\mathcal{X}, X_0) \\ g \downarrow & & \downarrow f \\ (B, 0) & \xrightarrow{\phi} & (C, 0). \end{array}$$

Then α is given by

$$\alpha = \frac{1}{\deg\{\phi: (B, 0) \rightarrow (C, 0)\}} \left\{ \rho(g, \Sigma_g) - \frac{1}{12} \kappa(g, \Sigma_g, \mathfrak{K}_{(\mathcal{Y}, Y_0)}) \right\}.$$

Proof. By the definition of semi-stable reduction [13, Chap. II], \mathcal{Y} is a smooth projective fourfold and B is a compact Riemann surface such that $\mathcal{Y}|_{B^*} \cong \mathcal{X}|_{C^*} \times_{C^*} B^*$ and the divisor $Y_0 = g^{-1}(0)$ is reduced and normal crossing. Here we set $B^* := B \setminus \{0\}$ and $C^* := C \setminus \{0\}$. By choosing an appropriate local parameter s of $(B, 0)$, we may assume that $\phi(s) = s^\nu$. Since $Y_s \cong X_{\phi(s)} = X_{s^\nu}$, the result follows from Theorem 3.11 applied to the semi-stable degeneration of Calabi-Yau threefolds $g: (\mathcal{Y}, Y_0) \rightarrow (B, 0)$. \square

In [7, Th. 9.1], a weaker version of Theorem 3.12 was proved, where α was inexplicit and real.

4. A LOCALITY OF THE LOGARITHMIC SINGULARITY

In this section, we prove a certain locality of the coefficient α in Theorem 3.12.

Set up Let \mathcal{X} and \mathcal{X}' be normal irreducible projective fourfolds. Let C and C' be compact Riemann surfaces. Let $f: \mathcal{X} \rightarrow C$ and $f': \mathcal{X}' \rightarrow C'$ be surjective holomorphic maps. Let $\overline{\Sigma}_f|_{\mathcal{X} \setminus \text{Sing } \mathcal{X}}$ (resp. $\overline{\Sigma}_{f'}|_{\mathcal{X}' \setminus \text{Sing } \mathcal{X}'}$) be the closure of the critical locus of $f|_{\mathcal{X} \setminus \text{Sing } \mathcal{X}}$ (resp. $f'|_{\mathcal{X}' \setminus \text{Sing } \mathcal{X}'}$) in \mathcal{X} (resp. \mathcal{X}'). Define the critical loci of f and f' as

$$\Sigma_f := \text{Sing } \mathcal{X} \cup \overline{\Sigma}_f|_{\mathcal{X} \setminus \text{Sing } \mathcal{X}}, \quad \Sigma_{f'} := \text{Sing } \mathcal{X}' \cup \overline{\Sigma}_{f'}|_{\mathcal{X}' \setminus \text{Sing } \mathcal{X}'}$$

and the discriminant loci of f and f' as

$$\Delta_f := f(\Sigma_f), \quad \Delta_{f'} := f'(\Sigma_{f'}).$$

Let $0 \in \Delta_f$ and $0' \in \Delta_{f'}$. Let V (resp. V') be a neighborhood of 0 (resp. $0'$) in C (resp. C') such that $V \cong \Delta$ and $V \cap \Delta_f = \{0\}$ (resp. $V' \cong \Delta$ and $V' \cap \Delta_{f'} = \{0'\}$).

In the rest of this section, we make the following:

Assumption

- (A1) $\Delta_f \neq C$, $\Delta_{f'} \neq C'$, $\dim \Sigma_f \leq 2$, $\dim \Sigma_{f'} \leq 2$, and X_0 and X'_0 are irreducible.
- (A2) X_t and X'_t are Calabi-Yau threefolds for all $t \in C \setminus \Delta_f$ and $t' \in C' \setminus \Delta_{f'}$.
- (A3) $f^{-1}(V) \setminus \Sigma_f$ carries a nowhere vanishing canonical form Ξ . Similarly, $(f')^{-1}(V') \setminus \Sigma_{f'}$ carries a nowhere vanishing canonical form Ξ' .
- (A4) The function germ of f near $\Sigma_f \cap f^{-1}(V)$ and the function germ of f' near $\Sigma_{f'} \cap (f')^{-1}(V')$ are isomorphic. Namely, there exist a neighborhood O of $\Sigma_f \cap f^{-1}(V)$ in $f^{-1}(V)$, a neighborhood O' of $\Sigma_{f'} \cap (f')^{-1}(V')$ in $(f')^{-1}(V')$, and an isomorphism $\varphi: O \rightarrow O'$ such that $f|_O = f' \circ \varphi|_{O'}$.

By (A3), (A4), the ratio $\varphi^*(\Xi'|_{O'})/(\Xi|_O)$ is a nowhere vanishing holomorphic function on $O \setminus \Sigma_f$. By (A1) and the normality of \mathcal{X} , the ratio $\varphi^*(\Xi'|_{O'})/(\Xi|_O)$ extends to a nowhere vanishing holomorphic function on O . Hence we have the following equality of divisors on O

$$(4.1) \quad \operatorname{div}(\Xi) = \varphi^* \operatorname{div}(\Xi').$$

For $z \in C$ and $z' \in C'$, we set $X_z := f^{-1}(z)$ and $X'_{z'} := (f')^{-1}(z')$. For $z \in C \setminus \Delta_f$ and $z' \in C' \setminus \Delta_{f'}$, the BCOV invariants $\tau_{\text{BCOV}}(X_z)$ and $\tau_{\text{BCOV}}(X'_{z'})$ are well defined. Let $0 \in \Delta_f$ and $0' \in \Delta_{f'}$. A local parameter of C (resp. C') centered at 0 (resp. $0'$) is denoted by t . Hence t is a generator of the maximal ideal of $\mathcal{O}_{C,0}$ and $\mathcal{O}_{C',0'}$. By Theorem 3.12, the functions $t \mapsto \log \tau_{\text{BCOV}}(X_t)$ and $t \mapsto \log \tau_{\text{BCOV}}(X'_t)$ have logarithmic singularities at 0 and $0'$, respectively.

Theorem 4.1. *Under (A1)–(A4), $\log \tau_{\text{BCOV}}(X_t)$ and $\log \tau_{\text{BCOV}}(X'_t)$ have the same logarithmic singularities at $t = 0$:*

$$\lim_{t \rightarrow 0} \frac{\log \tau_{\text{BCOV}}(X_t)}{\log |t|} = \lim_{t \rightarrow 0} \frac{\log \tau_{\text{BCOV}}(X'_t)}{\log |t|}.$$

In particular,

$$\log \tau_{\text{BCOV}}(X_t) - \log \tau_{\text{BCOV}}(X'_t) = O(\log(-\log |t|)) \quad (t \rightarrow 0).$$

Proof. (Step 1) By Hironaka, there exists a succession of blowing-ups $\sigma: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ inducing an isomorphism between $\tilde{\mathcal{X}} \setminus \sigma^{-1}(\Sigma_f)$ and $\mathcal{X} \setminus \Sigma_f$ such that $\tilde{X}_0 := (f \circ \sigma)^{-1}(0)$ is a normal crossing divisor of $\tilde{\mathcal{X}}$. Let $\tilde{X}_0 = D_0 \cup D_1 \cup \cdots \cup D_l$ be the irreducible decomposition. We may and will assume that all D_α 's are smooth hypersurfaces of $\tilde{\mathcal{X}}$ and that D_0 is the proper transform of X_0 . Then $D_1 \cup \cdots \cup D_l = \sigma^{-1}(\Sigma_f) \subset \sigma^{-1}(O)$, and σ induces an isomorphism from $D_0 \setminus \bigcup_{\alpha > 0} D_\alpha$ to $X_0 \setminus \Sigma_f$.

Identify the pairs (O, Σ_f) and $(O', \Sigma_{f'})$ via φ . We set

$$\tilde{\mathcal{X}}' := (\mathcal{X}' \setminus \Sigma_{f'}) \cup \sigma^{-1}(O),$$

where $\sigma^{-1}(O \setminus \Sigma_f)$ and $O' \setminus \Sigma_{f'}$ are identified by the isomorphism $\varphi \circ \sigma$. Then $\tilde{\mathcal{X}}'$ is a smooth fourfold equipped with the projection $\sigma': \tilde{\mathcal{X}}' \rightarrow \mathcal{X}'$ defined by $\sigma' := \text{id}$ on $\mathcal{X}' \setminus \Sigma_{f'}$ and by $\varphi \circ \sigma$ on $\sigma^{-1}(O)$. Since $\sigma: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ is a succession of blowing-ups, so is $\sigma': \tilde{\mathcal{X}}' \rightarrow \mathcal{X}'$. We define $\tilde{f} := f \circ \sigma$ and $\tilde{f}' := f' \circ \sigma'$, whose critical loci are denoted by $\Sigma_{\tilde{f}}$ and $\Sigma_{\tilde{f}'}$, respectively. Then $\Sigma_{\tilde{f}} \subset \sigma^{-1}(O)$ and $\Sigma_{\tilde{f}'} \subset (\sigma')^{-1}(O')$.

We set $\tilde{\varphi} := \text{id}_{\sigma^{-1}(O)}$. Then φ lifts to an isomorphism $\tilde{\varphi}: \sigma^{-1}(O) \cong (\sigma')^{-1}(O')$ such that $\tilde{f}' \circ \tilde{\varphi} = \tilde{f}$ on $\sigma^{-1}(O)$. Let $D'_0 \subset \tilde{\mathcal{X}}'$ be the proper transform of $X'_0 \subset \mathcal{X}'$.

Since $\tilde{f}^{-1}(0) = D_0 \cup D_1 \cup \cdots \cup D_l$ and $\sigma^{-1}(\Sigma_f) = D_1 \cup \cdots \cup D_l$, we have the irreducible decomposition $(\tilde{f}')^{-1}(0') = D'_0 \cup D'_1 \cup \cdots \cup D'_l$ with $(\sigma')^{-1}(\Sigma_{f'}) = D'_1 \cup \cdots \cup D'_l \subset (\sigma')^{-1}(O)$, where we set $D'_\alpha := \tilde{\varphi}(D_\alpha)$. By (4.1) and the equality $\sigma' \circ \tilde{\varphi} = \varphi \circ \sigma$, we get

$$(4.2) \quad \operatorname{div}(\sigma^* \Xi) = \tilde{\varphi}^* \operatorname{div}((\sigma')^* \Xi') \subset \sigma^{-1}(O).$$

(Step 2) Let $d \in \mathbf{Z}_{>0}$. Let $\pi_d: (C_d, 0_d) \rightarrow (C, 0)$ (resp. $\pi'_d: (C'_d, 0'_d) \rightarrow (C', 0')$) be a ramified covering with ramification index d at $0_d \in C_d$ (resp. $0'_d \in C'_d$). Let $\tilde{\mathcal{X}}_d$ (resp. $\tilde{\mathcal{X}}'_d$) be the normalization of the fibered product $\tilde{\mathcal{X}} \times_C C_d$ (resp. $\tilde{\mathcal{X}}' \times_{C'} C'_d$) and set $\tilde{f}_d := \operatorname{pr}_2: \tilde{\mathcal{X}}_d \rightarrow C_d$ (resp. $\tilde{f}'_d := \operatorname{pr}_2: \tilde{\mathcal{X}}'_d \rightarrow C'_d$). Let \tilde{O}_d (resp. \tilde{O}'_d) be the open subset of $\tilde{\mathcal{X}}_d$ (resp. $\tilde{\mathcal{X}}'_d$) defined as $\operatorname{pr}_1^{-1}(\sigma^{-1}(O))$ (resp. $(\operatorname{pr}'_1)^{-1}((\sigma')^{-1}(O'))$). Then $\tilde{\varphi}: \sigma^{-1}(O) \cong (\sigma')^{-1}(O')$ lifts to an isomorphism $\tilde{\varphi}_d: \tilde{O}_d \cong \tilde{O}'_d$ such that $\tilde{f}_d = \tilde{f}'_d \circ \tilde{\varphi}_d$.

Define $U_d := \tilde{\mathcal{X}}_d \setminus \tilde{f}_d^{-1}(0_d)$ and $U'_d := \tilde{\mathcal{X}}'_d \setminus (\tilde{f}'_d)^{-1}(0'_d)$. By [13, Chap. II, §3], the pairs $(\tilde{\mathcal{X}}_d, U_d)$ and $(\tilde{\mathcal{X}}'_d, U'_d)$ are toroidal embeddings. (See [13, Chap. II §1] for the notion of toroidal embeddings.) Let $\tilde{f}_d^{-1}(0_d) = E_0 \cup E_1 \cup \cdots \cup E_m$ (resp. $(\tilde{f}'_d)^{-1}(0'_d) = E'_0 \cup E'_1 \cup \cdots \cup E'_m$) be the irreducible decomposition, where E_0 (resp. E'_0) is the component corresponding to the proper transforms of X_0 (resp. X'_0) in $\tilde{\mathcal{X}}$ (resp. $\tilde{\mathcal{X}}'$). Then $m \geq n$. Since $X_0 \setminus \Sigma_f \cong D_0 \setminus \bigcup_{\alpha>0} D_\alpha$ is reduced and smooth, we have $E_0 \setminus \bigcup_{\beta>0} E_\beta \cong D_0 \setminus \bigcup_{\alpha>0} D_\alpha \cong X_0 \setminus \Sigma_f$. When $\beta > 0$, we deduce from [13, Chap. II §3] that $\operatorname{pr}_1(E_\beta) = D_{\alpha(\beta)}$ for some $\alpha(\beta) > 0$. Similarly, $E'_0 \setminus \bigcup_{\beta>0} E'_\beta \cong X'_0 \setminus \Sigma_{f'}$ and $\operatorname{pr}_1(E'_\beta) = D'_{\alpha(\beta)}$ ($\alpha(\beta) > 0$) for $\beta > 0$.

(Step 3) By an appropriate choice of $d \in \mathbf{Z}_{>0}$, there exists a sheaf of ideals $\mathcal{I}_d \subset \mathcal{O}_{\tilde{\mathcal{X}}_d}$ with $\mathcal{I}_d|_{U_d} = \mathcal{O}_{U_d}$, whose blowing-up $\varpi: \mathcal{Y}_d \rightarrow \tilde{\mathcal{X}}_d$ provides a semi-stable reduction of $f: (\mathcal{X}, X_0) \rightarrow (C, 0)$, i.e., the following commutative diagram (cf. [13, Chap. II])

$$\begin{array}{ccccccc} \mathcal{Y}_d & \xrightarrow{\varpi} & \tilde{\mathcal{X}}_d & \xrightarrow{\operatorname{pr}_1} & \tilde{\mathcal{X}} & \xrightarrow{\sigma} & \mathcal{X} \\ g_d \downarrow & & \tilde{f}_d \downarrow & & \tilde{f} \downarrow & & f \downarrow \\ C_d & \xrightarrow{\operatorname{id}} & C_d & \xrightarrow{\pi_d} & C & \xrightarrow{\operatorname{id}} & C, \end{array}$$

where \mathcal{Y}_d is smooth and $g_d^{-1}(0_d)$ is a *reduced*, normal crossing divisor of \mathcal{Y}_d . We define $E_\beta^\circ := E_\beta \setminus \bigcup_{\beta' \neq \beta} E_{\beta'}$. Since the ideal sheaf \mathcal{I}_d is of the form as in [13, p. 91 last line], we deduce from [13, Th. 9*] that there exists $\nu_0 \in \mathbf{Z}$ with

$$(4.3) \quad \mathcal{I}_d|_{U_d \cup E_0^\circ} \cong \mathcal{O}_{U_d \cup E_0^\circ}(\nu_0 E_0^\circ).$$

Since E_0° is a smooth divisor of $\tilde{\mathcal{X}}_d \setminus \bigcup_{\beta>0} E_\beta = U_d \cup E_0^\circ$, we deduce from (4.3) and the definition of blowing-up of sheaf of ideals that the maps $\varpi: \varpi^{-1}(E_0^\circ) \rightarrow E_0^\circ$ and $\varpi: \mathcal{Y}_d \setminus \varpi^{-1}(E_1 \cup \cdots \cup E_m) \rightarrow \tilde{\mathcal{X}}_d \setminus (E_1 \cup \cdots \cup E_m)$ are isomorphisms. Write $g_d^{-1}(0_d) = F_0 + \cdots + F_n$, where every F_i is irreducible and F_0 is the proper transform of E_0 . Then $n \geq m$ and $F_1 \cup \cdots \cup F_n = \varpi^{-1}(E_1 \cup \cdots \cup E_m) \subset \varpi^{-1}(\tilde{O}_d)$.

(Step 4) We define the sheaf of ideals $\mathcal{I}'_d \subset \mathcal{O}_{\tilde{\mathcal{X}}'_d}$ by

$$\mathcal{I}'_d|_{\tilde{\mathcal{X}}'_d \setminus \bigcup_{\beta>0} E'_\beta} = \mathcal{O}_{\tilde{\mathcal{X}}'_d \setminus \bigcup_{\beta>0} E'_\beta}(\nu_0 E'_0), \quad \mathcal{I}'_d|_{O'_d} = (\varphi_d)_* \mathcal{I}_d.$$

Then $\mathcal{I}'_d|_{U'_d} = \mathcal{O}_{U'_d}$. Let $\varpi': \mathcal{Y}' \rightarrow \tilde{\mathcal{X}}'_d$ be the blowing-up of \mathcal{I}'_d and set $g'_d := \tilde{f}'_d \circ \varpi'$. Since the map $\varpi': (\varpi')^{-1}(\tilde{O}'_d) \rightarrow \tilde{O}'_d$ is identified with $\varpi: \varpi^{-1}(\tilde{O}_d) \rightarrow \tilde{O}_d$ via the

identification $\tilde{\varphi}_d: \tilde{O}_d \cong \tilde{O}'_d$, we get the isomorphism of divisors

$$(g'_d)^{-1}(0'_d) \cap (\varpi')^{-1}(\tilde{O}'_d) \cong (F_0 \cap \tilde{O}_d) + F_1 + \cdots + F_n,$$

which is a reduced normal crossing divisor of $\varpi^{-1}(\tilde{O}_d)$. Let $F'_0 \subset (g'_d)^{-1}(0_d)$ be the proper transform of $E'_0 \subset \tilde{\mathcal{X}}'_d$ and let $F'_\gamma \subset (g'_d)^{-1}(0_d)$ be the irreducible component corresponding to F_γ for $\gamma > 0$. Then $F'_0 \cap (\varpi')^{-1}(\tilde{O}'_d) \cong F_0 \cap \tilde{O}_d$. Since the map $\varpi': \mathcal{Y}' \setminus (\varpi')^{-1}(\tilde{O}'_d) \rightarrow \tilde{\mathcal{X}}'_d \setminus \tilde{O}'_d$ is an isomorphism, $F'_0 \setminus (\varpi')^{-1}(\tilde{O}'_d) \cong X'_0 \setminus \tilde{O}'$ is a smooth divisor of $\mathcal{Y}' \setminus (\varpi')^{-1}(\tilde{O}'_d)$. Hence $(g'_d)^{-1}(0_d) = F'_0 + \cdots + F'_n$ is a reduced normal crossing divisor of \mathcal{Y}' . Thus $g'_d: \mathcal{Y}'_d \rightarrow C_d$ is a semi-stable reduction of $f': \mathcal{X}' \rightarrow C'$. We have the following commutative diagram

$$\begin{array}{ccccccc} \mathcal{Y}'_d & \xrightarrow{\varpi'} & \tilde{\mathcal{X}}'_d & \xrightarrow{\text{pr}_1} & \tilde{\mathcal{X}}' & \xrightarrow{\sigma'} & \mathcal{X}' \\ g'_d \downarrow & & \tilde{f}'_d \downarrow & & \tilde{f}' \downarrow & & f' \downarrow \\ C'_d & \xrightarrow{\text{id}} & C'_d & \xrightarrow{\pi_d} & C' & \xrightarrow{\text{id}} & C'. \end{array}$$

(Step 5) Set $O_d := \varpi^{-1}(\tilde{O}_d)$ and $O'_d := (\varpi')^{-1}(\tilde{O}'_d)$. Let $\varphi_d: O_d \cong O'_d$ be the isomorphism induced by $\tilde{\varphi}_d: \tilde{O}_d \cong \tilde{O}'_d$. Since

$$\Sigma_{g_d} \subset O_d, \quad \Sigma_{g'_d} = \varphi_d(\Sigma_{g_d}) \subset O'_d, \quad (g_d, O_d) = (g'_d \circ \varphi_d, O_d)$$

by construction, we get the following equality by the definition in Section 3.8

$$(4.4) \quad \rho(g_d, \Sigma_{g_d}) = \rho(g'_d, \Sigma_{g'_d}).$$

(Step 6) Set $\psi := \sigma \circ \text{pr}_1 \circ \varpi: \mathcal{Y}_d \rightarrow \mathcal{X}$ and $\psi' := \sigma' \circ \text{pr}_1 \circ \varpi': \mathcal{Y}'_d \rightarrow \mathcal{X}'$. Let Υ (resp. Υ') be a canonical form defined near $g_d^{-1}(0_d)$ (resp. $(g'_d)^{-1}(0'_d)$) and satisfying (3.12), (3.13). Then there exist $a_\gamma, a'_\gamma \in \mathbf{Z}_{\geq 0}$ for $0 \leq \gamma \leq n$ such that

$$(4.5) \quad \text{div}(\Upsilon) = \sum_{\gamma=0}^n a_\gamma F_\gamma, \quad \text{div}(\Upsilon') = \sum_{\gamma=0}^n a'_\gamma F'_\gamma.$$

Since $\psi^* \Xi$ (resp. $(\psi')^* \Xi'$) is a (possibly meromorphic) 4-form defined on a neighborhood of $g_d^{-1}(0_d)$ (resp. $(g'_d)^{-1}(0'_d)$), whose possible zeros and poles are supported on $g_d^{-1}(0_d)$ (resp. $(g'_d)^{-1}(0'_d)$), we can express

$$(4.6) \quad \text{div}(\psi^* \Xi) = \sum_{\gamma=0}^n b_\gamma F_\gamma, \quad \text{div}((\psi')^* \Xi') = \sum_{\gamma=0}^n b'_\gamma F'_\gamma,$$

where $b_\gamma, b'_\gamma \in \mathbf{Z}$ for $0 \leq \gamma \leq n$. Since Ξ (resp. Ξ') is nowhere vanishing on $f^{-1}(V) \setminus O$ (resp. $(f')^{-1}(V') \setminus O'$) by assumption and since ψ (resp. ψ') has ramification index d along $F_0 \setminus \bigcup_{\gamma>0} F_\gamma$ (resp. $F'_0 \setminus \bigcup_{\gamma>0} F'_\gamma$), $\psi^* \Xi$ (resp. $(\psi')^* \Xi'$) has zeros of order $d-1$ on the proper transform of E_0 (resp. E'_0). Hence

$$(4.7) \quad b_0 = b'_0 = d - 1.$$

Since the map $\varpi': O'_d \rightarrow \tilde{O}'_d$ is identified with $\varpi: O_d \rightarrow \tilde{O}_d$ via the identification $\tilde{\varphi}_d: \tilde{O}_d \cong \tilde{O}'_d$, we get by (4.2)

$$\text{div}(\psi^* \Xi) \cap O_d = \varphi_d^*(\text{div}((\psi')^* \Xi')) \cap O_d.$$

Hence

$$(4.8) \quad b_\gamma = b'_\gamma \quad (\gamma > 0).$$

Since $\varphi^*(\Xi' \wedge \overline{\Xi}')/(\Xi \wedge \overline{\Xi})$ is a nowhere vanishing positive function on O by (4.1), there exist constants $C_1, C_2 > 0$ such that for all t with $0 < |t| \ll 1$,

$$(4.9) \quad C_1 \int_{X_t \cap O} \sqrt{-1} \frac{\Xi \wedge \overline{\Xi}}{df \wedge d\overline{f}} \leq \int_{X'_t \cap O'} \sqrt{-1} \frac{\Xi' \wedge \overline{\Xi}'}{df' \wedge d\overline{f}'} \leq C_2 \int_{X_t \cap O} \sqrt{-1} \frac{\Xi \wedge \overline{\Xi}}{df \wedge d\overline{f}}.$$

Since $\sqrt{-1}\Xi \wedge \overline{\Xi}$ (resp. $\sqrt{-1}\Xi' \wedge \overline{\Xi}'$) is a volume form on $f^{-1}(V) \setminus O$ (resp. $(f')^{-1}(V') \setminus O'$) and since f (resp. f') has no critical points on $f^{-1}(V) \setminus O$ (resp. $(f')^{-1}(V') \setminus O'$), there exist constants $C_3, C_4 > 0$ such that for all t with $0 < |t| \ll 1$,

$$(4.10) \quad C_3 \int_{X_t \setminus O} \sqrt{-1} \frac{\Xi \wedge \overline{\Xi}}{df \wedge d\overline{f}} \leq \int_{X'_t \setminus O'} \sqrt{-1} \frac{\Xi' \wedge \overline{\Xi}'}{df' \wedge d\overline{f}'} \leq C_4 \int_{X_t \setminus O} \sqrt{-1} \frac{\Xi \wedge \overline{\Xi}}{df \wedge d\overline{f}}.$$

By (4.9), (4.10), there exist constants $C_5, C_6 > 0$ such that for all t with $0 < |t| \ll 1$,

$$(4.11) \quad C_5 \left\| \frac{\Xi}{df} \Big|_{X_t} \right\|_{L^2}^2 \leq \left\| \frac{\Xi'}{df'} \Big|_{X'_t} \right\|_{L^2}^2 \leq C_6 \left\| \frac{\Xi}{df} \Big|_{X_t} \right\|_{L^2}^2.$$

(Step 7) Let s be a local parameter of C_d and C'_d centered at 0_d and $0'_d$. Since $\pi_d: C_d \rightarrow C$ and $\pi'_d: C'_d \rightarrow C'$ has ramification index d at $s = 0$, we may assume

$$(4.12) \quad \pi_d(s) = s^d.$$

By the definition of Υ (resp. Υ'), the map $s \mapsto (\Upsilon/dg_d)|_{Y_s} \in H^0(Y_s, K_{Y_s})$ (resp. $s \mapsto (\Upsilon'/dg'_d)|_{Y'_s} \in H^0(Y'_s, K_{Y'_s})$) is a nowhere vanishing holomorphic section of $(g_d)_*K_{Y_d/C_d}$ (resp. $(g'_d)_*K_{Y'_d/C'_d}$) near 0_d (resp. $0'_d$). By (3.14), we get

$$(4.13) \quad \log \left\| \frac{\Upsilon}{dg_d} \Big|_{Y_s} \right\|_{L^2}^2 = O(\log(-\log|s|)), \quad \log \left\| \frac{\Upsilon'}{dg'_d} \Big|_{Y'_s} \right\|_{L^2}^2 = O(\log(-\log|s|))$$

as $s \rightarrow 0$, where we set $Y_s := g_d^{-1}(s)$ and $Y'_s := (g'_d)^{-1}(s)$.

Since the fibers Y_s and Y'_s are Calabi-Yau threefolds for $s \neq 0$, the map $s \mapsto \psi^*\Xi/dg_d|_{Y_s}$ (resp. $s \mapsto (\psi')^*\Xi'/dg'_d|_{Y'_s}$) is a holomorphic section of $(g_d)_*K_{Y_d/C_d}$ (resp. $(g'_d)_*K_{Y'_d/C'_d}$) near 0_d (resp. $0'_d$). Hence there exist $c, c' \in \mathbf{Z}$ and $\epsilon(s), \epsilon'(s) \in \mathcal{O}(\Delta)$ such that

$$(4.14) \quad \frac{\psi^*\Xi}{\Upsilon} \Big|_{Y_s} = s^c \epsilon(s), \quad \frac{(\psi')^*\Xi'}{\Upsilon'} \Big|_{Y'_s} = s^{c'} \epsilon'(s), \quad \epsilon(0) \neq 0, \quad \epsilon'(0) \neq 0.$$

By (4.5), (4.6), (4.7), (4.8), (4.14), we get

$$(4.15) \quad a_\gamma = b_\gamma + c, \quad a'_\gamma = b'_\gamma + c' = b_\gamma + c'.$$

Since $Y_s = X_{\pi_d(s)} = X_{s^d}$ and $Y'_s = X'_{s^d}$ for $s \neq 0$ by (4.12), we get by (4.13), (4.14), (4.16)

$$\begin{aligned} \log \left\| \frac{\Xi}{df} \Big|_{X_{s^d}} \right\|_{L^2}^2 &= \log \left\| \psi^* \left\{ \frac{\Xi}{df} \Big|_{X_{s^d}} \right\} \right\|_{L^2}^2 = \log \left\| \frac{\Xi}{d(g_d^d)} \Big|_{Y_s} \right\|_{L^2}^2 \\ &= \log \left(|s|^{-2(d-1)} \left\| \frac{\Xi}{dg_d} \Big|_{Y_s} \right\|_{L^2}^2 \right) + O(1) \\ &= -(d-1) \log |s|^2 + \log \left| \frac{\psi^* \Xi}{\Upsilon} \Big|_{Y_s} \right|^2 + \log \left\| \frac{\Upsilon}{dg_d} \Big|_{Y_s} \right\|_{L^2}^2 + O(1) \\ &= (c-d+1) \log |s|^2 + O(\log(-\log |s|)) \quad (s \rightarrow 0). \end{aligned}$$

Similarly, we get

$$(4.17) \quad \log \left\| \frac{\Xi'}{df'} \Big|_{X'_{s^d}} \right\|_{L^2}^2 = (c' - d + 1) \log |s|^2 + O(\log(-\log |s|)) \quad (s \rightarrow 0).$$

Comparing (4.11) and (4.16), (4.17), we get

$$(4.18) \quad c = c'.$$

By (4.15), (4.18), we get $a_\gamma = a'_\gamma$ for all $0 \leq \gamma \leq n$. Hence

$$(4.19) \quad \operatorname{div}(\Upsilon) = \sum_{\gamma=0}^n a_\gamma F_\gamma, \quad \operatorname{div}(\Upsilon') = \sum_{\gamma=0}^n a_\gamma F'_\gamma, \quad a_\gamma \in \mathbf{Z}_{\geq 0}, \quad (\forall \gamma \geq 0).$$

Since $F_0 \cap O_d$ (resp. F_γ ($\gamma > 0$)) is identified with $F'_0 \cap O'_d$ (resp. F'_γ ($\gamma > 0$)) via φ_d , we get by (4.19) and the definition of normalized canonical divisor in Section 3.8 the following equality of divisors via the identification $\varphi_d: O_d \cong O'_d$:

$$(4.20) \quad \mathfrak{K}_{(Y_d, Y_0)} \cap O_d = \varphi_d^*(\mathfrak{K}_{(Y'_d, Y'_0)} \cap O'_d).$$

Since $(g_d, \Sigma_{g_d}) = (g'_d \circ \varphi_d, \Sigma_{g_d})$ and $\Sigma_d = \varphi_d(\Sigma_{g'_d})$ via $\varphi_d: O_d \cong O'_d$, we deduce from (4.20) and the definition in Section 3.8 the equality

$$(4.21) \quad \kappa(g_d, \Sigma_{g_d}, \mathfrak{K}_{(Y_d, Y_0)}) = \kappa(g'_d, \Sigma_{g'_d}, \mathfrak{K}_{(Y'_d, Y'_0)}).$$

By Theorem 3.11 and (4.4), (4.21), we get

$$(4.22) \quad \lim_{s \rightarrow 0} \frac{\log \tau_{\text{BCOV}}(Y_s)}{\log |s|^2} = \lim_{s \rightarrow 0} \frac{\log \tau_{\text{BCOV}}(Y'_s)}{\log |s|^2}.$$

Since $Y_s = X_{s^d}$ and $Y'_s = X'_{s^d}$ for $s \neq 0$, the result follows from (4.22). \square

5. DEGENERATIONS TO CALABI-YAU VARIETIES WITH ORDINARY DOUBLE POINTS

In this section, we determine the asymptotic behavior of BCOV invariants for the simplest degenerations of Calabi-Yau threefolds, i.e., degenerations to Calabi-Yau varieties with at most ordinary double points (cf. [7, §2]). Recall that an n -dimensional singularity is an *ordinary double point* if it is isomorphic to the hypersurface singularity at $0 \in \mathbf{C}^n$ defined by the equation $z_0^2 + \cdots + z_n^2 = 0$.

Definition 5.1. A complex projective variety X of dimension 3 is a *Calabi-Yau variety with at most ordinary double points* if the following are satisfied:

- (1) There exists a nowhere vanishing canonical form on $X \setminus \text{Sing}(X)$.
- (2) X is connected and $H^q(X, \mathcal{O}_X) = 0$ for $0 < q < 3$.
- (3) $\text{Sing}(X)$ consists of at most ordinary double points.

Theorem 5.2. *Let $f: \mathcal{X} \rightarrow C$ be a surjective morphism from a smooth projective fourfold \mathcal{X} to a compact Riemann surface C . Let $\Delta_f \subset C$ be the discriminant locus of $f: \mathcal{X} \rightarrow C$ and assume that $X_t := f^{-1}(t)$ is a Calabi-Yau threefold for all $t \in C \setminus \Delta_f$. Let $0 \in \Delta_f$ and let t be a local parameter of C centered at 0. If X_0 is a Calabi-Yau variety with at most ordinary double points, then*

$$\log \tau_{\text{BCOV}}(X_t) = \frac{\#\text{Sing } X_0}{6} \log |t|^2 + O(\log(-\log |t|)) \quad (t \rightarrow 0).$$

Proof. (Step 1) Since the deformation germ $f: (\mathcal{X}, X_0) \rightarrow (C, 0)$ is a smoothing of X_0 , we have $h^{1,2}(X_0) = h^{2,1}(X_0) \geq 1$. Since $t \mapsto h^2(X_t, \Omega_{X_t}^1)$ is a locally constant function on C by [7, Th. 2.11], we get $h^{1,2}(X_t) = h^{2,1}(X_t) \geq 1$. When $h^{1,2}(X_t) = h^{2,1}(X_t) = 1$ and $\#\text{Sing } X_0 = 1$, the result was proved in [7, Th. 8.2]. Since there does exist a family of Calabi-Yau threefolds $f': \mathcal{X}' \rightarrow C'$ with $h^{1,2}(X'_t) = 1$ such that $\text{Sing } X'_0$ consists of a unique ordinary double point, e.g. the family of quintic mirror threefolds (cf. [7, §12]), we get the result by Theorem 4.1 and [7, Th. 8.2] when $\#\text{Sing } X_0 = 1$.

(Step 2) Fix a family of Calabi-Yau threefolds over a compact Riemann surface $\mathfrak{f}: \mathfrak{X} \rightarrow \mathfrak{C}$ such that $\mathfrak{X}_0, 0 \in \mathfrak{C}$, is a Calabi-Yau variety with a unique ordinary double point as its singular set. Fix its semi-stable reduction $\mathfrak{g}: (\mathfrak{Y}, \mathfrak{Y}_0) \rightarrow (\mathfrak{B}, 0)$: We have a commutative diagram:

$$\begin{array}{ccc} (\mathfrak{Y}, \mathfrak{Y}_0) & \xrightarrow{\Psi} & (\mathfrak{X}, \mathfrak{X}_0) \\ \mathfrak{g} \downarrow & & \downarrow \mathfrak{f} \\ (\mathfrak{B}, 0) & \xrightarrow{\psi} & (\mathfrak{C}, 0). \end{array}$$

Let ν be the ramification index of $\psi: (\mathfrak{B}, 0) \rightarrow (\mathfrak{C}, 0)$. Let $\mathfrak{Y}_0 = \mathfrak{E}_0 + \cdots + \mathfrak{E}_n$ be the irreducible decomposition such that $\mathfrak{E}_0 \setminus \bigcup_{\alpha>0} \mathfrak{E}_\alpha \cong \mathfrak{X}_0 \setminus \text{Sing } \mathfrak{X}_0$. Then Ψ ramifies along \mathfrak{E}_0 with ramification index ν . There exist a neighborhood \mathfrak{U} of $\text{Sing } \mathfrak{X}_0$ in \mathfrak{X} and an open subset \mathfrak{V} of \mathfrak{Y} such that $\mathfrak{V} = \Psi^{-1}(\mathfrak{U})$ and $\mathfrak{E}_1 \cup \cdots \cup \mathfrak{E}_n \subset \mathfrak{V}$. Then Ψ induces an isomorphism between $\mathfrak{E}_0 \setminus \mathfrak{V}$ and $\mathfrak{X}_0 \setminus \mathfrak{U}$ and the map Ψ has ramification index ν on $\mathfrak{E}_0 \setminus \bigcup_{\alpha>0} \mathfrak{E}_\alpha$.

(Step 3) By an appropriate choices of local parameters t of $(\mathfrak{C}, 0)$ and s of $(\mathfrak{B}, 0)$, we may assume $t = s^\nu$. By identifying \mathfrak{f} with $t \circ \mathfrak{f}$ and \mathfrak{g} with $s \circ \mathfrak{g}$, we have the following equality of functions defined near \mathfrak{Y}_0 :

$$\Psi^* \mathfrak{f} = \mathfrak{g}^\nu.$$

Since \mathfrak{X}_0 is a Calabi-Yau variety with a unique ordinary double point as its singular set and since \mathfrak{X} is smooth, there exists a nowhere vanishing holomorphic 4-form Ξ defined on a neighborhood of \mathfrak{X}_0 in \mathfrak{X} . Since \mathfrak{X}_0 has only canonical singularities, the function

$$t \mapsto \|(\Xi/d\mathfrak{f})|_{\mathfrak{X}_t}\|_{L^2}^2$$

is continuous around $0 \in \mathfrak{C}$ and $\|(\Xi/d\mathfrak{f})|_{\mathfrak{X}_0}\|_{L^2}^2 \neq 0$ by e.g. [20, Th. 7.2]. Since

$$\nu \Psi^*(\Xi/d\mathfrak{f}) = \Psi^* \Xi / (\mathfrak{g}^{\nu-1} d\mathfrak{g}),$$

the fact that the function $t \mapsto \|(\Xi/df)|_{\mathfrak{X}_t}\|_{L^2}^2$ is C^0 and does not vanish at $t = 0$ implies that the map $s \mapsto (\Psi^*\Xi/g^{\nu-1}dg)|_{\mathfrak{Y}_s} \in H^0(\mathfrak{Y}_s, K_{\mathfrak{Y}}|_{\mathfrak{Y}_s})$ is a nowhere vanishing holomorphic section of $\mathfrak{g}_*K_{\mathfrak{Y}}$ defined near $s = 0$. Hence $(\Psi^*\Xi)/g^{\nu-1}$ is a nowhere vanishing holomorphic 4-form defined near \mathfrak{Y}_0 and

$$0 \leq \mathfrak{K}_{(\mathfrak{Y}, \mathfrak{Y}_0)} = \operatorname{div}(\Psi^*\Xi/g^{\nu-1}) = \operatorname{div}(\Psi^*\Xi) - (\nu - 1)\mathfrak{Y}_0.$$

Write $\operatorname{div}(\Psi^*\Xi) = \sum_{\alpha=0}^n a_\alpha \mathfrak{E}_\alpha$, $a_\alpha \in \mathbf{Z}_{\geq 0}$. Since $\mathfrak{Y}_0 = \operatorname{div}(\mathfrak{g}) = \sum_{\alpha=0}^n \mathfrak{E}_\alpha$ by the reducedness of \mathfrak{Y}_0 , we get by the effectivity of $\operatorname{div}(\Psi^*\Xi) - (\nu - 1)\mathfrak{Y}_0$

$$a_\alpha \geq \nu - 1 \quad (\forall \alpha \in \{0, 1, \dots, n\}).$$

On the other hand, since Ψ has ramification index ν on $\mathfrak{E}_0 \setminus \bigcup_{\alpha>0} \mathfrak{E}_\alpha$, we have $a_0 = \nu - 1$. Hence we can express

$$(5.1) \quad \mathfrak{K}_{(\mathfrak{Y}, \mathfrak{Y}_0)} = \operatorname{div}(\Psi^*\Xi/g^{\nu-1}|_{\mathfrak{Y}}) = \sum_{\alpha=1}^n b_\alpha \mathfrak{E}_\alpha \quad (b_\alpha \in \mathbf{Z}_{\geq 0}).$$

(Step 4) Consider the general case. Set $m := \#\operatorname{Sing} X_0 \geq 1$. Then $\operatorname{Sing} X_0 = \{p_1, \dots, p_m\}$. Since every germ $f \in \mathcal{O}_{X_0, p_i}$ is isomorphic to the germ $(z_0)^2 + (z_1)^2 + (z_2)^2 + (z_3)^2$ at $0 \in \mathbf{C}^4$, it follows from the construction of semi-stable reduction [13, Chap. II §3] that there exists a semi-stable reduction

$$\begin{array}{ccc} (\mathcal{Y}, Y_0) & \xrightarrow{\Phi} & (\mathcal{X}, X_0) \\ g \downarrow & & \downarrow f \\ (B, 0) & \xrightarrow{\phi} & (C, 0) \end{array}$$

with the following properties:

- (i) The ramification index of $\phi: (B, 0) \rightarrow (C, 0)$ is given by ν .
- (ii) For every $p_i \in \operatorname{Sing} X_0$, there is a neighborhood U_i of p_i in \mathcal{X} such that $(f, U_i) \cong (f, \mathfrak{U})$ and $(g, \Phi^{-1}(U_i)) \cong (g, \mathfrak{Y})$.
- (iii) Set $V_i := \Phi^{-1}(U_i)$. Then $Y_0 \setminus \bigcup_{i=1}^m V_i \cong X_0 \setminus \bigcup_{i=1}^m U_i$.

By (ii), the irreducible component of $Y_0 = g^{-1}(0)$ contained in V_i can be expressed as $E_1^{(i)} + \dots + E_n^{(i)}$ and satisfy $E_0 \cap V_i + E_1^{(i)} + \dots + E_n^{(i)} \cong \mathfrak{E}_0 \cap \mathfrak{Y} + \mathfrak{E}_1 + \dots + \mathfrak{E}_n$, where $E_\alpha^{(i)} \cong \mathfrak{E}_\alpha$ for all $i = 1, \dots, m$ and $1 \leq \alpha \leq n$. This implies that

$$(5.2) \quad \rho(g, \Sigma_g) = \sum_{i=1}^m \rho(g|_{V_i}, \Sigma_{g|_{V_i}}) = m \rho(\mathfrak{g}, \Sigma_{\mathfrak{g}}).$$

(Step 5) Let t be a local parameter of $(C, 0)$ and let s be a local parameter of $(B, 0)$. As in Step 3, we may assume $\phi^*t = s^\nu$ and hence

$$\Phi^*f = g^\nu.$$

Let ω be a nowhere vanishing holomorphic 4-form defined near X_0 . Since X_0 has only canonical singularities, the section of the Hodge bundle $t \mapsto \omega/df|_{X_t} \in H^0(X_t, K_{X_t})$ is holomorphic and nowhere vanishing. By the same reason as in Step 3, the section of Hodge bundle $s \mapsto \Phi^*\omega/(g^{\nu-1}dg)|_{Y_s} \in H^0(Y_s, K_{Y_s})$ is holomorphic and nowhere vanishing around $s = 0$ and we get

$$(5.3) \quad \mathfrak{K}_{(\mathcal{Y}, Y_0)} = \operatorname{div}(\Phi^*\omega/g^{\nu-1}), \quad \operatorname{Supp} \mathfrak{K}_{(\mathcal{Y}, Y_0)} \subset \bigcup_{i=1}^m \bigcup_{\alpha=1}^n E_\alpha^{(i)}.$$

We may assume that, under the identification $(f, U_i) \cong (f, \mathfrak{U})$ in (ii), the ratio Ξ/ω is a nowhere vanishing holomorphic function on \mathfrak{U} . Then we get by (5.1), (5.3)

$$(5.4) \quad \mathfrak{K}_{(\mathcal{Y}, Y_0)} = \sum_{i=1}^m \sum_{\alpha=1}^n b_\alpha E_\alpha^{(i)}.$$

By (ii), (5.4) and the definition of $\kappa(g, \Sigma_g, \mathfrak{K}_{(\mathcal{Y}, Y_0)})$, we get

$$(5.5) \quad \kappa(g, \Sigma_g, \mathfrak{K}_{(\mathcal{Y}, Y_0)}) = m \kappa(\mathfrak{g}, \Sigma_{\mathfrak{g}}, \mathfrak{K}_{(\mathfrak{Y}, \mathfrak{Y}_0)}).$$

Since $\mathfrak{X}_{s^\nu} = \mathfrak{Y}_s$, we get by Theorem 3.11 and Step 1

$$\rho(\mathfrak{g}, \Sigma_{\mathfrak{g}}) - \frac{1}{12} \kappa(\mathfrak{g}, \Sigma_{\mathfrak{g}}, \mathfrak{K}_{(\mathfrak{Y}, \mathfrak{Y}_0)}) = \lim_{s \rightarrow 0} \frac{\log \tau_{\text{BCOV}}(\mathfrak{Y}_s)}{\log |s|^2} = \lim_{s \rightarrow 0} \frac{\log \tau_{\text{BCOV}}(\mathfrak{X}_{s^\nu})}{\log |s|^2} = \frac{\nu}{12}.$$

Thus we get by (5.2), (5.5)

$$\rho(g, \Sigma_g) - \frac{1}{12} \kappa(g, \Sigma_g, \mathfrak{K}_{(\mathcal{Y}, Y_0)}) = m \left\{ \rho(\mathfrak{g}, \Sigma_{\mathfrak{g}}) - \frac{1}{12} \kappa(\mathfrak{g}, \Sigma_{\mathfrak{g}}, \mathfrak{K}_{(\mathfrak{Y}, \mathfrak{Y}_0)}) \right\} = \frac{m\nu}{12}.$$

By Theorem 3.11 again and the relation $Y_s = X_{s^\nu}$, we get

$$\begin{aligned} \log \tau_{\text{BCOV}}(X_{s^\nu}) &= \log \tau_{\text{BCOV}}(Y_s) \\ &= \left\{ \rho(g, \Sigma_g) - \frac{1}{12} \kappa(g, \Sigma_g, \mathfrak{K}_{(\mathcal{Y}, Y_0)}) \right\} \log |s|^2 + O(\log(-\log |s|)) \\ &= \frac{m\nu}{12} \log |s|^2 + O(\log(-\log |s|)) \\ &= \frac{m}{12} \log |s^\nu|^2 + O(\log(-\log |s|)) \quad (s \rightarrow 0). \end{aligned}$$

This completes the proof. \square

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