

THE SCHUR-CLIFFORD SUBGROUP OF THE BRAUER-CLIFFORD GROUP

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ABSTRACT. We define a Schur-Clifford subgroup of Turull's Brauer-Clifford group, similar to the Schur subgroup of the Brauer group. The Schur-Clifford subgroup contains exactly the equivalence classes coming from the intended application to Clifford theory of finite groups. We show that the Schur-Clifford subgroup is indeed a subgroup of the Brauer-Clifford group, as are certain naturally defined subsets. We also show that this Schur-Clifford subgroup behaves well with respect to restriction and corestriction maps between Brauer-Clifford groups.

1. INTRODUCTION

The Brauer-Clifford group has been introduced by Turull [12, 13] to handle Clifford theory in finite groups in a way that respects fields of values and other rationality properties of the characters involved. Let $\kappa: \widehat{G} \rightarrow G$ be a surjective homomorphism of finite groups with kernel $N \trianglelefteq \widehat{G}$. In other words,

$$1 \longrightarrow N \hookrightarrow \widehat{G} \xrightarrow{\kappa} G \longrightarrow 1$$

is an exact sequence of groups. Let $\vartheta \in \text{Irr } N$, where $\text{Irr } N$ denotes, as usual, the set of irreducible complex characters of the group N . For convenient reference, we call the pair (ϑ, κ) a **Clifford pair**. Note that N , \widehat{G} and G are determined by (ϑ, κ) as the kernel, the domain and the image of κ , respectively. We will often consider different Clifford pairs which have the group G in common, while \widehat{G} and N vary. To describe this conveniently, we also say that (ϑ, κ) is a **Clifford pair over G** in the above situation.

Let $\mathbb{F} \subseteq \mathbb{C}$ be a field. Turull has shown how to associate a commutative \mathbb{F} -algebra $\mathbf{Z}(\vartheta, \kappa, \mathbb{F})$, on which G acts, to a Clifford pair (ϑ, κ) and the field \mathbb{F} [12, Definition 7.1]. This is called the **center algebra** of (ϑ, κ) over \mathbb{F} . Then the Brauer-Clifford group

$$\text{BrCliff}(G, \mathbf{Z}(\vartheta, \kappa, \mathbb{F}))$$

is defined (We will review its precise definition below). Moreover, Turull showed [12, Definition 7.7] how to associate an element

$$[[\vartheta, \kappa, \mathbb{F}]] \in \text{BrCliff}(G, \mathbf{Z}(\vartheta, \kappa, \mathbb{F}))$$

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to (ϑ, κ) and \mathbb{F} . We call this the **Brauer-Clifford class** of the pair (ϑ, κ) over \mathbb{F} . This element determines in some sense the character theory of \widehat{G} over ϑ , including fields of values and Schur indices over the field \mathbb{F} [12, Theorem 7.12].

In this paper, we consider the subset of $\text{BrCliff}(G, Z)$ that consists of elements that come from this construction. To make this more precise, we need the fact that the Brauer-Clifford group is a covariant functor in the second variable. This means that if $\alpha: R \rightarrow S$ is a homomorphism of G -rings, then there is a corresponding homomorphism of abelian groups

$$\bar{\alpha} = \text{BrCliff}(G, \alpha): \text{BrCliff}(G, R) \rightarrow \text{BrCliff}(G, S).$$

If α happens to be an isomorphism, then $\bar{\alpha}$ is an isomorphism, too.

For a fixed commutative G -algebra Z over the field \mathbb{F} , we define $\mathcal{SC}^{(\mathbb{F})}(G, Z)$ as the set of all $\bar{\alpha}[[\vartheta, \kappa, \mathbb{F}]]$ such that (ϑ, κ) is a Clifford pair over G and $\alpha: \mathbf{Z}(\vartheta, \kappa, \mathbb{F}) \rightarrow Z$ is an isomorphism of G -algebras. In short,

$$\mathcal{SC}^{(\mathbb{F})}(G, Z) := \{\bar{\alpha}[[\vartheta, \kappa, \mathbb{F}]] \mid \mathbf{Z}(\vartheta, \kappa, \mathbb{F}) \cong Z \text{ via } \alpha\}.$$

Of course, for the wrong Z , it may happen that $\mathcal{SC}^{(\mathbb{F})}(G, Z)$ is empty. If not, we call $\mathcal{SC}^{(\mathbb{F})}(G, Z)$ the **Schur-Clifford subgroup** (over \mathbb{F}) of the Brauer-Clifford group $\text{BrCliff}(G, Z)$. The main result of this paper justifies this name:

Theorem A. *If $Z \cong \mathbf{Z}(\vartheta, \kappa, \mathbb{F})$ for some Clifford pair (ϑ, κ) , then $\mathcal{SC}^{(\mathbb{F})}(G, Z)$ is a subgroup of $\text{BrCliff}(G, Z)$.*

With respect to the Brauer-Clifford group, the Schur-Clifford subgroup is the same as is the Schur subgroup [14] with respect to the Brauer group.

We will usually write $\mathcal{SC}(G, Z)$ instead of $\mathcal{SC}^{(\mathbb{F})}(G, Z)$. This is justified by the following result:

Proposition B. *Let Z be a G -ring and assume that $Z \cong \mathbf{Z}(\vartheta, \kappa, \mathbb{F})$ for some Clifford pair (ϑ, κ) and some field $\mathbb{F} \subseteq \mathbb{C}$. Then Z^G , the centralizer of G in Z , is a field. There is a unique subfield $\mathbb{K} \subseteq \mathbb{C}$ with $\mathbb{F} \subseteq \mathbb{K}$ such that $Z^G \cong \mathbb{K}$ as algebra over \mathbb{F} . For this field \mathbb{K} , we have*

$$\mathcal{SC}^{(\mathbb{F})}(G, Z) = \mathcal{SC}^{(\mathbb{K})}(G, Z).$$

This means that we can always work with the field $\mathbb{K} \cong Z^G$ which is uniquely determined by the G -algebra Z .

We also consider certain subclasses \mathcal{X} of the class of all Clifford pairs. Possible examples are the class \mathcal{X} of all pairs (ϑ, κ) such that $N = \text{Ker } \kappa$ is abelian (nilpotent, solvable . . .), or the class of all pairs such that $\text{Ker } \kappa$ is a central subgroup of the domain of κ , or the class of all pairs such that ϑ is induced from a linear character of some subgroup, and so on. For any class \mathcal{X} , we define a **restricted Schur-Clifford subset**

$$\mathcal{SC}_{\mathcal{X}}(G, Z) = \{\bar{\alpha}[[\vartheta, \kappa, \mathbb{F}]] \mid \mathbf{Z}(\vartheta, \kappa, \mathbb{F}) \cong Z \text{ via } \alpha \text{ for some } (\vartheta, \kappa) \in \mathcal{X}\}.$$

Of course, depending on \mathcal{X} and Z , this may not be a subgroup, even if $\mathcal{SC}(G, Z) \neq \emptyset$. Our proof of Theorem A will show, however, that $\mathcal{SC}_{\mathcal{X}}(G, Z)$ is a subgroup for certain

classes \mathcal{X} . (See Theorem 5.7 for the case where Z is a field.) Finally, we mention the following: Assume that $Z \cong \mathbf{Z}(\vartheta, \kappa, \mathbb{F})$ for some Clifford pair (ϑ, κ) over G . Then it is known that $Z \cong \mathbb{F}(\vartheta) \times \cdots \times \mathbb{F}(\vartheta)$ is a direct product of isomorphic fields and that G permutes transitively the factors of Z . Let $e \in Z$ be a primitive idempotent and let $H \leq G$ be its stabilizer in G . Thus $K = Ze \cong \mathbb{F}(\vartheta)$ is an H -algebra. Turull [12, Theorem 5.3] has shown that $\text{BrCliff}(G, Z) \cong \text{BrCliff}(H, K)$ canonically.

Theorem C. *In the situation just described, the canonical isomorphism*

$$\text{BrCliff}(G, Z) \cong \text{BrCliff}(H, K)$$

restricts to an isomorphism

$$\mathcal{SC}(G, Z) \cong \mathcal{SC}(H, K).$$

We mention that it is not particularly difficult to see that the map $\text{BrCliff}(G, Z) \rightarrow \text{BrCliff}(H, K)$ maps $\mathcal{SC}(G, Z)$ into $\mathcal{SC}(H, K)$ (replace (ϑ, κ) by (ϑ, π) , where π is the restriction of κ to $\kappa^{-1}(H)$). The proof that every class in $\mathcal{SC}(H, K)$ comes from a class in $\mathcal{SC}(G, Z)$ is more complicated and involves a construction using wreath products.

The proofs are organized as follows: First, we prove Theorem A and Proposition B in the case where Z is a field. (This is done in Sections 5 and 6. In Sections 2–4 we review the necessary definitions and background we need, and define the Schur-Clifford group.) Then, in Section 8, we prove Theorem C in the sense that we show that the set $\mathcal{SC}(G, Z)$ and the subgroup $\mathcal{SC}(H, K)$ correspond under the canonical isomorphism of Brauer-Clifford groups. This then implies that Theorem A and Proposition B are true for arbitrary Z . In Section 7, we show that the Schur-Clifford group behaves well with respect to restriction of groups (this yields the easy part of Theorem C). In the more technical Section 9, we consider the relation of the Schur-Clifford group to the *corestriction* map as defined in our paper [8]. This will be needed in forthcoming work, but is included here since the proof is similar to the proof of Theorem C, and we can use some lemmas from this proof. Finally, in the last section, we describe the Schur-Clifford group $\mathcal{SC}(G, \mathbb{F})$, when G acts trivially on \mathbb{F} , in terms of the Schur subgroup and a certain subgroup of $H^2(G, \mathbb{F}^*)$, which has been studied by Dade [2]. Based on this special case, we propose a conjecture.

2. REVIEW OF THE BRAUER-CLIFFORD GROUP

The Brauer-Clifford group was first defined by Turull [12] for commutative simple G -algebras. Later, he gave a different, but equivalent definition [13]. Herman and Mitra [4] have shown how to extend the definition to arbitrary commutative G -rings, and that the Brauer-Clifford group is the same as the equivariant Brauer group defined earlier by Fröhlich and Wall [3].

Let G be a group. A **G -ring** is a ring R on which G acts by ring automorphisms. In this paper, “ring” always means “ring with one” and “ring homomorphism” always means “unital ring homomorphism”. We use exponential notation $r \mapsto r^g$ ($r \in R$, $g \in G$) to denote the action of G on R . A **G -ring homomorphism** (or **homomorphism**

of G -rings) is a ring homomorphism $\varphi: R \rightarrow S$ between G -rings such that $r^g\varphi = (r\varphi)^g$ for all $r \in R$ and $g \in G$.

Let R be a commutative G -ring. A **G -algebra A over R** is a G -ring A , together with an homomorphism of G -rings $\varepsilon: R \rightarrow \mathbf{Z}(A)$. In particular, A is an algebra over R . We usually suppress ε and simply write ra for $(r\varepsilon) \cdot a$. A G -algebra A over the commutative G -ring R is called an **Azumaya G -algebra over R** if it is Azumaya over R as an R -algebra. We will only need the cases where R is a field or a direct product of fields. If R is a field, A is Azumaya over R iff A is a finite dimensional central simple R -algebra. If R is a direct product of fields, then A is Azumaya iff the algebra unit induces an isomorphism $R \cong \mathbf{Z}(A)$, and for every primitive idempotent e of R , the algebra Ae is central simple over the field Re .

If A and B are Azumaya G -algebras over the G -ring R , then $A \otimes_R B$ is an Azumaya G -algebra over R .

The Brauer-Clifford group $\text{BrCliff}(G, R)$ consists of equivalence classes of Azumaya G -algebras over R . To define the equivalence relation, we need some definitions first. Let R be a G -ring. The **(skew) group algebra RG** is the set of formal sums

$$\sum_{g \in G} gr_g, \quad r_g \in R,$$

which form a free R -module with basis G , and multiplication defined by $gr \cdot hs = ghr^h s$ and extended distributively. Let R be commutative and P a right RG -module. Then $\text{End}_R(P)$ is a G -algebra over R . If P_R is an R -progenerator, then $\text{End}_R(P)$ is Azumaya over R . (Note that any finitely generated non-zero R -module is an R -progenerator if R is a field.) By definition, a **trivial Azumaya G -algebra** over R is an algebra of the form $\text{End}_R(P)$, where P is a module over the group algebra RG such that P_R is a progenerator over R .

Two Azumaya G -algebras A and B over the G -ring R are called **equivalent**, if there are trivial Azumaya G -algebras E_1 and E_2 such that

$$A \otimes_R E_1 \cong B \otimes_R E_2.$$

In other words, A and B are equivalent, if there are RG -modules P_1 and P_2 which are progenerators over R and such that

$$A \otimes_R \text{End}_R(P_1) \cong B \otimes_R \text{End}_R(P_2).$$

The **Brauer-Clifford group** $\text{BrCliff}(G, R)$ is the set of equivalence classes of Azumaya G -algebras over R together with the (well defined) multiplication induced by the tensor product \otimes_R . With this multiplication, $\text{BrCliff}(G, R)$ becomes an abelian torsion group [13, Theorem 6.2, 4, Theorem 5].

3. THE SCHUR-CLIFFORD GROUP

Let (ϑ, κ) be a Clifford pair, as defined in the introduction. Recall that this means that there is an exact sequence of groups

$$1 \longrightarrow N \longleftarrow \widehat{G} \xrightarrow{\kappa} G \longrightarrow 1,$$

and that $\vartheta \in \text{Irr } N$.

In this situation, $G \cong \widehat{G}/N$ acts on the set $\text{Irr } N$ of irreducible characters of N . Moreover, G acts on $\mathbf{Z}(\mathbb{C}N)$, the center of the group algebra. Recall that with every $\vartheta \in \text{Irr } N$ there is associated a central primitive idempotent

$$e_\vartheta = \frac{\vartheta(1)}{|N|} \sum_{n \in N} \vartheta(n^{-1})n.$$

The actions of G on $\text{Irr } N$ and $\mathbf{Z}(\mathbb{C}N)$ are compatible in the sense that $(e_\vartheta)^g = e_{\vartheta g}$.

Let $\mathbb{F} \leq \mathbb{C}$ be a subfield of \mathbb{C} . Let $\Gamma := \text{Gal}(\mathbb{C}/\mathbb{F})$ be the group of all field automorphisms of \mathbb{C} that leave every element of \mathbb{F} fixed. (For our purposes, \mathbb{C} can be replaced by any algebraically closed field containing \mathbb{F} , or by a Galois extension of \mathbb{F} that is a splitting field of \widehat{G} .) Then Γ acts on $\text{Irr } N$ by $\vartheta^\alpha(n) = \vartheta(n)^\alpha$ and on $\mathbb{C}N$ by acting on coefficients. Again, these actions are compatible in the sense that $(e_\vartheta)^\alpha = e_{\vartheta^\alpha}$. Observe, however, that if we view ϑ as a function defined on $\mathbb{C}N$, then $\vartheta^\alpha(\sum_n a_n n) = \sum_n a_n \vartheta(n)^\alpha = \vartheta(\sum_n a_n^{\alpha^{-1}} n)^\alpha$.

Since the actions of Γ and G commute, we get an action of $\Gamma \times G$ on $\text{Irr } N$ respective $\mathbf{Z}(\mathbb{C}N)$.

We write

$$e_{\vartheta, \mathbb{F}} = \sum_{\alpha \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})} (e_\vartheta)^\alpha.$$

Thus $e_{\vartheta, \mathbb{F}}$ is the central primitive idempotent of $\mathbb{F}N$ associated to ϑ , and $\mathbb{F}N e_{\vartheta, \mathbb{F}}$ the corresponding simple summand. It is well known that $\mathbf{Z}(\mathbb{F}N e_{\vartheta, \mathbb{F}}) \cong \mathbb{F}(\vartheta)$: an isomorphism is given by restricting the **central character**

$$\omega_\vartheta: \mathbf{Z}(\mathbb{F}N) \rightarrow \mathbb{F}(\vartheta), \quad \omega_\vartheta(z) = \frac{\vartheta(z)}{\vartheta(1)}$$

to $\mathbf{Z}(\mathbb{F}N e_{\vartheta, \mathbb{F}})$ [5, Proposition 38.15].

We write $(e_{\vartheta, \mathbb{F}})^{\widehat{G}}$ or $(e_{\vartheta, \mathbb{F}})^G$ for the sum of the different G -conjugates of $e_{\vartheta, \mathbb{F}}$. Then $(e_{\vartheta, \mathbb{F}})^G$ is a primitive idempotent in $(\mathbf{Z}(\mathbb{F}N))^G = \mathbf{C}_{\mathbf{Z}(\mathbb{F}N)}(G) = \mathbb{F}N \cap \mathbf{Z}(\mathbb{F}\widehat{G})$. It is the sum of the idempotents corresponding to irreducible characters in the $(\Gamma \times G)$ -orbit of ϑ . Following Turull [12], we define

$$\mathbf{Z}(\vartheta, \kappa, \mathbb{F}) = \mathbf{Z}(\mathbb{F}N)(e_{\vartheta, \mathbb{F}})^G.$$

This is called the **center algebra** of (ϑ, κ) over \mathbb{F} . It is a commutative G -algebra over \mathbb{F} , and a direct sum of the algebras $\mathbf{Z}(\mathbb{F}N(e_{\vartheta, \mathbb{F}})^g) \cong \mathbb{F}(\vartheta)$. These summands are permuted transitively by the group G . Thus

$$\mathbf{Z}(\vartheta, \kappa, \mathbb{F}) \cong \mathbb{F}(\vartheta) \times \cdots \times \mathbb{F}(\vartheta)$$

is simple in the sense that it has no nontrivial G -invariant ideals. The center algebra depends only on the $\Gamma \times G$ -orbit of ϑ , where $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{F})$.

Turull [12] has also shown how to associate an equivalence class of simple G -algebras with center $\mathbf{Z}(\vartheta, \kappa, \mathbb{F})$ to the triple $(\vartheta, \kappa, \mathbb{F})$. Namely, let V be any non-zero $\mathbb{F}\widehat{G}$ -module such that $V = V(e_{\vartheta, \mathbb{F}})^G$. Such a module is called **ϑ -quasihomogeneous**. Let $S = \text{End}_{\mathbb{F}N}(V)$. Then $G \cong \widehat{G}/N$ acts on S . For $z \in Z = \mathbf{Z}(\vartheta, \kappa, \mathbb{F})$, the map $v \mapsto vz$ is

in S . This defines an isomorphism of G -algebras $\mathbf{Z}(\vartheta, \kappa, \mathbb{F}) \cong \mathbf{Z}(S)$. The equivalence class of S in the Brauer-Clifford group $\text{BrCliff}(G, Z)$ depends only on $(\vartheta, \kappa, \mathbb{F})$, not on the choice of the $\mathbb{F}\widehat{G}$ -module V [12, Theorem 7.6]. We write $[[\vartheta, \kappa, \mathbb{F}]]$ to denote this element of the Brauer-Clifford group, and call it the **Brauer-Clifford class** of (ϑ, κ) over \mathbb{F} .

3.1. Definition. Let S be a G -algebra, with $\mathbb{F} \subseteq \mathbf{Z}(S)^G$. We call S a **Schur G -algebra** over \mathbb{F} , if there is a Clifford pair $(\vartheta, \kappa: \widehat{G} \rightarrow G)$ and a ϑ -quasihomogeneous $\mathbb{F}\widehat{G}$ -module V , such that $S \cong \text{End}_{\mathbb{F}N} V$ as G -algebras over \mathbb{F} .

This means in particular that $S \in [[\vartheta, \kappa, \mathbb{F}]]$. This definition is not completely analogous to the definition used in the theory of the Schur subgroup of the Brauer group [cf. 14]. Recall that a central simple algebra S over a field K is called a Schur algebra if S is isomorphic to a direct summand of the group algebra KN of some finite group N . In the situation above, $V_{\mathbb{F}N} \cong (\sum_i U_i)^n$, where the U_i are simple $\mathbb{F}N$ -modules which are conjugate in \widehat{G} . Thus $S \cong \prod_i \mathbf{M}_n(D)$, where $D \cong \text{End}_{\mathbb{F}N}(U_i)$. We also have $\mathbb{F}N e_{\vartheta, \mathbb{F}} \cong \mathbf{M}_k(D)$ and $\mathbb{F}N(e_{\vartheta, \mathbb{F}})^G \cong \prod_i \mathbf{M}_k(D)$ for some k . Thus S and $\mathbb{F}N(e_{\vartheta, \mathbb{F}})^G$ are Morita equivalent.

We mention that one can show that in fact every G -algebra in a Brauer-Clifford class $[[\vartheta, \kappa, \mathbb{F}]]$ is a Schur G -algebra over \mathbb{F} , if the class contains one Schur algebra.

3.2. Lemma (Turull [12, Proposition 7.2]). *Let S be a Schur G -algebra over \mathbb{F} . Then $\mathbf{Z}(S) \cong \mathbb{E} \times \cdots \times \mathbb{E}$, where \mathbb{E} is a field contained in a cyclotomic extension of \mathbb{F} .*

Proof. When $S \in [[\vartheta, \kappa, \mathbb{F}]]$, then $\mathbf{Z}(S) \cong \mathbf{Z}(\vartheta, \kappa, \mathbb{F})$, and $\mathbf{Z}(\vartheta, \kappa, \mathbb{F})$ is a direct product of fields of the form $\mathbf{Z}(\mathbb{F}N) e_{\vartheta^g, \mathbb{F}} \cong \mathbb{F}(\vartheta^g) = \mathbb{F}(\vartheta)$. \square

3.3. Definition. Let Z be a commutative simple G -algebra, and assume $Z \cong \mathbf{Z}(\vartheta, \kappa, \mathbb{F})$ for some Clifford pair (ϑ, κ) . The **Schur-Clifford group** of Z is the set of all equivalence classes of central simple G -algebras over Z that contain a Schur G -algebra over \mathbb{F} . We denote it by $\mathcal{SC}^{(\mathbb{F})}(G, Z)$ or simply $\mathcal{SC}(G, Z)$. In other words,

$$\mathcal{SC}(G, Z) = \{\bar{\alpha}[[\vartheta, \kappa, \mathbb{F}]] \mid \mathbf{Z}(\vartheta, \kappa, \mathbb{F}) \cong Z \text{ via } \alpha\},$$

where

$$\bar{\alpha}: \text{BrCliff}(G, \mathbf{Z}(\vartheta, \kappa, \mathbb{F})) \rightarrow \text{BrCliff}(G, Z)$$

is induced by α .

Omitting the field \mathbb{F} from the notation will be justified once we have proved Proposition B from the introduction.

Our aim is to show that the Schur-Clifford group $\mathcal{SC}(G, Z)$ is indeed a subgroup of the Brauer-Clifford group.

It will be convenient to have a definition of a restricted Schur-Clifford group, where not all pairs (ϑ, κ) are allowed. For example, we could consider only pairs where the kernel of κ is contained in the center of \widehat{G} .

3.4. Definition. Let G be a finite group and Z a commutative simple G -algebra over \mathbb{F} . Let \mathcal{X} be a class of Clifford pairs. Then

$$\mathcal{SC}_{\mathcal{X}}(G, Z) = \{\overline{\alpha}[\vartheta, \kappa, \mathbb{F}] \mid (\vartheta, \kappa) \in \mathcal{X} \text{ and } \mathbf{Z}(\vartheta, \kappa, \mathbb{F}) \stackrel{\alpha}{\cong} Z\}.$$

Of course, depending on \mathcal{X} , $\mathcal{SC}_{\mathcal{X}}(G, Z)$ might not be a subgroup of the Brauer-Clifford group.

3.5. Example. Let $Z = \mathbb{C}$ with trivial G -action. Then $\text{BrCliff}(G, \mathbb{C}) \cong H^2(G, \mathbb{C}^*)$. Every simple G -algebra S over \mathbb{C} is isomorphic to a matrix ring $\mathbf{M}_n(\mathbb{C})$ and defines a projective representation $G \rightarrow S^*$. By the classical theory of Schur, the projective representation can be lifted to an ordinary representation $\widehat{G} \rightarrow S^*$, where \widehat{G} has a cyclic central subgroup C such that $\widehat{G}/C \cong G$. Thus $\text{BrCliff}(G, \mathbb{C}) = \mathcal{SC}(G, \mathbb{C}) = \mathcal{SC}_{\mathcal{C}}(G, \mathbb{C})$, where \mathcal{C} denotes the class of pairs (ϑ, κ) , where $\text{Ker } \kappa$ is a central cyclic subgroup.

It is also known that there is a central extension

$$1 \longrightarrow H^2(G, \mathbb{C}^*) \longrightarrow \widehat{G} \xrightarrow{\kappa} G \longrightarrow 1$$

such that every projective representation of G can be lifted to an ordinary representation of \widehat{G} . The group \widehat{G} is then called a Schur representation group. Thus for $\mathcal{C} = \mathcal{C}_{\kappa} = \{(\vartheta, \kappa) \mid \vartheta \in \text{Lin}(\text{Ker } \kappa)\}$, where $\kappa: \widehat{G} \rightarrow G$ is a fixed Schur representation group of G , we also have $\text{BrCliff}(G, \mathbb{C}) = \mathcal{SC}_{\mathcal{C}}(G, \mathbb{C})$.

4. SEMI-INVARIANT CLIFFORD PAIRS

Let (ϑ, κ) be a Clifford pair and $\mathbb{F} \subseteq \mathbb{C}$ a field such that $\mathbf{Z}(\vartheta, \kappa, \mathbb{F})$ is a field. By the remarks above on the center algebra, this means that $(e_{\vartheta, \mathbb{F}})^G = e_{\vartheta, \mathbb{F}}$ is invariant in G or \widehat{G} (where, as before, $\kappa: \widehat{G} \rightarrow G$). In this case, we say that ϑ is **semi-invariant** in \widehat{G} over \mathbb{F} . We also say that (ϑ, κ) is a semi-invariant Clifford pair over \mathbb{F} . A character ϑ is semi-invariant over \mathbb{F} in \widehat{G} if and only if for every $g \in \widehat{G}$ there is $\alpha_g \in \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$ such that $\vartheta^{g\alpha_g} = \vartheta$. The map $g \mapsto \alpha_g$ is a group homomorphism from \widehat{G} to $\text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$ with kernel \widehat{G}_{ϑ} [6, Lemma 2.1]. Since $N = \text{Ker } \kappa \subseteq \widehat{G}_{\vartheta}$, the map $g \mapsto \alpha_g$ defines an action of G on $\mathbb{F}(\vartheta)$. Thus $\mathbb{F}(\vartheta)$ can be viewed as a G -ring. Note that a Galois conjugate ϑ^{σ} of ϑ yields the same homomorphism, since $\text{Gal}(\mathbb{Q}(\vartheta)/\mathbb{Q})$ is abelian.

The central character ω_{ϑ} defines an isomorphism $\mathbf{Z}(\vartheta, \kappa, \mathbb{F}) \cong \mathbb{F}(\vartheta)$. From

$$\vartheta(z^g) = \vartheta^{g^{-1}}(z) = \vartheta(z)^{\alpha_g}$$

we see that $\omega_{\vartheta}: \mathbf{Z}(\vartheta, \kappa, \mathbb{F}) \rightarrow \mathbb{F}(\vartheta)$ is an isomorphism of G -algebras. In fact, any other \mathbb{F} -isomorphism $\mathbf{Z}(\vartheta, \kappa, \mathbb{F}) \rightarrow \mathbb{F}(\vartheta)$ is an isomorphism of G -algebras, since $\text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$ is abelian.

When considering the Schur-Clifford group of a field Z with G -action, it is thus natural to assume that $Z = \mathbb{E}$ is a subfield of \mathbb{C} (or any algebraically closed field of characteristic zero fixed in advance and in which all characters are assumed to take values). Thus

$$\mathcal{SC}(G, \mathbb{E}) = \{\overline{\omega_{\vartheta}}[\vartheta, \kappa, \mathbb{F}] \mid \mathbf{Z}(\vartheta, \kappa, \mathbb{F}) \stackrel{\omega_{\vartheta}}{\cong} \mathbb{E} = \mathbb{F}(\vartheta)\}.$$

Let us mention here that the notation $\mathcal{SC}(G, \mathbb{E})$, as well as $\text{BrCliff}(G, R)$, does not reflect the dependence on the actual action of G on R . A ring may be equipped with different actions of the same group G . In particular, two different semi-invariant Clifford pairs (ϑ_1, κ_1) and (ϑ_2, κ_2) with $\mathbb{F}(\vartheta_1) = \mathbb{F}(\vartheta_2) = \mathbb{E}$ may yield elements of different Schur-Clifford groups which are both denoted by $\mathcal{SC}(G, \mathbb{E})$. This is due to the fact that it can happen that $\mathbf{Z}(\vartheta_1, \kappa_1, \mathbb{F})$ and $\mathbf{Z}(\vartheta_2, \kappa_2, \mathbb{F})$ are not isomorphic as G -algebras, although they are isomorphic fields. In the equation displayed above, we assume that an action of G on \mathbb{E} is fixed and that ω_ϑ yields an isomorphism of G -algebras from $\mathbf{Z}(\vartheta, \kappa, \mathbb{F})$ to \mathbb{E} .

5. PROOF OF THE SUBGROUP PROPERTIES

We work now to show that $\mathcal{SC}_{\mathcal{X}}(G, Z)$ is a subgroup of $\text{BrCliff}(G, Z)$, given some mild conditions on \mathcal{X} and Z . We do this first for the case where $Z = \mathbb{E}$ is a field. The general case will follow once we have proved Theorem C from the introduction.

Note that when $\mathbb{C} \supseteq \mathbb{E} \cong \mathbf{Z}(\vartheta, \kappa, \mathbb{F})$ for some Clifford pair (ϑ, κ) , then necessarily $\mathbb{E} = \mathbb{F}(\vartheta)$ is contained in $\mathbb{F}(\varepsilon)$ for some root of unity ε . The next lemma yields the converse.

5.1. Lemma. *Let G be a group which acts on the field $\mathbb{E} \subseteq \mathbb{C}$. Assume that \mathbb{F} is a subfield of the fixed field \mathbb{E}^G . If there is a root of unity ε such that $\mathbb{E} \subseteq \mathbb{F}(\varepsilon)$, then \mathbb{E} is a Schur G -algebra over \mathbb{F} .*

Proof. Set $\Gamma = \text{Gal}(\mathbb{F}(\varepsilon)/\mathbb{F})$. As Γ is abelian, every field between \mathbb{F} and $\mathbb{F}(\varepsilon)$ is normal over \mathbb{F} . In particular, Γ acts on \mathbb{E} . Since \mathbb{E} is a G -algebra by assumption, the group G acts on \mathbb{E} , too. We let U be the pullback in

$$\begin{array}{ccc} U & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ G & \longrightarrow & \text{Aut}(\mathbb{E}), \end{array}$$

that is,

$$U = \{(g, \sigma) \in G \times \Gamma \mid x^g = x^\sigma \text{ for all } x \in \mathbb{E}\}.$$

Let $C = \langle \varepsilon \rangle$. The group U acts on C by $c^{(g, \sigma)} = c^\sigma$. Let $\widehat{G} = UC$ be the semidirect product of U and C with respect to this action. The map $\kappa: \widehat{G} \rightarrow G$ sending $(g, \sigma)c$ to g is an epimorphism. (The surjectivity follows from the well known fact that every \mathbb{F} -automorphism of \mathbb{E} extends to one of $\mathbb{F}(\varepsilon)$.) Let N be the kernel of κ . Then $N = AC$, where

$$A = \{(1, \sigma) \mid x^\sigma = x \text{ for all } x \in \mathbb{E}\} \cong \text{Gal}(\mathbb{F}(\varepsilon)/\mathbb{E}).$$

Set $V = \mathbb{F}(\varepsilon)$ and define a \widehat{G} -module structure on V by

$$v(g, \sigma)c = v^\sigma c \quad \text{for } v \in \mathbb{F}(\varepsilon), (g, \sigma) \in U \text{ and } c \in C.$$

We claim that $\mathbb{E} \cong \text{End}_{\mathbb{F}N}(V)$ as G -algebras. For $x \in \mathbb{E}$, right multiplication $\rho_x: v \mapsto vx$ ($v \in V$) is an $\mathbb{F}N$ -endomorphism. The map $x \mapsto \rho_x$ defines an injection

of \mathbb{F} -algebras $\mathbb{E} \rightarrow \text{End}_{\mathbb{F}N}(V)$. Since $\mathbb{F}C$ acts as $\mathbb{F}(\varepsilon)$ on V from the right, $\text{End}_{\mathbb{F}N}(V)$ can be identified with a subfield of $\mathbb{F}(\varepsilon)$ (via the above map $x \mapsto \rho_x$), and since A acts as $\text{Gal}(\mathbb{F}(\varepsilon)/\mathbb{E})$ on V , this subfield is just \mathbb{E} .

It remains to show that $x \mapsto \rho_x$ commutes with the action of G . Remember that for $g \in G$, the endomorphism $(\rho_x)^g$ is defined by $v \mapsto v\hat{g}^{-1}\rho_x\hat{g}$, where $\hat{g} \in \widehat{G}$ is some element with $\hat{g}\kappa = g$. Here we can choose $\hat{g} = (g, \sigma)$ with suitable $\sigma \in \Gamma$. We get that

$$v(\rho_x)^g = v(g^{-1}, \sigma^{-1})x(g, \sigma) = v^{\sigma^{-1}}x(g, \sigma) = vx^\sigma = vx^g,$$

where the last equation follows from the fact that $(g, \sigma) \in U$ and thus $x^\sigma = x^g$. So $(\rho_x)^g = \rho_{x^g}$, and the proof is finished. \square

From the proof of the last lemma, we get the following more precise result:

5.2. Corollary. *Assume the hypotheses of Lemma 5.1. Then $[\mathbb{E}] = [\vartheta, \kappa, \mathbb{F}]$, where (ϑ, κ) has the following properties:*

- (a) $N := \text{Ker } \kappa = AC$ is the semidirect product of a cyclic group $C \cong \langle \varepsilon \rangle$ and a subgroup A of $\text{Aut}(C)$ such that $A \cong \text{Gal}(\mathbb{F}(\varepsilon)/\mathbb{E})$.
- (b) $\vartheta = \lambda^N$, where λ is a faithful linear character of C .

Proof. Let $C = \langle \varepsilon \rangle$, \widehat{G} and V be as in the proof of Lemma 5.1. View the embedding $\lambda: C \hookrightarrow \mathbb{F}(\varepsilon)$ as linear character of C . The $\mathbb{F}C$ -module $V = \mathbb{F}(\varepsilon)$ affords the character $\text{Tr}_{\mathbb{F}}^{\mathbb{F}(\varepsilon)}(\lambda) = \sum_{\sigma \in \Gamma} \lambda^\sigma$, where $\Gamma = \text{Gal}(\mathbb{F}(\varepsilon)/\mathbb{F})$. Since $\vartheta = \lambda^N \in \text{Irr } N$, the character of V_N contains ϑ as irreducible constituent. This proves the corollary. \square

In the next result, we only assume that Z is a commutative G -ring and that $\mathbb{F} \subseteq Z^G$ is a field.

5.3. Lemma. *If S is a Schur G -algebra over \mathbb{F} , then S^{op} is a Schur G -algebra over \mathbb{F} . In fact, when $[S] = [\vartheta, \kappa, \mathbb{F}]$, then $[S^{\text{op}}] = [\bar{\vartheta}, \kappa, \mathbb{F}]$.*

Proof. Suppose that $\widehat{G}/N \cong G$ and that V is a (right) $\mathbb{F}\widehat{G}$ -module such that $S = \text{End}_{\mathbb{F}N}(V)$. Let $V' = \text{Hom}_{\mathbb{F}}(V, \mathbb{F})$. Then V' becomes a right $\mathbb{F}\widehat{G}$ -module by $v(v'g) = (vg^{-1})v'$ for $v \in V$, $v' \in V'$. Define $\alpha: S^{\text{op}} \rightarrow \text{End}_{\mathbb{F}N}(V')$ by $v((v')s^\alpha) = (vs)v'$. Then α is an isomorphism of G -algebras. In fact, with the same definition one gets an isomorphism from $\text{End}_{\mathbb{F}}(V)^{\text{op}}$ onto $\text{End}_{\mathbb{F}}(V')$ as \widehat{G} -algebras, as is well known. The first part of the lemma follows from this fact.

For the second, recall that when V affords the character χ , then V' affords the character $g \mapsto \chi(g^{-1}) = \overline{\chi(g)}$. Thus if ϑ is an irreducible constituent of the character of V_N , then $\bar{\vartheta}$ is an irreducible constituent of the character of $(V')_N$. The proof is finished. \square

Our next goal is to show that when a_1 and $a_2 \in \mathcal{SC}(G, Z)$, then $a_1a_2 \in \mathcal{SC}(G, Z)$. For simplicity of notation, we assume again that $Z = \mathbb{E}$ is a subfield of \mathbb{C} . Let (ϑ_i, κ_i) be two Clifford pairs over the same group G such that $\mathbf{Z}(\vartheta_1, \kappa_1, \mathbb{F}) \cong \mathbb{E} \cong \mathbf{Z}(\vartheta_2, \kappa_2, \mathbb{F})$ as G -algebras. We want to construct a Clifford pair (ϑ, κ) such that

$$[\vartheta, \kappa, \mathbb{F}] = [\vartheta_1, \kappa_1, \mathbb{F}][\vartheta_2, \kappa_2, \mathbb{F}].$$

For $i = 1, 2$ let

$$1 \longrightarrow N_i \longrightarrow G_i \xrightarrow{\kappa_i} G \longrightarrow 1$$

be the exact sequence of groups belonging to the Clifford pair (ϑ_i, κ_i) . Let $\widehat{G} = \{(g_1, g_2) \in G_1 \times G_2 \mid g_1 \kappa_1 = g_2 \kappa_2\}$ be the pullback of the maps κ_1 and κ_2 and let $\kappa: \widehat{G} \rightarrow G$ be the composition $\widehat{G} \rightarrow G_i \rightarrow G$:

$$\begin{array}{ccc} \widehat{G} & \longrightarrow & G_1 \\ \downarrow & \searrow \kappa & \downarrow \kappa_1 \\ G_2 & \xrightarrow{\kappa_2} & G \end{array}$$

We also write $\kappa_1 \times_G \kappa_2$ for this map. ($G_1 \times_G G_2$ is a common notation for the pullback \widehat{G} .) Then $\kappa = \kappa_1 \times_G \kappa_2$ is surjective, since both κ_1 and κ_2 are, and the kernel is $N_1 \times N_2$. Thus we have an exact sequence

$$1 \longrightarrow N_1 \times N_2 \longrightarrow \widehat{G} \xrightarrow{\kappa} G \longrightarrow 1.$$

We write $\vartheta_1 \times \vartheta_2$ for the irreducible character of $N_1 \times N_2$ defined by $(\vartheta_1 \times \vartheta_2)(n_1, n_2) = \vartheta_1(n_1)\vartheta_2(n_2)$. We have now defined a new Clifford pair

$$(\vartheta_1 \times \vartheta_2, \kappa_1 \times_G \kappa_2)$$

over G .

5.4. Lemma. *In the above situation, we have*

$$[[\vartheta_1 \times \vartheta_2, \kappa_1 \times_G \kappa_2, \mathbb{F}]] = [[\vartheta_1, \kappa_1, \mathbb{F}]] [[\vartheta_2, \kappa_2, \mathbb{F}]]$$

in $\text{BrCliff}(G, \mathbb{E})$.

Proof. That $\mathbf{Z}(\vartheta_1, \kappa_1, \mathbb{F}) \cong \mathbf{Z}(\vartheta_2, \kappa_2, \mathbb{F}) \cong \mathbb{E}$ as G -algebras means that $\mathbb{F}(\vartheta_1) = \mathbb{F}(\vartheta_2) = \mathbb{E}$ and that for every $g \in G$ there is $\alpha_g \in \text{Gal}(\mathbb{E}/\mathbb{F})$ such that $\vartheta_1^{g\alpha_g} = \vartheta_1$ and $\vartheta_2^{g\alpha_g} = \vartheta_2$. Clearly, we also have $\mathbb{F}(\vartheta_1 \times \vartheta_2) = \mathbb{E}$. Since $(\vartheta_1 \times \vartheta_2)^{g\alpha_g} = \vartheta_1^{g\alpha_g} \times \vartheta_2^{g\alpha_g} = \vartheta_1 \times \vartheta_2$, we have that $\mathbf{Z}(\vartheta_1 \times \vartheta_2, \kappa_1 \times_G \kappa_2, \mathbb{F}) \cong \mathbb{E}$ as G -algebras.

Suppose that V_i are ϑ_i -quasihomogeneous \widehat{G}_i -modules with $S_i = \text{End}_{\mathbb{F}N_i}(V_i)$ as G -algebras. We have to show that

$$S_1 \otimes_{\mathbb{E}} S_2 \in [[\vartheta_1 \times \vartheta_2, \kappa_1 \times_G \kappa_2, \mathbb{F}]].$$

Since $\mathbb{E} \cong \mathbf{Z}(\mathbb{F}N_i e_{\vartheta_i, \mathbb{F}})$ for $i = 1, 2$, we can view V_1 and V_2 as vector spaces over \mathbb{E} . (The vector space structures depend on the concrete isomorphisms, which we can assume to be given by ω_{ϑ_i} , but in fact the arguments to follow work for any choices of isomorphisms $\mathbb{E} \cong \mathbf{Z}(\mathbb{F}N_i e_{\vartheta_i, \mathbb{F}})$.)

Let $V = V_1 \otimes_{\mathbb{E}} V_2$ and define an action of \widehat{G} on V by $(v_1 \otimes v_2)(g_1, g_2) = v_1 g_1 \otimes v_2 g_2$. This is well defined: Let $z \in \mathbb{E}$. Then

$$\begin{aligned}
(v_1 z \otimes v_2)(g_1, g_2) &= v_1 z g_1 \otimes v_2 g_2 \\
&= v_1 g_1 z^{g_1 \kappa_1} \otimes v_2 g_2 \\
&= v_1 g_1 \otimes z^{g_1 \kappa_1} (v_2 g_2) \\
&= v_1 g_1 \otimes (v_2 g_2) z^{g_2 \kappa_2} && \text{since } g_1 \kappa_1 = g_2 \kappa_2 \\
&= v_1 g_1 \otimes v_2 z g_2 \\
&= (v_1 \otimes z v_2)(g_1, g_2).
\end{aligned}$$

In this way, $V = V_1 \otimes_{\mathbb{E}} V_2$ becomes an $\mathbb{F}\widehat{G}$ -module.

As $\mathbb{E}N_i$ -module, the character of V_i is a multiple of ϑ_i . Thus V as an $\mathbb{E}[N_1 \times N_2]$ module has as character a multiple of $\vartheta_1 \times \vartheta_2$. It follows that V as $\mathbb{F}[N_1 \times N_2]$ -module is $(\vartheta_1 \times \vartheta_2)$ -quasihomogeneous.

Since \mathbb{E} is isomorphic to the centers of the various algebras involved, we have

$$\begin{aligned}
\text{End}_{\mathbb{F}[N_1 \times N_2]}(V) &= \text{End}_{\mathbb{E}[N_1 \times N_2]}(V_1 \otimes_{\mathbb{E}} V_2) \\
&\cong \text{End}_{\mathbb{E}N_1}(V_1) \otimes_{\mathbb{E}} \text{End}_{\mathbb{E}N_2}(V_2) \\
&= \text{End}_{\mathbb{F}N_1}(V_1) \otimes_{\mathbb{E}} \text{End}_{\mathbb{F}N_2}(V_2) \\
&= S_1 \otimes_{\mathbb{E}} S_2.
\end{aligned}$$

The isomorphism sends $s_1 \otimes s_2 \in S_1 \otimes_{\mathbb{E}} S_2$ to the map $(v_1 \otimes v_2) \mapsto (v_1 s_1) \otimes (v_2 s_2)$. It is easy to see that this is an isomorphism of G -algebras. Thus $S_1 \otimes_{\mathbb{E}} S_2 \cong \text{End}_{\mathbb{E}[N_1 \times N_2]}(V) \in \llbracket \vartheta_1 \times \vartheta_2, \kappa_1 \times \kappa_2, \mathbb{F} \rrbracket$ as was to be shown. \square

5.5. Remark. The construction of the Clifford pair $(\vartheta_1 \times \vartheta_2, \kappa_1 \times_G \kappa_2)$ does not depend on the fact that the center algebras of (ϑ_1, κ_1) and (ϑ_2, κ_2) are fields and isomorphic. Without this assumption, one can show the following: The center algebra $Z := \mathbf{Z}(\vartheta_1 \times \vartheta_2, \kappa_1 \times_G \kappa_2, \mathbb{F})$ is a direct summand of $Z_1 \otimes_{\mathbb{F}} Z_2$, where $Z_i = \mathbf{Z}(\vartheta_i, \kappa_i, \mathbb{F})$. The isomorphism type of Z depends on the choice of ϑ_i in its orbit under the action of $G \times \text{Gal}(\mathbb{F}(\vartheta_i)/\mathbb{F})$. This is different from the situation in Lemma 5.4, where $\mathbf{Z}(\vartheta_1 \times \vartheta_2^\alpha, \kappa_1 \times \kappa_2, \mathbb{F}) \cong \mathbb{E}$ as G -algebras for all $\alpha \in \text{Gal}(\mathbb{E}/\mathbb{F})$.

In the general situation, it follows that there are G -algebra homomorphisms $Z_i \rightarrow Z_1 \otimes_{\mathbb{F}} Z_2 \rightarrow Z$. If S_i is a G -algebra in $\llbracket \vartheta_i, \kappa_i, \mathbb{F} \rrbracket$, then $(S_1 \otimes_{Z_1} Z) \otimes_Z (S_2 \otimes_{Z_2} Z)$ is in $\llbracket \vartheta_1 \times \vartheta_2, \kappa_1 \times_G \kappa_2, \mathbb{F} \rrbracket$. Since we do not need these more general results, we do not pause to prove them.

We are now ready to prove Theorem A for $Z = \mathbb{E}$ a field.

5.6. Theorem. *Let \mathbb{E} be a field on which a group G acts. Suppose \mathbb{E} is contained in a cyclotomic extension of $\mathbb{F} \subseteq \mathbb{E}^G$. Then the Schur-Clifford group $\text{SC}(G, \mathbb{E})$ is a subgroup of the Brauer-Clifford group $\text{BrCliff}(G, \mathbb{E})$.*

We prove this together with a theorem about restricted Schur-Clifford groups.

5.7. Theorem. *In the situation of Theorem 5.6, let \mathcal{X} be a class of Clifford pairs (ϑ, κ) (over G), such that the following conditions hold:*

- (a) $\mathcal{SC}_{\mathcal{X}}(G, \mathbb{E}) \neq \emptyset$,
- (b) *When $(\vartheta, \kappa) \in \mathcal{X}$, then $(\bar{\vartheta}, \kappa) \in \mathcal{X}$,*
- (c) *When (ϑ_1, κ_1) and $(\vartheta_2, \kappa_2) \in \mathcal{X}$, then $(\vartheta_1 \times \vartheta_2, \kappa_1 \times_G \kappa_2) \in \mathcal{X}$.*

Then $\mathcal{SC}_{\mathcal{X}}(G, \mathbb{E})$ is a subgroup of $\text{BrCliff}(G, \mathbb{E})$.

Proof of Theorem 5.6 and 5.7. Recall that

$$\mathcal{SC}_{\mathcal{X}}(G, \mathbb{E}) = \{[\vartheta, \kappa, \mathbb{F}] \mid (\vartheta, \kappa) \in \mathcal{X} \text{ and } \mathbf{Z}(\vartheta, \kappa, \mathbb{F}) \cong \mathbb{E}\}.$$

Lemma 5.1 yields that $[\mathbb{E}] \in \mathcal{SC}(G, \mathbb{E})$. Lemma 5.3 yields that when $[\vartheta, \kappa, \mathbb{F}] \in \mathcal{SC}_{\mathcal{X}}(G, \mathbb{E})$, then

$$[\vartheta, \kappa, \mathbb{F}]^{-1} = [\bar{\vartheta}, \kappa, \mathbb{F}] \in \mathcal{SC}_{\mathcal{X}}(G, \mathbb{E}).$$

Finally, when $[\vartheta_1, \kappa_1, \mathbb{F}]$ and $[\vartheta_2, \kappa_2, \mathbb{F}] \in \mathcal{SC}_{\mathcal{X}}(G, \mathbb{E})$, then $\mathbf{Z}(\vartheta_1, \kappa_1, \mathbb{F}) \cong \mathbb{E} \cong \mathbf{Z}(\vartheta_2, \kappa_2, \mathbb{F})$ as G -algebras, and Lemma 5.4 yields that

$$[\vartheta_1, \kappa_1, \mathbb{F}][\vartheta_2, \kappa_2, \mathbb{F}] = [\vartheta_1 \times \vartheta_2, \kappa_1 \times_G \kappa_2, \mathbb{F}] \in \mathcal{SC}_{\mathcal{X}}(G, \mathbb{E}). \quad \square$$

The conditions in Theorem 5.7 for \mathcal{X} are not necessary for $\mathcal{SC}_{\mathcal{X}}(G, \mathbb{E})$ to form a subgroup. The following is an example:

5.8. Corollary. *Let \mathcal{C} be the class of semi-invariant Clifford pairs (ϑ, κ) such that $N = \text{Ker } \kappa$ is cyclic. Suppose that $\mathbb{E} = \mathbb{F}(\varepsilon)$ for some root of unity ε . Then $\mathcal{SC}_{\mathcal{C}}(G, \mathbb{E})$ is a subgroup of $\text{BrCliff}(G, \mathbb{E})$.*

Proof. We begin by showing $[\mathbb{E}] \in \mathcal{SC}_{\mathcal{C}}(G, \mathbb{E})$. The action of G on $\mathbb{E} = \mathbb{F}(\varepsilon)$ yields an action of G on $C = \langle \varepsilon \rangle$. Let $\hat{G} = GC$ be the semidirect product and $\kappa: \hat{G} \rightarrow G$ the corresponding epimorphism with kernel C . View the inclusion $C \subset \mathbb{E}$ as a linear character λ . Make $V = \mathbb{E}$ into a \hat{G} -module by defining $v(gc) = v^g c$. Then V is λ -quasihomogeneous and $\text{End}_{\mathbb{F}C}(V) = \text{End}_{\mathbb{F}(\varepsilon)}(V) \cong \mathbb{E}$ as G -algebras. Thus $[\mathbb{E}] = [\lambda, \kappa, \mathbb{F}] \in \mathcal{SC}_{\mathcal{C}}(G, \mathbb{E})$.

Now let \mathcal{A} be the class of semi-invariant Clifford pairs over G such that $\text{Ker } \kappa$ is abelian. By Theorem 5.7 and the preceding paragraph, $\mathcal{SC}_{\mathcal{A}}(G, \mathbb{E})$ is a subgroup. Pick a Clifford pair $(\lambda, \kappa) \in \mathcal{A}$ with corresponding exact sequence

$$1 \longrightarrow N \hookrightarrow \hat{G} \xrightarrow{\kappa} G \longrightarrow 1.$$

Let $K = \text{Ker } \lambda$. Since λ is assumed to be semi-invariant over \mathbb{F} , it follows that K is normal in \hat{G} . View λ as character λ_1 of N/K and let $\kappa_1: \hat{G}/K \rightarrow G$ be the map induced by κ . From the definitions, it follows that $[\lambda, \kappa, \mathbb{F}] = [\lambda_1, \kappa_1, \mathbb{F}]$. Obviously, $(\lambda_1, \kappa_1) \in \mathcal{C}$. Thus $\mathcal{SC}_{\mathcal{A}}(G, \mathbb{E}) \subseteq \mathcal{SC}_{\mathcal{C}}(G, \mathbb{E})$. The reverse inclusion is trivial, and thus $\mathcal{SC}_{\mathcal{C}}(G, \mathbb{E}) = \mathcal{SC}_{\mathcal{A}}(G, \mathbb{E})$ is a subgroup. \square

6. DEPENDENCE ON THE FIELD

In this section, we prove Proposition B from the introduction in the case where $Z = \mathbb{E}$ is a field. However, let us first note that the uniqueness of the field $\mathbb{K} \subseteq \mathbb{C}$ in Proposition B follows in any case, because when $Z \cong \mathbf{Z}(\vartheta, \kappa, \mathbb{F})$, then Z^G is a finite abelian Galois extension of \mathbb{F} and thus isomorphic (as algebra over \mathbb{F}) to exactly one field $\mathbb{K} \subseteq \mathbb{C}$.

6.1. Lemma. *Let (ϑ, κ) be a Clifford pair over G that is semi-invariant over the field \mathbb{F} . Let \mathbb{K} be a field such that $\mathbb{F} \subseteq \mathbb{K} \subseteq \mathbb{F}(\vartheta)^G$. Then $[[\vartheta, \kappa, \mathbb{F}]] = [[\vartheta, \kappa, \mathbb{K}]]$.*

Proof. Note that the condition on \mathbb{K} implies that $\mathbb{F}(\vartheta) = \mathbb{K}(\vartheta)$, and thus both $[[\vartheta, \kappa, \mathbb{F}]]$ and $[[\vartheta, \kappa, \mathbb{K}]]$ are elements of the same Brauer-Clifford group $\text{BrCliff}(G, \mathbb{F}(\vartheta))$.

Let V be a ϑ -quasihomogeneous module over $\mathbb{F}\widehat{G}$, where, as usual, \widehat{G} is the domain of κ and $N = \text{Ker } \kappa$. Since \mathbb{K} is isomorphic to a subfield of $\mathbf{Z}(\mathbb{F}N e_{\vartheta, \mathbb{F}})^G$, we can view V as a $\mathbb{K}\widehat{G}$ -module. Then $\text{End}_{\mathbb{K}N}(V) = \text{End}_{\mathbb{F}N}(V)$, and the result follows. \square

The next result contains Proposition B in the case where $Z = \mathbb{E}$ is a field.

6.2. Proposition. *Let G be a group which acts on the field \mathbb{E} , and let*

$$\mathbb{F} \subseteq \mathbb{K} \subseteq \mathbb{E}^G \subseteq \mathbb{E} \subseteq \mathbb{F}(\varepsilon)$$

be a chain of fields, where ε is a root of unity. Then

$$\mathcal{SC}^{(\mathbb{F})}(G, \mathbb{E}) = \mathcal{SC}^{(\mathbb{K})}(G, \mathbb{E}).$$

Proof. The inclusion $\mathcal{SC}^{(\mathbb{F})}(G, \mathbb{E}) \subseteq \mathcal{SC}^{(\mathbb{K})}(G, \mathbb{E})$ is a direct consequence of Lemma 6.1: If $[[\vartheta, \kappa, \mathbb{F}]] \in \mathcal{SC}^{(\mathbb{F})}(G, \mathbb{E})$, then $\mathbb{E} = \mathbb{F}(\vartheta)$, and thus $\mathbb{K} \subseteq \mathbb{F}(\vartheta)^G$, so Lemma 6.1 applies.

Conversely, pick $[[\vartheta, \kappa, \mathbb{K}]] \in \mathcal{SC}^{(\mathbb{K})}(G, \mathbb{E})$. Then $\mathbb{E} = \mathbb{K}(\vartheta)$, but it may happen that $\mathbb{F}(\vartheta) < \mathbb{E}$ (equivalently, $\mathbb{K} \not\subseteq \mathbb{F}(\vartheta)$). However, by Lemma 5.1 there is a Clifford pair (φ, π) such that

$$[\mathbb{E}] = [[\varphi, \pi, \mathbb{F}]] = [[\varphi, \pi, \mathbb{K}]],$$

where the second equality follows from Lemma 6.1. We have $\mathbb{E} = \mathbb{F}(\varphi) = \mathbb{F}(\vartheta \times \varphi)$, and in particular $\mathbb{K} \subseteq \mathbb{F}(\vartheta \times \varphi)^G$. Applying first Lemma 5.4, then Lemma 6.1 again, we get

$$\begin{aligned} [[\vartheta, \kappa, \mathbb{K}]] &= [[\vartheta \times \varphi, \kappa \times_G \pi, \mathbb{K}]] \\ &= [[\vartheta \times \varphi, \kappa \times_G \pi, \mathbb{F}]] \in \mathcal{SC}^{(\mathbb{F})}(G, \mathbb{E}). \end{aligned} \quad \square$$

From the proof, we get a statement about restricted Schur-Clifford groups.

6.3. Corollary. *In the situation of Proposition 6.2, let \mathcal{X} be a class of Clifford pairs. Then*

$$\mathcal{SC}_{\mathcal{X}}^{(\mathbb{F})}(G, \mathbb{E}) \subseteq \mathcal{SC}_{\mathcal{X}}^{(\mathbb{K})}(G, \mathbb{E}).$$

If there is a Clifford pair (φ, π) such that $[\mathbb{E}] = [[\varphi, \pi, \mathbb{F}]]$ and such that $(\vartheta \times \varphi, \kappa \times_G \varphi) \in \mathcal{X}$ for all $(\vartheta, \kappa) \in \mathcal{X}$, then

$$\mathcal{SC}_{\mathcal{X}}^{(\mathbb{F})}(G, \mathbb{E}) = \mathcal{SC}_{\mathcal{X}}^{(\mathbb{K})}(G, \mathbb{E}).$$

As an example consider the class \mathcal{A} of Clifford pairs (ϑ, κ) such that $N = \text{Ker } \kappa$ is abelian. If there is a root of unity ζ such that $\mathbb{E} = \mathbb{F}(\zeta)$, then $\mathcal{SC}_{\mathcal{A}}^{(\mathbb{F})}(G, \mathbb{E}) = \mathcal{SC}_{\mathcal{A}}^{(\mathbb{K})}(G, \mathbb{E})$. On the other hand, it may happen that $\mathbb{E} = \mathbb{K}(\zeta)$ for some root of unity ζ , but \mathbb{E} over \mathbb{F} is not cyclotomic. (Example: $\mathbb{F} = \mathbb{Q}$, $\mathbb{K} = \mathbb{Q}(\sqrt{2})$ and $\mathbb{E} = \mathbb{Q}(\sqrt{2}, \zeta)$ where ζ is a primitive third root of unity.) Then $\mathcal{SC}_{\mathcal{A}}^{(\mathbb{F})}(G, \mathbb{E})$ is empty, while $\mathcal{SC}_{\mathcal{A}}^{(\mathbb{K})}(G, \mathbb{E})$ is a subgroup of $\text{BrCliff}(G, \mathbb{E})$ (see the proof of Corollary 5.8).

7. RESTRICTION

Let G and H be finite groups and $\varepsilon: H \rightarrow G$ a group homomorphism. Let R be a G -ring. Then R can be viewed as H -ring and ε induces a group homomorphism $\text{BrCliff}(G, R) \rightarrow \text{BrCliff}(H, R)$. Let Z be a commutative simple G -algebra over \mathbb{F} . Recall that this means that Z has no nontrivial G -invariant ideal, and that such a Z is a direct product of fields which are permuted transitively by G [13, Proposition 2.12]. Then Z may not be simple as an H -algebra (with the notable exception of the case where Z is a field). But if e is a primitive idempotent in $Z^H = \mathbf{C}_Z(H)$, then Ze is simple as H -algebra. The map $Z \rightarrow Ze$ is a homomorphism of H -algebras and induces a homomorphism of Brauer-Clifford groups $\text{BrCliff}(H, Z) \rightarrow \text{BrCliff}(H, Ze)$.

7.1. Proposition. *Assume the situation just described. Then the induced homomorphism $\text{BrCliff}(G, Z) \rightarrow \text{BrCliff}(H, Ze)$ maps $\mathcal{SC}(G, Z)$ into $\mathcal{SC}(H, Ze)$.*

Proof. Let (ϑ, κ) be a Clifford pair over G . Consider

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & \widehat{H} & \xrightarrow{\rho} & H & \longrightarrow & 1 \\ & & \parallel & & \downarrow \pi & & \downarrow \varepsilon & & \\ 1 & \longrightarrow & N & \longrightarrow & \widehat{G} & \xrightarrow{\kappa} & G & \longrightarrow & 1, \end{array}$$

where \widehat{H} is the pull-back of the morphisms $\kappa: \widehat{G} \rightarrow G$ and $\varepsilon: H \rightarrow G$. Then (ϑ, ρ) is a Clifford pair over H . (In the special case where $\varepsilon: H \rightarrow G$ is an inclusion of a subgroup, the group \widehat{H} is simply $\kappa^{-1}(H)$ and ρ the restriction of κ to \widehat{H} .) The idempotent $(e_{\vartheta, \mathbb{F}})^H$ is a sum of H -conjugates of $e_{\vartheta, \mathbb{F}}$ and thus $(e_{\vartheta, \mathbb{F}})^H (e_{\vartheta, \mathbb{F}})^G = (e_{\vartheta, \mathbb{F}})^H$. It follows that $\mathbf{Z}(\vartheta, \rho, \mathbb{F})$ is a direct summand of $\mathbf{Z}(\vartheta, \kappa, \mathbb{F})$. Conversely, any direct summand of $\mathbf{Z}(\vartheta, \kappa, \mathbb{F})$ as H -algebra is of the form $\mathbf{Z}(\vartheta^g, \rho, \mathbb{F})$ for some $g \in G$.

Let V be a ϑ -quasihomogeneous module over $\mathbb{F}\widehat{G}$ such that $S = \text{End}_{\mathbb{F}N}(V)$ is a G -algebra in $\llbracket \vartheta, \kappa, \mathbb{F} \rrbracket \in \mathcal{SC}(G, Z)$, and let $e \in Z^H$ be a primitive idempotent. Then Ve becomes a module over $\mathbb{F}\widehat{H}$ via $\pi: \widehat{H} \rightarrow \widehat{G}$. Thus $\text{End}_{\mathbb{F}N}(Ve) \cong \text{End}_{\mathbb{F}N}(V)e = Se$ is an H -algebra in $\llbracket \vartheta^g, \rho, \mathbb{F} \rrbracket$ for some $g \in G$. Since the map $\text{BrCliff}(G, Z) \rightarrow \text{BrCliff}(H, Ze)$ sends the equivalence class of S to the equivalence class of the H -algebra Se , this proves the result. \square

8. THE SCHUR-CLIFFORD GROUP OVER SIMPLE G -RINGS THAT ARE NOT A FIELD

Let Z be a simple G -ring. Then Z is isomorphic to a direct product of fields which are permuted transitively by the group G [13, Proposition 2.12]. More precisely, let $e \in Z$ be a primitive idempotent. Then $\mathbb{E} = Ze$ is a field. Let $H = G_e = \{g \in G \mid e^g = e\}$ be the stabilizer of e . Then $e^g e = 0$ for all $g \in G \setminus H$. Choose $T \subseteq G$ with $G = \bigcup_{t \in T} Ht$. Then $1 = \sum_{t \in T} e^t$ is a primitive idempotent decomposition of 1 in Z . Thus

$$Z = \prod_{t \in T} Ze^t \cong \prod_{t \in T} \mathbb{E}.$$

The group H acts on the field $\mathbb{E} = Ze$. We recall the following result of Turull [12, Theorem 5.3]:

8.1. Proposition. *With the notation just introduced, the map sending a G -algebra S over Z to the H -algebra Se over $\mathbb{E} = Ze$ defines an isomorphism*

$$\text{BrCliff}(G, Z) \cong \text{BrCliff}(H, \mathbb{E}).$$

The following is Theorem C from the introduction:

8.2. Theorem. *The isomorphism of Proposition 8.1 restricts to an isomorphism*

$$\mathcal{SC}(G, Z) \cong \mathcal{SC}(H, \mathbb{E}).$$

(As mentioned earlier, this theorem also completes the proofs of Theorem A and Proposition B in the general case.)

It follows from Proposition 7.1 that $\mathcal{SC}(G, Z)$ is mapped into $\mathcal{SC}(H, \mathbb{E})$. The nontrivial part is to show that the restriction $\mathcal{SC}(G, Z) \rightarrow \mathcal{SC}(H, \mathbb{E})$ is onto. To do that, we need some facts about wreath products and coset action which we recall now. Choose a right transversal T of H in G , so that $G = \bigcup_{t \in T} Ht$. The action of G on the right cosets Hg defines an action of G on T , which we denote by $t \circ g$. Thus for $t \in T$ and $g \in G$, the element $t \circ g$ is the unique element in $Htg \cap T$. We may write $tg = h(t, g)(t \circ g)$ with $h(t, g) \in H$. The map $t \mapsto t \circ g$ is a permutation $\sigma = \sigma(g) \in S_T$, where S_T denotes the group of permutations of T .

The symmetric group S_T acts on H^T , the set of maps $T \rightarrow H$, by $((h_t)_{t \in T})^\sigma = (h_{t\sigma^{-1}})_{t \in T}$. The semidirect product of S_T and H^T is also known as the wreath product of S_T and H and denoted by $H \wr S_T$. For every $g \in G$, set

$$g\varphi = (h(t, g))_{t \in T} \sigma(g) = \sigma(g)(h(t \circ g^{-1}, g))_{t \in T} \in H \wr S_T.$$

It turns out that the map $\varphi: G \rightarrow H \wr S_T$ is a group homomorphism [1, Lemma 13.3]. For later reference, we summarize:

8.3. Lemma. *Let G be a group, $H \leq G$ and suppose $G = \bigcup_{t \in T} Ht$. Then there is a group homomorphism $\varphi: G \rightarrow H \wr S_T$ such that*

$$\sigma(h_t)_{t \in T} = g\varphi \iff Ht\sigma = Htg \text{ and } tg = h_{t\sigma}t\sigma \text{ for all } t \in T.$$

One may identify T with $\Omega = \{Hg \mid g \in G\}$ and S_T with S_Ω , but even then φ does depend on the choice of the right transversal T .

Suppose that $\kappa: \widehat{H} \rightarrow H$ is a surjective homomorphism with kernel M . We want to construct a group \widehat{G} and a surjective homomorphism $\widehat{G} \rightarrow G$, where $H \leq G$ as before. The homomorphism κ yields, in an obvious way, a surjective homomorphism $\widehat{H} \wr S_T \rightarrow H \wr S_T$ with kernel M^T . We write $\kappa \wr 1$ for this homomorphism. We define \widehat{G} to be the pullback of

$$\begin{array}{ccc} & & G \\ & & \downarrow \varphi \\ \widehat{H} \wr S_T & \xrightarrow{\kappa \wr 1} & H \wr S_T \end{array}$$

that is, we set

$$\widehat{G} = \{(\sigma(h_t)_{t \in T}, g) \in (\widehat{H} \wr S_T) \times G \mid g\varphi = \sigma(h_t\kappa)_{t \in T}\}.$$

Since $\kappa \wr 1$ is surjective, the canonical homomorphism $\widehat{G} \rightarrow G$ is surjective and has kernel canonically isomorphic with M^T . Thus:

8.4. Lemma. *Let*

$$1 \longrightarrow M \longrightarrow \widehat{H} \xrightarrow{\kappa} H \longrightarrow 1$$

be an exact sequence of groups and suppose $H \leq G$. Let T and $\varphi: G \rightarrow H \wr S_T$ be as in Lemma 8.3. Then there exists a group \widehat{G} and a homomorphism $\kappa^{\otimes G}: \widehat{G} \rightarrow G$ such that the following diagram is commutative and has exact rows:

$$\begin{array}{ccccccc} 1 & \longrightarrow & M^T & \longrightarrow & \widehat{G} & \xrightarrow{\kappa^{\otimes G}} & G \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \varphi \\ 1 & \longrightarrow & M^T & \longrightarrow & \widehat{H} \wr S_T & \xrightarrow{\kappa \wr 1} & H \wr S_T \longrightarrow 1 \end{array}$$

The construction depends on the choice of the transversal T . It can be shown that the isomorphism type of the extension in the first row is independent of T . Maybe this justifies the notation $\kappa^{\otimes G}$, but otherwise we will not need this.

We need another piece of notation. Let Z be a G -ring and W a Z -module, and fix some $g \in G$. We call a Z -module X a g -**conjugate** of W , if there is an isomorphism of abelian groups $\alpha: W \rightarrow X$ such that $(wz)\alpha = w\alpha z^g$ for all $w \in W$ and $z \in Z$. The map α is called a g -isomorphism. Such a conjugate always exists, for example we can take $X = W$ as abelian group and define a new multiplication of Z on W by $w \bullet z = wz^{g^{-1}}$. Or we could take $X = W \otimes g \subseteq W \otimes_Z Z[G]$, where $Z[G]$ denotes the skew group ring. It is not difficult to show that all g -conjugates of a given module W are isomorphic. (This terminology is due to Riehm [9].)

Proof of Theorem 8.2. That $\mathcal{SC}(G, Z)$ is mapped into $\mathcal{SC}(H, \mathbb{E})$ follows from Proposition 7.1, as we mentioned already.

Conversely, assume that (ϑ, κ) is a Clifford pair over the group H such that $\mathbf{Z}(\vartheta, \kappa, \mathbb{F}) \cong \mathbb{E}$. Let M be the kernel of κ and \widehat{H} its domain. Let W be a ϑ -quasi-homogeneous $\mathbb{F}\widehat{H}$ -module and $A = \text{End}_{\mathbb{F}M}(W)$. Thus $[\vartheta, \kappa, \mathbb{F}] \in \mathcal{SC}(H, \mathbb{E})$ is the equivalence class of A . To prove Theorem 8.2, we have to find a Clifford pair over G such that the isomorphism of Proposition 8.1 maps its Brauer-Clifford class to $[\vartheta, \kappa, \mathbb{F}]$.

In the following, we use the notation of Lemma 8.4. In particular, let $\kappa^{\otimes G}: \widehat{G} \rightarrow G$ be the homomorphism of Lemma 8.4 with kernel $N := M^T$. We will show that the isomorphism of Proposition 8.1 maps

$$[\vartheta \times \underbrace{1_M \times \cdots \times 1_M}_{|T|-1}, \kappa^{\otimes G}, \mathbb{F}] \in \mathcal{SC}(G, Z)$$

to $[\vartheta, \kappa, \mathbb{F}]$, if $\vartheta \neq 1_M$. The case $\vartheta = 1_M$ will be treated separately.

First we define an $\mathbb{F}\widehat{G}$ -module V . Recall that $G = \cup_{t \in T} Ht$. Since $\mathbb{E} \cong \mathbf{Z}(\vartheta, \kappa, \mathbb{F})$, we may view W as an \mathbb{E} -module. Via the homomorphism $Z \rightarrow \mathbb{E}$, we can view W as a Z -module. Since G acts on Z , it makes sense to speak of g -conjugates for $g \in G$. We now choose a t -conjugate W_t and a t -isomorphism $\gamma_t: W \rightarrow W_t$ for every $t \in T$. Every γ_t is an isomorphism of vector spaces over $\mathbb{F} \subseteq \mathbb{E}^H$. Let V be the direct sum of the W_t 's:

$$V := \bigoplus_{t \in T} W_t.$$

Every element $v \in V$ can be written uniquely as $v = \sum_{t \in T} w_t \gamma_t$ with $w_t \in W$. The wreath product $\widehat{H} \wr S_T$ acts on V by

$$\left(\sum_{t \in T} w_t \gamma_t \right) (\sigma \cdot (h_t)_{t \in T}) = \sum_{t \in T} w_{t\sigma^{-1}} h_t \gamma_t.$$

Using the homomorphism $\widehat{G} \rightarrow \widehat{H} \wr S_T$, we can view V as $\mathbb{F}\widehat{G}$ -module.

Let S be a simple G -algebra over Z such that $Se \cong A = \text{End}_{\mathbb{F}M}(W)$. Every element of S can be written uniquely as $\sum_{t \in T} a_t^t$ with $a_t \in Se \cong A$. The algebra S is determined up to isomorphism by this property. (As a module over the skew group ring $Z[G]$, the algebra S is isomorphic to $A \otimes_{Z[H]} Z[G]$.) The algebra S acts on V by

$$\left(\sum_{t \in T} w_t \gamma_t \right) \left(\sum_{t \in T} a_t^t \right) = \sum_{t \in T} w_t a_t \gamma_t.$$

It is routine to verify that this action yields an G -algebra homomorphism $S \rightarrow \text{End}_{\mathbb{F}N}(V)$, and this homomorphism is injective. Next we want to show that it is onto. Here we will need to assume that $\vartheta \neq 1$.

Let $\beta \in \text{End}_{\mathbb{F}N}(V)$. For $t, u \in T$ and $w \in W$, there are elements $w\beta_{u,t} \in W$ such that

$$w\gamma_u\beta = \sum_{t \in T} w\beta_{u,t}\gamma_t.$$

This defines maps $\beta_{u,t}: W \rightarrow W$ for each $u, t \in T$. Let $(m_t)_{t \in T} \in N = M^T$ be arbitrary. From

$$w\gamma_u\beta(m_t)_{t \in T} = w\gamma_u(m_t)_{t \in T}\beta = wm_u\gamma_u\beta$$

it follows that $w\beta_{u,t}m_t = wm_u\beta_{u,t}$. For $u = t$ this yields $\beta_{u,u} \in \text{End}_{\mathbb{F}M}(W) = A$. For $u \neq t$ this yields $w\beta_{u,t} = w\beta_{u,t}m'$ for all $m, m' \in M$. We know that $W \cong tU$ for some simple $\mathbb{F}M$ -module U , and U is not the trivial $\mathbb{F}N$ -module, since we assume $\vartheta \neq 1_M$. Thus no non-zero element of U is fixed by all of M . It follows that $\beta_{u,t} = 0$ for $u \neq t$. Thus $\beta = \sum_{t \in T} (\beta_{t,t})^t \in S$. We have shown that $S \cong \text{End}_{\mathbb{F}N}(V)$.

Since $S \cong \text{End}_{\mathbb{F}N}(V)$ is a simple G -algebra, it follows that V is quasihomogeneous over some character in $\text{Irr } M^T$. Now $W_1 = W$ is a direct summand of V as $\mathbb{F}N$ -module, and the action of $(m_t)_{t \in T}$ on this summand is given by $w(m_t)_t = wm_1$. Since ϑ is a constituent of the character of $W_{\mathbb{F}M}$, it follows that $\vartheta \times 1_M \times \cdots \times 1_M$ is a constituent of the character of V . Thus S is a simple G -algebra in $[[\vartheta \times 1_M \times \cdots \times 1_M, \kappa^{\otimes G}, \mathbb{F}]]$ as claimed.

In the remaining case where $\vartheta = 1_M$ is the trivial character, it follows that M acts trivially on W and we may view W as an $\mathbb{F}H$ -module. Thus $A \cong \mathbf{M}_n(\mathbb{F})$ for some positive integer n and $\mathbb{F} = \mathbb{E}$. Then A is equivalent to the trivial H -algebra \mathbb{E} . We need to show that $Z = \times_{t \in T} \mathbb{E}$ is a Schur G -algebra, where G acts simply by permuting the factors.

For this, let M be a finite group such that there is a nontrivial, absolutely simple $\mathbb{E}M$ -module U . (For example, take a cyclic group of order 2 for M .) Let $\widehat{G} = M \wr G$ be the wreath product with respect to the action of G on T , and $N = M^T \trianglelefteq \widehat{G}$. Then \widehat{G} acts on $V = U^T$ by

$$(u_t)_{t \in T} \sigma (m_t)_{t \in T} = (u_{t\sigma^{-1}} m_t)_{t \in T},$$

and we have $\text{End}_{\mathbb{E}N}(V) \cong Z$ as desired. \square

9. CORESTRICTION

Let G be a finite group and $H \leq G$. Let \mathbb{F} be a field of characteristic zero and Z a commutative G -algebra over \mathbb{F} , so that $\mathbb{F} \leq Z^G$. Given an H -algebra A over Z , we showed in [8] how to construct a certain G -algebra $\text{Cores}_H^G(A)$ over Z . This defines a group homomorphism $\text{Cores}_H^G: \text{BrCliff}(H, Z) \rightarrow \text{BrCliff}(G, Z)$ called **corestriction**. If Z is simple as H -algebra, then it is also simple as G -algebra, and we may ask if the corestriction map maps $\mathcal{SC}^{(\mathbb{F})}(H, Z)$ into $\mathcal{SC}^{(\mathbb{F})}(G, Z)$. It is important to fix the field \mathbb{F} here. For example, assume that $Z = \mathbb{E}$ is a field. Then it may happen that the Galois extension \mathbb{E}/\mathbb{E}^H is abelian, but the Galois extension \mathbb{E}/\mathbb{F} (with $\mathbb{F} = \mathbb{E}^G$, say) is not. Then $\mathcal{SC}(H, \mathbb{E}) = \mathcal{SC}^{(\mathbb{E}^H)}(H, \mathbb{E})$ is not empty by Lemma 5.1 and Kronecker-Weber, while $\mathcal{SC}(G, \mathbb{E}) = \mathcal{SC}^{(\mathbb{F})}(G, \mathbb{E})$ and $\mathcal{SC}^{(\mathbb{F})}(H, \mathbb{E})$ are empty. The next result has content only when $\mathcal{SC}^{(\mathbb{F})}(H, Z)$ is non-empty.

9.1. Theorem. *With the notation just introduced, the corestriction homomorphism maps $\mathcal{SC}^{(\mathbb{F})}(H, Z)$ into $\mathcal{SC}^{(\mathbb{F})}(G, Z)$.*

Proof. Let (ϑ, κ) be a Clifford pair, where $\vartheta \in \text{Irr } M$ with $Z \cong \mathbf{Z}(\vartheta, \kappa, \mathbb{F})$ as H -algebras. Suppose that W is an $\widehat{\mathbb{F}H}$ -module which is ϑ -quasihomogeneous, and let $S = \text{End}_{\mathbb{F}M}(W)$. We will use the notation introduced in Lemmas 8.3 and 8.4. In

particular, let $G = \bigcup_{t \in T} Ht$ and recall the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & M^T & \longrightarrow & \widehat{G} & \xrightarrow{\kappa^{\otimes G}} & G & \longrightarrow & 1 \\ & & \parallel & & \downarrow & & \downarrow \varphi & & \\ 1 & \longrightarrow & M^T & \longrightarrow & \widehat{H} \wr S_T & \xrightarrow{\kappa \wr 1} & H \wr S_T & \longrightarrow & 1 \end{array}$$

with exact rows from Lemma 8.4. We are going to show that $\text{Cores}_H^G(S) = S^{\otimes G} \cong \text{End}_{\mathbb{F}[M^T]}(V)$ for some $\mathbb{F}\widehat{G}$ -module V .

Since $\text{End}_{\mathbb{F}M}(W) = S$, we can view W as a module over $Z = \mathbf{Z}(S)$. Recall that G acts on Z . We now choose a t -conjugate W_t (see p. 16) and a t -isomorphism $\gamma_t: W \rightarrow W_t$ for every $t \in T$. Every γ_t is an isomorphism of vector spaces over $\mathbb{F} \subseteq Z^H$. We set

$$V = \bigotimes_{t \in T} W_t \quad (\text{tensor product over } Z).$$

We want to define an action of \widehat{G} on V . So let $(\sigma(h_t)_{t \in T}, g) \in \widehat{G}$ and let $w_t \in W$. Define

$$\left(\bigotimes_{t \in T} w_t \gamma_t \right) (\sigma(h_t)_{t \in T}, g) = \bigotimes_{t \in T} w_{t \circ g^{-1}} h_t \gamma_t.$$

First we show that this yields a well defined endomorphism of V . So let $w_t \in W$ and $z \in Z$, and fix $t_0 \in T$. Suppose

$$u_t = \begin{cases} w_t, & \text{if } t \neq t_0, \\ w_t z^{t_0^{-1}}, & \text{if } t = t_0. \end{cases}$$

Then $u_{t_0} \gamma_{t_0} = w_{t_0} \gamma_{t_0} z$ and

$$\bigotimes_{t \in T} u_t \gamma_t = \left(\bigotimes_{t \in T} w_t \gamma_t \right) z.$$

Applying the definition to the left side yields

$$\left(\bigotimes_{t \in T} u_t \gamma_t \right) (\sigma(h_t)_{t \in T}, g) = \bigotimes_{t \in T} u_{t \circ g^{-1}} h_t \gamma_t = \bigotimes_{t \in T} v_t \gamma_t,$$

where

$$v_t = \begin{cases} w_{t \circ g^{-1}} h_t & \text{if } t \neq t_0 \circ g, \\ w_{t_0} z^{t_0^{-1}} h_{t_0 \circ g} & \text{if } t = t_0 \circ g. \end{cases}$$

Thus

$$\begin{aligned} v_{t_0 \circ g} \gamma_{t_0 \circ g} &= w_{t_0} z^{t_0^{-1}} h_{t_0 \circ g} \gamma_{t_0 \circ g} = w_{t_0} h_{t_0 \circ g} \gamma_{t_0 \circ g} z^{t_0^{-1} (h_{t_0 \circ g} \kappa) t_0 \circ g} \\ &= w_{t_0} h_{t_0 \circ g} \gamma_{t_0 \circ g} z^g, \end{aligned}$$

where the last equation follows from $(h_{t_0 \circ g} \kappa) t_0 \circ g = t_0 g$, which follows from the fact that $\sigma(h_t \kappa)_{t \in T} = g \varphi$ (see Lemma 8.3). It follows that

$$\bigotimes_{t \in T} v_t \gamma_t = \left(\bigotimes_{t \in T} w_{t \circ g^{-1}} h_t \gamma_t \right) z^g.$$

This does not depend on t_0 . Thus multiplication with $(\sigma(h_t)_{t \in T}, g)$ yields a well-defined endomorphism of V .

Having established this, it is routine to check that we actually have an action of \widehat{G} on V .

Our next goal is to show that $\text{End}_{\mathbb{F}N} V \cong S^{\otimes G}$ as G -algebra over Z . First recall that $S^{\otimes G} \cong \bigotimes_{t \in T} S^{\otimes t}$, where the tensor product is over Z and each $S^{\otimes t}$ is a t -conjugate of S . We define an action of $S^{\otimes G}$ on the module $V = \bigotimes_{t \in T} W_t$ by

$$\left(\bigotimes_{t \in T} w_t \gamma_t \right) \left(\bigotimes_{t \in T} s_t^{\otimes t} \right) = \bigotimes_{t \in T} w_t s_t \gamma_t.$$

It is routine to see that this yields a well-defined homomorphism $S^{\otimes G} \rightarrow \text{End}_{\mathbb{F}N} V$ of Z -algebras. We claim that it is actually an isomorphism and commutes with the action of G .

The homomorphism is injective, since $S^{\otimes G}$ is simple. We know that $S \cong \text{End}_{\mathbb{F}M} W = \text{End}_{ZM} W$. Clearly $Z \subseteq \text{End}_{\mathbb{F}N} V$, so that $\text{End}_{\mathbb{F}N} V = \text{End}_{ZN} V$. But $ZN \cong ZM \otimes_Z \cdots \otimes_Z ZM = (ZM)^{\otimes T}$ (tensor product over Z of $|T|$ factors), and so

$$\begin{aligned} \text{End}_{ZN} V &= \mathbf{C}_{\text{End}_Z(V)}(ZN) \\ &= \mathbf{C}_{\text{End}_Z(V)}(ZM \otimes_Z \cdots \otimes_Z ZM) \\ &\cong S \otimes_Z \cdots \otimes_Z S \cong S^{\otimes G} \quad (\text{as } Z\text{-algebra}). \end{aligned}$$

This shows that the homomorphism above is an isomorphism of Z -algebras.

Finally we have to show that this isomorphism is compatible with the action of G on $S^{\otimes G}$ and $\text{End}_{\mathbb{F}N} V$. Let $g \in G$ and $\bigotimes_{t \in T} s_t^{\otimes t} \in S^{\otimes G}$. Then

$$\left(\bigotimes_{t \in T} s_t^{\otimes t} \right)^g = \bigotimes_{t \in T} s_t^{\otimes tg} = \bigotimes_{t \in T} \left(s_{t \circ g^{-1}}^{\otimes (t \circ g^{-1})gt^{-1}} \right)^{\otimes t}.$$

Thus

$$\left(\bigotimes_{t \in T} w_t \gamma_t \right) \left(\bigotimes_{t \in T} s_t^{\otimes t} \right)^g = \bigotimes_{t \in T} w_t s_{t \circ g^{-1}}^{\otimes (t \circ g^{-1})gt^{-1}} \gamma_t.$$

Now choose $\sigma \in S_T$ and $h_t \in \widehat{H}$ ($t \in T$) such that $(\sigma(h_t)_{t \in T}, g) \in \widehat{G}$. Recall that this means that $t\sigma = t \circ g$ and $h_t \kappa = (t \circ g^{-1})gt^{-1}$. We compute

$$\begin{aligned} &\left(\bigotimes_{t \in T} w_t \gamma_t \right) (\sigma(h_t)_{t \in T}, g)^{-1} \left(\bigotimes_{t \in T} s_t^{\otimes t} \right) (\sigma(h_t)_{t \in T}, g) \\ &= \left(\bigotimes_{t \in T} w_{t\sigma} h_{t\sigma}^{-1} s_t \gamma_t \right) (\sigma(h_t)_{t \in T}, g) \\ &= \bigotimes_{t \in T} w_t h_t^{-1} s_{t\sigma^{-1}} h_t \gamma_t \\ &= \bigotimes_{t \in T} w_t s_{t \circ g^{-1}}^{h_t \kappa} \gamma_t \\ &= \bigotimes_{t \in T} w_t s_{t \circ g^{-1}}^{\otimes (t \circ g^{-1})gt^{-1}} \gamma_t. \end{aligned}$$

This finishes the proof. \square

9.2. Corollary. *Let Z be a commutative G -algebra and $H \leq G$. Assume that Z has no nontrivial H -invariant ideals. Let*

$$1 \longrightarrow M \longrightarrow \widehat{H} \xrightarrow{\kappa} H \longrightarrow 1$$

be an exact sequence of groups and $\vartheta \in \text{Irr } M$ with $Z \cong \mathbf{Z}(\vartheta, \kappa, \mathbb{F})$ as H -algebras. Write $B = \{\varphi \in \text{Irr } M \mid \varphi|_Z \neq 0\}$. Then

$$\text{Cores}[\vartheta, \kappa, \mathbb{F}] = \llbracket \times_{t \in T} \vartheta_t, \kappa^{\otimes G}, \mathbb{F} \rrbracket,$$

where each $\vartheta_t \in B$.

Proof. Note that B is the orbit of ϑ under the action of $\Gamma \times H$, where $\Gamma = \text{Gal}(\mathbb{F}(\vartheta)/\mathbb{F})$. Let $\beta = \sum_{\varphi \in B} \varphi$. Let W be a ϑ -quasihomogeneous module. Then the character of W_M is an integer multiple of β . View $W^{\otimes |T|} = \otimes_{t \in T} W$ (tensor product over \mathbb{F}) as module over $\mathbb{F}[M^T]$. This module affords the character $\times_{t \in T} \beta$. Let V be the $\widehat{\mathbb{F}G}$ -module constructed in the proof of Theorem 9.1. Consider the map

$$W^{\otimes |T|} \ni \otimes_{t \in T} w_t \mapsto \otimes_{t \in T} w_t \gamma_t \in V.$$

This maps a tensor product over \mathbb{F} to a tensor product over Z . It is easy to see that this map is a homomorphism of $\mathbb{F}[M^T]$ -modules. Thus V_N affords a character which is a sum of characters of the form $\times_{t \in T} \vartheta_t$ with $\vartheta_t \in B$. This is the claim. \square

9.3. Remark. The corollary does not tell the full story, namely what the ϑ_t actually are. Note that since Z is assumed to be a G -algebra, we have an action of G on B . The restriction of this action to H agrees with the usual action of $H \cong \widehat{H}/M$ on $B \subseteq \text{Irr } M$. With somewhat more effort, one can show that the character of the module V in the proof of Theorem 9.1 has the character $\times_{t \in T} \vartheta^{t^{-1}}$ as constituent. Thus

$$\text{Cores}[\vartheta, \kappa, \mathbb{F}] = \llbracket \times_{t \in T} \vartheta^{t^{-1}}, \kappa^{\otimes G}, \mathbb{F} \rrbracket.$$

But for the applications we have in mind, it is enough to know the weaker statement of Corollary 9.2.

Recall that the Brauer-Clifford group is an abelian torsion group. It is thus the direct sum of its p -torsion parts where p runs through the primes. The same is true for the Schur-Clifford group, if defined. Denote the p -part of an abelian group A by A_p .

9.4. Corollary. *Let Z be a simple G -algebra over \mathbb{F} , P a Sylow p -subgroup of G for some prime p and assume that Z is simple as P -algebra. If $\mathcal{SC}^{(\mathbb{F})}(G, Z)$ is not the empty set, then $\mathcal{SC}^{(\mathbb{F})}(G, Z)_p$ is isomorphic to a subgroup of $\mathcal{SC}^{(\mathbb{F})}(P, Z)_p$. Moreover, an equivalence class $a = [S] \in \text{BrCliff}(G, Z)_p$ belongs to $\mathcal{SC}^{(\mathbb{F})}(G, Z)$ if and only if $\text{Res}_P^G(a) \in \mathcal{SC}^{(\mathbb{F})}(P, Z)$.*

Proof. By Theorem 6.3 in [8] the composition

$$\mathrm{BrCliff}(G, Z) \xrightarrow{\mathrm{Res}_P^G} \mathrm{BrCliff}(P, Z) \xrightarrow{\mathrm{Cores}_P^G} \mathrm{BrCliff}(G, Z)$$

maps an element $a = [S]$ to $a^{|G:P|} = [S]^{|G:P|}$. Thus Res_P^G is injective on the p -torsion part. By Proposition 7.1, it maps $\mathcal{SC}^{(\mathbb{F})}(G, Z)_p$ into $\mathcal{SC}^{(\mathbb{F})}(P, Z)_p$.

Conversely, if $\mathrm{Res}_P^G(a) \in \mathcal{SC}^{(\mathbb{F})}(P, Z)$ for some $a \in \mathrm{BrCliff}(G, Z)$, then $a^{|G:P|} \in \mathcal{SC}^{(\mathbb{F})}(G, Z)$ by Theorem 9.1. Thus if the order of a is a power of p , then $a \in \langle a^{|G:P|} \rangle$ is in $\mathcal{SC}^{(\mathbb{F})}(G, Z)$, too. \square

10. A SPECIAL CASE AND A CONJECTURE

The results of this paper suggest that one can carry through a study of the Schur-Clifford subgroup for various groups G and simple commutative G -algebras Z . For example, we have seen that for $Z = \mathbb{C}$ with trivial action of the group G , we have $\mathrm{BrCliff}(G, \mathbb{C}) = \mathcal{SC}(G, \mathbb{C})$ and that $\mathcal{SC}(G, \mathbb{C}) \cong H^2(G, \mathbb{C}^*)$ (Example 3.5). Moreover, every element in $\mathcal{SC}(G, \mathbb{C})$ can be realized as the class of a Clifford pair (ϑ, κ) such that $\mathrm{Ker} \kappa$ is a cyclic central subgroup of \hat{G} , the domain of κ . In a follow-up paper [7], we prove a similar result for arbitrary fields \mathbb{E} with G -action, showing that every element in $\mathcal{SC}(G, \mathbb{E})$ comes from a Clifford pair (ϑ, κ) such that the normal subgroup $\mathrm{Ker} \kappa$ (which is the domain of ϑ) is cyclic by abelian.

We might also discuss how G and Z bear upon group theoretic properties of $\mathcal{SC}(G, Z)$. We discuss now a slightly more general case than the case \mathbb{C} . First, observe the following. Given an element of the Brauer-Clifford group $\mathrm{BrCliff}(G, \mathbb{E})$, where \mathbb{E} is a field on which G acts, we may take its equivalence class in the Brauer group $\mathrm{Br}(\mathbb{E})$. This defines a group homomorphism $\mathrm{BrCliff}(G, \mathbb{E}) \rightarrow \mathrm{Br}(\mathbb{E})$. It is clear that this homomorphism maps the Schur-Clifford group into the Schur subgroup of the Brauer group. (In fact, this is the case $H = \{1\}$ of Proposition 7.1.)

Turull has shown [11, Theorem 3.12] that the kernel of the homomorphism is isomorphic to the second cohomology group $H^2(G, \mathbb{E}^*)$. In the case where G acts trivially on $\mathbb{E} = \mathbb{F}$, Turull has shown [11, Corollary 3.13] that $\mathrm{BrCliff}(G, \mathbb{F}) \cong \mathrm{Br}(\mathbb{F}) \times H^2(G, \mathbb{F}^*)$. The Schur-Clifford group decomposes accordingly:

10.1. Proposition. *Suppose that \mathbb{F} is a field on which the group G acts trivially. Let $\mathcal{S}(\mathbb{F}) \leq \mathrm{Br}(\mathbb{F})$ be the Schur subgroup of the Brauer group of \mathbb{F} . Then*

$$\mathcal{SC}(G, \mathbb{F}) \cong \mathcal{S}(\mathbb{F}) \times A \quad \text{for some } A \leq H^2(G, \mathbb{F}^*).$$

Proof. The homomorphism $\beta: \mathrm{BrCliff}(G, \mathbb{F}) \rightarrow \mathrm{Br}(\mathbb{F})$ maps $\mathcal{SC}(G, \mathbb{F})$ into $\mathcal{S}(\mathbb{F})$.

Conversely, there is a homomorphism $\mathrm{Br}(\mathbb{F}) \rightarrow \mathrm{BrCliff}(G, \mathbb{F})$ which is induced by viewing a central simple \mathbb{F} -algebra as a G -algebra with trivial action of G . This homomorphism shows that $\mathrm{BrCliff}(G, \mathbb{F})$ is a direct product of $\mathrm{Br}(\mathbb{F})$ and some other group. The other group is isomorphic to the second cohomology group $H^2(G, \mathbb{F}^*)$ by the results of Turull mentioned before.

To finish the proof, it remains to show that the homomorphism $\mathrm{Br}(\mathbb{F}) \rightarrow \mathrm{BrCliff}(G, \mathbb{F})$ maps the Schur group $\mathcal{S}(\mathbb{F})$ into the Schur-Clifford group $\mathcal{SC}(G, \mathbb{F})$. Suppose that φ

is the character of some group N with $\mathbb{F}(\varphi) = \mathbb{F}$, so that φ defines an element $[[\varphi]]$ of the Schur subgroup $\mathcal{S}(\mathbb{F})$. More precisely, there is an $\mathbb{F}N$ -module V whose character is a multiple of φ , and then $\text{End}_{\mathbb{F}N}(V)$ is a central simple \mathbb{F} -algebra whose equivalence class defines an element $[[\varphi]]$.

Let $\widehat{G} = N \times G$ and $\kappa: \widehat{G} \rightarrow G$ be the projection on the second component. We identify N with the kernel of κ . Then (φ, κ) is a Clifford pair over G with $\beta[[\varphi, \kappa, \mathbb{F}]] = [[\varphi]]$. It is easy to see that the map $[[\varphi]] \mapsto [[\varphi, \kappa, \mathbb{F}]]$ is the restriction of the map $\text{Br}(\mathbb{F}) \rightarrow \text{BrCliff}(G, \mathbb{F})$ defined above. The result follows. \square

Given a Clifford pair (ϑ, κ) over G , one can construct directly a cocycle in $Z^2(G, \mathbb{F}^*)$, and then show that its cohomology class depends only on $[[\vartheta, \kappa, \mathbb{F}]]$. This yields an alternative proof of the proposition.

The subgroup $A \leq H^2(G, \mathbb{F})$ occurring in the last proposition has been studied by Dade [2]. He proved that if m is the exponent of the group A , then \mathbb{F} contains a primitive m -th root of unity. For the Schur subgroup of the Brauer group, the same result is true by a result of Benard-Schacher [14, Proposition 6.2]. Thus we have:

10.2. Corollary. *Suppose that G acts trivially on the field \mathbb{F} and let $m = \exp(\mathcal{SC}(G, \mathbb{F}))$. Then \mathbb{F} contains a primitive m -th root of unity.*

Schmid [10] shows how to associate a cohomology class to a semi-invariant Clifford pair (ϑ, κ) over \mathbb{F} in a slightly more general situation. (The assumption is that the Schur index of ϑ over \mathbb{F} and the index $|G : G_{\vartheta}|$ of the inertia group of ϑ in G are coprime.) Moreover, Schmid shows that $\mathbb{F}(\vartheta)$ contains a primitive m -th root of unity if this cohomology class has order m [10, Theorem 7.3]. We conclude this paper with the conjecture that Corollary 10.2 is true for all Schur-Clifford subgroups:

10.3. Conjecture. *Let $\mathbb{E} \subseteq \mathbb{C}$ be a field on which the group G acts, and suppose $\mathcal{SC}(G, \mathbb{E}) \neq \emptyset$. If m is the exponent of $\mathcal{SC}(G, \mathbb{E})$, then \mathbb{E} contains a primitive m -th root of unity.*

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