# On the strength of connectedness of a random hypergraph

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#### Abstract

Bollobás and Thomason (1985) proved that for each  $k = k(n) \in [1, n-1]$ , with high probability, the random graph process, where edges are added to vertex set  $V = [n]$  uniformly at random one after another, is such that the stopping time of having minimal degree k is equal to the stopping time of becoming  $k$ -(vertex-)connected. We extend this result to the d-uniform random hypergraph process, where k and d are fixed. Consequently, for  $m = \frac{n}{d}(\ln n + (k-1)\ln \ln n + c)$  and where k and a are fixed. Consequently, for  $m = \frac{1}{d}(\ln h + (k-1)\ln \ln h + c)$  and  $p = (d-1)! \frac{\ln n + (k-1)\ln \ln n + c}{n^{d-1}}$ , the probability that the random hypergraph models  $H_d(n,m)$  and  $H_d(n,p)$  are k-connected tends to  $e^{-e^{-c}/(k-1)!}$ .

Keywords: random hypergraph; vertex connectivity

#### 1 Introduction

Let  $H_d(n, p)$  denote the random d-uniform hypergraph with vertex set  $[n] := \{1, 2, ..., n\},\$ where each of the  $\binom{n}{d}$  $\binom{n}{d}$  potential (hyper)edges of cardinality d is present with probability p, independently of all other potential edges. Likewise, let  $H_d(n,m)$  be the random duniform hypergraph on  $[n]$ , where m edges are chosen uniformly at random among all sets of m potential edges. The model  $H_d(n,m)$  can be gainfully viewed as a snapshot of the random hypergraph process  $\{H_d(n,\mu)\}_{\mu=0}^{n \choose d}$ , where  $H_d(n,\mu+1)$  is obtained from  $H_d(n,\mu)$ by inserting an extra edge chosen uniformly at random among all  $\binom{n}{d}$  $\binom{n}{d} - \mu$  remaining potential edges. For  $d = 2$ , these models are the typical random graph models,  $G(n, p)$ ,  $G(n, m)$  and  ${G(n, \mu)}_{\mu=0}^{\binom{n}{2}}$ .

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As customary, we say that for a given  $m = m(n)$  (p resp.) some graph property Q holds with high probability, denoted w.h.p., if the probability that  $H_d(n,m)$  ( $(H_d(n,p)$ ) resp.) has property Q tends to 1 as  $n \to \infty$ . Further,  $m(n)$  is the *sharp threshold* for Q if for each  $\epsilon > 0$  (fixed), w.h.p.  $H_d(n,(1-\epsilon)m)$  does not have Q and w.h.p  $H_d(n,(1+\epsilon)m)$ does have Q. For the random hypergraph process, the *stopping time* of Q, denoted  $\tau(Q)$  is the first moment that the process has this property  $\mathcal{Q}$ ; we denote the hypergraph process stopped at this time by  $H_d(n, \tau(Q))$ .

In one of the first papers on random graphs, Erdős and Rényi [\[4\]](#page-15-0) showed that  $m =$ 1  $\frac{1}{2}n \ln n$  is a sharp threshold for connectivity in  $G(n, m)$ . Later, Stepanov [\[7\]](#page-15-1) established the sharp threshold of connectivity for  $G(n, p)$  among other results. More recently, Bollobás and Thomason [3] proved the stronger result for the random graph process that w.h.p. the moment the graph process loses its last isolated vertex is also the moment that the process becomes connected; in other words, w.h.p.  $\tau$  (no isolated vertices) =  $\tau$  (connected). We prove the analogous result for the the random d-uniform hypergraph process; a consequence of this result is that  $m = \frac{n}{d}$  $\frac{n}{d} \ln n$  is a sharp threshold of connectivity for  $H_d(n,m)$ .

There are various measures for the *strength* of connectedness of a connected graph, but here we will focus on k-(vertex-)connectivity. For  $k \in \mathbb{N}$ , a hypergraph with more than k vertices is k-connected if whenever  $k-1$  vertices are deleted, along with their incident edges, the remaining hypergraph is connected. Note that the definition of 1-connectedness coincides with connectedness. Necessarily, for a hypergraph to be  $k$ -connected, each vertex must have degree at least k, because if a vertex v has degree less than k, then we can delete a neighbor from each incident edge to isolate  $v$ . However, as commonly seen in these types of results, the main barrier to  $k$ -connectivity in these random graph models arises from such vertices that can be separated from the rest of the graph by the deletion of their neighbors (see for instance Erdős-Rényi [5], Ivchenko  $[6]$ , Bollobás  $[1], [2]$ ). Here, we extend this idea to random d-uniform hypergraphs; in particular, we find that if  $m_0 = \frac{n}{d}$  $\frac{n}{d}(\ln n + (k-1)\ln \ln n - \omega)$  and  $m_1 = \frac{n}{d}$  $\frac{n}{d}(\ln n + (k-1)\ln \ln n + \omega), \omega \to \infty$ , then w.h.p.  $H_d(n, m_0)$  is not k-connected and w.h.p.  $H_d(n, m_1)$  is k-connected; also we find an analogous threshold value for  $H_d(n, p)$ .

A stronger result concerns the random graph process where edges are added one after another. Let  $\tau_k := \tau$  (min degree at least k) and  $T_k := \tau$  (k-connected); note that  $\tau_k \leq T_k$ . In [3], Bollobás and Thomason showed that for  $d = 2$  (the graph case) and any  $k = k(n) \in$ [1, n−1],  $P(\tau_k = T_k) \to 1$ . We extend this result for d-uniform random hypergraphs albeit for fixed d and k.

<span id="page-1-0"></span>**Theorem 1.1.** W.h.p, at the moment the d-uniform hypergraph process loses its last vertex with degree less than k, this process becomes k-connected. Formally, for  $d \geq 3$  and  $k \geq 1$  (both fixed),  $P(\tau_k = T_k) \to 1$ .

To prove this result, we begin by determining the likely range of  $\tau_k$ , and further that just prior to this window, at some  $m_0$  edges, w.h.p. there are not many vertices of degree less than k. Then, we prove that w.h.p.  $H_d(n, m_0)$  is almost k-connected in the sense that whenever  $k-1$  vertices are deleted, there is a massive component using almost all leftover vertices. Third, we show that w.h.p. to isolate a vertex of  $H_d(n, \tau_k)$ , you would have to delete at least k of its neighbors (this is trivially true for graphs, but not so for  $d \geqslant 3$ ). In particular, we show that w.h.p. edges incident to degree  $k - 1$  vertices have trivial intersection (just the vertex itself). Finally, we show that the probability that  $\tau_k < T_k$ , but these three previous likely events also hold tends to zero, which completes the proof of the theorem. The following corollary is nearly immediate in light of the theorem.

<span id="page-2-2"></span>Corollary 1.2. *(i)* Let  $m = \frac{n}{d}$  $\frac{n}{d}(\ln n + (k-1)\ln \ln + c_n), \text{ where } c_n \to c \in \mathbb{R}.$  W.h.p.  $H_d(n,m)$  is  $(k-1)$ -connected, but not  $(k+1)$ -connected. Further, the probability that  $H_d(n,m)$  is k-connected tends to  $e^{-e^{-c}/(k-1)!}$ .

(ii) Let  $p = (d-1)! \frac{\ln n + (k-1) \ln \ln n + c_n}{n^{d-1}}$ , where  $c_n \to c \in \mathbb{R}$ . W.h.p.  $H_d(n, p)$  is  $(k-1)$ . connected, but not  $(k + 1)$ -connected. Further, the probability that  $H_d(n, p)$  is k-connected tends to  $e^{-e^{-c}/(k-1)!}$ .

For the remainder of this paper, let  $d \geq 3$  and  $k \geq 1$  be fixed numbers.

#### 2 Likely range of  $\tau_k$

<span id="page-2-1"></span>**Lemma 2.1.** Let  $\omega = \omega(n) \to \infty$ , but  $\omega = o(\ln \ln n)$ ,  $m_0 = \frac{n}{d}$  $\frac{n}{d}$  (ln  $n + (k-1)$ ) ln ln  $n - \omega$ ) and  $m_1 = \frac{n}{d}$  $\frac{n}{d}(\ln n + (k-1)\ln \ln n + \omega)$ . Then w.h.p.,

(i) the minimum degree of  $H_d(n, m_0)$  is  $k-1$  and the number of vertices with degree  $k-1$  is in the interval

<span id="page-2-0"></span>
$$
\left[\frac{1}{2}\frac{e^{\omega}}{(k-1)!}, \frac{3}{2}\frac{e^{\omega}}{(k-1)!}\right].
$$
\n(2.1)

(ii) there are no vertices of degree  $k-1$  in  $H_d(n, m_1)$ . Consequently, w.h.p.  $\tau_k \in [m_0, m_1]$ .

*Proof.* We prove that the number of vertices with degree  $k - 1$ , denoted by X, is in the interval [\(2.1\)](#page-2-0) by Chebyshev's Inequality. Note that a given vertex can be in  $\binom{n-1}{d-1}$  $_{d-1}^{n-1}$ ) possible edges, so

$$
E[X] = nP[\deg(1) = k - 1] = n\frac{\binom{\binom{n-1}{d-1}}{\binom{n}{d-1}\binom{\binom{n}{d}-\binom{n-1}{d-1}}{m_0-k+1}}{\binom{\binom{n}{d}}{m_0}}.
$$

Here and elsewhere in this paper, we use the identity  $\binom{N}{m-\ell} = \binom{N}{m} \frac{(m)\ell}{(N-m+\ell)\ell}$ , where  $(j)_{\ell} =$  $j(j-1)\cdots(j-\ell+1)$ , and later, we use the inequality  $\binom{N}{m-\ell} \leqslant \binom{N}{m} \left(\frac{m}{N-m}\right)^{\ell}$ . Now

$$
E[X] = (1 + O(1/n)) \frac{n \cdot n^{(d-1)(k-1)}}{(k-1)!((d-1)!)^{k-1}} \left(\frac{d! \, m_0}{n^d}\right)^{k-1} \frac{\binom{\binom{n}{d} - \binom{n-1}{d-1}}{m_0^0}}{\binom{\binom{n}{d}}{m_0}}.
$$

This latter fraction can be sharply approximated.

$$
\frac{\binom{\binom{n}{d}-\binom{n-1}{d-1}}{m_0}}{\binom{\binom{n}{d}}{m_0}} = \prod_{i=0}^{m_0-1} \left(1 - \frac{\binom{n-1}{d-1}}{\binom{n}{d}-i}\right) = \prod_{i=0}^{m_0-1} \left(1 - \frac{d}{n} + O\left(\frac{i}{n^{d+1}}\right)\right)
$$

$$
= \exp\left(\sum_{i=0}^{m_0-1} \left[\frac{-d}{n} + O\left(\frac{1}{n^2}\right) + O\left(\frac{i}{n^{d+1}}\right)\right]\right)
$$

$$
= \left(1 + O\left(\frac{\ln n}{n}\right)\right) \exp\left(-\frac{dm_0}{n}\right) = \left(1 + O\left(\frac{\ln n}{n}\right)\right) \frac{e^{\omega}}{n(\ln n)^{k-1}}.\tag{2.2}
$$

Hence

<span id="page-3-0"></span>
$$
E[X] = \left(1 + O\left(\frac{\ln n}{n}\right)\right) \frac{e^{\omega}}{(k-1)!} \left(\frac{dm_0/n}{\ln n}\right)^{k-1}
$$

$$
= \left(1 + O\left(\frac{\ln \ln n}{\ln n}\right)\right) \frac{e^{\omega}}{(k-1)!}.
$$

For the second factorial moment, we have that

$$
E[X(X-1)] = n(n-1)P(\deg(1) = \deg(2) = k - 1).
$$

We break this latter probability over  $i$ , the number of edges that include both vertices 1 and 2. In particular, vertex 1 is in  $k - 1 - i$  edges that do not contain vertex 2 and vice versa; further there are  $m_0 - 2(k-1) + i$  edges that include neither vertex 1 or 2. Since there are  $\binom{n-2}{d-2}$  $\binom{n-2}{d-2}$  potential hyperedges containing both vertices and  $\binom{n-1}{d-1}$  $\binom{n-1}{d-1} - \binom{n-2}{d-2}$  $\binom{n-2}{d-2}$  potential hyperedges containing one vertex but not the other, we have that

<span id="page-3-1"></span>
$$
P(\deg(1) = \deg(2) = k - 1) = \sum_{i=0}^{k-1} \binom{\binom{n-2}{d-2}}{i} \binom{\binom{n-1}{d-1} - \binom{n-2}{d-2}}{k-1-i}^2 \frac{\binom{\binom{n}{d} - 2\binom{n-1}{d-1} + \binom{n-2}{d-2}}{m_0 - 2(k-1)+i}}{\binom{\binom{n}{d}}{m_0}}.
$$
(2.3)

Just as in [\(2.2\)](#page-3-0), we can estimate this latter fraction

$$
\frac{\binom{\binom{n}{d}-2\binom{n-1}{d-1}+\binom{n-2}{d-2}}{m_0-2(k-1)+i}}{\binom{\binom{n}{d}}{m_0}} = \frac{(m_0)_{2(k-1)-i}}{\left(\binom{n}{d}-2\binom{n-1}{d-1}+\binom{n-2}{d-2}-m_0+2(k-1)-i\right)_{2(k-1)-i}} \times \frac{\binom{\binom{n}{d}-2\binom{n-1}{d-1}+\binom{n-2}{d-2}}{m_0}}{\binom{\binom{n}{d}}{m_0}} = \left(1+O\left(\frac{\ln\ln n}{\ln n}\right)\right)\left(\frac{m_0}{\binom{n}{d}}\right)^{2(k-1)-i} \frac{e^{2\omega}}{n^2\left(\ln n\right)^{2(k-1)}}.
$$

Using these asymptotics, one can show that the  $i$ 'th term in  $(2.3)$  is on the order of  $e^{2\widetilde{\omega}}$  $\frac{e^{2\omega}}{n^{2+i}(\ln n)^i}$ . In particular, the sum of the terms over  $i \in [1, k-1]$ , is  $O(n^{-3})$ . Therefore

$$
P(\deg(1) = \deg(2) = k - 1) = {\binom{\binom{n-1}{d-1} - \binom{n-2}{d-2}}{k-1}}^2 \frac{\binom{\binom{n}{d} - 2\binom{n-1}{d-1} + \binom{n-2}{d-2}}{\binom{n}{d}}}{\binom{\binom{n}{d}}{\binom{n}{d}}} + O(n^{-3})
$$

$$
= \left(1 + O\left(\frac{\ln \ln n}{\ln n}\right)\right) \frac{e^{2\omega}}{((k-1)!)^2 n^2};
$$

whence

$$
E[X(X-1)] = \left(1 + O\left(\frac{\ln \ln n}{\ln n}\right)\right)E[X]^2.
$$

Consequently,

$$
var[X] = E[X] + O\left((E[X])^{2} \frac{\ln \ln n}{\ln n}\right).
$$

By Chebyshev's Inequality,  $X$  is concentrated around its mean and in particular, w.h.p. X is in the interval  $(2.1)$ . To finish the proof of part  $(i)$ , it remains to show that w.h.p. there are no vertices of degree less than  $k-1$ , which can be done by a first moment argument using similar techniques to the asymptotics of  $E[X]$ . Similarly, for part (ii), one can easily show that the expected number of vertices of degree  $k - 1$  in  $H_d(n, m_1)$  tends to zero as well. tends to zero as well.

### 3  $H_d(n, m_0)$  is almost k-connected

Now we will establish that w.h.p.  $H_d(n, m_0)$  is almost k-connected in the sense that if  $k-1$  vertices are deleted, then there remains a massive component containing almost all left-over vertices. To this end, we prove an analogous statement for the random Bernoulli hypergraph  $H_d(n, p)$  and use a standard conversion lemma to obtain the desired result for  $H_d(n, m_0)$ . In this next lemma, we pick a specific version of  $m_0$ , one where  $\omega = \ln \ln \ln n$ .

<span id="page-4-1"></span>Lemma 3.1. Let  $m'_0 = \frac{n}{d}$  $\frac{n}{d}(\ln n + (k-1)\ln \ln n - \ln \ln \ln n)$  and  $p = m'_0/(n_d)$  $\binom{n}{d}$ . With high probability,

(i)  $H_d(n, p)$  has the property "whichever  $k-1$  vertices are deleted, there remains a giant component which includes all but up to ln n leftover vertices."

(ii)  $H_d(n, m'_0)$  has the property "whichever  $k-1$  vertices are deleted, there remains a giant component which includes all but up to ln n leftover vertices."

*Proof.* (i) Given a set of  $k-1$  vertices,  $\mathbf{v} = \{v_1, \ldots, v_{k-1}\}\)$ , let  $\mathcal{F}(\mathbf{v})$  be the event that if the vertices **v** are deleted from  $H_d(n, p)$  along with their incident edges, then there remains no components of size at least  $n - (k - 1) - \ln n$ . In particular, we wish to show that w.h.p.  $H_d(n, p)$  is not in  $\mathcal{F}(v)$  for any v. Using the union bound over all  $k - 1$ element sets of  $[n]$  as well as symmetry, we find that

<span id="page-4-0"></span>
$$
P\left(\bigcup_{\mathbf{v}} \mathcal{F}(\mathbf{v})\right) \leqslant {n \choose k-1} P(\mathcal{F}(\mathbf{v}^*)),\tag{3.1}
$$

where  $\mathbf{v}^* = \{n - (k-1) + 1, \ldots, n-1, n\}$ . Note that the remaining hypergraph left after deleting  $\mathbf{v}^*$  from  $H_d(n, p)$  is distributed as  $H_d(n', p), n' := n - (k - 1)$  (this is the primary reason that we consider the Bernoulli hypergraph  $H_d(n, p)$  rather than  $H_d(n, m)$ . Therefore  $P(\mathcal{F}(v^*))$  is precisely the probability that  $H_d(n', p)$  does not have a component of size at least  $n' - \ln n$ .

To bound  $P(\mathcal{F}(v^*))$ , we note that any hypergraph on n' vertices without a component of size at least  $n' - \ln n$  has a set of vertices S such that there are no edges between S

and  $[n'] \setminus S$  where  $|S| \in [\ln n, n' - \ln n]$ . To see this fact, consider a hypergraph H on  $n'$  vertices without such a large component and let  $L_1, \ldots, L_\ell$  be the vertex sets of the components of  $H$  in increasing order by their cardinalities. Then there is some minimal j so that

$$
\ln n \leqslant |U_{i=1}^j L_i| < \ln n + n' - \ln n = n'.
$$

Further,  $L_{j+1}$  is not empty and since  $|L_j| \leq |L_{j+1}|$ , we have that  $|L_j| \leq n'/2$  and

$$
\ln n \leqslant |U_{i=1}^j L_i| < \ln n + n'/2 < n' - \ln n.
$$

Clearly, there are no edges including a vertex of  $S := \bigcup_{i=1}^{j} L_i$  and  $[n'] \setminus S$ .

Therefore,

$$
P(\mathcal{F}(\mathbf{v}^*)) \leqslant \sum_{s=\ln n}^{n'-\ln n} P(\exists S \subset [n'], |S| = s, \text{ no edge between } S \text{ and } [n'] \setminus S),
$$

and by symmetry over all such vertex sets  $S$ ,

$$
P(\mathcal{F}(\mathbf{v}^*)) \leqslant \sum_{s=\ln n}^{n'-\ln n} \binom{n'}{s} P(\text{no edge between } [s] \text{ and } [n'] \setminus [s]).
$$

Further, this latter probability is symmetric about  $s = n'/2$  (i.e. the probabilities corresponding to s and  $n' - s$  are equal). Hence

$$
P(\mathcal{F}(\mathbf{v}^*)) \leq 2 \sum_{s=\ln n}^{\lfloor n'/2 \rfloor} {n' \choose s} P(\text{no edge between } [s] \text{ and } [n'] \setminus [s]).
$$

The number of potential edges that contain at least one vertex from [s] and at least one vertex from  $[n'] \setminus [s]$  is  $\binom{n'}{d}$  $\binom{n'}{d} - \binom{s}{d}$  $\binom{s}{d} - \binom{n'-s}{d}$  $\binom{-s}{d}$ . Hence

$$
P(\mathcal{F}(\mathbf{v}^*)) \leq 2 \sum_{s=\ln n}^{\lfloor n'/2 \rfloor} {n' \choose s} (1-p)^{{n \choose d} - {s \choose d} - {n'-s \choose d}} \leq 2 \sum_{s=\ln n}^{\lfloor n'/2 \rfloor} {n' \choose s} e^{-p \left( {n' \choose d} - {s \choose d} - {n'-s \choose d} \right)}
$$
  

$$
=: 2E_1 + 2E_2,
$$

where  $E_1$  and  $E_2$  are the sums over  $S_1 := [\ln n, n/(\ln n)]$  and  $S_2 := (n/(\ln n), \lfloor n/2 \rfloor]$ respectively. We begin with analyzing  $E_2$  since these bounds will be cruder and simpler.

Trivially,

<span id="page-5-0"></span>
$$
E_2 \leqslant \sum_{s \in S_2} \binom{n'}{s} e^{-p\binom{n'}{d} + p \max_{t \in S_2} \left(\binom{t}{d} + \binom{n'-t}{d}\right)} \leqslant 2^{n'} e^{-p\binom{n'}{d} + p \max_{t \in S_2} \left(\binom{t}{d} + \binom{n'-t}{d}\right)}.
$$
 (3.2)

Now let's take on these binomial coefficient terms. Trivially  $\binom{\nu}{d}$  $\binom{d}{d} \leqslant \frac{\nu^d}{d!}$  $\frac{\partial u}{\partial x}$ , and so

$$
\binom{t}{d} + \binom{n'-t}{d} \leqslant \frac{t^d + (n'-t)^d}{d!}.
$$

Further, the function  $f(t) = t^d + (n'-t)^d$  is decreasing for  $t \in S_2$ , so we have that

$$
\max_{t \in S_2} \left( \binom{t}{d} + \binom{n'-t}{d} \right) \leq \frac{1}{d!} \left( \left( \frac{n}{\ln n} \right)^d + \left( n' - \frac{n}{\ln n} \right)^d \right).
$$

Now our bound in [\(3.2\)](#page-5-0) becomes

$$
E_2 \leqslant 2^{n'} \exp\left(-p\binom{n'}{d} + \frac{p}{d!} \left(\frac{n}{\ln n}\right)^d + \frac{p}{d!} \left(n' - \frac{n}{\ln n}\right)^d\right).
$$

In the previous expression, the leading order terms in the first and third terms will cancel and the middle term is absorbed in the error. Namely, we have that

$$
E_2 \leqslant 2^{n'} \exp\left(-\frac{p(n')^d}{d!} + O(\ln n) + O\left(\frac{n \ln n}{(\ln n)^d}\right) + \frac{p(n')^d}{d!} - \frac{p d}{d!} \frac{n^d}{\ln n} + O\left(\frac{n}{\ln n}\right)\right),
$$

or equivalently

<span id="page-6-2"></span>
$$
E_2 \leq 2^{n'} \exp\left(-\frac{p n^d}{(d-1)! \ln n} + O\left(\frac{n}{\ln n}\right)\right)
$$
  
=  $\exp((\ln 2 - 1)n + o(n)) \leq e^{-n/4}.$  (3.3)

Now let's take on the sum  $E_1$ . We begin with taking the leading order terms in the exponent.

<span id="page-6-0"></span>
$$
E_1 \leqslant \sum_{s \in S_1} \binom{n'}{s} \exp\left(-p \binom{n'}{d} \left(1 - \frac{\binom{n'-s}{d}}{\binom{n'}{d}} - O\left(\frac{s^d}{n^d}\right)\right)\right). \tag{3.4}
$$

Uniformly over  $s \in S_1$ , we have that

$$
\frac{\binom{n'-s}{d}}{\binom{n'}{d}} = \left(1 - \frac{s}{n'} + O\left(\frac{s}{n^2}\right)\right)^d = 1 - \frac{ds}{n'} + O\left(\frac{s^2}{n^2}\right).
$$

Consequently, the exponent in [\(3.4\)](#page-6-0) is

$$
p\binom{n'}{d}\left(\frac{ds}{n}+O\left(\frac{s^2}{n^2}\right)\right)=(\ln n + (k-1)\ln\ln n - \ln\ln\ln n)\left(s+O(s^2/n)\right)+O(1);
$$

whence there is some (fixed)  $\gamma > 0$  such that for all  $s \in S_1$ ,

<span id="page-6-1"></span>
$$
p\binom{n'}{d}\left(\frac{ds}{n} + O\left(\frac{s^2}{n^2}\right)\right) \ge (\ln n - \ln \ln n)(s - \gamma s^2/n) - \gamma. \tag{3.5}
$$

Using the bound  $\binom{n'}{s}$  $s'$   $\leq$   $(en/s)^s$  as well as  $(3.5)$ ,  $(3.4)$  becomes

$$
E_1 \leqslant \sum_{s \in S_1} \left(\frac{en}{s}\right)^s \exp\left(-(\ln n - \ln \ln \ln n)(s - \gamma s^2/n) + \gamma\right)
$$
  
=  $e^{\gamma} \sum_{s \in S_1} \exp\left(s\left(1 - \ln s + \ln \ln \ln n + \gamma s(\ln n - \ln \ln \ln n)/n\right)\right).$ 

However, for  $s \in S_1$ , we have that

$$
s \leqslant \frac{n}{\ln n} \leqslant \frac{2n}{\ln n - \ln \ln \ln n} \implies s(\ln n - \ln \ln \ln n)/n \leqslant 2.
$$

Therefore

$$
E_1 \leqslant e^{\gamma} \sum_{s \in S_1} \exp \left( s \left( 1 - \ln s + \ln \ln \ln n + 2\gamma \right) \right).
$$

This sum is dominated by the first term  $(s = \ln n)$  because ratios of consecutive terms uniformly tend to zero. Consequently, we have that

<span id="page-7-0"></span>
$$
E_1 = O\left[\exp\left(-\ln n \ln \ln n + o(\ln n \ln \ln n)\right)\right].\tag{3.6}
$$

Summing our bounds for  $E_1$  and  $E_2$  ([\(3.6\)](#page-7-0) and [\(3.3\)](#page-6-2), respectively), we have that

$$
P(\mathcal{F}(\mathbf{v}^*)) \leqslant \exp(-\ln n \ln \ln n + o(\ln n \ln \ln n)), \qquad (3.7)
$$

which most definitely is  $o(n^{-(k-1)})$  and so by the bound [\(3.1\)](#page-4-0), part (i) of the lemma is proved.

Part (ii) is established by using a standard conversion technique between  $H_d(n, m)$ and  $H_d(n, p)$ . For any hypergraph property  $A$ , we have that

<span id="page-7-1"></span>
$$
P(H_d(n, p) \in \mathcal{A}) = \sum_{m=0}^{\binom{n}{d}} P(H_d(n, m) \in \mathcal{A}) P(e(H_d(n, p)) = m), \tag{3.8}
$$

where  $e(H)$  is the number of edges of H. Therefore, for any (possible) m,

$$
P(H_d(n,p) \in \mathcal{A}) \geqslant P(H_d(n,m) \in \mathcal{A})P(e(H_d(n,p)) = m),
$$

whence

$$
P(H_d(n,m) \in \mathcal{A}) \leqslant \frac{P(H_d(n,p) \in \mathcal{A})}{P(e(H_d(n,p) = m))}.
$$

For  $m = \Theta(n \ln n)$  and  $p = m / {n \choose d}$  $\binom{n}{d}$ , one can show that

$$
P(e(H_d(n,p)) = m) = \binom{\binom{n}{d}}{m} p^m (1-p)^{\binom{n}{d}-m} = \Theta(m^{-1/2}).
$$

Hence in our case,

$$
P(H_d(n, m'_0) \in \bigcup_{\mathbf{v}} \mathcal{F}(\mathbf{v})) = O\left(\sqrt{n \ln n} P(H_d(n, p) \in \bigcup_{\mathbf{v}} \mathcal{F}(\mathbf{v}))\right).
$$

In the proof of part (i), we found that this latter probability tends to zero superpolynomially fast.  $\Box$ 

#### 4 Quasi-disjoint Edges

For the random graph process  $(d = 2)$ , it was found that the main barrier to k-connectivity is the presence of vertices of degree less than  $k$ , which could be isolated with the deletion of their neighbors (see Erdős-Rényi [5], Ivchenko [\[6\]](#page-15-2), Bollobás [\[1\]](#page-15-3),[2]). We will find a similar situation for the random hypergraph process.

However, we run into an additional issue here for hypergraphs. Even if the degree of a vertex v is k, we could isolate v with the deletion of less than k vertices. For instance, if all of v's edges also include vertex w, then the deletion of just w from the hypergraph (along with its incident edges) will isolate v from the rest of the hypergraph. Our ultimate goal in this section is to show that w.h.p. each vertex of  $H_d(n, \tau_k)$  has at least k edges whose pairwise intersections are precisely  $\{v\}$ ; in this case, for any vertex, you would need to delete at least k of its neighbors to isolate it. To this end, we first prove that w.h.p.  $H_d(n, m_0)$  has this property for vertices with degree at least k and as nearly as could be expected for vertices with degree  $k - 1$ .

A set of edges E incident to vertex v is quasi-disjoint if all pairwise intersections of these edges are  $\{v\}$ ; formally, if  $e, f \in E$ ,  $e \neq f$ , then  $e \cap f = \{v\}$ .

<span id="page-8-0"></span>**Lemma 4.1.** Let  $m_0 = \frac{n}{d}$  $\frac{n}{d}(\ln n + (k-1)\ln \ln n - \omega)$ , where  $\omega \to \infty$ , but  $\omega = o(\ln \ln n)$ . W.h.p.,  $H_d(n, m_0)$  is such that

(i) the incident edges of a degree  $k-1$  vertex form a quasi-disjoint set,

 $(ii)$  vertices with degree at least k have a quasi-disjoint set of incident edges with size at least k.

*Proof.* Note that both parts of this lemma are trivially true for  $k = 1$  and part (i) is also trivially true for  $k = 2$ . Let  $X(j, \ell)$  be the number of vertices whose maximum quasidisjoint set has size j and whose degree is  $j + \ell$ . To prove this lemma, it suffices to show that w.h.p. for  $j \leq k - 1$  and  $\ell \geq 1$ , we have that  $X(j, \ell) = 0$ , which is shown by a first moment argument. Now

$$
E[X(j, \ell)] = nP(j, \ell),
$$

where  $P(j, \ell)$  is the probability that a generic vertex v has a maximum quasi-disjoint set of size j and whose degree is  $j + \ell$ . To bound this probability, note that v has a set of j quasi-disjoint edges and each of the remaining  $\ell$  edges must have at least one vertex from the  $j(d-1)$  neighbors from the quasi-disjoint edges; further the remaining  $m_0 - j - \ell$ edges do not include v. Hence

$$
P(j,\ell) \leq \binom{\binom{n-1}{d-1}}{j} \binom{\binom{j(d-1)}{1}\binom{n-2}{d-2}}{\ell} \frac{\binom{\binom{n}{d}-\binom{n-1}{d-1}}{m_0-j-\ell}}{\binom{\binom{n}{d}}{m_0}} \leq n^{(d-1)j} \left(\frac{ej(d-1)n^{d-2}}{\ell(d-2)!}\right)^{\ell} \left(\frac{m_0}{\binom{n}{d}} - \binom{n-1}{d-1} - m_0\right)^{j+\ell} \frac{\binom{\binom{n}{d}-\binom{n-1}{d-1}}{m_0}}{\binom{\binom{n}{d}}{m_0}}.
$$

We gave sharp asymptotics for the last fraction in [\(2.2\)](#page-3-0). Here and throughout the rest of the paper, we will use  $f \leq_b g$  for  $f = O(g)$  when the formula for g becomes too bulky.

Therefore

$$
P(j,\ell) \leq b (\ln n)^j \frac{e^{\omega}}{n(\ln n)^{k-1}} \left( \frac{e \, j (d-1) n^{d-2} m_0}{\ell (d-2)! \left( {n \choose d} - {n-1 \choose d-1} - m_0 \right)} \right)^{\ell};
$$

whence

$$
P(j,\ell) \leqslant_b \frac{e^{\omega}}{n} \left( C \frac{\ln n}{n} \right)^{\ell},
$$

for  $C = 2(k-1)(d-1)$  (independent of  $j \leq k-1$  and  $\ell \geq 1$ ). Thus

$$
\sum_{j=0}^{k-1} \sum_{\ell \ge 1} E[X(j,\ell)] \le_b e^{\omega} \sum_{\ell \ge 1} \left( C \frac{\ln n}{n} \right)^{\ell} \le_b e^{\omega} \frac{\ln n}{n} \to 0,
$$

which completes the proof of the lemma.

<span id="page-9-0"></span>**Lemma 4.2.** W.h.p. each vertex of  $H_d(n, \tau_k)$  has a quasi-disjoint set of incident edges with size at least k.

*Proof.* This lemma is trivially true for  $k = 1$ . Suppose that  $k \geq 2$ . Let  $A_n$  be the event that  $H_d(n, \tau_k)$  has a vertex that does not have a quasi-disjoint set of edges with size at least k; we wish to show that  $P(A_n) \to 0$ . Let  $m_0, m_1$  be as defined in Lemma [2.1.](#page-2-1) We have proved that w.h.p.  $\tau_k \in [m_0, m_1]$  and that  $H_d(n, m_0)$  does not have vertices of degree less than k − 1. Further, w.h.p. the number of degree k − 1 vertices in  $H_d(n, m_0)$  is less than  $\frac{3e^{\omega}}{2(k-1)!}$  (Lemma [2.1\)](#page-2-1). In addition, w.h.p.  $H_d(n,m_0)$  has the two properties of the previous lemma (Lemma [4.1\)](#page-8-0). Let  $B_n$  be the intersection of these four likely events. To prove the lemma, it suffices to show that  $P(A_n \cap B_n) \to 0$ .

Let  $\tilde{V}_0$  be the vertex set of vertices of degree  $k-1$  in  $H_d(n, m_0)$ . Note that

$$
P(A_n \cap B_n) = \sum_{V_0 \subset [n], |V_0| \le 3e^{\omega}/(2(k-1)!)} P(A_n \cap B_n \cap {\tilde{V}_0} = V_0 \}).
$$

On the event that  $A_n$  and  $B_n$  occur and  $\tilde{V}_0 = V_0$ , necessarily some edge  $e_m$  is added in the hypergraph process at some step  $m \in [m_0, m_1]$  such that  $e_m$  includes both a vertex  $v \in V_0$  and one of v's  $(k-1)(d-1)$  neighbors in  $H_d(n, m_0)$ . Thus

$$
P(A_n \cap B_n \cap \{\tilde{V}_0 = V_0\}) \leq \sum_{m=m_0}^{m_1} \frac{\binom{|V_0|}{1}\binom{(k-1)(d-1)}{1}\binom{n-2}{d-2}}{\binom{n}{d}-m} P(\tilde{V}_0 = V_0)
$$
  

$$
\leq_b (m_1 - m_0) e^{\omega} \frac{\binom{n-2}{d-2}}{\binom{n}{d}-m_1} P(\tilde{V}_0 = V_0) \leq_b \frac{\omega e^{\omega}}{n} P(\tilde{V}_0 = V_0).
$$

Therefore

$$
P(A_n \cap B_n) \leq b \frac{\omega e^{\omega}}{n} \sum_{V_0 \subset [n], |V_0| \leq 3e^{\omega}/(2(k-1)!)} P(\tilde{V}_0 = V_0) \leq \frac{\omega e^{\omega}}{n} \to 0.
$$



 $\Box$ 

# 5 W.h.p.  $H_d(n, \tau_k)$  is k-connected

Now that we have sufficient knowledge about the structure of  $H_d(n, m_0)$  and low-degree vertices in  $H_d(n, \tau_k)$ , we can prove our main Theorem.

**Theorem [1.1.](#page-1-0)** W.h.p.  $H_d(n, \tau_k)$  is k-connected. In short, w.h.p.  $\tau_k = T_k$ .

*Proof.* Let  $m'_i = \frac{n}{d}$  $\frac{n}{d}$  (ln n + (k – 1) ln ln n + (-1)<sup>i+1</sup> ln ln ln n), for  $i = 0, 1$ . By Lemma [2.1,](#page-2-1) we have shown that w.h.p.  $\tau_k \in [m'_0, m'_1]$  and by Lemma [4.2,](#page-9-0) each vertex of  $H_d(n, \tau_k)$  has a quasi-disjoint edge set of size at least k. Further, the property "whichever  $k-1$  vertices are deleted, there remains a giant component which includes all but up to  $\ln n$  leftover vertices," denoted  $Q$ , is an increasing property (closed under the addition of edges). Therefore, by Lemma [3.1,](#page-4-1)  $H_d(n, \tau_k)$  has property Q as well. To prove this theorem, it suffices to show that the probability that these three likely events hold yet  $\tau_k < T_k$  tends to zero.

To this end, for  $m \in [m'_0, m'_1]$ , let  $C_m$  be the event that  $H_d(n, m)$  is not k-connected, but each vertex has a quasi-disjoint edge set of size at least k and  $H_d(n,m)$  has property Q. To prove this theorem, it suffices to prove that  $P(\cup C_m) \to 0$ . We will in fact show that

<span id="page-10-0"></span>
$$
P(C_m) \leqslant_b \frac{(\ln n)^{dk+k+1}}{n^{d-1}},\tag{5.1}
$$

uniformly over  $m \in [m'_0, m'_1]$ . In this case,  $P(\cup C_m) \leq b \frac{(\ln n)^{dk+k+2}}{n^{d-2}} \to 0$ , as desired. All that remains is to prove the bound [\(5.1\)](#page-10-0).

On the event  $C_m$ , there are  $k-1$  vertices,  $w_1, ..., w_{k-1}$  such that upon their deletion, there is a component of size  $n' - s$  for some  $s \in [1, \ln n)$ . In fact, since each remaining vertex must have at least one incident edge, we must have that  $s \geq d$ . Let S be the set of vertices not in this large component. By the union bound over all  $k - 1$  element sets of [n] and sets  $S, |S| = s$ , as well as symmetry, we have that

$$
P(C_m) \leqslant {n \choose k-1} \sum_{s=d}^{\ln n} {n-(k-1) \choose s} P_s,
$$

where  $P_s$  is the probability that each vertex of  $H_d(n,m)$  has a quasi-disjoint edge set of size at least k and that after the deletion of  $\{n' + 1, ..., n' + (k - 1) = n\}$  from  $H_d(n, m)$ , the vertices  $[n'-s]$  form a component; in this case,  $S = \{n'-s+1, ..., n'\}$ . We now turn to showing that  $P_s$  tends to zero sufficiently fast.

Suppose that H is some hypergraph in the event corresponding to  $P_s$ . After the deletion of the  $k-1$  vertices, we know that vertex  $w := \{n'\}$  from S has at least one incident edge, which necessarily must reside completely within S. Further, before deletion, any incident edge to  $w$  must be completely contained within  $S$  or this edge must contain one of the  $k-1$  to-be-deleted vertices. Moreover, there are at least k edges incident to w before the deletion. Therefore

$$
P_s \leqslant \sum_{i=1}^k P_s(i),
$$

where  $P_s(i)$  is the corresponding probability to when there are (at least) i incident edges to w contained within S and (at least)  $k - i$  incident edges to w that contain at least one of the to-be-deleted vertices. To bound the number of hypergraphs contributing to  $P_s(i)$ , we choose i potential edges within S containing w,  $k - i$  potential edges that include w and at least one to-be-deleted vertex; then we choose the remaining  $m - k$  edges among all potential edges except those that include a vertex of S and  $d-1$  vertices of  $[n'-s]$ (which necessarily can not be present). Note that these last chosen edges can include  $w$ as well. Therefore

$$
P_s(i) \leqslant \binom{\binom{s-1}{d-1}}{i} \binom{\binom{1}{1}\binom{k-1}{1}\binom{n-2}{d-2}}{k-i} \binom{\binom{n}{d}-\binom{s}{1}\binom{n'-s}{d-1}}{m-k} \frac{1}{\binom{\binom{n}{d}}{m}}.
$$

First, we use trivial bounds on the first two binomial terms. Then we use the inequality  $\binom{N-\ell}{j} \leqslant \binom{N}{j} e^{-j\ell/N}$ . Namely, note that

$$
P_s(i) \leqslant s^{di} k^k n^{(d-2)(k-i)} \left( \frac{m}{\binom{n}{d} - s\binom{n'-s}{d-1} - m} \right)^k \frac{\binom{\binom{n}{d} - s\binom{n'-s}{d-1}}{m}}{\binom{\binom{n}{d}}{m}} \leqslant_b s^{dk} n^{(d-2)(k-1)} \left( \frac{\ln n}{n^{d-1}} \right)^k \exp\left( -\frac{s\binom{n'-s}{d-1}m}{\binom{n}{d}} \right).
$$

Further, for  $s \leq \ln n$ , we have that

$$
\frac{s\binom{n'-s}{d-1}m}{\binom{n}{d}} = \frac{s \, d \, m}{n} + O\left(\frac{(\ln n)^2}{n}\right) \geqslant \frac{s \, d \, m'_0}{n} + o(1).
$$

Hence

$$
P_s(i) \leq_b (\ln n)^{dk} n^{(d-2)(k-1)} \left(\frac{\ln n}{n^{d-1}}\right)^k \left(\frac{\ln \ln n}{n(\ln n)^{k-1}}\right)^s
$$
  

$$
\leq (\ln n)^{dk+k} n^{2-d-k} \left(\frac{\ln \ln n}{n(\ln n)^{k-1}}\right)^s,
$$

which no longer depends on  $i$ . Therefore

$$
P_s \leqslant_b (\ln n)^{dk+k} n^{2-d-k} \left( \frac{\ln \ln n}{n(\ln n)^{k-1}} \right)^s,
$$

and

$$
P(C_m) \leqslant_b n^{k-1} \sum_{s=d}^{\ln n} \frac{n^s}{s!} (\ln n)^{dk+k} n^{2-d-k} \left( \frac{\ln \ln n}{n (\ln n)^{k-1}} \right)^s.
$$

Now taking on this sum, we find that

$$
P(C_m) \leqslant_b \frac{(\ln n)^{dk+k}}{n^{d-1}} \sum_{s=d}^{\ln n} \frac{1}{s!} \left( \frac{\ln \ln n}{(\ln n)^{k-1}} \right)^s \leqslant \frac{(\ln n)^{dk+k}}{n^{d-1}} \exp\left( \frac{\ln \ln n}{(\ln n)^{k-1}} \right),
$$

and we find that  $P(C_m) \leq b \frac{(\ln n)^{dk+k+1}}{n^{d-1}}$ , as desired.



#### 6 Sharp Threshold of  $k$ -connectivity

As a consequence of Theorem [1.1,](#page-1-0) for any  $m$ , we have that

<span id="page-12-0"></span>
$$
P(H_d(n,m) \text{ is } k\text{-connected}) = P(T_k \leqslant m) = P(\tau_k \leqslant m) + o(1)
$$

$$
= P(\min\text{-deg } H_d(n,m) \geqslant k) + o(1). \tag{6.1}
$$

We use this fact to determine the probability that  $H_d(n,m)$  and  $H_d(n,p)$  is k-connected in the critical window.

Corollary [1.2.](#page-2-2) *(i)* Let  $m = \frac{n}{d}$  $\frac{n}{d}(\ln n + (k-1)\ln \ln n + c_n), \text{ where } c_n \to c \in \mathbb{R}.$  W.h.p.  $H_d(n,m)$  is  $(k-1)$ -connected, but not  $(k+1)$ -connected. Further the probability that  $H_d(n,m)$  is k-connected tends to  $e^{-e^{-c}/(k-1)!}$ .

(ii) Let  $p = (d-1)! \frac{\ln n + (k-1) \ln \ln n + c_n}{n^{d-1}}$ , where  $c_n \to c \in \mathbb{R}$ . W.h.p.  $H_d(n, p)$  is  $(k-1)$ . connected, but not  $(k + 1)$ -connected. Further the probability that  $H_d(n, p)$  is k-connected tends to  $e^{-e^{-c}/(k-1)!}$ .

*Proof.* (i) First, note that w.h.p.  $\tau_{k-1} < m$  and  $\tau_{k+1} > m$  by Lemma [2.1.](#page-2-1) Therefore, by Theorem [1.1,](#page-1-0) w.h.p.  $H_d(n,m)$  is  $(k-1)$ -connected, but not  $(k+1)$ -connected. In the lemma following this proof, we show that X, the number of vertices of degree  $k - 1$  in  $H_d(n,m)$  is asymptotically Poisson with parameter  $e^{-c}/(k-1)!$ . Thus

$$
P(\min-\deg H_d(n,m) \ge k) = P(Poi(e^{-c}/(k-1)!)) = 0) + o(1) = e^{-e^{-c}/(k-1)!} + o(1).
$$

Using the equation [\(6.1\)](#page-12-0) finishes off the proof.

(ii) This part will be proved from (i) using a standard conversion technique similar to the one used in Lemma [3.1.](#page-4-1) Since the number of edges in  $H_d(n, p)$ , denoted  $e(H_d(n, p))$ , is binomially distributed on  $N := \binom{n}{d}$  $\binom{n}{d}$  trials with success probability p, we have that

$$
e(H_d(n,p)) = Np + O_p\left(\sqrt{Np(1-p)}\right) = \frac{n}{d}\left(\ln n + (k-1)\ln\ln n + c_n\right) + O_p\left(\sqrt{n\ln n}\right).
$$

Therefore, if  $m_{+,-} := \frac{n}{d} (\ln n + (k-1) \ln \ln n + c_n^{+,-})$ , where  $c_n^{+,-} = c_n \pm \ln n / \sqrt{n}$ , then w.h.p.  $m_$  ≤  $e(H_d(n, p))$  ≤  $m_+$ . Using this fact, [\(3.8\)](#page-7-1) becomes

$$
P(H_d(n,p) \in \mathcal{A}) = \sum_{m=m_-}^{m_+} P(H_d(n,m) \in \mathcal{A}) P(e(H_d(n,p)) = m) + o(1).
$$

Notice that  $c_n^{+,-} \to c$ . By part (i), if A is  $\{(k-1)$ -connected} or  $\{\text{not } (k+1)$ -connected}, then  $P(H_d(n, m_{+,-}) \in \mathcal{A}) \to 1$ ; also, if  $\mathcal{A} = \{k$ -connected $\}$ , then  $P(H_d(n, m_{+,-}) \in \mathcal{A}) \to$  $e^{-e^{-c}/(k-1)!}$ . To finish off the proof that  $P(H_d(n, p) \in \mathcal{A})$  has the same limits, we will use the fact that these properties are monotone.

Now an *increasing* (*decreasing*) property is a property that is closed under the addition (deletion resp.) of edges. For any increasing property A, we have that  $P(H_d(n, m) \in \mathcal{A}) \leq$  $P(H_d(n, m') \in \mathcal{A})$  for any  $m \leq m'$ . This fact is obvious when you consider  $H_d(n, m)$  and

 $H_d(n,m')$  to be snapshots of the random hypergraph process  $\{H_d(n,\mu)\}_{\mu=0}^N$ . Moreover, if  $\mathcal A$  is an increasing property, then

$$
P(H_d(n, m_-) \in \mathcal{A}) + o(1) \leqslant P(H_d(n, p) \in \mathcal{A}) \leqslant P(H_d(n, m_+) \in \mathcal{A}) + o(1);
$$

further, if  $A$  is decreasing, then the inequalities above are reversed. Consequently, for a monotone property A such that both  $P(H_d(n, m_{+,-}) \in \mathcal{A})$  tend to the same number, then  $P(H_d(n, p) \in \mathcal{A})$  does as well. □ then  $P(H_d(n, p) \in \mathcal{A})$  does as well.

**Lemma 6.1.** Let  $m = \frac{n}{d}$  $\frac{n}{d}(\ln n + (k-1)\ln \ln n + c_n)$ , where  $c_n \to c \in \mathbb{R}$ . W.h.p. the number of vertices of degree  $k-1$ , denoted by X, converges in distribution to a Poisson random variable with parameter  $e^{-c}/(k-1)!$ .

Proof. We prove this lemma using the method of moments (see [2] for a description of this method). In order to prove the lemma, it suffices to show that for each  $r \in \mathbb{N}$  (fixed),

$$
\lim_{n \to \infty} E[(X)_r] = \left(\frac{e^{-c}}{(k-1)!}\right)^r.
$$

To compute the r'th factorial moment, note that

$$
E[(X)_r] = E_0 + E_1 + \ldots + E_{(k-1)r},
$$

where  $E_j$  is the expected number of ordered r-tuples of vertices of degree at most  $k-1$ such that there are exactly  $(k-1)r-j$  edges containing at least one of these vertices. We will see that the terms other than  $E_0$  are negligible.

Let's first consider  $E_0$ . Since the number of edges is  $(k-1)r$ , each of these r vertices have degree  $k - 1$  and must necessarily not be adjacent; so

$$
E_0 = (n)_r \binom{\binom{n-r}{d-1}}{k-1}^r \frac{\binom{\binom{n-r}{d-1}}{m-r(k-1)}}{\binom{\binom{n}{m}}{m}} = (1+O\left(n^{-1}\right)) n^r \left(\frac{n^{(k-1)(d-1)}}{(k-1)!((d-1)!)^{k-1}}\right)^r \frac{\binom{\binom{n-r}{d-1}}{m-r(k-1)}}{\binom{\binom{n}{m}}{m}}.
$$

Taking on this last factor, we have that

$$
\binom{\binom{n-r}{d}}{m-r(k-1)} = \binom{\binom{n-r}{d}}{m} \frac{(m)_{r(k-1)}}{\left(\binom{n-r}{d}-m+r(k-1)\right)_{r(k-1)}}.
$$

By sharply approximating this last fraction, we have that

$$
E_0 = \left(1 + O\left(\frac{\ln \ln n}{\ln n}\right)\right) \left(\frac{n (\ln n)^{k-1}}{(k-1)!}\right)^r \frac{\binom{n-r}{d}}{\binom{n}{d}}.
$$

Note that

$$
\binom{\binom{n-r}{d}}{m} = \frac{1}{m!} \left( \binom{n-r}{d} \right)^m \prod_{i=0}^{m-1} \left( 1 - \frac{i}{\binom{n-r}{d}} \right) = \frac{1}{m!} \left( \binom{n-r}{d} \right)^m \left( 1 + O\left( \frac{m^2}{n^d} \right) \right).
$$

We also have  $d \geqslant 3$  so that

$$
\binom{\binom{n-r}{d}}{m} = \frac{1}{m!} \binom{n-r}{d}^m \left(1 + O\left(\frac{(\ln n)^2}{n}\right)\right);
$$

similarly, we have that

$$
\binom{\binom{n}{d}}{m} = \frac{1}{m!} \binom{n}{d}^m \left( 1 + O\left(\frac{(\ln n)^2}{n}\right) \right).
$$

Further

$$
\left(\frac{\binom{n-r}{d}}{\binom{n}{d}}\right)^m = \left(\left(\frac{n-r}{n} + O\left(\frac{1}{n^2}\right)\right)^d\right)^m = \exp\left(d\,m\left(-\frac{r}{n} + O(n^{-2})\right)\right),
$$

and

<span id="page-14-0"></span>
$$
\frac{\binom{\binom{n-r}{d}}{m}}{\binom{\binom{n}{m}}{m}} = \left(1 + O\left(\frac{(\ln n)^2}{n}\right)\right) e^{-rdm/n} = \left(1 + O\left(\frac{(\ln n)^2}{n}\right)\right) \left(\frac{e^{-c_n}}{n(\ln n)^{k-1}}\right)^r. \tag{6.2}
$$

Thus

$$
E_0 = \left(1 + O\left(\frac{\ln \ln n}{\ln n}\right)\right) \left(\frac{e^{-c_n}}{(k-1)!}\right)^r.
$$

Now we turn to  $E_j$ , for  $j \geqslant 1$ . Note that  $E_j$  is less than the expected number of such r-tuples where these r vertices have exactly  $r(k-1) - j$  adjacent edges (we dropped the degree condition). Then, for  $j \geq 1$ , we have that

$$
E_j \leqslant (n)_r \binom{\binom{n}{d} - \binom{n-r}{d}}{\left(k-1\right)r - j} \frac{\binom{\binom{n-r}{d}}{m - \left(k-1\right)r + j}}{\binom{\binom{n}{d}}{m}}.
$$

In particular, we have that

$$
E_j \leqslant n^r \left( \binom{n}{d} - \binom{n-r}{d} \right)^{(k-1)r-j} \left( \frac{m}{\binom{n-r}{d} - m} \right)^{(k-1)r-j} \frac{\binom{n-r}{d}}{\binom{n}{m}}.
$$

Using the fact that

$$
\binom{n}{d} - \binom{n-r}{d} = r\binom{n-r}{d-1} + O(n^{d-2}) \leqslant rn^{d-1}
$$

along with the bound [\(6.2\)](#page-14-0), we have that

$$
E_j \leqslant_b n^{r + (d-1)[(k-1)r - j]} \left(\frac{\ln n}{n^{d-1}}\right)^{(k-1)r - j} \left(\frac{1}{n(\ln n)^{k-1}}\right)^r = \frac{1}{(\ln n)^j},
$$

which completes the proof of the lemma.

 $\Box$ 

# <span id="page-15-3"></span>References

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