

# DEFORMATION QUANTIZATION FOR ACTIONS OF $\mathbb{Q}_p^d$

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**ABSTRACT.** The main objective of this article is to develop the theory of deformation of  $C^*$ -algebras endowed with a group action, from the perspective of non-formal equivariant quantization. This program, initiated in [2], aims to extend Rieffel's deformation theory [27] for more general groups than  $\mathbb{R}^d$ . In [2], we have constructed such a theory for a class of non-Abelian Lie groups. In the present article, we study the somehow opposite situation of Abelian but non-Lie groups. More specifically, we construct here a deformation theory of  $C^*$ -algebras endowed with an action of a finite dimensional vector space over a non-Archimedean local field of characteristic different from 2. At the root of our construction stands the  $p$ -adic version of the Weyl quantization introduced by Haran [12] and further extended by Bechata [1] and Unterberger [34].

**Keywords:** Deformation of  $C^*$ -algebras, Equivariant quantization, Local fields,  $p$ -Adic pseudo-differential analysis

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## 1. INTRODUCTION

When formulated in the setting of operator algebras, equivariant quantization interconnects both with deformation theory and with quantum groups. These interconnections originate in the work of Rieffel [27], where it is shown that Weyl's pseudo-differential calculus can be used to design a deformation theory for  $C^*$ -algebras equipped with a continuous action of  $\mathbb{R}^d$ . Applying this deformation process to  $C_0(G)$ , where  $G$  is a locally compact group possessing a copy of  $\mathbb{R}^d$  as a closed subgroup and for the action  $\rho \otimes \lambda$  of  $G \times G$ , in [29] Rieffel was further able to produce a large class of examples of quantum groups in the  $C^*$ -algebraic setting. In [2], Bieliavsky and one of us have successfully extended Rieffel's

deformation theory for actions of negatively curved Kählerian Lie groups on  $C^*$ -algebras. This was the first explicit example of a deformation theory for  $C^*$ -algebras coming from actions of non-Abelian groups and it was based in an essential way on a generalization (to all negatively curved Kählerian Lie groups) of the  $ax + b$ -equivariant quantization due to Unterberger [33]. (See also [3] for an extension of Rieffel's to construction to actions of the Heisenberg supergroup.)

There is another approach to quantization, due to Landstad and Raeburn [17, 18, 19, 20], which also connects to quantum groups. At the conceptual level, the starting point there is that the twisted group  $C^*$ -algebra associated with a unitary 2-cocycle should be considered as a quantization of the virtual dual group. This approach to quantization has been further developed by Kasprzak in [14] to design a deformation theory for  $C^*$ -algebras endowed with a continuous action of a locally compact Abelian group, from a unitary 2-cocycle on the dual group. It was then observed by Bhowmick, Neshveyev and Sangha in [6] that Kasprzak's construction still makes sense for actions of non-Abelian locally compact groups, provided that the unitary 2-cocycle is now chosen in the dual quantum group (i.e. the group von Neumann algebra). An important point is that unless the group used to deform is Abelian, the symmetries of the deformed objects are now given by a quantum group. All this suggests that quantum groups are naturally present in the context of equivariant quantizations and in the associated deformation theories.

Very recently, Neshveyev and Tuset gave in [23] a great clarification of the role of quantum groups in deformations, by providing a beautiful theory holding with the most imaginable degree of generality, namely for continuous actions of locally compact quantum groups on  $C^*$ -algebras and from a unitary 2-cocycle on the dual quantum group. Their starting point is the work of De Commer [7], which shows that given a locally compact quantum group  $(G, \Delta)$  (in the von Neumann algebraic setting [15, 16]) together with a dual measurable unitary 2-cocycle<sup>1</sup>  $F$  on  $(G, \Delta)$ , the pair  $(\hat{G}, F\hat{\Delta}(\cdot)F^*)$  is again a locally compact quantum group. The dual quantum group, denoted by  $(G_F, \Delta)$ , is thought as the deformation of  $(G, \Delta)$  and it is that quantum group which acts on the deformed  $C^*$ -algebras.

However, already when  $G$  is an ordinary non-Abelian group, constructing a nontrivial and concrete dual unitary 2-cocycle can be a very difficult task. For instance, in [23] the only example given is the one canonically attached (see below) to the equivariant quantization map constructed in [2]. Moreover, even at the level of  $C_0(G)$ , it is not clear whether the constructions of [23] and of [2] agree, while it is known [22] to hold for of actions of  $\mathbb{R}^d$ . We should also mention that the framework of [27] and [2] comes naturally with parameters and that Rieffel's methods are perfectly well adapted to study the question of continuity of the associated field of deformed  $C^*$ -algebras. In contrast, it is uncertain whether the methods of [23] (and already those of [6, 14]) applied in a parametric situation can lead to results about continuity. Moreover, oppositely to Rieffel's type methods, it is unclear whether those of [23] are well suited in view of applications in noncommutative geometry, for instance in spectral triple theory [8, 9].

For all these reasons, and even if there exists now a satisfactory and completely general deformation theory of  $C^*$ -algebras by use of its symmetries [23], we believe that constructing deformation theories directly from equivariant quantizations is important in its own right.

The main goal of this paper is to continue the program initiated in [2] and which consists in extending Rieffel's approach to deformation for more general groups than  $\mathbb{R}^d$ . In [2], even if we were in the relatively simple situation of solvable simply connected real Lie groups, we faced serious

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<sup>1</sup>A dual measurable unitary 2-cocycle  $F$  on  $G$ , is an unitary element of  $L^\infty(\hat{G})\bar{\otimes}L^\infty(\hat{G})$  which satisfies the cocycle condition  $(F \otimes 1)(\hat{\Delta} \otimes \text{Id})(F) = (1 \otimes F)(\text{Id} \otimes \hat{\Delta})(F)$ .

analytical difficulties underlying the non-commutativity of the group. Here we study the somehow opposite situation of Abelian but non-Lie groups. More specifically, the groups we consider here are finite dimensional vector spaces over a non-Archimedean local field of characteristic different from 2. At the root of our construction stands the  $p$ -adic version of the Weyl quantization introduced by Haran [12] and further extended by Bechata [1] and Unterberger [34]. Even if our framework is already covered by Kasprzak's approach (in fact, it is this one of our results), the primary interest of the present approach is to design new analytical tools adapted to the non-Lie case. In a forthcoming paper, we treat a non-Abelian and non-Lie example, given by the affine group of a non-Archimedean local field. Another important feature of the case studied here, is that the deformation parameter is no longer a real number. Instead, our parameter space is the ring of integers of the field. This affects substantially the answer we are able to give about the continuity of the field of deformed  $C^*$ -algebras. To conclude with general features, we should also mention that here, part of the analytical arguments are even simpler than their Archimedean analogues in [27]. This is a somehow recurrent phenomenon in  $p$ -adic harmonic analysis. Here, this comes from the following reason. The  $p$ -adic pseudo-differential calculus is controlled by two operators  $I$  and  $J$  [1, 12], which are the natural non-Archimedean substitutes for the operator of multiplication by the function  $[x \in \mathbb{R}^n \mapsto (1 + \langle x, x \rangle)^{1/2}]$  and for the flat Laplacian. But here they do commute! However,  $p$ -adic pseudo-differential operators do not commute in general!

Let us now be more precise about the program we wish to develop. In the differentiable setting, to define a non-formal equivariant quantization, one generally starts from a symplectic manifold  $(M, \omega)$  together with a Lie subgroup  $\tilde{G}$  of the group of symplectomorphisms. An equivariant quantization is a map

$$\Omega : C_c^\infty(M) \rightarrow \mathcal{B}(\mathcal{H}_\pi),$$

associated with a projective unitary representation  $(\mathcal{H}_\pi, \pi)$  of  $\tilde{G}$ , satisfying the covariance property:

$$\pi(g) \Omega(f) \pi(g)^* = \Omega(f^g), \quad \forall f \in C_c^\infty(M), \forall g \in \tilde{G},$$

where  $f^g := [x \in M \mapsto f(g^{-1}.x)]$ . There is a paradigm of such equivariant quantizations, which covers most of the quantizations known, called ‘‘Moyal-Stratonowich quantization’’ by Cariñena, Gracia-Bondía and Várilly in [5] (see also [11, section 3.5]). It is associated with a family of bounded (to simplify a little bit the picture) selfadjoint operators  $\{\Omega(x)\}_{x \in M}$  on  $\mathcal{H}_\pi$  satisfying the covariance property  $\pi(g) \Omega(x) \pi(g)^* = \Omega(g.x)$  (plus two other properties that are not very relevant for the following discussion). The associated quantization map is then defined by

$$\Omega(f) := \int_M f(x) \Omega(x) d\mu(x), \quad \forall f \in C_c^\infty(M),$$

where  $d\mu$  is the Liouville measure on  $M$ . Now, to connect equivariant quantization to deformation and to quantum groups, we need to restrict ourselves to the situation where  $\tilde{G}$  possesses a subgroup  $G$  which acts simply transitively on  $M$ . Under the identification  $G \simeq M$ , the Lie group  $G$  is then endowed with a symplectic structure that is invariant under left translations. Hence, what we are looking for is a non-formal quantization map on a the symplectic Lie group  $G$  which is equivariant under left translations. In the context of Moyal-Stratonowich quantization, with  $e$  the neutral element of  $G$ , we then have:

$$\Omega(g) = \pi(g) \Omega(e) \pi(g)^*,$$

and the Liouville measure  $d\mu(x)$  on  $M$  becomes a left invariant Haar measure  $d^\lambda(g)$  on  $G$ . Hence, setting  $\Sigma := \Omega(e)$ , a  $G$ -equivariant Moyal-Stratonowich quantization on  $G$  is always of the form

$$\Omega_{\pi,\Sigma}(f) := \int_G f(g) \pi(g) \Sigma \pi(g)^* d^\lambda(g), \quad \forall f \in C_c^\infty(G).$$

What is important with the formula above is that symplectic differential geometry disappeared from the picture and provides an ansatz to construct left-invariant quantizations on general groups.

Assume now that  $G$  is an arbitrary locally compact second countable group, pick a projective representation  $(\mathcal{H}_\pi, \pi)$  and let  $\Sigma \in \mathcal{B}(\mathcal{H}_\pi)$ . In general, there is no reason why the associated quantization map behaves well. A natural assumption is that the quantization map  $\Omega_{\pi,\Sigma} : C_c(G) \rightarrow \mathcal{B}(\mathcal{H}_\pi)$  extends to a unitary operator from  $L^2(G)$  to  $\mathcal{L}^2(\mathcal{H}_\pi)$ , the Hilbert space of Hilbert-Schmidt operators on  $\mathcal{H}_\pi$ . In this case we talk about unitary quantizations. In the existing examples, the representation space  $\mathcal{H}_\pi$  is of the form  $L^2(Q, \nu)$  and the basic operator  $\Sigma$  is of the form  $m \circ T_\sigma$ , where  $m$  is an operator of multiplication by a Borelian function on  $Q$  and  $T_\sigma$  is the operator of composition by a Borelian involution  $\sigma : Q \rightarrow Q$ .

For unitary quantizations, one can transfer the algebraic structure of  $\mathcal{L}^2(\mathcal{H}_\pi)$  to  $L^2(G)$  and define an associative left equivariant deformed product:

$$\star_{\pi,\Sigma} : L^2(G) \times L^2(G) \rightarrow L^2(G), \quad (f_1, f_2) \mapsto \Omega_{\pi,\Sigma}^*(\Omega_{\pi,\Sigma}(f_1) \Omega_{\pi,\Sigma}(f_2)),$$

where  $\Omega_{\pi,\Sigma}^* : \mathcal{L}^2(\mathcal{H}_\pi) \rightarrow L^2(G)$  denotes the adjoint map, which is traditionally called the symbol map. Note that on the trace-class ideal  $\mathcal{L}^1(\mathcal{H}_\pi) \subset \mathcal{L}^2(\mathcal{H}_\pi)$ , it is given by

$$\Omega_{\pi,\Sigma}^*(S) = [g \mapsto \text{Tr}(S \pi(g) \Sigma \pi(g)^*)], \quad \forall S \in \mathcal{L}^1(\mathcal{H}_\pi),$$

so that the deformed product is then associated with a distributional (in the sense of Bruhat [4]) tri-kernel:

$$f_1 \star_{\pi,\Sigma} f_2(g_0) = \int_{G \times G} K_{\pi,\Sigma}(g_0, g_1, g_2) f_1(g_1) f_2(g_2) d^\lambda(g_1) d^\lambda(g_2),$$

where  $K_{\pi,\Sigma}$  is (formally) given by

$$K_{\pi,\Sigma}(g_0, g_1, g_2) = \text{Tr}(\Sigma \pi(g_0^{-1} g_1) \Sigma \pi(g_1^{-1} g_2) \Sigma \pi(g_2^{-1} g_0)).$$

In general,  $K_{\pi,\Sigma}$  is not a singular object but rather a regular function (in the sense of Bruhat [4]).

There is then a natural candidate for a dual unitary 2-cocycle  $F_{\pi,\Sigma}$  on  $G$ , namely

$$F_{\pi,\Sigma} := \int_{G \times G} \overline{K_{\pi,\Sigma}(e, g_1, g_2)} \lambda_{g_1^{-1}} \otimes \lambda_{g_2^{-1}} d^\lambda(g_1) d^\lambda(g_2) \in W^*(G \times G).$$

The 2-cocyclicity property is automatic from the construction since this property is equivalent to left-equivariance and associativity of the deformed product  $\star_{\pi,\Sigma}$ . The only remaining task is to check that  $F_{\pi,\Sigma}$  is well defined as a unitary element of the group von Neumann algebra  $W^*(G \times G)$ . As observed in [23], this is the case for the quantization map considered in [2].

There is also a natural candidate for a deformation theory. Consider now a  $C^*$ -algebra  $A$  endowed with a continuous action  $\alpha$  of  $G$ . Then, we may try to define a new multiplication on  $A$  by the formula:

$$(1) \quad a \star_{\pi,\Sigma}^\alpha b := \int_{G \times G} K_{\pi,\Sigma}(e, g_1, g_2) \alpha_{g_1}(a) \alpha_{g_2}(b) d^\lambda(g_1) d^\lambda(g_2), \quad a, b \in A.$$

Of course, there no reason why this integral should be well defined since the map  $[g \mapsto \alpha_g(a)]$  is constant in norm and since  $K_{\pi,\Sigma}$  is typically unbounded (at least when the group is non-unimodular). Rieffel's approach to deformations consists then in two steps:

- Find  $A_{\text{reg}}$ , a dense  $\alpha$ -stable Fréchet subalgebra of  $A$ , on which the multiplication (1) is inner.
- Embed continuously the deformed Fréchet algebra  $(A_{\text{reg}}, \star_{\pi,\Sigma}^\alpha)$  into a  $C^*$ -algebra.

The  $C^*$ -deformation of  $A$  is then defined as the  $C^*$ -completion of  $A_{\text{reg}}$  and is denoted  $A_{\pi,\Sigma}$ .

To deal with the first step, one usually works with oscillatory integrals. Roughly speaking, it boils down to find a countable family of operators  $\underline{\mathbf{D}} := \{\mathbf{D}_j\}_{j \in J}$ , where  $J$  is the index set of the seminorms of  $A_{\text{reg}}$ , acting on the space of regular functions (in the sense of Bruhat)  $\mathcal{E}(G \times G)$ , which leave invariant the two-point kernel  $\mathbf{D}_j K_{\pi,\Sigma}(e, \cdot, \cdot) = K_{\pi,\Sigma}(e, \cdot, \cdot)$  and such that for all  $a, b \in A_{\text{reg}}$ , the transposed operator  $\mathbf{D}_j^t$  sends the mapping  $[(g_1, g_2) \mapsto \alpha_{g_1}(a) \alpha_{g_2}(b)]$  to an element of  $L^1(G \times G, A_{\text{reg},j})$ , where  $A_{\text{reg},j}$  denotes the semi-normed space associated with the  $j$ -th seminorm of  $A_{\text{reg}}$ . One then gets a continuous bilinear map defined by

$$\star_{\pi,\Sigma}^\alpha : A_{\text{reg}} \times A_{\text{reg}} \rightarrow A_{\text{reg},j}, \quad (a, b) \mapsto \int_{G \times G} K_{\pi,\Sigma}(e, g_1, g_2) \mathbf{D}_j^t(\alpha_{g_1}(a) \alpha_{g_2}(b)) d^\lambda(g_1) d^\lambda(g_2).$$

Then, it remains to show that the associativity is preserved by the regularization scheme underlying the introduction of the operators  $\underline{\mathbf{D}}$ .

For the second point, one usually starts by proving an  $A$ -valued version of the Calderon-Vaillancourt Theorem. It basically says that if you consider  $A$  to be the  $C^*$ -algebra of right uniformly continuous bounded functions on  $G$  with action given by right translation, then the quantization map  $\Omega_{\pi,\Sigma}$  should send continuously  $A_{\text{reg}}$  to  $\mathcal{B}(\mathcal{H})$ . When the projective representation  $\pi$  is square integrable<sup>2</sup>, from the Duflo-Moore theory [10] we can construct a weak resolution of identity from  $\pi$ . It allows to use general methods based on Wigner functions as first introduced in [32].

The paper is organized as follow. In section 2, we fix notations and we review the  $p$ -adic Weyl pseudo-differential calculus on  $\mathbf{k}^d$ , where  $\mathbf{k}$  is a non-Archimedean local field of characteristic different from 2. In fact, we consider a family of  $p$ -adic quantization maps, indexed by a parameter  $\theta$  in  $\mathcal{O}_{\mathbf{k}}$ , the ring of integers of  $\mathbf{k}$ . Section 3 contains the most technical part of the paper. It is in that section that we construct the space  $A_{\text{reg}}$  of regular elements of a  $C^*$ -algebra  $A$  for a given continuous action  $\alpha$  of  $\mathbf{k}^{2d}$  (Definition 3.10). We then define a deformation theory at the Fréchet level (Theorem 3.19), using oscillatory integrals methods. In section 4, we extend the  $p$ -adic Calderon-Vaillancourt Theorem of [1] in the case of  $C^*$ -valued symbols (Proposition 4.4). This yields an embedding of the deformed Fréchet algebra into a  $C^*$ -algebra and consequently a deformation theory at the  $C^*$ -level (Theorem 4.6). We call  $A_\theta$  the  $C^*$ -deformation of  $A$ . We also prove that our deformed  $C^*$ -norm can be realized as the  $C^*$ -norm of  $A$ -linear adjointable bounded endomorphisms of a  $C^*$ -module (Proposition 4.11) and that our construction coincides with those of [14] and [6] (Theorem 4.15). In the final section 5, we establish the basic properties of the deformation. In particular, we show that contrary to the Archimedean case, the  $K$ -theory is not an invariant of the deformation and that the fields of deformed  $C^*$ -algebras  $(A_{\gamma\theta^2})_{\theta \in \mathcal{O}_{\mathbf{k}}}$ , for  $\gamma \in \mathcal{O}_{\mathbf{k}}$  arbitrary, are continuous.

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<sup>2</sup>By a square integrable projective representation, we mean a representation of the associated central extension which is square integrable modulo its center.

## 2. A $p$ -ADIC PSEUDO-DIFFERENTIAL CALCULUS

**2.1. Framework and notations.** Let  $\mathbf{k}$  be a non-Archimedean local field, that is a non-Archimedean non-discrete locally compact topological field.  $\mathbf{k}$  is complete for the ultrametric associated with the absolute value  $|\cdot|_{\mathbf{k}}$ , given by the restriction to dilations of the module function (and extended to zero on 0). Non-Archimedean local fields are classified. In characteristic zero,  $\mathbf{k}$  is isomorphic to a finite extension of  $\mathbb{Q}_p$ , the field of  $p$ -adic numbers. In positive characteristic,  $\mathbf{k}$  is isomorphic to  $\mathbb{F}_q((X))$ , the field of Laurent series with coefficients in a finite field. For important technical reasons, we will (mostly) assume that the characteristic of  $\mathbf{k}$  different from 2. The additive group  $(\mathbf{k}, +)$  is self-dual, with isomorphism  $\mathbf{k} \simeq \text{Hom}(\mathbf{k}, \mathbb{U}(1))$  given by  $x \mapsto \Psi(x)$ , where  $\Psi$  is a fixed non-trivial character. We denote by  $\mathcal{O}_{\mathbf{k}} := \{x \in \mathbf{k} : |x|_{\mathbf{k}} \leq 1\}$  the ring of integers, by  $\varpi$  the generator its unique maximal ideal ( $\varpi$  is called the uniformizer and satisfies  $|\varpi|_{\mathbf{k}} = (\text{Card}(\mathcal{O}_{\mathbf{k}}/\varpi\mathcal{O}_{\mathbf{k}}))^{-1}$ ) and by  $\mathcal{O}_{\mathbf{k}}^o$  the conductor of  $\Psi$ , that is to say the largest ideal<sup>3</sup> of  $\mathcal{O}_{\mathbf{k}}$  on which  $\Psi$  is constant. We normalize the Haar measure of  $(\mathbf{k}, +)$  (denoted by  $dx$ ) by requiring it to be selfdual with respect to the duality associated with  $\Psi$  or, equivalently, by requiring that  $\text{Vol}(\mathcal{O}_{\mathbf{k}}) \times \text{Vol}(\mathcal{O}_{\mathbf{k}}^o) = 1$ .

For example, if  $\mathbf{k} = \mathbb{Q}_p$ , we have  $\mathcal{O}_{\mathbf{k}} = \mathbb{Z}_p$ , the ring of  $p$ -adic integers,  $\varpi = p$  and  $|x|_{\mathbb{Q}_p} = p^{-k}$  if  $x = p^k \frac{m}{n} \in \mathbb{Q}$  (where  $m$  and  $n$  are integers non-divisible by  $p$ ). If one chooses the (standard) character:

$$\Psi_0(x) := \exp \left\{ 2\pi i \sum_{-n_0 \leq n < 0} a_n p^n \right\} \quad \text{if } x = \sum_{n \geq -n_0} a_n p^n,$$

we find  $\mathcal{O}_{\mathbf{k}}^o = \mathbb{Z}_p$  and our normalization for the Haar measure reads  $\text{Vol}(\mathbb{Z}_p) = 1$ .

We let  $|\cdot|_{\mathbf{k}}^{\vee}$  be the second ultra-metric norm on  $\mathbf{k}$ , given by  $|x|_{\mathbf{k}}^{\vee} = |x\varpi^{-n(\Psi)}|_{\mathbf{k}}$ . More generally, we let  $|\cdot|_{\mathbf{k}^d}$ ,  $|\cdot|_{\mathbf{k}^d}^{\vee}$  be the associated sup-norms on  $\mathbf{k}^d$ ,  $d \in \mathbb{N}$ :

$$|x|_{\mathbf{k}^d} = \max_{1 \leq i \leq d} |x_i|_{\mathbf{k}} \quad \text{and} \quad |\xi|_{\mathbf{k}^d}^{\vee} = \max_{1 \leq i \leq d} |\xi_i|_{\mathbf{k}}^{\vee}.$$

For  $x, y, \xi, \eta \in \mathbf{k}^d$  we set  $X = (x, \xi), Y = (y, \eta) \in \mathbf{k}^{2d}$  and we consider the symplectic structure:

$$(2) \quad [.,.] : \mathbf{k}^{2d} \times \mathbf{k}^{2d} \rightarrow \mathbf{k}, \quad (X, Y) \mapsto \langle y, \xi \rangle - \langle x, \eta \rangle,$$

where  $\langle x, y \rangle = \sum_{j=1}^d x_j y_j$ .

The following numerical function plays a decisive role in our analysis:

$$(3) \quad \mu_0(X) := \max\{1, |2x|_{\mathbf{k}^d}, |2\xi|_{\mathbf{k}^d}^{\vee}\}, \quad X = (x, \xi) \in \mathbf{k}^{2d}.$$

From the ultrametric inequality, one sees that  $\mu_0$  is invariant under translations in  $(\frac{1}{2}\mathcal{O}_{\mathbf{k}})^d \times (\frac{1}{2}\mathcal{O}_{\mathbf{k}}^o)^d$ . It is known (see [12]) that  $\mu_0^{-1}$  belongs to  $L^p(\mathbf{k}^{2d})$  for all  $p > 2d$  and satisfies a Peetre type inequality:

$$(4) \quad \mu_0(X + Y) \leq \mu_0(X) \mu_0(Y), \quad \forall X, Y \in \mathbf{k}^{2d}.$$

Let  $\mathcal{E}(\mathbf{k}^{2d})$  (resp.  $\mathcal{D}(\mathbf{k}^{2d})$ ,  $\mathcal{D}'(\mathbf{k}^{2d})$ ) be the set of smooth functions (resp. smooth and compactly supported functions, distributions) in the sense of Bruhat [4]. Since  $\mathbf{k}^{2d}$  is totally disconnected,  $\mathcal{E}(\mathbf{k}^{2d})$  consists in locally constant functions and  $\mathcal{D}(\mathbf{k}^{2d})$  consists in locally constant compactly supported

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<sup>3</sup> $\mathcal{O}_{\mathbf{k}}^o$  is of the form  $\varpi^{n(\Psi)}\mathcal{O}_{\mathbf{k}}$ , with  $n(\Psi) \in \mathbb{Z}$  uniquely defined by the character  $\Psi$ .



functions. In particular,  $\mu_0$  belongs to  $\mathcal{E}(\mathbf{k}^{2d}) \subset \mathcal{D}'(\mathbf{k}^{2d})$ .  $\mathcal{D}(\mathbf{k}^{2d})$  and  $\mathcal{D}'(\mathbf{k}^{2d})$  are stabilized by the (selfdual) Fourier transform and by its symplectic variant:

$$(5) \quad (\mathcal{G}f)(X) = |2|_{\mathbf{k}}^d \int_{\mathbf{k}^{2d}} f(Y) \Psi(2[Y, X]) dY.$$

The symplectic Fourier transform extends to a unitary operator on  $L^2(\mathbf{k}^{2d})$  and is its own inverse.

**Remark 2.1.** *The normalization chosen in the definition of  $\mu_0$  and  $\mathcal{G}$  (i.e. the factor 2) allows to simplify some computations but is by no mean the reason why we have to exclude the characteristic 2.*

From our perspectives, the Schwartz-Bruhat space<sup>4</sup>  $\mathcal{D}(\mathbf{k}^{2d})$  is not suitable. Instead, we consider a variant of it, that we may define with the help of two unbounded operators on  $L^2(\mathbf{k}^{2d})$ . Let  $I$  be the operator of point-wise multiplication by the function  $\mu_0$ :

$$(6) \quad I\varphi(X) := \mu_0(X) \varphi(X),$$

and  $J$  the convolution operator by the Bruhat distribution  $\mathcal{G}(\mu_0)$ :

$$(7) \quad J := \mathcal{G} \circ I \circ \mathcal{G}.$$

More generally, we denote by  $J^s$ ,  $s \in \mathbb{R}$ , the convolution operator by  $\mathcal{G}(\mu_0^s)$ . The Bruhat distributions  $\mathcal{G}(\mu_0^s)$  are known to be supported in  $(\frac{1}{2}\mathcal{O}_{\mathbf{k}})^d \times (\frac{1}{2}\mathcal{O}_{\mathbf{k}}^o)^d$  (we will give an elementary proof of this fact in Lemma 2.2). The operator  $J$  has to be considered as a substitute of an order one elliptic differential operator, in the dual sense that  $\mu_0$  has to be considered as a substitute for a radial coordinate function on  $\mathbf{k}^{2d}$ . Since  $\mu_0$  is  $(\frac{1}{2}\mathcal{O}_{\mathbf{k}})^d \times (\frac{1}{2}\mathcal{O}_{\mathbf{k}}^o)^d$ -locally constant and  $\mathcal{G}(\mu_0)$  is supported on  $(\frac{1}{2}\mathcal{O}_{\mathbf{k}})^d \times (\frac{1}{2}\mathcal{O}_{\mathbf{k}}^o)^d$ , as continuous operators on  $\mathcal{D}'(\mathbf{k}^{2d})$ ,  $I$  and  $J$  commute! As unbounded operators on  $L^2(\mathbf{k}^{2d})$  (with initial domain  $\mathcal{D}(\mathbf{k}^{2d})$ ) they are essentially selfadjoint and positive. Following Haran [12], we introduce the another analogue of the Schwartz space:

$$\mathcal{S}(\mathbf{k}^{2d}) := \bigcap_{n, m \in \mathbb{N}} \text{Dom}(I^n J^m) := \{\varphi \in L^2(\mathbf{k}^{2d}) : \forall n, m \in \mathbb{N}, I^n J^m \varphi \in L^2(\mathbf{k}^{2d})\}.$$

Equipped with the seminorms:

$$(8) \quad \varphi \mapsto \|I^n J^m \varphi\|_2, \quad n, m \in \mathbb{N},$$

$\mathcal{S}(\mathbf{k}^{2d})$  becomes Fréchet and nuclear and of course

$$\mathcal{D}(\mathbf{k}^{2d}) \subset \mathcal{S}(\mathbf{k}^{2d}) \subset C(\mathbf{k}^{2d}),$$

continuously. Moreover,  $\mathcal{D}(\mathbf{k}^{2d})$  is dense in  $\mathcal{S}(\mathbf{k}^{2d})$  but the inclusion is proper since an element in  $\mathcal{S}(\mathbf{k}^{2d})$  does not need to be locally constant nor compactly supported. Since  $\mu_0^s$  belongs to  $L^1(\mathbf{k}^{2d})$  for  $s < -2d$ , in the seminorms (8), we can change the  $L^2$ -norm with any other  $L^p$ -norm,  $p \in [1, \infty]$ , while keeping the same topology<sup>5</sup>. We let  $\mathcal{S}'(\mathbf{k}^{2d})$  be the strong dual of  $\mathcal{S}(\mathbf{k}^{2d})$ , that we call the space of tempered distributions. The operators  $I$  and  $J$  extend to continuous endomorphisms of  $\mathcal{S}'(\mathbf{k}^{2d})$  and, of course, still commute there. We can also define the  $d$ -dimensional versions of the Schwartz space  $\mathcal{S}(\mathbf{k}^d)$  and of its dual  $\mathcal{S}'(\mathbf{k}^d)$  by considering the  $d$ -dimensional version of the function  $\mu_0$  and the  $d$ -dimensional ordinary Fourier transform to define  $d$ -dimensional version of the operators  $I$  and  $J$ .

The following (almost obvious) properties will be used repeatedly:

<sup>4</sup>Indeed, since  $\mathbf{k}^{2d}$  is totally disconnected,  $\mathcal{D}(\mathbf{k}^{2d})$  coincides with the Schwartz space, as defined in [4, numéro 9].

<sup>5</sup>In section 3 we will consider the seminorms (8) with  $p = \infty$  instead of 2, see (22).

**Lemma 2.2.** (i) For  $Y \in \mathbf{k}^{2d}$ , set  $\mu_Y(X) := \mu_0(X - Y)$ . Then for all  $s, t_1, \dots, t_n \in \mathbb{R}$ ,  $Y_1, \dots, Y_n \in \mathbf{k}^{2d}$ , we have:

$$I^s \mathcal{G}(\mu_{Y_1}^{t_1} \dots \mu_{Y_n}^{t_n}) = \mathcal{G}(\mu_{Y_1}^{t_1} \dots \mu_{Y_n}^{t_n}) \quad \text{and} \quad J^s(\mu_{Y_1}^{t_1} \dots \mu_{Y_n}^{t_n}) = \mu_{Y_1}^{t_1} \dots \mu_{Y_n}^{t_n},$$

(ii) For  $Y \in \mathbf{k}^{2d}$ , set  $\Psi_Y(X) := \Psi(2[X, Y])$ . Then for all  $s \in \mathbb{R}$ , we have  $J^s \Psi_Y = \mu_0^s(Y) \Psi_Y$ .

*Proof.* (i) Of course, the two equalities we have to prove are equivalent. Set  $O := (\frac{1}{2}\mathcal{O}_{\mathbf{k}})^d \times (\frac{1}{2}\mathcal{O}_{\mathbf{k}}^o)^d$ . Fix  $t > 2d$  and observe that  $\mu_0^{-t} \in L^1(\mathbf{k}^{2d})$  so that  $\mathcal{G}(\mu_0^{-t}) \in C_0(\mathbf{k}^{2d})$ . Since moreover  $\mu_0^{-t}$  is  $O$ -invariant, we have

$$\mathcal{G}(\mu_0^{-t})(X) = \overline{\Psi}(2[X, Y]) \mathcal{G}(\mu_0^{-t})(X), \quad \forall (Y, X) \in O \times \mathbf{k}^{2d}.$$

This implies that  $\mathcal{G}(\mu_0^{-t})$  is supported in  $O$  since one can easily construct a pair  $(Y, X) \in O \times \mathbf{k}^{2d} \setminus O$  such that  $\Psi(2[X, Y]) \neq 1$ . As  $\mu_0$  equals one on  $O$ , we get for all  $s > 0$ ,  $I^s \mathcal{G}(\mu_0^{-t}) = \mathcal{G}(\mu_0^{-t})$  which is equivalent to  $J^s(\mu_0^{-t}) = \mu_0^{-t}$ . Using that  $J$  commutes with  $I$  and translations  $\tau_Y f(X) = f(X + Y)$ , the result follows by applying  $J^s$  to the identity

$$\mu_{Y_1}^{t_1} \dots \mu_{Y_n}^{t_n} = \tau_{Y_1} I^{t_1} \tau_{Y_2 - Y_1} I^{t_2} \dots \tau_{Y_n - Y_{n-1}} I^{t_n} (\mu_0^{-t}).$$

(ii) We denote by  $\langle \cdot, \cdot \rangle$  the (bilinear) duality pairing between  $\mathcal{S}(\mathbf{k}^{2d})$  and  $\mathcal{S}'(\mathbf{k}^{2d})$ . Fix  $X \in \mathbf{k}^{2d}$ . Since  $\Psi_X \in C_b(\mathbf{k}^{2d})$ , we may view it as an element of  $\mathcal{S}'(\mathbf{k}^{2d})$ . Then, we have for all  $\varphi \in \mathcal{S}(\mathbf{k}^{2d})$ :

$$\mathcal{G}\varphi(X) = |2|_{\mathbf{k}}^d \langle \Psi_X, \varphi \rangle \quad \text{and thus} \quad \varphi(X) = |2|_{\mathbf{k}}^d \langle \Psi_X, \mathcal{G}\varphi \rangle = |2|_{\mathbf{k}}^d \langle \mathcal{G}\Psi_X, \varphi \rangle.$$

From this and the fact that  $J^s = (\mathcal{G}I\mathcal{G})^s = \mathcal{G}I^s\mathcal{G}$ , we get

$$\langle J^s \Psi_X, \varphi \rangle = \langle \mathcal{G}I^s \mathcal{G}\Psi_X, \varphi \rangle = \langle \mathcal{G}\Psi_X, I^s \mathcal{G}\varphi \rangle = |2|_{\mathbf{k}}^{-d} I^s \mathcal{G}\varphi(X) = |2|_{\mathbf{k}}^{-d} \mu_0^s(X) \mathcal{G}\varphi(X) = \mu_0^s(X) \langle \Psi_X, \varphi \rangle.$$

This completes the proof.  $\square$

**2.2. Weyl quantization on local fields.** In this subsection we recall some facts about the  $p$ -adic pseudo-differential calculus introduced by Haran in [12] and further studied by Bechata and Unterberger [1, 34] (see also [30, 35] for a completely general construction of the Weyl quantization). We assume that the characteristic of  $\mathbf{k}$  is different from 2. We fix  $\theta \in \mathbf{k}^\times$ . It will play the role of the deformation parameter. For any tempered distribution  $F \in \mathcal{S}'(\mathbf{k}^{2d})$ , we denote by  $\Omega_\theta(F)$  the continuous linear operator from  $\mathcal{S}(\mathbf{k}^d)$  to  $\mathcal{S}'(\mathbf{k}^d)$  defined (with a little abuse of notation) by:

$$\Omega_\theta(F) : \mathcal{S}(\mathbf{k}^d) \rightarrow \mathcal{S}'(\mathbf{k}^d)$$

$$(9) \quad \varphi \mapsto \left[ \phi \in \mathcal{S}(\mathbf{k}^d) \mapsto |\theta|_{\mathbf{k}}^{-d} \int_{\mathbf{k}^d} \left( \int_{\mathbf{k}^{2d}} F\left(\frac{1}{2}(x+y), \eta\right) \varphi(y) \Psi(\theta^{-1}\langle x-y, \eta \rangle) d\eta dy \right) \phi(x) dx \right].$$

The distribution  $F$  is called the symbol of the pseudo-differential operator  $\Omega_\theta(F)$ . This Weyl type pseudo-differential calculus is covariant under the action of the additive group  $\mathbf{k}^{2d}$  by translations, in the sense that

$$(10) \quad U_\theta(Y) \Omega_\theta(F) U_\theta(Y)^* = \Omega_\theta(\tau_{-Y} F), \quad \forall F \in \mathcal{S}'(\mathbf{k}^{2d}), \quad \forall X \in \mathbf{k}^{2d},$$

where  $\tau_Y F(X) := F(X + Y)$  and where  $U_\theta$  is the projective unitary (Schrödinger) representation of  $\mathbf{k}^{2d}$  on  $L^2(\mathbf{k}^d)$  given, for  $X = (x, \xi) \in \mathbf{k}^d \times \mathbf{k}^d$ , by

$$(11) \quad U_\theta(X) \varphi(y) := \Psi(\theta^{-1}\langle \xi, y - \frac{1}{2}x \rangle) \varphi(y - x).$$



The covariance property is obvious for a symbol  $F \in L^1(\mathbf{k}^{2d})$ , as seen from the absolutely convergent integral representation:

$$(12) \quad \Omega_\theta(F) = |\frac{2}{\theta}|_{\mathbf{k}}^d \int_{\mathbf{k}^{2d}} F(X) \Omega_\theta(X) dX,$$

where

$$\Omega_\theta(X) := U_\theta(X) \Sigma U_\theta(X)^*, \quad \forall X \in \mathbf{k}^{2d},$$

and where  $\Sigma$  is the selfadjoint involution on  $L^2(\mathbf{k}^d)$  given by  $\Sigma\varphi(x) = \varphi(-x)$ . Note the scaling relations

$$U_\theta(x, \xi) = U_1(x, \theta^{-1}\xi), \quad \Omega_\theta(x, \xi) = \Omega_1(x, \theta^{-1}\xi), \quad \forall x, \xi \in \mathbf{k}^d.$$

The integral representation given above implies that when  $F \in L^1(\mathbf{k}^{2d})$ , the pseudo-differential operator  $\Omega_\theta(F)$  is bounded, with<sup>6</sup>:

$$(13) \quad \|\Omega_\theta(F)\| \leq |\theta|_{\mathbf{k}}^{-d} \|F\|_1.$$

Of course, this inequality blows up in the limit  $\theta \rightarrow 0$ . By Fourier theory (on selfdual locally compact Abelian groups), and when characteristic of  $\mathbf{k}$  is different from 2, one sees that the associated quantization map

$$\Omega_\theta : \mathcal{S}'(\mathbf{k}^{2d}) \rightarrow \mathcal{L}(\mathcal{S}(\mathbf{k}^d), \mathcal{S}'(\mathbf{k}^d)), \quad F \mapsto \Omega_\theta(F),$$

restricts to  $|\theta|_{\mathbf{k}}^{-d/2}$  times a unitary operator from  $L^2(\mathbf{k}^{2d})$  to the Hilbert space of Hilbert-Schmidt operators on  $L^2(\mathbf{k}^d)$ . Hence, we also have the bound:

$$(14) \quad \|\Omega_\theta(F)\| \leq \|\Omega_\theta(F)\|_2 = |\theta|_{\mathbf{k}}^{-d/2} \|F\|_2.$$

One can then transport the algebraic structure of the Hilbert-Schmidt operators to  $L^2(\mathbf{k}^{2d})$ , by setting

$$f_1 \star_\theta f_2 := \Omega_\theta^{-1}(\Omega_\theta(f_1) \Omega_\theta(f_2)), \quad \forall f_1, f_2 \in L^2(\mathbf{k}^{2d}).$$

At the level of the Schwartz space, this deformed product has a familiar form:

$$(15) \quad f_1 \star_\theta f_2(X) = |\frac{2}{\theta}|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(\frac{2}{\theta}[Y - X, Z - X]) f_1(Y) f_2(Z) dY dZ, \quad \forall f_1, f_2 \in \mathcal{S}(\mathbf{k}^{2d}).$$

Indeed, this is the  $p$ -adic version of the Moyal product in its integral form. Note that this relation can be rewritten as a functional identity:

$$f_1 \star_\theta f_2 = |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z]) \tau_{\theta Y}(f_1) \tau_Z(f_2) dY dZ, \quad \forall f_1, f_2 \in \mathcal{S}(\mathbf{k}^{2d}),$$

where  $\tau$  is the translation operator  $\tau_{\theta Y}(f_1)(X) = f_1(X + Y)$ . The important observation is that this formula makes sense even when  $\theta = 0$ . Indeed, in this case we have  $f_1 \star_{\theta=0} f_2 = f_1 \mathcal{G}^2(f_2) = f_1 f_2$ .

**Remark 2.3.** *In characteristic 2, one can formally change the character  $\Psi$  to  $\Psi(\frac{1}{2}\cdot)$  in (15) while preserving its fundamental properties of associativity and covariance. The corresponding modification in (9) is to suppress the ill-defined factor  $\frac{1}{2}$ . But then, the operator kernel of  $\Omega_\theta(F)$  will up to a constant<sup>7</sup> be  $(\mathcal{F}_2 F)(x + y, \theta^{-1}(x - y))$ . Since the matrix  $(\frac{1}{\theta} \frac{1}{-\theta})$  is not invertible in characteristic 2, we loose the crucial property of unitarity (from  $L^2$ -symbols to Hilbert-Schmidt operators) of the quantization map.*

<sup>6</sup>In characteristic different from 2,  $|2|_{\mathbf{k}} = 1$  or  $|2|_{\mathbf{k}} = \frac{1}{2}$ .

<sup>7</sup> $\mathcal{F}_2$  is the partial Fourier transform on the second set of variables.

Set further

$$\mathcal{B}(\mathbf{k}^{2d}) := \{F \in \mathcal{S}'(\mathbf{k}^{2d}) : J^n F \in L^\infty(\mathbf{k}^{2d}), \forall n \in \mathbb{N}\}.$$

Using coherent states and Wigner functions methods, Bechata proved in [1] an analogue of the Calderon-Vaillancourt Theorem for the space  $\mathcal{B}(\mathbf{k}^{2d})$ . Namely, he proved the following estimate:

$$(16) \quad \|\Omega_\theta(F)\| \leq \|\mu_0^{-2d-1}\|_1 \|J^{2d+1}F\|_\infty, \quad F \in \mathcal{B}(\mathbf{k}^{2d}),$$

where the norm on the left hand side denotes the operator norm on  $L^2(\mathbf{k}^d)$ . Contrarily to (13) and (14), this inequality does not blow up in the limit  $\theta \rightarrow 0$ . The methods leading to this key result rely on a clever redefinition of  $\Omega_\theta(F)$  in term of a quadratic form constructed out of specific coherent states and Wigner functions. Since we will borrow part of Bechata's techniques, we recall some ingredients of his construction.

For  $\varphi \in L^2(\mathbf{k}^d)$ ,  $\theta \in \mathbf{k}^\times$  and  $X \in \mathbf{k}^{2d}$ , set  $\varphi_X^\theta := U_\theta(X)\varphi$ , where  $U_\theta$  is the projective representation of  $\mathbf{k}^{2d}$  given in (11). It is known that  $U_\theta$  is square integrable modulo its center and that the following reproducing formula holds:

$$(17) \quad \langle \phi, \psi \rangle = |\theta|_{\mathbf{k}}^{-d} \|\varphi\|^{-2} \int_{\mathbf{k}^{2d}} \langle \phi, \varphi_X \rangle \langle \varphi_X, \psi \rangle dX, \quad \forall \phi, \psi, \varphi \in L^2(\mathbf{k}^d), \varphi \neq 0.$$

Let then

$$(18) \quad W_{\phi, \psi}^\theta(X) := \langle \phi, \Omega_\theta(X)\psi \rangle, \quad X \in \mathbf{k}^{2d},$$

be the Wigner function<sup>8</sup> associated with the pair of vectors  $\phi, \psi \in L^2(\mathbf{k}^d)$ . Let  $\eta$  be the characteristic function of  $\mathcal{O}_{\mathbf{k}}^d$ , normalized by  $\|\eta\|_2 = 1$ . By [1, eq. (1.9)], we have for  $X = (x, \xi) \in \mathbf{k}^{2d}$  and  $\alpha, \beta \in \mathbb{R}$ :

$$\tilde{I}^\alpha \tilde{J}^\beta \eta_X^\theta = \mu_0^\alpha(x, 0) \mu_0^\beta(0, \theta^{-1}\xi) \eta_X^\theta,$$

where  $\tilde{I}, \tilde{J}$  denote the  $d$ -dimensional versions of the operators  $I$  and  $J$ . In particular, the relation above entails that  $\eta_X^\theta \in \mathcal{S}(\mathbf{k}^d)$ . For  $X, Y \in \mathbf{k}^{2d}$ , we set  $W_{X, Y}^\theta$  for the Wigner function associated with the pair of coherent states  $\eta_X^\theta, \eta_Y^\theta$ :

$$(19) \quad W_{X, Y}^\theta(Z) := W_{\eta_X^\theta, \eta_Y^\theta}^\theta(Z) = \langle \eta_X^\theta, \Omega_\theta(Z)\eta_Y^\theta \rangle = \langle U_\theta(X)\eta, \Omega_\theta(Z)U_\theta(Y)\eta \rangle.$$

The next statement is extracted from [1, Proposition 2.10 and Lemme 3.1].

**Lemma 2.4.** *Let  $\theta \in \mathbf{k}^\times$ . For  $X = (x, \xi) \in \mathbf{k}^{2d}$ , set  $X_\theta := (x, \theta^{-1}\xi) \in \mathbf{k}^{2d}$ . Then for all  $X, Y, Z \in \mathbf{k}^{2d}$  and with  $\Phi$  the characteristic function of  $(\frac{1}{2}\mathcal{O}_{\mathbf{k}})^d \times (\frac{1}{2}\mathcal{O}_{\mathbf{k}}^o)^d$ , we have:*

$$|W_{X, Y}^\theta(Z)| = |2|_{\mathbf{k}}^d \Phi(Z_\theta - \frac{1}{2}(X_\theta + Y_\theta)).$$

Moreover,  $W_{X, Y}^\theta \in \mathcal{S}(\mathbf{k}^{2d})$  and for all  $n, m \in \mathbb{Z}$  we have:

$$I^m J^n W_{X, Y}^\theta = \mu_0^m(\frac{1}{2}(X + Y)) \mu_0^n(\frac{1}{2\theta}(X - Y)) W_{X, Y}^\theta.$$

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<sup>8</sup>The Wigner function  $W_{\phi, \psi}^\theta$  is the symbol of the rank-one operator  $\varphi \mapsto \langle \psi, \varphi \rangle \phi$ .

3. THE FRÉCHET DEFORMATION OF A  $C^*$ -ALGEBRA

In this section, we fix a  $C^*$ -algebra  $A$ , together with a continuous action  $\alpha$  of the additive group  $\mathbf{k}^{2d}$ . This yields a map

$$(20) \quad \tilde{\alpha} : A \rightarrow C_b(\mathbf{k}^{2d}, A), \quad a \mapsto [X \mapsto \alpha_X(a)].$$

Fixing a faithful representation  $\pi$  of  $A$  on  $\mathcal{B}(\mathcal{H})$ , we will frequently identify  $A$  with its image on  $\mathcal{B}(\mathcal{H})$ .

Our first goal is to find  $A_{\text{reg}}$ , a dense and  $\alpha$ -stable Fréchet subalgebra of  $A$ , on which we can give a meaning to the natural generalization of the deformed product (15):

$$(21) \quad a \star_{\theta}^{\alpha} b := |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z]) \alpha_{\theta Y}(a) \alpha_Z(b) dY dZ, \quad \forall a, b \in A_{\text{reg}}.$$

Having in mind Bechata's version of the Calderon-Vaillancourt estimate (16), there is an obvious candidate for  $A_{\text{reg}}$ , namely the set of elements  $a$  in  $A$  which are such that  $\tilde{\alpha}(a) \in \mathcal{B}(\mathbf{k}^{2d}, A)$  (see Definition 3.1).

**3.1. Spaces of  $A$ -valued functions and distributions.** Set  $C_b(\mathbf{k}^{2d}, A)$  for the  $C^*$ -algebra of  $A$ -valued continuous and bounded functions on  $\mathbf{k}^{2d}$ , with norm:

$$\mathfrak{P}_0^A(F) := \sup_{X \in \mathbf{k}^{2d}} \|F(X)\|_A,$$

and let  $C_u(\mathbf{k}^{2d}, A)$  be the  $C^*$ -algebra of  $A$ -valued uniformly continuous and bounded functions on  $\mathbf{k}^{2d}$ . The latter space is the maximal sub- $C^*$ -algebra of  $C_b(\mathbf{k}^{2d}, A)$  on which the action  $\tau \otimes \text{Id}$  of  $\mathbf{k}^{2d}$  is continuous. Set then  $\mathcal{S}(\mathbf{k}^{2d}, A) := \mathcal{S}(\mathbf{k}^{2d}) \widehat{\otimes} A$  for the  $A$ -valued version of the Schwartz space (recall that  $\mathcal{S}(\mathbf{k}^{2d})$  is nuclear). We naturally embed  $\mathcal{S}(\mathbf{k}^{2d}, A)$  into  $C_u(\mathbf{k}^{2d}, A)$ . Since the (pair-wise commuting) linear maps  $I^n J^m$ ,  $n, m \in \mathbb{Z}$ , are continuous on  $\mathcal{S}(\mathbf{k}^{2d})$ ,  $I^n J^m \otimes \text{Id}$  (originally defined on the algebraic tensor product  $\mathcal{S}(\mathbf{k}^{2d}) \otimes A$ ) extends to a continuous linear map on  $\mathcal{S}(\mathbf{k}^{2d}, A)$ . To lighten our notations, and when no confusion can occur, we will denote their extensions by  $I^n J^m$ . In a similar way, we will use the symbol  $\mathcal{G}$  to denote the continuous extension of  $\mathcal{G} \otimes \text{Id}$  on  $\mathcal{S}(\mathbf{k}^{2d}, A)$ . As already mentioned, the Fréchet topology of  $\mathcal{S}(\mathbf{k}^{2d}, A)$  can be alternatively described via the seminorms:

$$(22) \quad \mathfrak{P}_{n,m}^A(f) := \sup_{X \in \mathbf{k}^{2d}} \|(I^n J^m f)(X)\|_A, \quad n, m \in \mathbb{N}.$$

Since  $\mathcal{S}(\mathbf{k}^{2d})$  is Fréchet and nuclear, its strong dual  $\mathcal{S}'(\mathbf{k}^{2d})$  is also nuclear (see for instance [31, Proposition 50.6]). Therefore, we shall denote by  $\mathcal{S}'(\mathbf{k}^{2d}, A)$  the completed tensor product  $\mathcal{S}'(\mathbf{k}^{2d}) \widehat{\otimes} A$ . Note that by [31, P. 525],  $\mathcal{S}'(\mathbf{k}^{2d}, A)$  identifies isometrically with the space of continuous linear mappings from  $\mathcal{S}(\mathbf{k}^{2d})$  to  $A$ . Under this identification, we get an embedding of  $C_b(\mathbf{k}^{2d}, A)$  into  $\mathcal{S}'(\mathbf{k}^{2d}, A)$ . Since the operators  $I^n J^m$ ,  $n, m \in \mathbb{Z}$ , act continuously (by transposition) on  $\mathcal{S}'(\mathbf{k}^{2d})$ , they extend continuously on  $\mathcal{S}'(\mathbf{k}^{2d}, A)$  and we still denote them by  $I^n J^m$ . Similarly, we denote by  $\mathcal{G}$  the continuous extension of the symplectic Fourier transform on  $\mathcal{S}'(\mathbf{k}^{2d}, A)$ . The next space we introduce is of our principal tools:

**Definition 3.1.** For  $A$  a  $C^*$ -algebra, we set

$$\mathcal{B}(\mathbf{k}^{2d}, A) := \{F \in \mathcal{S}'(\mathbf{k}^{2d}, A) : \forall n \in \mathbb{N}, J^n F \in L^\infty(\mathbf{k}^{2d}, A)\}.$$

We endow  $\mathcal{B}(\mathbf{k}^{2d}, A)$  with the topology associated with the following family of seminorms:

$$(23) \quad \mathfrak{P}_n^A(F) := \sup_{X \in \mathbf{k}^{2d}} \|(J^n F)(X)\|_A, \quad \forall n \in \mathbb{N}.$$

When  $A = \mathbb{C}$ , we denote these seminorms by  $\mathfrak{P}_n$ . Identifying in a natural way the algebraic tensor product  $\mathcal{B}(\mathbf{k}^{2d}) \otimes A$  with a subspace of  $\mathcal{B}(\mathbf{k}^{2d}, A)$ , it is easy to see that  $\mathfrak{P}_n^A$  is a cross-seminorm:  $\mathfrak{P}_n^A(F \otimes a) = \mathfrak{P}_n(F) \|a\|_A$ .

Last, we introduce  $C_u^\infty(\mathbf{k}^{2d}, A)$  to be the subspace of smooth (in the sense of Bruhat) elements in  $C_u(\mathbf{k}^{2d}, A)$  for the regular representations  $\tau \otimes \text{Id}$  of  $\mathbf{k}^{2d}$  (see [21] for more details):

$$C_u^\infty(\mathbf{k}^{2d}, A) := \{F \in C_u(\mathbf{k}^{2d}, A) : \tilde{\tau}(F) := [X \mapsto \tau_X(F)] \in \mathcal{E}(\mathbf{k}^{2d}, C_u(\mathbf{k}^{2d}, A))\}.$$

**Lemma 3.2.** *The space  $\mathcal{B}(\mathbf{k}^{2d}, A)$  is Fréchet and  $C_u^\infty(\mathbf{k}^{2d}, A) \subset \mathcal{B}(\mathbf{k}^{2d}, A) \subset C_u(\mathbf{k}^{2d}, A)$  with dense inclusions.*

*Proof.* That  $\mathcal{B}(\mathbf{k}^{2d}, A)$  is Fréchet follows from standard arguments.

To prove that  $\mathcal{B}(\mathbf{k}^{2d}, A) \subset C_u(\mathbf{k}^{2d}, A)$ , we assume first that  $A = \mathbb{C}$ . Since  $L^1(\mathbf{k}^{2d}) * L^\infty(\mathbf{k}^{2d}) = C_u(\mathbf{k}^{2d})$  (see for instance [13, (32.45) (b), p. 283]), it suffices to show that  $\mathcal{B}(\mathbf{k}^{2d}) \subset L^1(\mathbf{k}^{2d}) * L^\infty(\mathbf{k}^{2d})$ . So, let  $F \in \mathcal{B}(\mathbf{k}^{2d})$  and set  $G := J^{2p+1}F$ . We have  $F = J^{-2p-1}G = \mathcal{G}(\mu_0^{-2p-1}) * G$ , which is the desired factorization. Indeed,  $G \in \mathcal{B}(\mathbf{k}^{2d}) \subset L^\infty(\mathbf{k}^{2d})$  and  $\mathcal{G}(\mu_0^{-2p-1}) \in L^1(\mathbf{k}^{2d})$  because  $\mu_0^{-2p-1} \in L^1(\mathbf{k}^{2d})$  and because  $\mathcal{G}(\mu_0^{-2p-1})$  is compactly supported by Lemma 2.2 (i). For a generic  $C^*$ -algebra  $A$ , we deduce that the algebraic tensor product  $\mathcal{B}(\mathbf{k}^{2d}) \otimes A$  is contained in  $C_u(\mathbf{k}^{2d}, A)$ . Since  $\mathcal{B}(\mathbf{k}^{2d}) \otimes A$  is dense in  $\mathcal{B}(\mathbf{k}^{2d}, A)$  and since  $\mathfrak{P}_0^A$  is the  $C^*$ -norm of  $C_u(\mathbf{k}^{2d}, A)$ , we conclude that  $\mathcal{B}(\mathbf{k}^{2d}, A)$  is contained in the norm closure of  $\mathcal{B}(\mathbf{k}^{2d}) \otimes A$  in  $C_u(\mathbf{k}^{2d}, A)$ .

Next, for  $F \in C_u(\mathbf{k}^{2d}, A)$  and  $\varphi \in \mathcal{S}(\mathbf{k}^{2d})$ , we set

$$(24) \quad \tau_\varphi(F) := \int_{\mathbf{k}^{2d}} \varphi(X) \tau_X(F) dX,$$

where  $\tau_X$  is the operator of translation by  $X \in \mathbf{k}^{2d}$ . By isometry of  $\tau_X$  for the norm  $\mathfrak{P}_0^A$ , we get  $\mathfrak{P}_0^A(\tau_\varphi F) \leq \|\varphi\|_1 \mathfrak{P}_0^A(F)$ . Hence, the integral in (24) converges in  $C_u(\mathbf{k}^{2d}, A)$  since the latter is closed in  $C_b(\mathbf{k}^{2d}, A)$ . The operator  $J$  commuting with  $\tau_X$ , we get for all  $n \in \mathbb{N}$   $J^n \tau_\varphi(F) = \tau_{J^n \varphi}(F)$  which entails that  $\mathfrak{P}_n^A(\tau_\varphi(F)) \leq \|J^n \varphi\|_1 \mathfrak{P}_0^A(F)$ . Hence,  $\tau_\varphi(F) \in \mathcal{B}(\mathbf{k}^{2d}, A)$ . Chose next a positive sequence  $\{\varphi_k\}_{k \in \mathbb{N}}$  in  $\mathcal{S}(\mathbf{k}^{2d})$  such that  $\|\varphi_k\|_1 = 1$  and such that  $\varphi_k$  is supported in  $B(0, k^{-1})$ , the open ball centered at 0 of radius  $k^{-1}$ . Then we have

$$F - \tau_{\varphi_k}(F) = \int_{\mathbf{k}^{2d}} \varphi_k(X) (F - \tau_X(F)) dX,$$

which entails that

$$\mathfrak{P}_0^A(F - \tau_{\varphi_k}(F)) \leq \sup_{X \in B(0, k^{-1})} \mathfrak{P}_0^A(F - \tau_X(F)) = \sup_{X \in B(0, k^{-1})} \sup_{Y \in \mathbf{k}^{2d}} \|F(Y) - F(Y - X)\|_A,$$

which goes to zero when  $k$  goes to infinity due to the uniform continuity of  $F$ . In particular, the set of finite sums of elements of the form  $\tau_\varphi(F)$ ,  $\varphi \in \mathcal{S}(\mathbf{k}^{2d})$ ,  $F \in C_u(\mathbf{k}^{2d}, A)$  is dense in  $\mathcal{B}(\mathbf{k}^{2d}, A)$ . Since  $\mathcal{D}(\mathbf{k}^{2d})$  is dense in  $\mathcal{S}(\mathbf{k}^{2d})$ , we deduce that the set of finite sums of elements of the form  $\tau_\varphi(F)$ ,  $\varphi \in \mathcal{D}(\mathbf{k}^{2d})$ ,  $F \in C_u(\mathbf{k}^{2d}, A)$  is also dense in  $\mathcal{B}(\mathbf{k}^{2d}, A)$ . But by the extension of the Dixmier-Malliavin theorem for arbitrary locally compact groups, as stated in [21, Theorem 4.16], the former space

coincides with  $C_u^\infty(\mathbf{k}^{2d}, A)$ . Hence  $C_u^\infty(\mathbf{k}^{2d}, A)$  is a dense subspace of  $\mathcal{B}(\mathbf{k}^{2d}, A)$  but since  $C_u^\infty(\mathbf{k}^{2d}, A)$  is also dense in  $C_u(\mathbf{k}^{2d}, A)$ , we get that  $\mathcal{B}(\mathbf{k}^{2d}, A)$  is dense in  $C_u(\mathbf{k}^{2d}, A)$  too.  $\square$

**Remark 3.3.** *Observe that  $C_u^\infty(\mathbf{k}^{2d}, A) = C_u(\mathbf{k}^{2d}, A) \cap \mathcal{E}(\mathbf{k}^{2d}, A)$ . Since an element in  $\mathcal{B}(\mathbf{k}^{2d}, A)$  does not need to be locally constant, the dense inclusion  $C_u^\infty(\mathbf{k}^{2d}, A) \subset \mathcal{B}(\mathbf{k}^{2d}, A)$  is proper.*

Next, we come to the crucial fact that  $\mathcal{B}(\mathbf{k}^{2d}, A)$  is stable under point-wise multiplication, which contrary to the case of  $C_u^\infty(\mathbf{k}^{2d}, A)$ , is not obvious at all. This essentially follows from the integral representation of elements in  $\mathcal{B}(\mathbf{k}^{2d}, A)$ :

**Lemma 3.4.** *Let  $n \in \mathbb{N}$  and  $F \in \mathcal{B}(\mathbf{k}^{2d}, A)$ . Then, for all  $N \geq n + 2d + 1$ , we have the uniformly (in  $X \in \mathbf{k}^{2d}$ ) absolutely convergent integral representation:*

$$(25) \quad J^n F(X) = |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z]) \mu_0^n(Y - Z) \mu_0^{-N}(Y) \mu_0^{-N}(Z) (J^N F)(Y + X) dY dZ.$$

*Proof.* Thanks to the Peetre inequality, the integral on the right hand side of (25) is absolutely convergent in  $A$  and the convergence is uniform in  $X \in \mathbf{k}^{2d}$  as it should be. Assume first that the result is proven for  $n = 0$ . The invariance of the Haar measure by translation gives then:

$$F(X) = |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y - X, Z - X]) \mu_0^{-N}(Y - X) \mu_0^{-N}(Z - X) (J^N F)(Y) dY dZ.$$

Applying  $J^n$  on both sides when  $N \geq n + 2d + 1$ , we get from Lemma (2.2) (ii) and since  $J$  commutes with  $I$  and with the translations:

$$J^n F(X) = |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y - X, Z - X]) \mu_0^n(Y - Z) \mu_0^{-N}(Y - X) \mu_0^{-N}(Z - X) (J^N F)(Y) dY dZ,$$

which gives the result. Hence, it is enough to prove the result for  $n = 0$ . In this case, the statement is immediate for  $F \in \mathcal{S}(\mathbf{k}^{2d}, A)$ : Define  $S_F(X)$  to be the right hand side of (25) for  $n = 0$ . Since  $\mu_0^{-N} \in L^1(\mathbf{k}^{2d}) \cap L^\infty(\mathbf{k}^{2d})$ , it also belongs to  $L^2(\mathbf{k}^{2d})$  and  $S_F(X)$  can be rewritten as

$$\begin{aligned} S_F(X) &= \langle \mathcal{G}(\mu_0^{-N}), I^{-N} J^N \tau_X(F) \rangle = \langle \mathcal{G}(\mu_0^{-N}), J^N \tau_X(F) \rangle \\ &= \langle \mu_0^{-N}, \mu_0^N \mathcal{G}(\tau_X(F)) \rangle = \mathcal{G}\mathcal{G}(\tau_X(F))(0) = \tau_X(F)(0) = F(X), \end{aligned}$$

where the second equality follows by Lemma 2.2 (i), third equality follows by Plancherel and the last three are immediate. Now, the general case of  $F \in \mathcal{B}(\mathbf{k}^{2d}, A)$  follows easily by duality: Taking  $\varphi \in \mathcal{S}(\mathbf{k}^{2d}, A)$  arbitrary, one sees by Fubini that  $\langle S_F, \varphi \rangle = \langle F, \overline{S_\varphi} \rangle$  which (from the preceding case) reads  $\langle F, \varphi \rangle$ . Identifying  $\mathcal{B}(\mathbf{k}^{2d}, A)$  with a subspace of  $\mathcal{S}'(\mathbf{k}^{2d}, A)$ , we are done.  $\square$

**Remark 3.5.** *Form a slight modification of the arguments given in the above lemma, we also deduce:*

$$(26) \quad F = |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z]) \mu_0^{-2d-1}(Y) \mu_0^{-2d-1}(Z) \tau_Y(J^{2d+1} F) dY dZ, \quad \forall F \in \mathcal{B}(\mathbf{k}^{2d}, A),$$

where now the integral is absolutely convergent for all the seminorms of  $\mathcal{B}(\mathbf{k}^{2d}, A)$ .

**Corollary 3.6.**  *$\mathcal{B}(\mathbf{k}^{2d}, A)$  is a Fréchet algebra under the point-wise product. More precisely, for all  $n \in \mathbb{N}$  and all  $F_1, F_2 \in \mathcal{B}(\mathbf{k}^{2d}, A)$  we have:*

$$\mathfrak{P}_n^A(F_1 F_2) \leq \|\mu_0^{-2d-1}\|_1^4 \mathfrak{P}_{n+2d+1}^A(F_1) \mathfrak{P}_{n+2d+1}^A(F_2).$$

*Proof.* Fix  $n \in \mathbb{N}$ . Using Lemma 3.4 twice, we get for  $N = n + 2d + 1$ :

$$F_1 F_2(X) = |2|_{\mathbf{k}}^{4d} \int \overline{\Psi}(2[X, Y_1 - Z_1 + Y_2 - Z_2]) \overline{\Psi}(2[Y_1, Z_1]) \overline{\Psi}(2[Y_2, Z_2]) \\ \times \mu_0^{-N}(Y_1 - X) \mu_0^{-N}(Z_1 - X) \mu_0^{-N}(Y_2 - X) \mu_0^{-N}(Z_2 - X) (J^N F)(Y_1) (J^N F)(Y_2) dY_1 dZ_1 dY_2 dZ_2.$$

Applying  $J^n$  on both sides, we deduce since  $J$  commutes with the operator of multiplication by  $\mu_0$  and by its translates:

$$J^n(F_1 F_2)(X) = |2|_{\mathbf{k}}^{4d} \int \overline{\Psi}(2[X, Y_1 - Z_1 + Y_2 - Z_2]) \overline{\Psi}(2[Y_1, Z_1]) \overline{\Psi}(2[Y_2, Z_2]) \\ \times \mu_0^n(Y_1 - Z_1 + Y_2 - Z_2) \mu_0^{-N}(Y_1 - X) \mu_0^{-N}(Z_1 - X) \mu_0^{-N}(Y_2 - X) \mu_0^{-N}(Z_2 - X) \\ \times (J^N F)(Y_1) (J^N F)(Y_2) dY_1 dZ_1 dY_2 dZ_2. \quad (27)$$

One concludes using the Peetre inequality together with  $|2|_{\mathbf{k}} \leq 1$ .  $\square$

**Remarks 3.7.** Since  $I^n(fF) = (I^n f)F$ , for  $f \in \mathcal{S}(\mathbf{k}^{2d}, A)$  and  $F \in \mathcal{B}(\mathbf{k}^{2d}, A)$  we also get from (27)

$$\mathfrak{P}_{m,n}^A(fF) \leq \|\mu_0^{-2d-1}\|_1^4 \mathfrak{P}_{m,n+2d+1}^A(f) \mathfrak{P}_{n+2d+1}^A(F).$$

Hence  $\mathcal{S}(\mathbf{k}^{2d}, A)$  is an ideal of  $\mathcal{B}(\mathbf{k}^{2d}, A)$  for the point-wise product.

Last, we need to prove that the space  $\mathcal{B}(\mathbf{k}^{2d}, A)$  behaves well under certain dilations. For this, we need to introduce some more notations. For  $\theta \in \mathbf{k}$ , we let  $D_\theta$  be the operator of dilation by  $\theta$ :  $D_\theta F(X) := F(\theta X)$ . Also, we let  $I_\theta$  to be the operator of multiplication by  $D_\theta \mu_0$  and  $J_\theta := \mathcal{G} I_\theta \mathcal{G}$ . Note that for  $\theta = 0$ ,  $I_\theta = J_\theta = \text{Id}$ .

**Lemma 3.8.** Let  $\theta \in \mathcal{O}_{\mathbf{k}}$  and retain the notations given above.

(i) As operators on  $\mathcal{S}'(\mathbf{k}^{2d}, A)$ , we have

$$[I_\theta, J] = [I, J_\theta] = 0.$$

(ii) The operator  $J_\theta^n$ ,  $n \in \mathbb{N}$ , maps continuously  $\mathcal{B}(\mathbf{k}^{2d}, A)$  to  $C_b(\mathbf{k}^{2d}, A)$  with

$$\mathfrak{P}_0^A(J_\theta^n F) \leq \|\mu_0^{-2d-1}\|_1^2 \mathfrak{P}_{n+2d+1}^A(F).$$

(iii) We have  $J^n D_\theta = D_\theta J_\theta^n$  and consequently, the operator  $D_\theta$  is continuous on  $\mathcal{B}(\mathbf{k}^{2d}, A)$  with

$$\mathfrak{P}_n^A(D_\theta F) \leq \|\mu_0^{-2d-1}\|_1^2 \mathfrak{P}_{n+2d+1}^A(F).$$

*Proof.* (i) The vanishing of the first commutator follows because when  $\theta \in \mathcal{O}_{\mathbf{k}}$ ,  $D_\theta \mu_0$  is also invariant by translations in  $(\frac{1}{2}\mathcal{O}_{\mathbf{k}})^d \times (\frac{1}{2}\mathcal{O}_{\mathbf{k}}^o)^d$ . The vanishing of the second commutator follows from the first, after conjugation by the symplectic Fourier transform.

(ii) A minor adaptation of Lemma 3.4, which uses a minor adaptation of Lemma 2.2 (ii) and (i), entails that for  $N = n + 2d + 1$ :

$$J_\theta^n F(X) = |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z]) \mu_0^n(\theta Y - \theta Z) \mu_0^{-N}(Y) \mu_0^{-N}(Z) (J^N F)(Y + X) dY dZ.$$

The estimate then follows from the Peetre inequality together with the estimate  $\mu_0(\theta X) \leq \mu_0(X)$ , valid when  $|\theta|_{\mathbf{k}} \leq 1$ .

(iii) The equality  $J^n D_\theta = D_\theta J_\theta^n$  follows by direct computation and implies the last inequality from the one obtained in (ii).  $\square$



**Remark 3.9.** *The lemma above is false for  $\theta \in \mathbf{k} \setminus \mathcal{O}_{\mathbf{k}}$ . This is the (technical) reason why we have restricted the range of the deformation parameter to be  $\mathcal{O}_{\mathbf{k}}$ .*

**Definition 3.10.** *The space  $A_{\text{reg}}$  of regular elements in  $A$  for the action  $\alpha$  is given by:*

$$(28) \quad A_{\text{reg}} := \{a \in A : \tilde{\alpha}(a) \in \mathcal{B}(\mathbf{k}^{2d}, A)\},$$

where the map  $\tilde{\alpha} : A \rightarrow C_u(\mathbf{k}^{2d}, A)$  is described in (20).

We endow  $A_{\text{reg}}$  with the topology associated with the transported seminorms:

$$(29) \quad \|\cdot\|_n^A : A_{\text{reg}} \rightarrow \mathbb{R}_+, \quad a \mapsto \mathfrak{P}_n^A(\tilde{\alpha}(a)), \quad n \in \mathbb{N}.$$

Observe that  $A_{\text{reg}}$  depends on the action  $\alpha$ . When we need to stress this dependence, we will denote the space of regular elements by  $A_{\text{reg}}^\alpha$ .

We also need the space  $A^\infty$ , consisting in smooth vectors of  $A$  in the sense of Bruhat, as considered in [21]:

$$(30) \quad A^\infty := \{a \in A : \tilde{\alpha}(a) \in \mathcal{E}(\mathbf{k}^{2d}, A)\}.$$

Since  $\mathbf{k}^{2d}$  is totally disconnected,  $\tilde{\alpha}(a) \in \mathcal{E}(\mathbf{k}^{2d})$  if and only if it is locally constant. Hence, an element  $a \in A$  belongs to  $A^\infty$  if and only if there exists an open neighborhood  $U$  of 0 in  $\mathbf{k}^{2d}$  such that for all  $x \in U$ , we have  $\alpha_x(a) = a$ . As expected, we have:

**Proposition 3.11.**  *$A_{\text{reg}}$  is a dense and  $\alpha$ -stable Fréchet subalgebra of  $A$ . Moreover the action  $\alpha$  is isometric for each seminorm (29) and  $A^\infty \subset A_{\text{reg}}$  with a dense inclusion.*

*Proof.*  $A_{\text{reg}}$  is clearly a linear subspace of  $A$ . Moreover, by Corollary 3.6, we have for all  $a, b \in A_{\text{reg}}$ :

$$\begin{aligned} \|ab\|_n^A &= \mathfrak{P}_n^A(\tilde{\alpha}(ab)) = \mathfrak{P}_n^A(\tilde{\alpha}(a)\tilde{\alpha}(b)) \\ &\leq \|\mu_0^{-2d-1}\|_1^4 \mathfrak{P}_{n+2d+1}^A(\tilde{\alpha}(a)) \mathfrak{P}_{n+2d+1}^A(\tilde{\alpha}(b)) = \|\mu_0^{-2d-1}\|_1^4 \|a\|_{n+2d+1}^A \|b\|_{n+2d+1}^A, \end{aligned}$$

hence  $A_{\text{reg}}$  is an algebra. Let now  $X \in \mathbf{k}^{2d}$  and  $a \in A_{\text{reg}}$ . Since  $J$  commutes with translations, we have:

$$\|\alpha_X(a)\|_n^A = \mathfrak{P}_n^A(\tilde{\alpha}(\alpha_X(a))) = \mathfrak{P}_n^A(\tau_X(\tilde{\alpha}(a))) = \mathfrak{P}_n^A(\tilde{\alpha}(a)) = \|a\|_n^A.$$

Hence  $\alpha$  is isometric for each seminorm  $\|\cdot\|_n^A$  and thus  $A_{\text{reg}}$  is preserved by  $\alpha$ . The restriction to  $A_{\text{reg}}$  of the map  $\tilde{\alpha} : A \rightarrow C_u(\mathbf{k}^{2d}, A)$  identifies  $A_{\text{reg}}$  with a closed subspace of  $\mathcal{B}(\mathbf{k}^{2d}, A)$ . Since the topology of  $A_{\text{reg}}$  is inherited from those of  $\mathcal{B}(\mathbf{k}^{2d}, A)$  via this identification,  $A_{\text{reg}}$  is Fréchet. That  $A_{\text{reg}}$  is dense in  $A$  follows from an argument almost identical to those of Lemma 3.2 : by considering for every  $a \in A$  the sequence in  $A_{\text{reg}}$  given by  $\alpha_{\varphi_k}(a) := \int_{\mathbf{k}^{2d}} \varphi_k(X) \alpha_X(a) dX$ , where  $0 \leq \varphi_k \in \mathcal{S}(\mathbf{k}^{2d})$  has integral one and support within  $B(0, k^{-1})$ . Finally, that  $A^\infty$  is dense in  $A_{\text{reg}}$  follows by the Dixmier-Malliavin Theorem [21, Theorem 4.16] which shows that  $A^\infty$  coincides with the finite linear sums of elements of the form  $\alpha_\varphi(a)$ ,  $\varphi \in \mathcal{D}(\mathbf{k}^{2d})$  and  $a \in A$ , which is dense in the set of finite linear sums of elements of the form  $\alpha_\varphi(a)$ ,  $\varphi \in \mathcal{S}(\mathbf{k}^{2d})$  and  $a \in A$ .  $\square$

**3.2. The deformed product.** Our goal is to give a meaning to the formula (21) on  $A_{\text{reg}}$ . Since  $\tilde{\alpha} : A_{\text{reg}} \rightarrow \mathcal{B}(\mathbf{k}^{2d}, A)$  is a continuous (indeed isometric for each seminorm) embedding of Fréchet spaces, we will first work on  $\mathcal{B}(\mathbf{k}^{2d}, A)$  and then pull back our results to  $A_{\text{reg}}$ . Until soon, that  $A$  carries an action of  $\mathbf{k}^{2d}$  is unimportant. Let  $K(X, Y) := \overline{\Psi}(2[X, Y])$ . Seen as an element of  $\mathcal{S}'(\mathbf{k}^{2d} \times \mathbf{k}^{2d})$ , the content of Lemma 2.2 (ii) is that

$$J \otimes I^{-1}K = K \quad \text{and} \quad I^{-1} \otimes JK = K.$$

Hence, using further the commutativity of  $I$  and  $J$ , we find for all  $N \in \mathbb{N}$ :

$$K = (I^{-N} \otimes J^N)(J^N \otimes I^{-N})K = (J^N \otimes J^N)(I^{-N} \otimes I^{-N})K = J^N \otimes J^N(\mu_0^{-N} \otimes \mu_0^{-N})K.$$

In particular, for  $F \in \mathcal{S}(\mathbf{k}^{2d} \times \mathbf{k}^{2d}, A)$ , we get the equality for all  $N \in \mathbb{N}$ :

$$(31) \quad \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z])F(X, Y) dX dY = \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z]) \mu_0^{-N}(Y) \mu_0^{-N}(Z) (J^N \otimes J^N F)(X, Y) dX dY.$$

The point is that since  $\mu_0^{-N} \in L^1(\mathbf{k}^{2d})$ ,  $N \geq 2d + 1$ , the right hand side of (31) still makes sense for  $F \in \mathcal{B}(\mathbf{k}^{2d} \times \mathbf{k}^{2d}, A)$  when  $N$  is large enough. In the following, we refer to the identity (31) as the oscillatory trick.

For  $F \in \mathcal{B}(\mathbf{k}^{2d}, A)$ , we observe that the map  $\tilde{\tau}(F) := [(X, Y) \in \mathbf{k}^{2d} \times \mathbf{k}^{2d} \mapsto (\tau_X F)(Y) \in A]$ , belongs to  $\mathcal{B}(\mathbf{k}^{2d} \times \mathbf{k}^{2d}, A)$  and that

$$(32) \quad \tilde{\tau}(J^s F) = J^s \otimes \text{Id} \tilde{\tau}(F), \quad s \in \mathbb{R}.$$

The oscillatory trick (31), Lemma 3.8 (iii) and the equality (32) suggest to extend the star-product  $\star_\theta$  from  $\mathcal{S}(\mathbf{k}^{2d})$  to  $\mathcal{B}(\mathbf{k}^{2d}, A)$  as follows:

**Proposition 3.12.** *Let  $\theta \in \mathcal{O}_{\mathbf{k}}$ . Then the bilinear map*

$$(33) \quad \star_\theta : \mathcal{B}(\mathbf{k}^{2d}, A) \times \mathcal{B}(\mathbf{k}^{2d}, A) \rightarrow \mathcal{B}(\mathbf{k}^{2d}, A),$$

$$(F_1, F_2) \mapsto |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z]) \mu_0^{-2d-1}(Y) \mu_0^{-2d-1}(Z) \tau_{\theta Y}(J_\theta^{2d+1} F_1) \tau_Z(J_\theta^{2d+1} F_2) dY dZ,$$

*is continuous and associative. Moreover when  $\theta = 0$ , we have  $F_1 \star_{\theta=0} F_2 = F_1 F_2$ .*

*Proof.* For  $F_1, F_2 \in \mathcal{B}(\mathbf{k}^{2d}, A)$  and  $n \in \mathbb{N}$ , we have

$$J^n(F_1 \star_\theta F_2) = |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z]) \mu_0^{-2d-1}(Y) \mu_0^{-2d-1}(Z) J^n(\tau_{\theta Y}(J_\theta^{2d+1} F_1) \tau_Z(J_\theta^{2d+1} F_2)) dY dZ.$$

Hence we get

$$\mathfrak{P}_n^A(F_1 \star_\theta F_2) \leq \|\mu_0^{-2d-1}\|_1^2 \sup_{Y, Z \in \mathbf{k}^{2d}} \mathfrak{P}_n^A(\tau_{\theta Y}(J_\theta^{2d+1} F_1) \tau_Z(J_\theta^{2d+1} F_2)).$$

Therefore, by Corollary 3.6 and Lemma 3.8 (ii) (and the fact that  $J$  and  $J_\theta$  commute), we deduce

$$\begin{aligned}
(34) \quad \mathfrak{P}_n^A(F_1 \star_\theta F_2) &\leq \|\mu_0^{-2d-1}\|_1^6 \sup_{Y, Z \in \mathbf{k}^{2d}} \mathfrak{P}_{2d+1+n}^A(\tau_{\theta Y}(J_\theta^{2d+1} F_1)) \mathfrak{P}_{2d+1+n}^A(\tau_Z(J^{2d+1} F_2)) \\
&= \|\mu_0^{-2d-1}\|_1^6 \mathfrak{P}_{2d+1+n}^A(J_\theta^{2d+1} F_1) \mathfrak{P}_{2d+1+n}^A(J^{2d+1} F_2) \\
&= \|\mu_0^{-2d-1}\|_1^8 \mathfrak{P}_{6d+3+n}^A(F_1) \mathfrak{P}_{4d+2+n}^A(F_2),
\end{aligned}$$

which proves continuity.

Associativity is obvious when  $A = \mathbb{C}$ : it is the shadow of the associativity of the algebra of bounded operators on  $L^2(\mathbf{k}^d)$  (see [1, Théorème 3.3] from which it follows that the quantization map  $\Omega_\theta^{\mathbb{C}} : \mathcal{B}(\mathbf{k}^{2d}) \rightarrow \mathcal{B}(L^2(\mathbf{k}^d))$  is injective). It immediately implies the associativity at the level of the algebraic tensor product  $\mathcal{B}(\mathbf{k}^{2d}) \otimes A$ . We conclude by density of the former in  $\mathcal{B}(\mathbf{k}^{2d}, A)$ : Fix  $\varepsilon > 0$  and  $n \in \mathbb{N}$ . For  $F_j \in \mathcal{B}(\mathbf{k}^{2d}, A)$ , we let  $F_j^\varepsilon \in \mathcal{B}(\mathbf{k}^{2d}) \otimes A$  be such that  $\mathfrak{P}_k^A(F_j - F_j^\varepsilon) \leq \varepsilon$  for any  $j = 1, 2, 3$ , and  $k \in \{6d + 3 + n, 8d + 4 + n, 10d + 5 + n, 12d + 6 + n\}$ . Then we get

$$\begin{aligned}
&F_1 \star_\theta (F_2 \star_\theta F_3) - (F_1 \star_\theta F_2) \star_\theta F_3 = \\
&(F_1 - F_1^\varepsilon) \star_\theta (F_2 \star_\theta F_3) + F_1^\varepsilon \star_\theta ((F_2 - F_2^\varepsilon) \star_\theta F_3) + F_1^\varepsilon \star_\theta (F_2^\varepsilon \star_\theta (F_3 - F_3^\varepsilon)) \\
&- ((F_1 - F_1^\varepsilon) \star_\theta F_2) \star_\theta F_3 - (F_1^\varepsilon \star_\theta (F_2 - F_2^\varepsilon)) \star_\theta F_3 - (F_1^\varepsilon \star_\theta F_2^\varepsilon) \star_\theta (F_3 - F_3^\varepsilon),
\end{aligned}$$

and from (34):

$$\begin{aligned}
&\|\mu_0^{-2d-1}\|_1^{-16} \mathfrak{P}_n^A(F_1 \star_\theta (F_2 \star_\theta F_3) - (F_1 \star_\theta F_2) \star_\theta F_3) \\
&\leq \mathfrak{P}_{6d+3+n}^A(F_1 - F_1^\varepsilon) \mathfrak{P}_{10d+5+n}^A(F_2) \mathfrak{P}_{8d+4+n}^A(F_3) + \mathfrak{P}_{6d+3+n}^A(F_1^\varepsilon) \mathfrak{P}_{10d+5+n}^A(F_2 - F_2^\varepsilon) \mathfrak{P}_{8d+4+n}^A(F_3) \\
&+ \mathfrak{P}_{6d+3+n}^A(F_1^\varepsilon) \mathfrak{P}_{10d+5+n}^A(F_2^\varepsilon) \mathfrak{P}_{8d+4+n}^A(F_3 - F_3^\varepsilon) + \mathfrak{P}_{12d+6+n}^A(F_1 - F_1^\varepsilon) \mathfrak{P}_{10d+5+n}^A(F_2) \mathfrak{P}_{8d+4+n}^A(F_3) \\
&+ \mathfrak{P}_{12d+6+n}^A(F_1^\varepsilon) \mathfrak{P}_{10d+5+n}^A(F_2 - F_2^\varepsilon) \mathfrak{P}_{8d+4+n}^A(F_3) + \mathfrak{P}_{12d+6+n}^A(F_1^\varepsilon) \mathfrak{P}_{10d+5+n}^A(F_2^\varepsilon) \mathfrak{P}_{8d+4+n}^A(F_3 - F_3^\varepsilon) \\
&\leq \varepsilon \left( \mathfrak{P}_{10d+5+n}^A(F_2) \mathfrak{P}_{8d+4+n}^A(F_3) + \mathfrak{P}_{6d+3+n}^A(F_1^\varepsilon) \mathfrak{P}_{8d+4+n}^A(F_3) + \mathfrak{P}_{6d+3+n}^A(F_1^\varepsilon) \mathfrak{P}_{10d+5+n}^A(F_2^\varepsilon) \right. \\
&\left. + \mathfrak{P}_{10d+5+n}^A(F_2) \mathfrak{P}_{8d+4+n}^A(F_3) + \mathfrak{P}_{12d+6+n}^A(F_1^\varepsilon) \mathfrak{P}_{8d+4+n}^A(F_3) + \mathfrak{P}_{12d+6+n}^A(F_1^\varepsilon) \mathfrak{P}_{10d+5+n}^A(F_2^\varepsilon) \right).
\end{aligned}$$

Using last  $\mathfrak{P}_k^A(F_j^\varepsilon) \leq \varepsilon + \mathfrak{P}_k^A(F_j)$  ( $j = 1, 2, 3$ ,  $k \in \{6d + 3 + n, 8d + 4 + n, 10d + 5 + n, 12d + 6 + n\}$ ), we deduce that for all  $n \in \mathbb{N}$ ,  $F_1 \star_\theta (F_2 \star_\theta F_3) - (F_1 \star_\theta F_2) \star_\theta F_3$  can be rendered as small as one wishes in the seminorms  $\mathfrak{P}_n^A$ , hence this associator vanishes.

The fact that the deformed product coincides with the point-wise product when  $\theta = 0$  follows directly from Lemma 3.4.  $\square$

**Remark 3.13.** Obviously, we have

$$F_1 \star_\theta F_2 = |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z]) \mu_0^{-N}(Y) \mu_0^{-N}(Z) \tau_{\theta Y}(J_\theta^N F_1) \tau_Z(J^N F_2) dY dZ,$$

for any  $N \in \mathbb{N}$  such that  $N \geq 2d + 1$ . Using moreover the commutation of  $J$  with translations, we also deduce the point-wise expression:

$$F_1 \star_\theta F_2(X) = |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z]) \mu_0^{-N}(Y) \mu_0^{-N}(Z) (J_\theta^N F_1)(X + \theta Y) (J^N F_2)(X + Z) dY dZ.$$

Last, when  $F_1, F_2 \in \mathcal{S}(\mathbf{k}^{2d}, A)$ , we can undo the oscillatory trick to get:

$$F_1 \star_\theta F_2(X) = |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z]) F_1(X + \theta Y) F_2(X + Z) dY dZ.$$

The following representation of the product  $\star_\theta$  will be useful to handle the deformed product in a rather simple way.

**Lemma 3.14.** *Let  $\theta \in \mathcal{O}_{\mathbf{k}}$ . For  $F_1, F_2 \in \mathcal{B}(\mathbf{k}^{2d}, A)$  and  $N \in \mathbb{N}$ , set*

$$F_N := |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z]) \tau_{\theta Y}(F_1) \tau_Z(F_2) e^{-\mu_0(Y)\mu_0(Z)/N} dY dZ.$$

Then, the sequence  $(F_N)_{N \in \mathbb{N}}$  belongs to  $\mathcal{B}(\mathbf{k}^{2d}, A)$  and converges to  $F_1 \star_\theta F_2$  for the topology of  $\mathcal{B}(\mathbf{k}^{2d}, A)$ .

*Proof.* That  $F_N$ ,  $N \in \mathbb{N}$ , belongs to  $\mathcal{B}(\mathbf{k}^{2d}, A)$  follows from arguments almost identical to those given in the first part of the proof of Proposition 3.12. Next, using the oscillatory trick together with the commutativity of  $I$  and  $J$ , we get

$$F_N = |2|_{\mathbf{k}}^{2d} \int \overline{\Psi}(2[Y, Z]) \mu_0^{-2d-2}(Y) \mu_0^{-2d-2}(Z) \tau_{\theta Y}(J_\theta^{2d+2} F_1) \tau_Z(J^{2d+2} F_2) e^{-\mu_0(Y)\mu_0(Z)/N} dY dZ,$$

and thus (using Remark 3.13)

$$F_1 \star_\theta F_2 - F_N =$$

$$|2|_{\mathbf{k}}^{2d} \int \overline{\Psi}(2[Y, Z]) \mu_0^{-2d-2}(Y) \mu_0^{-2d-2}(Z) \tau_{\theta Y}(J_\theta^{2d+2} F_1) \tau_Z(J^{2d+2} F_2) \left(1 - e^{-\mu_0(Y)\mu_0(Z)/N}\right) dY dZ.$$

Using Corollary 3.6 and Lemma 3.8 (ii), we then deduce

$$\mathfrak{P}_n^A(F_1 \star_\theta F_2 - F_N) \leq \|\mu_0^{-2d-1}\|_1^8 \mathfrak{P}_{n+6d+4}^A(F_1) \mathfrak{P}_{n+4d+3}^A(F_2) \sup_{Y, Z \in \mathbf{k}^{2d}} \frac{1 - e^{-\mu_0(Y)\mu_0(Z)/N}}{\mu_0(Y)\mu_0(Z)}.$$

Observing then that

$$\sup_{Y, Z \in \mathbf{k}^{2d}} \frac{1 - e^{-\mu_0(Y)\mu_0(Z)/N}}{\mu_0(Y)\mu_0(Z)} \leq \sup_{x>0} \frac{1 - e^{-x/N}}{x} \leq \frac{1}{N},$$

we get the result.  $\square$

We also note:

**Lemma 3.15.** *Let  $\theta \in \mathcal{O}_{\mathbf{k}}$ . Then,  $(\mathcal{S}(\mathbf{k}^{2d}, A), \star_\theta)$  is an ideal of  $(\mathcal{B}(\mathbf{k}^{2d}, A), \star_\theta)$ .*

*Proof.* Let  $f \in \mathcal{S}(\mathbf{k}^{2d}, A)$ ,  $F \in \mathcal{B}(\mathbf{k}^{2d}, A)$  and  $n, m \in \mathbb{N}$ . For  $M, N$  arbitrary integers satisfying  $M, N \geq 2d + 1$ . By Remark 3.13 we have

$$\mathfrak{P}_{m,n}^A(f \star_\theta F) \leq \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \mu_0(Y)^{-N} \mu_0(Z)^{-M} \mathfrak{P}_{m,n}^A(\tau_{\theta Y}(J_\theta^M f) \tau_Z(J^N F)) dY dZ.$$

Using Corollary 3.6 and Lemma 3.8 again and the Peetre inequality, we deduce

$$\begin{aligned} \mathfrak{P}_{m,n}^A(\tau_{\theta Y}(J_\theta^M f) \tau_Z(J^N F)) &= \mathfrak{P}_n^A(\mu_0^m \tau_{\theta Y}(J_\theta^M f) \tau_Z(J^N F)) \\ &\leq \|\mu_0^{-2d-1}\|_1^4 \mathfrak{P}_{2d+1+n}^A(\mu_0^m \tau_{\theta Y}(J_\theta^M f)) \mathfrak{P}_{2d+1+n}^A(\tau_Z(J^N F)) \\ &\leq \|\mu_0^{-2d-1}\|_1^6 \mu_0^m(Y) \mathfrak{P}_{m,4d+2+n+M}^A(f) \mathfrak{P}_{2d+1+n+N}^A(F). \end{aligned}$$

Choosing  $M = 2d + 1$  and  $N = 2d + 1 + m$ , we deduce

$$\mathfrak{P}_{m,n}^A(f \star_\theta F) \leq \|\mu_0^{-2d-1}\|_1^8 \mathfrak{P}_{m,6d+3+n}^A(f) \mathfrak{P}_{4d+2+m+n}^A(F).$$

The case of  $F \star_\theta f$  is similar.  $\square$

**Lemma 3.16.** *Let  $\theta \in \mathcal{O}_{\mathbf{k}}$ . With  $*$  the involution of  $A$ , we set  $F^*(X) := F(X)^*$ . Then we have  $(F_1 \star_\theta F_2)^* = F_2^* \star_\theta F_1^*$  for all  $F_1, F_2 \in \mathcal{B}(\mathbf{k}^{2d}, A)$ .*

*Proof.* Observe that the involution defined on  $\mathcal{B}(\mathbf{k}^{2d}, A)$  is continuous and commutes with  $J$  and  $\tau$ . Therefore, we get from Lemma 3.14

$$\begin{aligned} (F_1 \star_\theta F_2)^* &= \lim_{N \rightarrow \infty} |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \Psi(2[Y, Z]) e^{-\mu_0(Y)\mu_0(Z)/N} (\tau_{\theta Y}(F_1) \tau_Z(F_2))^* dY dZ \\ &= \lim_{N \rightarrow \infty} |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z]) e^{-\mu_0(\theta Y)\mu_0(\theta^{-1}Z)/N} \tau_{\theta Y}(F_2^*) \tau_Z(F_1^*) dY dZ. \end{aligned}$$

But from the same reasoning that the one given in Lemma 3.14, and using Lemma 3.8 (i) for the commutativity of  $I_\theta$  and  $J$ , one sees that the expression above is exactly  $F_2^* \star_\theta F_1^*$ .  $\square$

**Lemma 3.17.** *Let  $\theta \in \mathcal{O}_{\mathbf{k}}$ . The action of  $\mathbf{k}^{2d}$  by translation on  $\mathcal{B}(\mathbf{k}^{2d}, A)$  is still an automorphism for the deformed product  $\star_\theta$ .*

*Proof.* This follows from the defining relation (33) of  $\star_\theta$  on  $\mathcal{B}(\mathbf{k}^{2d}, A)$  together with the fact that  $\tau$  is continuous and commutes with  $J$  on  $\mathcal{B}(\mathbf{k}^{2d}, A)$ .  $\square$

**Definition 3.18.** *The deformed product of the Fréchet algebra  $A_{\text{reg}}$  is given by the map:*

$$\star_\theta^\alpha : A_{\text{reg}} \times A_{\text{reg}} \rightarrow A, \quad (a, b) \mapsto \tilde{\alpha}(a) \star_\theta \tilde{\alpha}(b)(0),$$

which by Remark 3.13 can be rewritten as:

$$(35) \quad a \star_\theta^\alpha b = |2|_{\mathbf{k}}^{2d} \int \overline{\Psi}(2[X, Y]) \mu_0^{-2d-1}(X) \mu_0^{-2d-1}(Y) (J_\theta^{2d+1} \tilde{\alpha}(a))(\theta X) (J_\theta^{2d+1} \tilde{\alpha}(b))(Y) dX dY.$$

We arrive to our first main result.

**Theorem 3.19.** *Let  $\mathbf{k}$  be a non-Archimedean local field of characteristic different from 2 and  $\theta \in \mathcal{O}_{\mathbf{k}}$ . Let also  $A$  be a  $C^*$ -algebra endowed with a continuous action  $\alpha$  of  $\mathbf{k}^{2d}$ . Then (keeping the notations displayed above)  $(A_{\text{reg}}, \star_\theta^\alpha)$  is an associative Fréchet algebra that we call the Fréchet deformation of the  $C^*$ -algebra  $A$ . Moreover, the original action  $\alpha$  is still by automorphisms and the original involution is still an involution.*

*Proof.* By construction, the action  $\alpha$  yields an isometric embedding of  $A_{\text{reg}}$  in  $\mathcal{B}(\mathbf{k}^{2d}, A)$ , Proposition 3.12 entails then that  $\star_\theta : \mathcal{B}(\mathbf{k}^{2d}, A) \times \mathcal{B}(\mathbf{k}^{2d}, A) \rightarrow \mathcal{B}(\mathbf{k}^{2d}, A)$  continuously. Note also that the evaluation map  $\mathcal{B}(\mathbf{k}^{2d}, A) \rightarrow A$ ,  $F \mapsto F(0)$  is continuous too. Hence,  $\star_\theta^\alpha : A_{\text{reg}} \times A_{\text{reg}} \rightarrow A$  is continuous and from the inequality (34), we deduce

$$\|a \star_\theta^\alpha b\|_A \leq \|\mu_0^{-2d-1}\|_1^2 \|a\|_{2d+1}^A \|b\|_{2d+1}^A.$$

Next, we need to show that the map  $\star_\theta^\alpha$  takes values in  $A_{\text{reg}}$  (and not only in  $A$ ). To show this, let  $a \in A_{\text{reg}}$  and  $X, Y \in \mathbf{k}^{2d}$ . Observe first that  $\tau_X \circ \tilde{\alpha}(a) = \tilde{\alpha}(\alpha_X(a))$ . Consider then the action  $\hat{\alpha}$  of  $\mathbf{k}^{2d}$  on  $\mathcal{B}(\mathbf{k}^{2d}, A)$  given by  $(\hat{\alpha}_X(F))(Y) = \alpha_X(F(Y))$ . Then we have  $\hat{\alpha}_X(\tilde{\alpha}(a)) = \tau_X(\tilde{\alpha}(a))$ . Since  $\tau$

commutes with  $J$ , we therefore get  $\hat{\alpha}_X(J^n \tilde{\alpha}(a)) = \tau_X(J^n \tilde{\alpha}(a))$ , from which we easily deduce by (35) that the map  $\tilde{\alpha}$  intertwines  $\star_\theta^\alpha$  and  $\star_\theta$ :

$$(36) \quad \tilde{\alpha}(a \star_\theta^\alpha b) = \tilde{\alpha}(a) \star_\theta \tilde{\alpha}(b), \quad \forall a, b \in A_{\text{reg}}.$$

Eq. (36) immediately implies that  $\star_\theta^\alpha$  takes values in  $A_{\text{reg}}$ . Moreover, it also implies the associativity of  $\star_\theta^\alpha$  on  $A_{\text{reg}}$  from the associativity of  $\star_\theta$  on  $\mathcal{B}(\mathbf{k}^{2d}, A)$ :

$$(a \star_\theta^\alpha b) \star_\theta^\alpha c = \tilde{\alpha}(a \star_\theta^\alpha b) \star_\theta \tilde{\alpha}(c)(0) = (\tilde{\alpha}(a) \star_\theta \tilde{\alpha}(b)) \star_\theta \tilde{\alpha}(c)(0), \quad \forall a, b, c \in A_{\text{reg}}.$$

Observe then that (36) (together with Lemma 3.17) also implies that the action  $\alpha$  on  $A_{\text{reg}}$  is still by automorphism of the deformed product:

$$\begin{aligned} \alpha_X(a \star_\theta^\alpha b) &= \tilde{\alpha}(a) \star_\theta \tilde{\alpha}(b)(X) = \tau_X(\tilde{\alpha}(a) \star_\theta \tilde{\alpha}(b))(0) = (\tau_X \tilde{\alpha}(a) \star_\theta \tau_X \tilde{\alpha}(b))(0) \\ &= \tilde{\alpha}(\alpha_X(a)) \star_\theta \tilde{\alpha}(\alpha_X(b))(0) = \alpha_X(a) \star_\theta^\alpha \alpha_X(b). \end{aligned}$$

Last, that the original involution is still an involution follows from Lemma 3.16.  $\square$

**Remark 3.20.** *Theorem 3.19 can be extended in two directions. Firstly, if  $\mathbf{k}$  is of characteristic 2, then all the statements of this section (including Theorem 3.19) continue to hold true provided we redefine the function  $\mu_0$  in (3), the symplectic Fourier transform  $\mathcal{G}$  in (5) and the deformed product  $\star_\theta$  in (33) without the factor 2. However, and as indicated earlier, we then lose the contact with the pseudo-differential calculus that we will intensively use in the next section in order to construct a  $C^*$ -norm on the deformed Fréchet algebra  $(A_{\text{reg}}, \star_\theta^\alpha)$ . Secondly, it is not difficult to extend Theorem 3.19 in the case of a Fréchet algebra  $\mathcal{A}$  (instead of a  $C^*$ -algebra  $A$ ). If the topology of  $\mathcal{A}$  comes from a countable set of seminorms  $\{\|\cdot\|_j\}_{j \in \mathbb{N}}$ , then we only need to require that it carries a continuous action  $\alpha$  of  $\mathbf{k}^{2d}$  which is tempered in the sense that for all  $j \in \mathbb{N}$ , there exist  $C > 0$  and  $k, n \in \mathbb{N}$  such that for all  $a \in \mathcal{A}$ ,  $\|\alpha_X(a)\|_j \leq C \mu_0(X)^n \|a\|_k$ . But again, to construct a deformed  $C^*$ -norm we have to restrict ourselves to  $C^*$ -algebras and isometric actions.*

**Remark 3.21.** *By equivariance of the deformed product and from the discussion which follows (30), we deduce that  $A^\infty$  is also stable under  $\star_\theta^\alpha$ . However it is not clear if we have continuity for the topology of  $A^\infty$ .*

#### 4. THE $C^*$ -DEFORMATION OF A $C^*$ -ALGEBRA

**4.1. The Wigner functions approach.** In this section, we assume that  $\mathbf{k}$  is of characteristic different from 2 and that  $\theta \in \mathcal{O}_{\mathbf{k}} \setminus \{0\}$ . Also, we identify our  $C^*$ -algebra  $A$  with a subalgebra of  $\mathcal{B}(\mathcal{H})$  for a separable Hilbert space  $\mathcal{H}$ .

By analogy with the integral representation (12), we may define for  $f \in L^1(\mathbf{k}^{2d}, A)$ :

$$(37) \quad \Omega_\theta^A(f) := \left| \frac{2}{\theta} \right|_{\mathbf{k}}^d \int_{\mathbf{k}^{2d}} \Omega_\theta(X) \otimes f(X) dX.$$

The map  $\Omega_\theta^A$  sends continuously  $L^1(\mathbf{k}^{2d}, A)$  to  $\mathcal{B}(L^2(\mathbf{k}^d) \otimes \mathcal{H})$ . Indeed, since

$$\|\Omega_\theta(X)\|_{\mathcal{B}(L^2(\mathbf{k}^d))} = \|U_\theta(X) \Sigma U_\theta(X)^*\|_{\mathcal{B}(L^2(\mathbf{k}^d))} = 1, \quad \forall X \in \mathbb{K}^{2d},$$



we get

$$(38) \quad \begin{aligned} \|\Omega_\theta^A(f)\|_{\mathcal{B}(L^2(\mathbf{k}^d) \otimes \mathcal{H})} &\leq |\theta|_{\mathbf{k}}^{-d} \int_{\mathbf{k}^{2d}} \|\Omega_\theta(X) \otimes f(X)\|_{\mathcal{B}(L^2(\mathbf{k}^d) \otimes \mathcal{H})} dX \\ &= |\theta|_{\mathbf{k}}^{-d} \int_{\mathbf{k}^{2d}} \|f(X)\|_A dX = |\theta|_{\mathbf{k}}^{-d} \|f\|_1. \end{aligned}$$

Since moreover  $\Omega_\theta(X)$  is selfadjoint, we get  $\Omega_\theta^A(f)^* = \Omega_\theta^A(f^*)$ , where  $f^* \in L^1(\mathbf{k}^{2d}, A)$  is defined by  $f^*(X) := f(X)^*$ . There is an obvious reason to introduce the map  $\Omega_\theta^A$ :

**Lemma 4.1.** *The map  $\Omega_\theta^A : (\mathcal{S}(\mathbf{k}^{2d}, A), \star_\theta) \rightarrow \mathcal{B}(L^2(\mathbf{k}^d) \otimes \mathcal{H})$  is a continuous  $*$ -homomorphism.*

*Proof.* That  $\Omega_\theta^A$  is continuous and involution preserving has already been proved. That  $\Omega_\theta^A$  is a homomorphism when  $A = \mathbb{C}$  follows by construction of the  $\star_\theta$ . Hence,  $\Omega_\theta^A$  is still a homomorphism at the level of the algebraic tensor product  $\mathcal{S}(\mathbf{k}^{2d}) \otimes A$ . For  $j = 1, 2$ , take  $f_j \in \mathcal{S}(\mathbf{k}^{2d}, A)$  and choose  $(f_{j,k})_{k \in \mathbb{N}} \subset \mathcal{S}(\mathbf{k}^{2d}) \otimes A$  converging to  $f_j$  in the topology of  $\mathcal{S}(\mathbf{k}^{2d}, A)$ . Then, we have in  $\mathcal{B}(L^2(\mathbf{k}^d) \otimes \mathcal{H})$ :

$$\begin{aligned} \|\Omega_\theta^A(f_1)\Omega_\theta^A(f_2) - \Omega_\theta^A(f_1 \star_\theta f_2)\| &\leq \|\Omega_\theta^A(f_1 - f_{1,k})\Omega_\theta^A(f_2)\| + \|\Omega_\theta^A(f_{1,k})\Omega_\theta^A(f_2 - f_{2,k})\| \\ &\quad + \|\Omega_\theta^A((f_1 - f_{1,k}) \star_\theta f_2)\| + \|\Omega_\theta^A(f_{1,k} \star_\theta (f_2 - f_{2,k}))\|, \end{aligned}$$

which by the estimates (34), (38) and  $\|f\|_1 \leq \|\mu_0^{-2d-1}\|_1 \mathfrak{P}_{2d+1,0}^A(f)$ , may be rendered as small as wished by choosing  $k \in \mathbb{N}$  large enough. Hence  $\|\Omega_\theta^A(f_1)\Omega_\theta^A(f_2) - \Omega_\theta^A(f_1 \star_\theta f_2)\| = 0$  and thus  $\Omega_\theta^A(f_1)\Omega_\theta^A(f_2) = \Omega_\theta^A(f_1 \star_\theta f_2)$ .  $\square$

Let now  $\eta$  be the characteristic function of  $\mathcal{O}_{\mathbf{k}}^d$ , normalized by  $\|\eta\|_2 = 1$  and let also  $W_{X,Y}^\theta$  the Wigner function associated with the pair  $(\eta_X^\theta, \eta_Y^\theta)$  of coherent states as in (19). For  $F \in \mathcal{B}(\mathbf{k}^{2d}, A)$ , we can then define the following  $A$ -valued function on  $\mathbf{k}^{2d} \times \mathbf{k}^{2d}$ :

$$(39) \quad W_{X,Y}^{\theta,A}(F) := \left| \frac{2}{\theta} \right|_{\mathbf{k}}^d \int_{\mathbf{k}^{2d}} W_{X,Y}^\theta(Z) F(Z) dZ.$$

**Lemma 4.2.** *Let  $F \in \mathcal{B}(\mathbf{k}^{2d}, A)$ . Then for all  $X, Y \in \mathbf{k}^{2d}$  and all  $n \in \mathbb{N}$ , we have:*

$$\|W_{X,Y}^{\theta,A}(F)\|_A \leq \mu_0^{-n} \left( \frac{1}{2\theta} (X - Y) \right) \mathfrak{P}_n^A(F).$$

*If moreover  $f \in \mathcal{S}(\mathbf{k}^{2d}, A)$  then for all  $m, n \in \mathbb{N}$  we have*

$$\|W_{X,Y}^{\theta,A}(f)\|_A \leq \mu_0^{-n} \left( \frac{1}{2} (X + Y) \right) \mu_0^{-m} \left( \frac{1}{2\theta} (X - Y) \right) \mathfrak{P}_{n,m}^A(f).$$

*Proof.* Note that if  $F \in \mathcal{B}(\mathbf{k}^{2d}, A)$ , we have

$$W_{X,Y}^{\theta,A}(F) = \left| \frac{2}{\theta} \right|_{\mathbf{k}}^d \int_{\mathbf{k}^{2d}} (J^{-n} W_{X,Y}^\theta)(Z) (J^n f)(Z) dZ, \quad \forall n \in \mathbb{N},$$

and if  $f \in \mathcal{S}(\mathbf{k}^{2d}, A)$ , we have

$$W_{X,Y}^{\theta,A}(f) = \left| \frac{2}{\theta} \right|_{\mathbf{k}}^d \int_{\mathbf{k}^{2d}} (I^{-n} J^{-m} W_{X,Y}^\theta)(Z) (I^n J^m f)(Z) dZ, \quad \forall m, n \in \mathbb{N}.$$

The result follows immediately from Lemma 2.4.  $\square$

Now, for  $\phi, \psi \in L^2(\mathbf{k}^d)$ , we denote by  $|\phi\rangle\langle\psi|$ , the rank one operator  $L^2(\mathbf{k}^d) \rightarrow L^2(\mathbf{k}^d)$ ,  $\rho \mapsto \langle\psi, \rho\rangle\phi$ .

**Definition 4.3.** For  $F \in \mathcal{B}(\mathbf{k}^{2d}, A)$ , define in the weak sense in  $L^2(\mathbf{k}^d) \otimes \mathcal{H}$ :

$$\mathbf{W}_\theta^A(F) := |\theta|_{\mathbf{k}}^{-2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} |\eta_Y^\theta \rangle \langle \eta_X^\theta| \otimes W_{X,Y}^{\theta,A}(F) dX dY.$$

The following may be thought as a variant of the Calderon-Vaillancourt Theorem for our  $A$ -valued Weyl pseudo-differential calculus on local fields. This result for  $A = \mathbb{C}$  is due to Bechata [1].

**Proposition 4.4.** The quadratic form associated with the weak integral operator  $\mathbf{W}_\theta^A(F)$  defines an element of  $\mathcal{B}(L^2(\mathbf{k}^d) \otimes \mathcal{H})$ , with

$$\|\mathbf{W}_\theta^A(F)\|_{\mathcal{B}(L^2(\mathbf{k}^d) \otimes \mathcal{H})} \leq \|\mu_0^{-2d-1}\|_1 \mathfrak{P}_{2d+1}^A(F).$$

Moreover, we have  $\mathbf{W}_\theta^A(F)^* = \mathbf{W}_\theta^A(F^*)$ , where  $F^* \in \mathcal{B}(\mathbf{k}^{2d}, A)$  is defined by  $F^*(X) := F(X)^*$ .

*Proof.* For  $\Phi \in L^2(\mathbf{k}^d) \otimes \mathcal{H}$  and  $\varphi \in L^2(\mathbf{k}^d)$  we denote by  $\langle \varphi, \Phi \rangle_{L^2(\mathbf{k}^d)}$  the vector in  $\mathcal{H}$  defined by  $\langle \langle \varphi, \Phi \rangle_{L^2(\mathbf{k}^d)}, \rho \rangle_{\mathcal{H}} := \langle \Phi, \varphi \otimes \rho \rangle_{L^2(\mathbf{k}^d) \otimes \mathcal{H}}$  for every  $\rho \in \mathcal{H}$ . With this in mind and with  $\eta$  the characteristic function of  $(\mathcal{O}_{\mathbf{k}})^d$  (normalized by  $\|\eta\|_2 = 1$ ), it is not difficult to see that the resolution of the identity (17) on  $L^2(\mathbf{k}^d)$  entails:

$$(40) \quad \|\Phi\|_{L^2(\mathbf{k}^d) \otimes \mathcal{H}}^2 = |\theta|_{\mathbf{k}}^{-d} \int_{\mathbf{k}^{2d}} \|\langle \eta_X^\theta, \Phi \rangle_{L^2(\mathbf{k}^d)}\|_{\mathcal{H}}^2 dX, \quad \forall \Phi \in L^2(\mathbf{k}^d) \otimes \mathcal{H}.$$

Take now  $\Phi_1, \Phi_2 \in L^2(\mathbf{k}^d) \otimes \mathcal{H}$ . We therefore get

$$|\langle \Phi_1, \mathbf{W}_\theta^A(F) \Phi_2 \rangle_{L^2(\mathbf{k}^d) \otimes \mathcal{H}}| \leq |\theta|_{\mathbf{k}}^{-3d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \|\langle \eta_Y^\theta, \Phi_1 \rangle_{L^2(\mathbf{k}^d)}\|_{\mathcal{H}} \|\langle \eta_X^\theta, \Phi_2 \rangle_{L^2(\mathbf{k}^d)}\|_{\mathcal{H}} \|W_{X,Y}^A(F)\| dX dY.$$

By the Cauchy-Schwarz inequality and (40), we deduce that the integral above is bounded by

$$|\theta|_{\mathbf{k}}^{-2d} \|\Phi_1\|_{L^2(\mathbf{k}^d) \otimes \mathcal{H}} \|\Phi_2\|_{L^2(\mathbf{k}^d) \otimes \mathcal{H}} \left( \sup_{X \in \mathbf{k}^{2d}} \int_{\mathbf{k}^{2d}} \|W_{X,Y}^{\theta,A}(F)\|_A dY \right)^{1/2} \left( \sup_{Y \in \mathbf{k}^{2d}} \int_{\mathbf{k}^{2d}} \|W_{X,Y}^{\theta,A}(F)\|_A dX \right)^{1/2},$$

Hence, by Lemma 4.2 we get

$$|\langle \Phi_1, \mathbf{W}_\theta^{\theta,A}(F) \Phi_2 \rangle_{L^2(\mathbf{k}^d) \otimes \mathcal{H}}| \leq \|\Phi_1\|_{L^2(\mathbf{k}^d) \otimes \mathcal{H}} \|\Phi_2\|_{L^2(\mathbf{k}^d) \otimes \mathcal{H}} \|\mu_0^{-2d-1}\|_1 \mathfrak{P}_{2d+1}^A(F),$$

which completes the proof.  $\square$

**Remark 4.5.** Observe that the bound of the norm of  $\mathbf{W}_\theta^{\theta,A}(F)$  we have obtained, is independent of parameter  $\theta$ .

**Corollary 4.6.** The map  $\mathbf{W}_\theta^A : (\mathcal{B}(\mathbf{k}^{2d}, A), \star_\theta) \rightarrow \mathcal{B}(L^2(\mathbf{k}^d) \otimes \mathcal{H})$  is a continuous  $*$ -homomorphism which extends  $\Omega_\theta^A : (\mathcal{S}(\mathbf{k}^{2d}, A), \star_\theta) \rightarrow \mathcal{B}(L^2(\mathbf{k}^d) \otimes \mathcal{H})$ .

*Proof.* By Proposition 4.4,  $\mathbf{W}_\theta^A$  is continuous and involution preserving. When  $A = \mathbb{C}$ , the relations  $\mathbf{W}_\theta^A(F_1) \mathbf{W}_\theta^A(F_2) = \mathbf{W}_\theta^A(F_1 \star_\theta F_2)$  and  $\Omega_\theta^A(f) = \mathbf{W}_\theta^A(f)$ , for  $f \in \mathcal{S}(\mathbf{k}^{2d})$  and  $F, F_1, F_2 \in \mathcal{B}(\mathbf{k}^{2d})$  are implicit in the work of Bechata [1] (they are almost tautological). Obviously, these relations are still valid at the level of algebraic tensor products. The general case follows from the same methods as those used in Lemma 4.1, using the estimates (34), (38) and Proposition 4.4.  $\square$

**Remark 4.7.** Since  $\mathcal{S}(\mathbf{k}^{2d}, A)$  is an ideal in  $\mathcal{B}(\mathbf{k}^{2d}, A)$ , we deduce from Corollary 4.6 that  $\Omega_\theta^A(f \star_\theta F) = \Omega_\theta^A(f) \mathbf{W}_\theta^A(F)$  whenever  $f \in \mathcal{S}(\mathbf{k}^{2d}, A)$  and  $F \in \mathcal{B}(\mathbf{k}^{2d}, A)$ .

We are now able to state the main result of this section, whose proof is an immediate consequence of Corollary 4.6.

**Theorem 4.8.** *Let  $\mathbf{k}$  be a non-Archimedean local field of characteristic different from 2, let  $\theta \in \mathcal{O}_{\mathbf{k}} \setminus \{0\}$  and let  $A \subset \mathcal{B}(\mathcal{H})$  be a  $C^*$ -algebra endowed with a continuous action  $\alpha$  of  $\mathbf{k}^{2d}$ . Then (keeping the notations displayed above) the norm:*

$$A_{\text{reg}} \rightarrow \mathbb{R}^+, \quad a \mapsto \|a\|_{\theta} := \|\mathbf{W}_{\theta}^A(\tilde{\alpha}(a))\|_{\mathcal{B}(L^2(\mathbf{k}^d) \otimes \mathcal{H})},$$

*endows  $(A_{\text{reg}}, \star_{\theta}^{\alpha}, *)$  with the structure of a pre- $C^*$ -algebra. We call its completion the  $C^*$ -deformation of the  $C^*$ -algebra  $A$  and denote it by  $A_{\theta}^{\alpha}$  (or by  $A_{\theta}$  when no confusion can occur).*

We first mention an important feature, namely that the deformed algebra  $A_{\theta}$  still carries a continuous action of  $\mathbf{k}^{2d}$ . We stress that this property heavily relies on the fact that our group is Abelian. For non-Abelian groups (e.g. [2, 23]), the only surviving action is the one of a quantum group.

**Proposition 4.9.** *The action  $\alpha$  of  $\mathbf{k}^{2d}$  on  $(A_{\text{reg}}, \star_{\theta}^{\alpha})$  extends to a continuous action on  $A_{\theta}$ , that we denote by  $\alpha_{\theta}$ .*

*Proof.* Once we will have shown that  $\alpha$  gives a continuous action of  $\mathbf{k}^{2d}$  on the Fréchet algebra  $(A_{\text{reg}}, \star_{\theta}^{\alpha})$ , the existence of the extension  $\alpha_{\theta}$  can be easily proven following the lines of [27, Proposition 5.11]. That  $\alpha_{\theta}$  is continuous on  $A_{\text{reg}}$  follows from the fact that the isometric embedding  $A_{\text{reg}} \rightarrow \mathcal{B}(\mathbf{k}^{2d}, A)$  intertwines  $\alpha$  with  $\tau$  and that  $\tau$  is continuous on  $\mathcal{B}(\mathbf{k}^{2d}, A)$  as the latter is a subspace of  $C_u(\mathbf{k}^{2d}, A)$ . That it is by automorphism on  $A_{\theta}$  follows from Lemma 3.17.  $\square$

**4.2. The  $C^*$ -module approach.** We now realize the deformed  $C^*$ -norm  $\|\cdot\|_{\theta}$  as the  $C^*$ -norm of bounded adjointable endomorphisms of a  $C^*$ -module for  $A$ , in a manner very similar to the  $\mathbb{R}^d$ -case [27]. However, this construction cannot substitute the previous one since lattice methods used in [27] are not available here. In fact, we are in the same situation than those of negatively curved Kählerian Lie groups [2].

Let  $\langle \cdot, \cdot \rangle_A$  be the  $A$ -valued sesquilinear pairing on  $\mathcal{S}(\mathbf{k}^{2d}, A)$  given by

$$\langle f_1, f_2 \rangle_A := \int_{\mathbf{k}^{2d}} f_1^*(X) f_2(X) dX.$$

This paring is clearly well defined. Testing this paring on elementary tensors, we deduce (by the density of products in a  $C^*$ -algebra) that  $\langle \mathcal{S}(\mathbf{k}^{2d}, A), \mathcal{S}(\mathbf{k}^{2d}, A) \rangle_A$  is dense in  $A$ . It is manifestly positive since  $\langle f, f \rangle_A = \int_{\mathbf{k}^{2d}} |f(X)|^2 dX \geq 0$  and  $\langle f_1, f_2 \rangle_A^* = \langle f_2, f_1 \rangle_A$ . If we endow further  $\mathcal{S}(\mathbf{k}^{2d}, A)$  with the right action of the undeformed  $C^*$ -algebra  $A$  given by juxtaposition:  $f \cdot a := [X \mapsto f(X)a]$ , then we get  $\langle f_1, f_2 a \rangle_A = \langle f_1, f_2 \rangle_A a$ . Hence,  $\mathcal{S}(\mathbf{k}^{2d})$  becomes a right pre- $C^*$ -module for the undeformed  $C^*$ -algebra  $A$ . Now, by Lemma 3.15, we know that  $(\mathcal{B}(\mathbf{k}^{2d}, A), \star_{\theta})$  acts continuously on  $\mathcal{S}(\mathbf{k}^{2d}, A)$  by

$$L_{\theta}(F) : \mathcal{S}(\mathbf{k}^{2d}, A) \rightarrow \mathcal{S}(\mathbf{k}^{2d}, A), \quad f \mapsto F \star_{\theta} f.$$

This action clearly commutes with the right action of  $A$ . That  $L_{\theta}(F)$  is also adjointable and bounded follows from the following alternative expression for the paring:

**Lemma 4.10.** *For  $f_1, f_2 \in \mathcal{S}(\mathbf{k}^{2d}, A)$  and  $\theta \in \mathcal{O}_{\mathbf{k}} \setminus \{0\}$ , we have:*

$$\langle f_1, f_2 \rangle_A = \int_{\mathbf{k}^{2d}} \langle \eta_X^{\theta}, \Omega_{\theta}^A(f_1^* \star_{\theta} f_2) \eta_X^{\theta} \rangle_{L^2(\mathbf{k}^d)} dX.$$

*Proof.* By polarization, we may assume without loss of generality that  $f_1 = f_2$ . Let  $\langle \cdot, \cdot \rangle'_A$  be the  $A$ -valued pairing given by the right hand side of the equality we have to prove. Let us first show that it is well defined. Note that for  $f \in \mathcal{S}(\mathbf{k}^{2d}, A)$ , we have

$$\langle \eta_X^\theta, \Omega_\theta^A(f^* \star_\theta f) \eta_X^\theta \rangle_{L^2(\mathbf{k}^d)} = \langle \eta_X^\theta, \left( \left| \frac{2}{\theta} \right|_{\mathbf{k}}^d \int_{\mathbf{k}^{2d}} f^* \star_\theta f(Y) \otimes \Omega_\theta(Y) dY \right) \eta_X^\theta \rangle_{L^2(\mathbf{k}^d)}.$$

Since the integral converges in the norm of  $\mathcal{B}(L^2(\mathbf{k}^d) \otimes \mathcal{H})$ , we get

$$\begin{aligned} \langle \eta_X^\theta, \Omega_\theta^A(f^* \star_\theta f) \eta_X^\theta \rangle_{L^2(\mathbf{k}^d)} &= \left| \frac{2}{\theta} \right|_{\mathbf{k}}^d \int_{\mathbf{k}^{2d}} f^* \star_\theta f(Y) \langle \eta_X^\theta, \Omega_\theta(Y) \eta_X^\theta \rangle_{L^2(\mathbf{k}^d)} dY \\ &= \left| \frac{2}{\theta} \right|_{\mathbf{k}}^d \int_{\mathbf{k}^{2d}} f^* \star_\theta f(Y) W_{X,X}^\theta(Y) dY = W_{X,X}^{\theta,A}(f^* \star_\theta f). \end{aligned}$$

We conclude using Lemma 4.2 which gives in that case

$$\|W_{X,X}^{\theta,A}(f^* \star_\theta f)\|_A \leq \mu_0(X)^{-2d-1} \mathfrak{P}_{2d+1,0}^A(f^* \star_\theta f),$$

and thus

$$\|\langle f, f \rangle'_A\| \leq \|\mu_0^{-2d-1}\|_1 \mathfrak{P}_{2d+1,0}^A(f^* \star_\theta f) < \infty.$$

From this inequality, we also deduce that it is enough to treat the case  $A = \mathbb{C}$ . Indeed, if the equality works on  $\mathcal{S}(\mathbf{k}^{2d})$  then it works on the algebraic tensor product  $\mathcal{S}(\mathbf{k}^{2d}) \otimes A$  and one concludes using a limiting argument based on  $\|\langle f, f \rangle_A\| \leq \|f\|_2^2$  and on the inequality given above for  $\|\langle f, f \rangle'_A\|$ .

In the case  $A = \mathbb{C}$ , note first that by unitarity of the quantization map, we have

$$\langle f, f \rangle_{\mathbb{C}} = |\theta|_{\mathbf{k}}^d \operatorname{Tr}(\Omega_\theta(f)^* \Omega_\theta(f)) = |\theta|_{\mathbf{k}}^d \operatorname{Tr}(\Omega_\theta(f^* \star_\theta f)).$$

The resolution of the identity (17), implies that for a positive trace-class operator  $S$  on  $L^2(\mathbf{k}^d)$ , we have  $\operatorname{Tr}(S) = |\theta|_{\mathbf{k}}^{-d} \int_{\mathbf{k}^{2d}} \langle \eta_X^\theta, S \eta_X^\theta \rangle dX$ . Indeed, since  $\|\eta\|_2 = 1$ , we get by (17) that for all  $\varphi \in L^2(\mathbf{k}^d)$ ,  $\langle \varphi, \varphi \rangle = |\theta|_{\mathbf{k}}^{-d} \int \langle \eta_X^\theta, \varphi \rangle \langle \varphi, \eta_X^\theta \rangle dX$ . Hence, for any orthonormal basis  $(\varphi_k)_{k \in \mathbb{N}}$ , using monotone convergence and  $\sum_{k \in \mathbb{N}} |\varphi_k\rangle \langle \varphi_k| = \operatorname{Id}$  in the weak sense, we get

$$\begin{aligned} \operatorname{Tr}(S) &= \sum_{k \in \mathbb{N}} \langle \varphi_k, S \varphi_k \rangle_{L^2(\mathbf{k}^d)} = \sum_{k \in \mathbb{N}} \langle S^{1/2} \varphi_k, S^{1/2} \varphi_k \rangle_{L^2(\mathbf{k}^d)} \\ &= |\theta|_{\mathbf{k}}^{-d} \sum_{k \in \mathbb{N}} \int_{\mathbf{k}^{2d}} \langle S^{1/2} \varphi_k, \eta_X^\theta \rangle_{L^2(\mathbf{k}^d)} \langle \eta_X^\theta, S^{1/2} \varphi_k \rangle_{L^2(\mathbf{k}^d)} dX \\ &= |\theta|_{\mathbf{k}}^{-d} \int_{\mathbf{k}^{2d}} \sum_{k \in \mathbb{N}} \langle S^{1/2} \eta_X^\theta, \varphi_k \rangle_{L^2(\mathbf{k}^d)} \langle \varphi_k, S^{1/2} \eta_X^\theta \rangle_{L^2(\mathbf{k}^d)} dX \\ &= |\theta|_{\mathbf{k}}^{-d} \int_{\mathbf{k}^{2d}} \langle \eta_X^\theta, S \eta_X^\theta \rangle_{L^2(\mathbf{k}^d)} dX, \end{aligned}$$

which completes the proof.  $\square$

From the expression of  $\langle \cdot, \cdot \rangle'_A$  given in Lemma 4.10, it is clear that the operator  $L_\theta(F)$ ,  $F \in \mathcal{B}(\mathbf{k}^{2d}, A)$ , is adjointable with adjoint  $L_\theta(F^*)$ . But the elementary operator inequality on  $\mathcal{B}(L^2(\mathbf{k}^d) \otimes \mathcal{H})$ :

$$\Omega_\theta^A(f^* \star_\theta F^* \star_\theta F \star_\theta f) = \Omega_\theta^A(f^*) |\mathbf{W}_\theta^A(F)|^2 \Omega_\theta^A(f) \leq \|\mathbf{W}_\theta^A(F)\|^2 \Omega_\theta^A(f^* \star_\theta f),$$

entails that  $\langle L_\theta(F)f, L_\theta(F)f \rangle_A \leq \|\mathbf{W}_\theta^A(F)\|^2 \langle f, f \rangle_A$ . Hence for  $F \in \mathcal{B}(\mathbf{k}^{2d}, A)$ ,  $L_\theta(F)$  is bounded adjointable  $A$ -linear endomorphism of  $\mathcal{S}(\mathbf{k}^{2d}, A)$ , with  $\|L_\theta(F)\| \leq \|\mathbf{W}_\theta^A(F)\|$ . In fact this inequality

is an equality. Indeed, by construction, the restriction to the algebraic tensor product  $\mathcal{B}(\mathbf{k}^{2d}) \otimes A$  of the deformed  $C^*$ -norm  $F \mapsto \|\mathbf{W}_\theta^A(F)\|$  coincides with the minimal  $C^*$ -norm on the algebraic tensor product of the  $C^*$ -completion of  $(\mathcal{B}(\mathbf{k}^{2d}), \star_\theta)$  by  $A$ . But the restriction to the algebraic tensor product  $\mathcal{B}(\mathbf{k}^{2d}) \otimes A$  of the  $C^*$ -norm  $F \mapsto \|L_\theta(F)\|$  extends to a  $C^*$ -cross norm on the  $C^*$ -completion of  $(\mathcal{B}(\mathbf{k}^{2d}), \star_\theta)$  by  $A$ . Hence, the two norms coincide. Restricting this to the image of  $A_{\text{reg}}$  in  $\mathcal{B}(\mathbf{k}^{2d}, A)$ , we deduce:

**Proposition 4.11.** *Let  $\theta \in \mathcal{O}_{\mathbf{k}} \setminus \{0\}$ . Then the deformed  $C^*$ -norm  $\|\cdot\|_\theta$  on the Fréchet  $*$ -algebra  $(A_{\text{reg}}, \star_\theta^\alpha, *)$  coincides with:*

$$A_{\text{reg}} \rightarrow \mathbb{R}^+, \quad a \mapsto \|L_\theta(\tilde{\alpha}(a))\|.$$

The main point with the realization of the deformed  $C^*$ -norm as the operator norm on the pre- $C^*$ -module  $\mathcal{S}(\mathbf{k}^{2d}, A)$  is that it still makes sense for the value  $\theta = 0$ , where there is no pseudo-differential calculus. Indeed, when  $\theta = 0$  the product on  $A_{\text{reg}}$  is the undeformed one (by Proposition 3.12) and thus  $A_{\text{reg}}$  acts on the left of  $\mathcal{S}(\mathbf{k}^{2d}, A)$  via  $f \mapsto \tilde{\alpha}(a) f$ . Since moreover

$$\|L_{\theta=0}(\tilde{\alpha}(a))\| = \sup_{X \in \mathbf{k}^{2d}} \|\alpha_X(a)\|_A = \|a\|_A, \quad \forall a \in A_{\text{reg}},$$

we deduce that  $A_{\theta=0} = A$ .

As an illustration of the interest of the  $C^*$ -module approach to the deformation, we clarify the relations between the deformations of  $C_0(\mathbf{k}^{2d}, A)$  and of  $C_u(\mathbf{k}^{2d}, A)$  for the action given by translations on the one hand and the  $C^*$ -closures of the Fréchet algebras  $(\mathcal{S}(\mathbf{k}^{2d}, A), \star_\theta)$  and of  $(\mathcal{B}(\mathbf{k}^{2d}, A), \star_\theta)$  induced by the representations  $\Omega_\theta^A$  and  $\mathbf{W}_\theta^A$  on the other hand.

**Proposition 4.12.** *Let  $\theta \in \mathcal{O}_{\mathbf{k}}$ . Consider the  $C^*$ -algebras  $C_0(\mathbf{k}^{2d}, A)$  and  $C_u(\mathbf{k}^{2d}, A)$  endowed with the action of  $\mathbf{k}^{2d}$  given by  $\tau \otimes \text{Id}$  where as usual  $\tau$  is the action by translations. Then we have the isomorphisms:*

$$C_0(\mathbf{k}^{2d}, A)_\theta \simeq \overline{(\mathcal{S}(\mathbf{k}^{2d}, A), \star_\theta)} \quad \text{and} \quad C_u(\mathbf{k}^{2d}, A)_\theta \simeq \overline{(\mathcal{B}(\mathbf{k}^{2d}, A), \star_\theta)}.$$

Moreover, when  $\theta \in \mathcal{O}_{\mathbf{k}} \setminus \{0\}$ , then

$$C_0(\mathbf{k}^{2d}, A)_\theta \simeq \mathcal{K}(L^2(\mathbf{k}^d)) \otimes A.$$

*Proof.* When  $A = \mathbb{C}$  and  $\theta \neq 0$ , the quantization map  $\Omega_\theta$  is a (multiple of a) unitary operator from  $L^2(\mathbf{k}^{2d})$  to the Hilbert-Schmidt operators on  $L^2(\mathbf{k}^d)$ . Since  $\mathcal{S}(\mathbf{k}^{2d})$  is densely contained in  $L^2(\mathbf{k}^{2d})$ , and since the Hilbert-Schmidt operators are norm-dense in the compacts, we get after completion  $\overline{(\mathcal{S}(\mathbf{k}^{2d}), \star_\theta)} \simeq \mathcal{K}(L^2(\mathbf{k}^d))$ . The associated isomorphism with  $A$  arbitrary then follows by nuclearity of the compact operators. The first two isomorphisms can be proven exactly as in [27, Proposition 4.15], by observing that  $\mathcal{B}(\mathbf{k}^{2d}, A) = C_u(\mathbf{k}^{2d}, A)_{\text{reg}}$  and that  $\mathcal{S}(\mathbf{k}^{2d}, A) \subset C_0(\mathbf{k}^{2d}, A)_{\text{reg}}$  densely, and using the  $C^*$ -module picture for the deformed  $C^*$ -norm.  $\square$

In the case of actions of  $\mathbb{R}^{2d}$ , Rieffel proved in [29] that the  $K$ -theory is an invariant of the deformation. In [2], we also proved the same result for actions of negatively curved Kählerian group. From the isomorphisms given in Proposition 4.12, we easily deduce this property no longer holds here:

**Corollary 4.13.** *The  $K$ -theory is not an invariant of the deformation.*

*Proof.* We give a counter example. Take  $A = C_0(\mathbf{k}^{2d})$ . As  $\mathbf{k}$  is a totally disconnected space,  $K_0(A) = C_c(\mathbf{k}^{2d}, \mathbb{Z})$ . On the other hand, Proposition 4.12 says that the deformation of  $A$  by the regular action is the  $C^*$ -algebra of compact operators. Hence  $K_0(A_\theta) = \mathbb{Z}$ . (Note that in this example the  $K_1$ -group is not deformed as it is trivial in both cases.)  $\square$

**4.3. The twisted crossed product approach.** There is a third way to realize the deformed  $C^*$ -norm on the deformed Fréchet algebra  $(A_{\text{reg}}, \star_\theta^\alpha)$ , which is based on the work of Kasprzak [14] and its development by Neshveyev et al. [6, 23, 22]. Kasprzak's original construction uses general results on crossed product and the notion of Landstad algebras. It applies to continuous actions of a locally compact Abelian groups on  $C^*$ -algebras and is parametrized by a continuous unitary 2-cocycle on the dual group. In fact, the deformed algebra  $A_\theta^K$  in Kasprzak's picture is abstractly characterized by a crossed product bi-decomposition  $\mathbf{k}^{2d} \rtimes_{\alpha_\theta} A_\theta^K = \mathbf{k}^{2d} \rtimes_\alpha A$ , where  $\alpha_\theta$  is the extension of  $\alpha$  from  $A_{\text{reg}}$  to  $A_\theta$ , as described in Proposition 4.9. An equivalent and more concrete approach (see below) has been given by Bhowmick, Neshveyev and Sangha in [6], which applies to continuous actions of locally compact groups (not necessarily Abelian) on  $C^*$ -algebras and is parametrized by a measurable unitary 2-cocycle on the dual, viewed as a quantum group. This approach to deformation had been extended in full generality in [23] to continuous actions of locally compact quantum groups (in the von Neumann algebraic setting) on  $C^*$ -algebras and is still parametrized by a measurable unitary 2-cocycle on the dual quantum group. Here we mostly follow the paper [22], we let  $\theta \in \mathcal{O}_{\mathbf{k}} \setminus \{0\}$  and we still assume that  $\mathbf{k}$  is of characteristic different from 2.

For  $X \in \mathbf{k}^{2d}$ , let  $V_X^\theta$  be the unitary operator on  $L^2(\mathbf{k}^{2d})$  given by

$$V_X^\theta f(Y) := \overline{\Psi}\left(\frac{2}{\theta}[X, Y]\right) f(X + Y).$$

The operators  $(V_X^\theta)_{X \in \mathbf{k}^{2d}}$  satisfies Weyl type relations  $V_{X+Y}^\theta = \overline{\Psi}\left(\frac{2}{\theta}[X, Y]\right) V_X^\theta V_Y^\theta$ . The  $C^*$ -sub-algebra of  $\mathcal{B}(L^2(\mathbf{k}^{2d}))$  generated by the operators

$$V_f^\theta := \int_{\mathbf{k}^{2d}} f(X) V_X^\theta dX, \quad f \in L^1(\mathbf{k}^{2d}),$$

is called the twisted group  $C^*$ -algebra and is denoted by  $C_\theta^*(\mathbf{k}^{2d})$ . For  $f_1, f_2 \in L^1(\mathbf{k}^{2d})$ , we have  $V_{f_1}^\theta V_{f_2}^\theta = V_{f_1 *_\theta f_2}^\theta$ , where  $*_\theta$  is the twisted convolution product, defined by

$$f_1 *_\theta f_2(X) := \int_{\mathbf{k}^{2d}} f_1(X - Y) f_2(Y) \overline{\Psi}\left(\frac{2}{\theta}[Y, X]\right) dY.$$

Moreover, we have for  $f_1, f_2 \in \mathcal{S}(\mathbf{k}^{2d})$ :

$$\mathcal{G}_\theta(f_1 *_\theta f_2) = \mathcal{G}_\theta(f_1) *_\theta \mathcal{G}_\theta(f_2),$$

where  $\mathcal{G}_\theta$  denotes the rescaled version of the symplectic Fourier transform:

$$(\mathcal{G}_\theta f)(X) := \left|\frac{2}{\theta}\right|^d \int_{\mathbf{k}^{2d}} \Psi\left(\frac{2}{\theta}[Y, X]\right) f(Y) dY.$$

Hence  $C_\theta^*(\mathbf{k}^{2d})$  is isomorphic to  $C_0(\mathbf{k}^{2d})_\theta$  thus (by Proposition 4.12) isomorphic to the  $C^*$ -algebra of compact operators. However, this does not implies directly that the 2-cocycle  $(X, Y) \mapsto \Psi\left(\frac{2}{\theta}[Y, X]\right)$  on the selfdual group  $\mathbf{k}^{2d}$  is regular in the sense of [23, Definition 2.9]. Since here the modular involution  $\hat{J}$  is the complex conjugation, regularity here means that  $C_\theta^*(\mathbf{k}^{2d})C_0(\mathbf{k}^{2d}) \subset \mathcal{K}(L^2(\mathbf{k}^{2d}))$ . But this is a trivial fact here since the operator kernel of  $V_{f_1}^\theta M_{f_2}$  ( $M_f$  stands for the operator of point-wise



multiplication by  $f$ ), for  $f_1, f_2 \in \mathcal{S}(\mathbf{k}^{2d})$  is given by  $\overline{\Psi}(2[X, Y])f_1(X - Y)f_2(Y)$  and so  $V_{f_1}^\theta M_{f_2}$  is Hilbert-Schmidt thus compact, and one concludes by density.

Following [22], we introduce the operator on  $\mathcal{S}(\mathbf{k}^{2d}, A_{\text{reg}})$  given by:

$$(41) \quad \Pi_\theta f(X) := |2|_{\mathbf{k}}^d \int_{\mathbf{k}^{2d}} \overline{\Psi}(2[X, Y]) \alpha_{\theta Y}(\mathcal{G}(f)(Y)) dY, \quad \theta \in \mathcal{O}_{\mathbf{k}}.$$

This operator is continuous. Indeed,  $\Pi_\theta = \mathcal{G} \circ S_\theta \circ \mathcal{G}$ , where  $S_\theta f(X) := \alpha_{\theta X}(f(X))$ , which is continuous on  $\mathcal{S}(\mathbf{k}^{2d}, A_{\text{reg}})$  (essentially by Lemma 3.8). Note however that  $S_\theta$  does not need to be continuous on  $\mathcal{S}(\mathbf{k}^{2d}, A)$ . Since  $S_\theta$  is invertible with inverse given by  $S_{-\theta}$ ,  $\Pi_\theta$  is also invertible with inverse  $\Pi_{-\theta}$ . Defining the faithful representations  $\pi$  and  $\pi_\theta$  of the crossed products  $\mathbf{k}^{2d} \rtimes_\alpha A$  and  $\mathbf{k}^{2d} \rtimes_{\alpha_\theta} A_\theta$  on the Hilbert module  $L^2(\mathbf{k}^{2d}, A)$  given for  $f, \xi \in \mathcal{S}(\mathbf{k}^{2d}, A_{\text{reg}})$  by:

$$\pi(f)\xi := \int_{\mathbf{k}^{2d}} \tilde{\alpha}(f(X)) (\tau_X \xi) dX, \quad \pi_\theta(f)\xi := \int_{\mathbf{k}^{2d}} \tilde{\alpha}(f(X)) \star_\theta (\tau_X \xi) dX.$$

The first main result of [22] is that the (Archimedean version of the) map  $\Pi_\theta$  extends to an isomorphism of crossed products. This is the most important step to prove equivalence between Rieffel's and Kasprzak's approaches to deformation. A quick inspection shows that the proofs of [22, Theorems 1.1 & 2.1] extend to our context without modification, this yields:

**Proposition 4.14.** *For  $f \in \mathcal{S}(\mathbf{k}^{2d}, A_{\text{reg}})$ , we have  $\pi_\theta(f) = \pi(\Pi_\theta(f))$ . Moreover,  $\Pi_\theta$  extends to an isomorphism of crossed products  $\mathbf{k}^{2d} \rtimes_{\alpha_\theta} A_\theta \simeq \mathbf{k}^{2d} \rtimes_\alpha A$ .*

From this, one deduces exactly as [14, Theorem 3.10] that  $A_\theta$  is nuclear if and only if  $A$  is nuclear. Mimicking the arguments of [28, Theorem 3.2] (see also [2, Corollary 7.49]), one can also prove that our deformed  $C^*$ -algebra  $A_\theta$  is strongly Morita equivalent to the crossed product

$$\mathbf{k}^{2d} \rtimes_{\text{Ad}(U_\theta) \otimes \alpha} (\mathcal{K}(L^2(\mathbf{k}^d)) \otimes A),$$

where  $U_\theta$  is the projective unitary irreducible representation of  $\mathbf{k}^{2d}$  on  $L^2(\mathbf{k}^d)$  given in (11). This gives an alternative proof of the property of preservation of nuclearity, which is how Rieffel proved the analogous result for actions of  $\mathbb{R}^{2d}$  [28, Theorem 4.1]. Note that Proposition 4.14 together with the Stone-von Neumann Theorem (see e.g. [25, Theorem C.34]) also implies that the deformed  $C^*$ -algebra is stably isomorphic to a double crossed product of the undeformed  $C^*$ -algebra (see [23, Theorem 3.6] for a more general statement in the context of regular cocycles for locally compact quantum groups):

$$\mathcal{K} \otimes A_\theta \simeq \mathbf{k}^{2d} \rtimes_{\beta_\theta} (\mathbf{k}^{2d} \rtimes_\alpha A),$$

where  $\beta_\theta$  is the image under  $\Pi_\theta$  of the action dual to  $\alpha_\theta$  of the selfdual group  $\mathbf{k}^{2d}$  on the crossed product  $\mathbf{k}^{2d} \rtimes_{\alpha_\theta} A_\theta$  (which is not the action dual to  $\alpha$  on  $\mathbf{k}^{2d} \rtimes_\alpha A$ ).

The deformed  $C^*$ -algebra constructed in [6] is based on the following ‘‘quantization maps’’:

$$T_{\theta, \nu} : C_0(\mathbf{k}^{2d}) \rightarrow C_\theta^*(\mathbf{k}^{2d}), \quad f \mapsto \text{Id} \otimes \nu(W \overline{\Psi}_\theta(\mathcal{G}_\theta f \mathcal{G}_\theta \otimes \text{id}) \Psi_\theta W^*).$$

Here the function  $f$  is viewed as an operator of multiplication on  $L^2(\mathbf{k}^{2d})$ ,  $\Psi_\theta$  is the operator of multiplication by the function  $[(X, Y) \mapsto \Psi(\frac{2}{\theta}[X, Y])]$  on  $L^2(\mathbf{k}^{2d} \times \mathbf{k}^{2d})$  and  $W$  is the multiplicative unitary on  $L^2(\mathbf{k}^{2d} \times \mathbf{k}^{2d})$  given by  $W\xi(X, Y) = \xi(X, Y - X)$ . Last,  $\nu$  is an element of the predual of the von Neumann algebra generated by  $C_\theta^*(\mathbf{k}^{2d})$  in  $L^2(\mathbf{k}^{2d})$ . Given that  $C_\theta^*(\mathbf{k}^{2d}) \simeq \mathcal{K}(L^2(\mathbf{k}^{2d}))$ ,  $\nu$  is

of the form  $\text{Tr}(A)$ , for  $A$  of trace-class on  $L^2(\mathbf{k}^{2d})$ . For  $f \in \mathcal{S}(\mathbf{k}^{2d})$ , a computation shows that (up to a constant)

$$T_{\theta,\nu}(f) = \int_{\mathbf{k}^{2d}} (\mathcal{G}_\theta f)(X) \overline{\nu(V_{-X}^\theta)} V_{-X}^\theta dX.$$

From this, we can easily show that the union of the images of the maps  $T_{\theta,\nu}$  is dense in  $C_\theta^*(\mathbf{k}^{2d})$  (see also [23, Lemma 3.2] for a more general statement). To simplify the discussion we assume that 2 is invertible in  $\mathcal{O}_\mathbf{k}$  (if not, the formulas are slightly different but the conclusion is unchanged). Let  $\gamma \in \mathbf{k}$  such that  $|\gamma|_\mathbf{k} > 1$ , define for  $n \in \mathbb{Z}$ ,  $\varphi_n := \chi_{\theta\gamma^n(\mathcal{O}_\mathbf{k} \times \mathcal{O}_\mathbf{k}^\circ)^d} \in \mathcal{S}(\mathbf{k}^{2d})$  and consider the element  $\nu_n := |\frac{\gamma^n}{\theta}|_\mathbf{k}^{2d} \langle \varphi_n, \cdot \varphi_{-n} \rangle$ . Then we have

$$\nu_n(V_{-X}^\theta) = |\frac{\gamma^n}{\theta}|_\mathbf{k}^{2d} \langle \varphi_n, V_{-X}^\theta \varphi_{-n} \rangle = |\frac{\gamma^{2n}}{\theta}|_\mathbf{k}^d \mathcal{G}_\theta(\varphi_n \tau_{-X}(\varphi_{-n}))(X).$$

An explicit computation then shows that  $\nu_n(V_{-X}^\theta) = \varphi_n(X)$  and thus

$$T_{\theta,\nu_n}(f) = \int_{\gamma^{n\theta}(\mathcal{O}_\mathbf{k} \times \mathcal{O}_\mathbf{k}^\circ)^d} (\mathcal{G}_\theta f)(X) V_{-X}^\theta dX,$$

which, by dominated convergence, converges to  $V_{\mathcal{G}_\theta(f)}^\theta$  when  $n \rightarrow \infty$ . One concludes using the fact that  $\mathcal{G}_\theta$  is an automorphism of  $\mathcal{S}(\mathbf{k}^{2d})$  (indeed  $\mathcal{G}_\theta$  is an involution).

The crucial observation in [6] is that for any  $C^*$ -algebra  $A$ , the map  $T_{\theta,\nu}$  extends to a map

$$T_{\theta,\nu} : M(C_0(\mathbf{k}^{2d}) \otimes A) \rightarrow M(C_\theta^*(\mathbf{k}^{2d}) \otimes A),$$

which is continuous on the unit ball for the strict topology. Composing this map with  $\tilde{\alpha} : A \rightarrow C_u(\mathbf{k}^{2d}, A) \subset M(C_0(\mathbf{k}^{2d}, A))$ , we get a family of maps

$$A \rightarrow M(C_\theta^*(\mathbf{k}^{2d}) \otimes A), \quad a \mapsto \text{Id} \otimes \nu(W \overline{\Psi}_\theta(\mathcal{G}_\theta \tilde{\alpha}(a) \mathcal{G}_\theta \otimes \text{id}) \Psi_\theta W^*).$$

By definition, the deformation of  $A$  in the sense [6] is the sub- $C^*$ -algebra  $A_\theta^{BNS}$  of  $M(C_0(\mathbf{k}^{2d}, A))$  generated by the images of the maps  $T_{\theta,\nu} \circ \tilde{\alpha}$ . The action  $\hat{\alpha}_\theta := [Y \mapsto \text{Ad}(\mathcal{G}_\theta \tau_Y \mathcal{G}_\theta)]$  of  $\mathbf{k}^{2d}$  on  $A_\theta^{BNS}$  induces a representation of the crossed product  $\mathbf{k}^{2d} \rtimes_{\hat{\alpha}_\theta} A_\theta^{BNS}$  on the Hilbert module  $L^2(\mathbf{k}^{2d}, A)$ . By [23, Theorem 3.9] (see also the detailed discussion in [22, pages 4-5]) we have  $\mathbf{k}^{2d} \rtimes_{\hat{\alpha}_\theta} A_\theta^{BNS} = \mathcal{G}_\theta(\mathbf{k}^{2d} \rtimes_\alpha A) \mathcal{G}_\theta$ . Hence,  $\mathcal{G}_\theta \hat{\alpha}_\theta(A_\theta^{BNS}) \mathcal{G}_\theta \subset M(\mathbf{k}^{2d} \rtimes_\alpha A)$ . Consider last the extension of the map (41) to the multipliers of crossed products:

$$\Pi_\theta : M(\mathbf{k}^{2d} \rtimes_{\alpha_\theta} A_\theta) \rightarrow M(\mathbf{k}^{2d} \rtimes_\alpha A).$$

The proof of [22, Theorem 2.3] (which is mainly based on general crossed product arguments) extends straightforwardly to our context and gives a third way to realize our deformed  $C^*$ -algebra:

**Theorem 4.15.** *The map  $\Pi_\theta$  establishes an isomorphism of  $A_\theta \simeq \tilde{\alpha}_\theta(A_\theta) \subset M(\mathbf{k}^{2d} \rtimes_{\alpha_\theta} A_\theta)$  with  $A_\theta^{BNS} \simeq \mathcal{G}_\theta \hat{\alpha}_\theta(A_\theta^{BNS}) \mathcal{G}_\theta \subset M(\mathbf{k}^{2d} \rtimes_\alpha A)$ .*

## 5. PROPERTIES OF THE DEFORMATION

In this final section we always assume that the characteristic of  $\mathbf{k}$  is different from 2 but otherwise specified, the deformation parameter  $\theta$  can be freely chosen in  $\mathcal{O}_\mathbf{k}$  (i.e. the value  $\theta = 0$  is also allowed). Our aim is to show that most of the structural properties of the deformation survive in the non-Archimedean context. In order to give the shortest possible proofs, we take advantages of the three different ways to realize our deformed  $C^*$ -algebra: as a subalgebra of  $\mathcal{B}(L^2(\mathbf{k}^d)) \otimes_{\min} A$  as initially

defined (see subsection 4.1), as a subalgebra of bounded adjointable  $A$ -linear endomorphisms of the Hilbert module  $L^2(\mathbf{k}^{2d}, A)$  (see subsection 4.2) or as a subalgebra of  $M(C_\theta^*(\mathbf{k}^{2d}) \otimes A)$  (see subsection 4.3). But it is important to mention that all the results that use the twisted group algebra approach can be alternatively proven by methods similar to those developed in [27, 28].

We first study the question of approximate unit. This result is a very important technical tool in numerous forthcoming statements. Here, we have to substantially modify Rieffel's original arguments and thus we provide a rather detailed proof (this is mostly due to the fact that the operator  $J$  does not satisfy any kind of Leibniz rule).

**Proposition 5.1.** *The deformed  $C^*$ -algebra  $A_\theta$  possesses an approximate unit (in the sense of [24]) consisting of elements of  $A_{\text{reg}}$ .*

*Proof.* Let  $\{e'_\lambda\}_{\lambda \in \Lambda}$  be a net of approximate unit for  $A$  ( $e'_\lambda \in A_+$ ,  $\|e'_\lambda\|_A \leq 1$ ,  $\lim_\lambda \|a - e'_\lambda a\|_A = 0$ ,  $\lim_\lambda \|a - ae'_\lambda\|_A = 0$  for all  $a \in A$ ) and let  $0 \leq \varphi \in \mathcal{S}(\mathbf{k}^{2d})$  with compact support and with  $\int \varphi = 1$ . Define then

$$e_\lambda := \alpha_\varphi(e'_\lambda) = \int_{\mathbf{k}^{2d}} \varphi(X) \alpha_X(e'_\lambda) dX \in A_{\text{reg}}.$$

Since for all  $a \in A$ , we have

$$\|a - e_\lambda a\|_A \leq \int_{\mathbf{k}^{2d}} \varphi(X) \|a - \alpha_X(e'_\lambda) a\|_A dX = \int_{\mathbf{k}^{2d}} \varphi(X) \|\alpha_{-X}(a) - e'_\lambda \alpha_{-X}(a)\|_A dX,$$

and similarly for  $\|a - ae_\lambda\|$ , a compactness argument over the support of  $\varphi$  entails that  $\{e_\lambda\}_{\lambda \in \Lambda}$  is an approximate unit for the undeformed  $C^*$ -algebra  $A$  consisting of elements of regular elements.

We are going to prove that  $\{e_\lambda\}_{\lambda \in \Lambda}$  is also an approximate unit for the deformed  $C^*$ -algebra  $A_\theta$ . Since  $A_{\text{reg}}$  is dense in  $A_\theta$ , it suffices to prove that  $\|a - a \star_\theta^\alpha e_\lambda\|_\theta$  and  $\|a - e_\lambda \star_\theta^\alpha a\|_\theta$  go to zero for all  $a \in A_{\text{reg}}$ . By Proposition 4.4  $\|\cdot\|_\theta \leq C\|\cdot\|_{2d+1}$  (on  $A_{\text{reg}}$ ) and thus it suffices to show that  $\mathfrak{B}_{2d+1}^A(\tilde{\alpha}(a - a \star_\theta^\alpha e_\lambda))$  and  $\mathfrak{B}_{2d+1}^A(\tilde{\alpha}(a - e_\lambda \star_\theta^\alpha a))$  go to zero for all  $a \in A_{\text{reg}}$ . For this, note that for  $F_1, F_2 \in \mathcal{B}(\mathbf{k}^{2d}, A)$ , we have

$$J^n(F_1 \star_\theta F_2) = |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z]) \mu_0^{-2d-1}(Y) \mu_0^{-2d-1}(Z) J^n(\tau_{\theta Y}(J_\theta^{2d+1} F_1) \tau_Z(J^{2d+1} F_2)) dY dZ.$$

Using the integral formula (27) applied to  $J^n(\tau_{\theta Y}(J_\theta^{2d+1} F_1) \tau_Z(J^{2d+1} F_2))$ , we deduce by commutativity of  $J$  and  $\tau$  and with  $N = 2d + 1 + n$ :

$$\begin{aligned} J^n(F_1 \star_\theta F_2)(X) &= |2|_{\mathbf{k}}^{6d} \int \overline{\Psi}(2[Y, Z]) \overline{\Psi}(2[X, Y_1 - Z_1 + Y_2 - Z_2]) \overline{\Psi}(2[Y_1, Z_1]) \overline{\Psi}(2[Y_2, Z_2]) \\ &\times \mu_0^{-2d-1}(Y) \mu_0^{-2d-1}(Z) \mu_0^N(Y_1 - Z_1 + Y_2 - Z_2) \mu_0^{-N}(Y_1 - X) \mu_0^{-N}(Z_1 - X) \mu_0^{-N}(Y_2 - X) \mu_0^{-N}(Z_2 - X) \\ &\times (J^N J_\theta^{2d+1} F_1)(Y_1 + \theta Y) (J^{N+2d+1} F_2)(Y_2 + Z) dY dZ dY_1 dZ_1 dY_2 dZ_2. \end{aligned}$$

On the other hand, by Lemma 3.4, we have for all  $F \in \mathcal{B}(\mathbf{k}^{2d}, A)$ :

$$J^n F = |2|_{\mathbf{k}}^{2d} \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[Y, Z]) \mu_0^{-2d-1}(Y) \mu_0^{-2d-1}(Z) J^n(\tau_{\theta Y}(J_\theta^{2d+1} F)) dY dZ.$$

But the integral formula (27) generalizes when  $F_1 \in \mathcal{B}(\mathbf{k}^{2d}, A)$  and  $F_2 \in \mathcal{B}(\mathbf{k}^{2d}, M(A))$ . Applying it for  $F_2 = 1$ , one deduces

$$\begin{aligned} J^n F(X) &= |2|_{\mathbf{k}}^{6d} \int \overline{\Psi}(2[Y, Z]) \overline{\Psi}(2[X, Y_1 - Z_1 + Y_2 - Z_2]) \overline{\Psi}(2[Y_1, Z_1]) \overline{\Psi}(2[Y_2, Z_2]) \\ &\times \mu_0^{-2d-1}(Y) \mu_0^{-2d-1}(Z) \mu_0^n(Y_1 - Z_1 + Y_2 - Z_2) \mu_0^{-N}(Y_1 - X) \mu_0^{-N}(Z_1 - X) \mu_0^{-N}(Y_2 - X) \mu_0^{-N}(Z_2 - X) \\ &\times (J^N J_\theta^{2d+1} F_1)(Y_1 + \theta Y) dY dZ dY_1 dZ_1 dY_2 dZ_2. \end{aligned}$$

These observations imply that for all  $a \in A_{\text{reg}}$ ,  $n \in \mathbb{N}$  and with  $N = n + 2d + 1$ , we have

$$\begin{aligned} J^n(\tilde{\alpha}(a) - \tilde{\alpha}(a \star_\theta^\alpha e_\lambda))(X) &= J^n(\tilde{\alpha}(a) - \tilde{\alpha}(a) \star_\theta \tilde{\alpha}(e_\lambda))(X) \\ &= |2|_{\mathbf{k}}^{6d} \int dY dZ dY_1 dZ_1 dY_2 dZ_2 \overline{\Psi}(2[Y, Z]) \overline{\Psi}(2[X, Y_1 - Z_1 + Y_2 - Z_2]) \overline{\Psi}(2[Y_1, Z_1]) \overline{\Psi}(2[Y_2, Z_2]) \\ &\times \mu_0^{-2d-1}(Y) \mu_0^{-2d-1}(Z) \mu_0^n(Y_1 - Z_1 + Y_2 - Z_2) \mu_0^{-N}(Y_1 - X) \mu_0^{-N}(Z_1 - X) \mu_0^{-N}(Y_2 - X) \mu_0^{-N}(Z_2 - X) \\ &\times \left( (J^N J_\theta^{2d+1} \tilde{\alpha}(a))(Y_1 + \theta Y) - (J^N J_\theta^{2d+1} \tilde{\alpha}(a))(Y_1 + \theta Y) (J^{N+2d+1} \tilde{\alpha}(e_\lambda))(Y_2 + Z) \right). \end{aligned}$$

Using the Peetre inequality, the fact that the action  $\alpha$  is isometric and the (almost tautological) relation

$$(J\tilde{\alpha}(a))(X) = \alpha_X(J\tilde{\alpha}(a)(0)),$$

we deduce

$$\begin{aligned} \mathfrak{P}_n^A(\tilde{\alpha}(a) - \tilde{\alpha}(a \star_\theta^\alpha e_\lambda)) &\leq \sup_{X \in \mathbf{k}^{2d}} \int \mu_0^{-2d-1}(Y) \mu_0^{-2d-1}(Z) \mu_0^{-2d-1}(Y_1 - X) \\ &\times \mu_0^{-2d-1}(Z_1 - X) \mu_0^{-2d-1}(Y_2 - X) \mu_0^{-2d-1}(Z_2 - X) \left\| (J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(Y_1 + \theta Y - Y_2 - Z) \right. \\ &\left. - (J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(Y_1 + \theta Y - Y_2 - Z) (J^{n+4d+2} \tilde{\alpha}(e_\lambda))(0) \right\|_A dY dZ dY_1 dZ_1 dY_2 dZ_2, \end{aligned}$$

Performing the translations  $Y_j \mapsto Y_j + X$ ,  $Z_j \mapsto Z_j + X$ , we see that the integral above does not depend on  $X$ . Performing the translation  $Y_1 \mapsto Y_1 - \theta Y + Y_2 + Z$  and using Fubini, we see that the integral above is of the form

$$(42) \quad \int_{\mathbf{k}^{2d}} \Phi(X) \left\| (J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(X) - (J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(X) (J^{n+4d+2} \tilde{\alpha}(e_\lambda))(0) \right\|_A dX,$$

where  $0 \leq \Phi \in L^1(\mathbf{k}^{2d})$ .

Next, we estimate the integral (42) as a sum of two terms

$$\begin{aligned} I_1^\lambda &:= \int_{\mathbf{k}^{2d}} \Phi(X) \left\| (J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(X) - (J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(X) e_\lambda \right\|_A dX, \\ I_2^\lambda &:= \int_{\mathbf{k}^{2d}} \Phi(X) \left\| (J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(X) ((J^{n+4d+2} - \text{Id}) \tilde{\alpha}(e_\lambda))(0) \right\|_A dX. \end{aligned}$$

Let now  $C_n$  be the ball in  $\mathbf{k}^{2d} \times \mathbf{k}^{2d}$  centered in 0 and of radius  $n$ . By absolute convergence (in norm) of the integral  $I_1^\lambda$  and since  $\|e_\lambda\|_A \leq \|\varphi\|_1 \|e'_\lambda\|_A \leq 1$ , we deduce that for each  $\varepsilon > 0$  there exists  $n_0 \in \mathbb{N}$  (independent of  $\lambda$ ) such that for all  $n \geq n_0$  we have

$$\int_{\mathbf{k}^{2d} \setminus C_n} \Phi(X) \left\| (J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(X) - (J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(X) e_\lambda \right\|_A dX \leq \varepsilon.$$

On the other hand, by a compactness argument and since  $\{e_\lambda\}_{\lambda \in \Lambda}$  is an approximate unit, it easily follows that for any  $n \in \mathbb{N}$ , we have

$$\lim_{\lambda} \int_{C_n} \Phi(X) \|(J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(X) - (J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(X) e_\lambda\|_A dX = 0,$$

hence  $\lim_{\lambda} I_1^\lambda = 0$ . For the second bit, we first come back to the definition of  $e_\lambda \in A_{\text{reg}}$  in terms of  $e'_\lambda \in A$ , to get

$$(J^N - 1)\tilde{\alpha}(e_\lambda)(0) = \int_{\mathbf{k}^{2d}} (J^N - 1)\varphi(Y) \alpha_Y(e'_\lambda) dY.$$

Since moreover  $\int_{\mathbf{k}^{2d}} (J^N - 1)\varphi(Y) dY = 0$ , we get for any  $b \in A$

$$b(J^N - 1)\tilde{\alpha}(e_\lambda)(0) = \int_{\mathbf{k}^{2d}} (J^N - 1)\varphi(Y) (b \alpha_Y(e'_\lambda) - b) dY.$$

Applying this formula to  $b = (J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(X)$ , we get the bound

$$\begin{aligned} I_2^\lambda &\leq \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \Phi(X) |(J^N - 1)\varphi(Y)| \\ &\quad \times \|(J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(X) \alpha_Y(e'_\lambda) - (J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(X)\|_A dX dY. \end{aligned}$$

Using one more time the isometricity of the action, we finally get,

$$\begin{aligned} I_2^\lambda &\leq \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \Phi(X) |(J^N - 1)\varphi(Y)| \\ &\quad \times \|(J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(X - Y) e'_\lambda - (J^{n+2d+1} J_\theta^{2d+1} \tilde{\alpha}(a))(X - Y)\|_A dX dY. \end{aligned}$$

Noting that  $(J^N - 1)\varphi \in \mathcal{S}(\mathbf{k}^{2d})$ , which can be approximated in  $\mathcal{D}(\mathbf{k}^{2d})$ , we conclude that  $\lim_{\lambda} I_2^\lambda = 0$  with the same compactness argument than the one we used for  $I_1^\lambda$ . The case of  $a - e_\lambda \star_\theta^\alpha a$  is entirely similar.  $\square$

In particular, we deduce that if  $A$  is  $\sigma$ -unital, so does  $A_\theta$  and, thanks to Theorem 5.5, we get that  $A$  is  $\sigma$ -unital if and only if  $A_\theta$  is.

**Remark 5.2.** *The proof of the proposition above gives the existence of a bounded approximate unit for the Fréchet algebra  $(A_{\text{reg}}, \star_\theta^\alpha)$ , in the sense that for all  $n \in \mathbb{N}$ ,  $\sup_{\lambda \in \Lambda} \|e_\lambda\|_n < \infty$  and for all  $a \in A_{\text{reg}}$ ,  $\lim_{\lambda} \|a \star_\theta e_\lambda - a\|_n = \lim_{\lambda} \|e_\lambda \star_\theta a - a\|_n = 0$ .*

We next study the question of compatibility of the deformation with ideals and morphisms. The following two results follow from minor modifications of the similar statements in [27]. (See [14, Proposition 3.8] for an alternative proof of Proposition 5.3.)

**Proposition 5.3.** *Let  $(A, \alpha)$  and  $(B, \beta)$  be two  $C^*$ -algebras endowed with continuous actions of  $\mathbf{k}^{2d}$  and let  $T : A \rightarrow B$  be a  $*$ -homomorphism which intertwines the actions  $\alpha$  and  $\beta$ . Then,  $T$  maps  $A_{\text{reg}}$  to  $B_{\text{reg}}$  and extends to a continuous homomorphism  $T_\theta : A_\theta \rightarrow B_\theta$  which intertwines the actions  $\alpha_\theta$  and  $\beta_\theta$ . If moreover  $T$  is injective (respectively surjective) then  $T_\theta$  is injective (respectively surjective) too.*

*Proof.* Let  $a \in A_{\text{reg}}$ . Then from the equality (in  $C_u(\mathbf{k}^{2d}, A)$ )  $\tilde{\beta}(T(a)) = \text{Id} \otimes T(\tilde{\alpha}(a))$  together with the fact that  $*$ -homomorphisms are norm decreasing, we get  $\|T(a)\|_n \leq \|a\|_n$ , which implies that  $T$  maps  $A_{\text{reg}}$  to  $B_{\text{reg}}$ . From the absolute convergence of the integral formula (35) for  $\star_\theta$  at the level of  $A_{\text{reg}}$  we deduce that  $T$  is a continuous  $*$ -homomorphism from  $(A_{\text{reg}}, \star_\theta^\alpha)$  to  $(B_{\text{reg}}, \star_\theta^\alpha)$ . From arguments identical to those of [27, Theorem 5.7] (using the  $C^*$ -module approach to the deformed  $C^*$ -norm as explained in Proposition 4.11), we see that  $T$  is continuous for the  $C^*$ -norms, hence it extends to the completions.  $T_\theta$ , the extension of  $T$ , also intertwines the actions because the actions have not changed. That  $T_\theta$  is injective (respectively surjective) when  $T$  is injective (respectively surjective) can be proven exactly as in [27, Proposition 5.8].  $\square$

**Proposition 5.4.** *Let  $(A, \alpha)$  be a  $C^*$ -algebra endowed with a continuous action of  $\mathbf{k}^{2d}$  and let  $I$  be an  $\alpha$ -invariant (essential) ideal of  $A$ . Then  $I_\theta$  is an (essential) ideal of  $A_\theta$ .*

*Proof.* This is exactly the arguments of [27, Proposition 5.9] except the fact that we need to show that if  $I$  is an  $\alpha$ -invariant (essential) ideal of  $A$  then,  $I_{\text{reg}}$  is an ideal of  $(A_{\text{reg}}, \star_\theta^\alpha)$ . But this fact again follows from the absolute convergence of the integral formula (35).  $\square$

Now we come to a very important point, namely that the deformation can be performed in stages.

**Theorem 5.5.** *Let  $\theta, \theta' \in \mathcal{O}_{\mathbf{k}}$ . Then,  $(A_\theta)_{\theta'} \simeq A_{\theta+\theta'}$  and moreover  $(A_\theta)_{\text{reg}} = A_{\text{reg}}$ .*

*Proof.* That  $(A_\theta)_{\theta'} \simeq A_{\theta+\theta'}$  follows from [14, Lemma 3.5] or [23, Theorem 3.10] and it remains to prove that  $(A_\theta)_{\text{reg}} = A_{\text{reg}}$ . The first step is to show that  $A_{\text{reg}} \subset (A_\theta)_{\text{reg}}$  with dense inclusion. By construction,  $A_{\text{reg}} \subset A_\theta$ , so that it makes sense to evaluate the seminorms  $\|\cdot\|_n^{A_\theta}$  (i.e. those giving the Fréchet topology of  $(A_\theta)_{\text{reg}}$ ) on  $A_{\text{reg}}$ :

$$\|a\|_n^{A_\theta} = \mathfrak{P}_n^{A_\theta}(\tilde{\alpha}(a)) = \sup_{X \in \mathbf{k}^{2d}} \|J^n \tilde{\alpha}(a)(X)\|_\theta \leq C \sup_{X \in \mathbf{k}^{2d}} \|J^n \tilde{\alpha}(a)(X)\|_{2d+1}^A = C \sup_{X \in \mathbf{k}^{2d}} \mathfrak{P}_{2d+1}^A(J^n \tilde{\alpha}(a)(X)),$$

but it is easy to see that the later expression coincides with  $\mathfrak{P}_{n+2d+1}^A(\tilde{\alpha}(a)) = \|a\|_{n+2d+1}^A$ , showing that  $A_{\text{reg}} \subset (A_\theta)_{\text{reg}}$ . That  $A_{\text{reg}}$  is dense in  $(A_\theta)_{\text{reg}}$ , follows from the Dixmier-Malliavin Theorem for general locally compact groups [21, Theorem 4.16]. Indeed, let  $a \in (A_\theta)_{\text{reg}}$ ,  $\varepsilon > 0$  and  $n \in \mathbb{N}$ . By Proposition 3.11,  $(A_\theta)^\infty \subset (A_\theta)_{\text{reg}}$  densely so that there exists  $b \in (A_\theta)^\infty$  with  $\|a - b\|_n^{A_\theta} \leq \varepsilon/2$ . Now, by [21, Theorem 4.16], there exists  $b_1, \dots, b_k \in A_\theta$  and  $\varphi_1, \dots, \varphi_k \in \mathcal{D}(\mathbf{k}^{2d}) \subset \mathcal{S}(\mathbf{k}^{2d})$  such that  $b = \sum_{j=1}^k \alpha_{\varphi_j}(b_j)$ . But by construction  $A_{\text{reg}}$  is dense in  $A_\theta$  so that there exists  $c_1, \dots, c_k \in A_{\text{reg}}$  with  $\|b_j - c_j\|_\theta \leq \varepsilon/(2k\|J^n \varphi_j\|_1)$ . Setting  $c := \sum_{j=1}^k \alpha_{\varphi_j}(c_j)$ , we finally deduce

$$\|a - c\|_n^{A_\theta} \leq \|a - b\|_n^{A_\theta} + \|b - c\|_n^{A_\theta} \leq \frac{\varepsilon}{2} + \sum_{j=1}^k \|\alpha_{\varphi_j}(b_j - c_j)\|_n^{A_\theta}.$$

Now, from the (already used) relation  $J^n \tilde{\alpha}(\alpha_\varphi(a)) = \tilde{\alpha}(\alpha_{J^n \varphi}(a))$ , valid for  $a \in A_\theta$  and  $\varphi \in \mathcal{S}(\mathbf{k}^{2d})$  and since the action  $\alpha$  is still isometric on  $A_\theta$ , we deduce that

$$\|\alpha_{\varphi_j}(b_j - c_j)\|_n^{A_\theta} = \|\alpha_{J^n \varphi_j}(b_j - c_j)\|_\theta \leq \|J^n \varphi_j\|_1 \|b_j - c_j\|_\theta,$$

and thus

$$\|a - c\|_n^{A_\theta} \leq \frac{\varepsilon}{2} + \sum_{j=1}^k \|J^n \varphi_j\|_1 \|b_j - c_j\|_\theta \leq \varepsilon,$$

as needed. The reversed inclusion follows from the first part: we have seen that  $A_{\text{reg}} \subset (A_\theta)_{\text{reg}}$  for any  $C^*$ -algebra  $A$  endowed with a continuous action of  $\mathbf{k}^{2d}$ . Applying this to the deformed  $C^*$ -algebra  $A_\theta$ , which still carries a continuous action of  $\mathbf{k}^{2d}$ , we deduce that for any  $\theta' \in \mathcal{O}_{\mathbf{k}}$ , we have  $(A_\theta)_{\text{reg}} \subset ((A_\theta)_{\theta'})_{\text{reg}}$  but since  $((A_\theta)_{\theta'})_{\text{reg}} = (A_{\theta+\theta'})_{\text{reg}}$ , we deduce for  $\theta' = -\theta$  that  $(A_\theta)_{\text{reg}} \subset A_{\text{reg}}$ , which completes the proof.  $\square$

With the help of the above theorem, we can use the same proof than [27, Theorem 7.7] to get that the deformation maps equivariant short exact sequences to short exact sequences. Alternatively, one can use [14, Theorem 3.9].

**Theorem 5.6.** *Let  $I$  be an  $\alpha$ -invariant ideal of  $A$  and let  $Q := A/I$  endowed with the quotient action. Then the equivariant short exact sequence*

$$0 \rightarrow I \rightarrow A \rightarrow Q \rightarrow 0,$$

*gives rise to a short exact sequence of deformed algebras:*

$$0 \rightarrow I_\theta \rightarrow A_\theta \rightarrow Q_\theta \rightarrow 0.$$

Our last result concerns continuity of the field of deformed  $C^*$ -algebras  $(A_\theta)_{\theta \in \mathcal{O}_{\mathbf{k}}}$ . Here, the continuity structure refers to the  $*$ -subalgebra  $A_{\text{reg}}$ , viewed as a subspace of constant sections. For the question of continuity, the twisted crossed product approach does not seem to be especially appropriate, contrary to the methods developed in [27]. Note however that due to particularity of non-Archimedean analysis, we are forced to consider a restricted range of parameters.

**Theorem 5.7.** *Let  $\gamma \in \mathcal{O}_{\mathbf{k}}$  fixed. Then, the field of deformed  $C^*$ -algebras  $(A_{\gamma\theta^2})_{\theta \in \mathcal{O}_{\mathbf{k}}}$  is continuous.*

*Proof.* Our proof mimics [27, Chapter 8], where continuity is obtained from combination of lower and upper semicontinuity. By an immediate adaptation of the arguments given in [27, page 55], lower semicontinuity of the field  $(A_\theta)_{\theta \in \mathcal{O}_{\mathbf{k}}}$  will follow if for all  $a \in A_{\text{reg}}$ , all  $f_1 \in \mathcal{D}(\mathbf{k}^{2d}, A)$  and  $f_2 \in \mathcal{S}(\mathbf{k}^{2d}, A)$  with  $\mathcal{G}(f_2) \in \mathcal{D}(\mathbf{k}^{2d}, A)$ , we have

$$\|\langle f_1, \tilde{\alpha}(a) \star_\theta f_2 \rangle_A - \langle f_1, \tilde{\alpha}(a) \star_{\theta'} f_2 \rangle_A\|_A \rightarrow 0, \quad \theta \rightarrow \theta'.$$

But this easily follows from the strong continuity of  $\alpha$  and a compactness argument once one has realized that

$$\begin{aligned} & \langle f_1, \tilde{\alpha}(a) \star_\theta f_2 \rangle_A - \langle f_1, \tilde{\alpha}(a) \star_{\theta'} f_2 \rangle_A \\ &= |2|_{\mathbf{k}}^d \int_{\mathbf{k}^{2d} \times \mathbf{k}^{2d}} \overline{\Psi}(2[X, Y]) f_1(X) \alpha_X(\alpha_{\theta Y}(a) - \alpha_{\theta' Y}(a)) \mathcal{G}(f_2)(Y) dX dY. \end{aligned}$$

In particular, the field  $(A_{\gamma\theta^2})_{\theta \in \mathcal{O}_{\mathbf{k}}}$  is lower-semicontinuous for any  $\gamma \in \mathcal{O}_{\mathbf{k}}$ .

Upper semicontinuity relies on [26, Proposition 1.2]. To be able to use this result, we must let the action  $\alpha$  variate. So, fix  $\gamma \in \mathcal{O}_{\mathbf{k}}$  and define a new action  $\alpha^\gamma$  of  $\mathbf{k}^{2d}$  on  $A$  by  $\alpha_X^\gamma(a) := \alpha_{\gamma X}(a)$ . It is clear that  $\alpha^\gamma$  is still continuous. To do not get confused, we need extra notations. We now let  $A_{\text{reg}}^\alpha$  to be our dense Fréchet subspace of  $A$  as given in (28) for a given action  $\alpha$ . Accordingly, we denote by  $\|\cdot\|_n^\alpha$  to be seminorms on  $A_{\text{reg}}^\alpha$  as defined in (29) and by  $A_\theta^\alpha$  the deformed  $C^*$ -algebra. We first observe that  $A_{\text{reg}}^\alpha \subset A_{\text{reg}}^{\alpha^\gamma}$  with continuous and dense inclusion. Indeed, for  $a \in A_{\text{reg}}^\alpha$ , using the notation  $D_\gamma F(X) := F(\gamma X)$ ,  $F \in \mathcal{B}(\mathbf{k}^{2d}, A)$ , we have  $\tilde{\alpha}^\gamma(a) = D_\gamma \tilde{\alpha}(a)$  and thus we deduce from Lemma 3.8 (iii):

$$\|a\|_n^{\alpha^\gamma} = \mathfrak{P}_n^A(\tilde{\alpha}^\gamma(a)) \leq \|\mu_0^{-2d-1}\|_1^2 \mathfrak{P}_{n+2d+1}^A(\tilde{\alpha}(a)) = \|\mu_0^{-2d-1}\|_1^2 \|a\|_{n+2d+1}^\alpha,$$



and the continuity follows. For the density, one observes that the spaces of smooth vectors (in the sense of Bruhat) for the actions  $\alpha$  and  $\alpha^\gamma$  (for  $\gamma \neq 0$ ) coincide since  $\tilde{\alpha}(a)$  is locally constant if and only if  $\tilde{\alpha}^\gamma(a)$  does. Thus  $A^\infty$  is dense in  $A_{\text{reg}}^{\alpha^\gamma}$  and  $A^\infty$  is contained in  $A_{\text{reg}}^\alpha$  so  $A_{\text{reg}}^{\alpha^\gamma}$  is dense in  $A_{\text{reg}}^\alpha$ . Next we compare the deformed  $C^*$ -algebras  $A_\theta^{\alpha^\gamma}$  and  $A_{\gamma 2\theta}^\alpha$ . Let  $F \in \mathcal{B}(\mathbf{k}^{2d})$  and  $f \in \mathcal{S}(\mathbf{k}^{2d})$ . Undoing partially the oscillatory trick in Eq. (33), we get after some rearrangements:

$$F \star_\theta f(X) = |2|_{\mathbf{k}}^d \int_{\mathbf{k}^{2d}} \overline{\Psi}(2[X, Y]) F(X + \theta Y) (\mathcal{G}f)(Y) dY.$$

From this and the scaling relation  $\mathcal{G}D_\gamma(f) = |\gamma|_{\mathbf{k}}^{-2d} D_\gamma \mathcal{G}(f)$ ,  $\gamma \in \mathcal{O}_{\mathbf{k}} \setminus \{0\}$ , we deduce that  $D_\gamma(F) \star_\theta D_\gamma(f) = D_\gamma(F \star_{\gamma 2\theta} f)$  in  $\mathcal{S}(\mathbf{k}^{2d}, A)$ . Introducing the unitary operator  $U_\gamma := |\gamma|_{\mathbf{k}}^d D_\gamma$  on the pre- $C^*$ -module  $\mathcal{S}(\mathbf{k}^{2d}, A)$ , the previous relation entails that for any  $a \in A_{\text{reg}}^\alpha$ , we have:

$$(43) \quad U_\gamma^* L_\theta(\tilde{\alpha}^\gamma(a)) U_\gamma = L_{\gamma 2\theta}(\tilde{\alpha}(a)),$$

where the equality holds in the  $C^*$ -algebra of adjointable bounded  $A$ -linear endomorphisms of the pre- $C^*$ -module  $\mathcal{S}(\mathbf{k}^{2d}, A)$ . The above relation also gives  $L_{\gamma 2\theta}(\tilde{\alpha}(a \star_\theta^{\alpha^\gamma} b)) = L_{\gamma 2\theta}(\tilde{\alpha}(a \star_{\gamma 2\theta}^\alpha b))$ , for  $a, b \in A_{\text{reg}}^\alpha$ , and thus

$$0 = \|L_{\gamma 2\theta}(\tilde{\alpha}(a \star_\theta^{\alpha^\gamma} b)) - L_{\gamma 2\theta}(\tilde{\alpha}(a \star_{\gamma 2\theta}^\alpha b))\| = \|L_{\gamma 2\theta}(\tilde{\alpha}(a \star_\theta^{\alpha^\gamma} b - a \star_{\gamma 2\theta}^\alpha b))\| = \|a \star_\theta^{\alpha^\gamma} b - a \star_{\gamma 2\theta}^\alpha b\|_{\gamma 2\theta}.$$

Hence,  $a \star_\theta^{\alpha^\gamma} b = a \star_{\gamma 2\theta}^\alpha b$  in  $A_{\gamma 2\theta}^\alpha$  but since  $a \star_{\gamma 2\theta}^\alpha b \in A_{\text{reg}}^\alpha$  (a priori,  $a \star_{\gamma 2\theta}^\alpha b \in A_{\text{reg}}^{\alpha^\gamma}$ ) the equality takes place within  $A_{\text{reg}}^\alpha$ . But the relation (43) also shows that

$$\|a\|_\theta^{\alpha^\gamma} = \|L_\theta(\tilde{\alpha}^\gamma(a))\| = \|L_{\gamma 2\theta}(\tilde{\alpha}(a))\| = \|a\|_{\gamma 2\theta}^\alpha, \quad \forall a \in A_{\text{reg}}^\alpha.$$

Since  $A_{\text{reg}}^\alpha$  is dense both in  $A_\theta^{\alpha^\gamma}$  and in  $A_{\gamma 2\theta}^\alpha$ , we deduce that  $A_\theta^{\alpha^\gamma} = A_{\gamma 2\theta}^\alpha$ . Hence (inverting the roles of  $\theta \in \mathcal{O}_{\mathbf{k}}$  and of  $\gamma \in \mathcal{O}_{\mathbf{k}}$ ), it suffices to show that the field  $(A_\theta^{\alpha^\gamma})_{\gamma \in \mathcal{O}_{\mathbf{k}}}$  is upper-semicontinuous. To this aim, consider on the  $C^*$ -algebra  $C_0(\mathcal{O}_{\mathbf{k}}, A)$  the action of  $\mathbf{k}^{2d}$  given by:

$$\beta_X(\Phi)(\gamma) := \alpha_{\gamma X}(\Phi(\gamma)).$$

The space  $\mathcal{O}_{\mathbf{k}}$  being compact, one easily sees that  $\beta$  is continuous. For fixed  $\gamma \in \mathcal{O}_{\mathbf{k}}$ , let  $e_\gamma : C_0(\mathcal{O}_{\mathbf{k}}, A) \rightarrow A$  be the evaluation map at  $\gamma$  and let  $C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A)$  the (norm closed) ideal of elements in  $C_0(\mathcal{O}_{\mathbf{k}}, A)$  vanishing at  $\gamma$ . The associated short exact sequence  $0 \rightarrow C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A) \rightarrow C_0(\mathcal{O}_{\mathbf{k}}, A) \rightarrow A \rightarrow 0$  being equivariant for  $\beta$  on  $C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A)$  and on  $C_0(\mathcal{O}_{\mathbf{k}}, A)$ , and for  $\alpha^\gamma$  on  $A$  (since  $e_\gamma$  intertwines  $\beta$  and  $\alpha^\gamma$ ), we deduce from Theorem 5.6 that we have a short exact sequence of deformed  $C^*$ -algebras:

$$0 \rightarrow C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta \rightarrow C_0(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta \rightarrow A_\theta^{\alpha^\gamma} \rightarrow 0.$$

Moreover, as  $C_0(\mathcal{O}_{\mathbf{k}})$  (seen as a subalgebra of  $M(C_0(\mathcal{O}_{\mathbf{k}}, A))$ ) is left invariant by the action  $\beta$ , it is its own space of regular elements and by Proposition 3.12  $\Phi \star_\theta^\beta \eta = \Phi \eta = \eta \star_\theta^\beta \Phi$  for all  $\Phi \in C_0(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta$  and all  $\eta \in C_0(\mathcal{O}_{\mathbf{k}})$ . Hence,  $C_0(\mathcal{O}_{\mathbf{k}})$  may also be viewed as a subalgebra of  $M(\underline{C_0(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta})$ . Let then  $C_0^\gamma(\mathcal{O}_{\mathbf{k}})$  be the norm closed ideal in  $C_0(\mathcal{O}_{\mathbf{k}})$  of elements vanishing at  $\gamma$  and let  $C_0(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta C_0^\gamma(\mathcal{O}_{\mathbf{k}})$  be the norm closure in the deformed  $C^*$ -algebra  $C_0(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta$  of the linear span of products. Then by [26, Proposition 1.2] the field of  $C^*$ -algebra

$$\left( C_0(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta / \overline{C_0(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta C_0^\gamma(\mathcal{O}_{\mathbf{k}})} \right)_{\gamma \in \mathcal{O}_{\mathbf{k}}},$$

is upper-semicontinuous. But since  $A_\theta^{\alpha\gamma} \simeq C_0(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta / C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta$ , it suffices to show that  $C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta$  coincides with  $\overline{C_0(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta C_0^\gamma(\mathcal{O}_{\mathbf{k}})}$ . On the one hand, we have  $C_0(\mathcal{O}_{\mathbf{k}}, A)C_0^\gamma(\mathcal{O}_{\mathbf{k}}) \subset C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A)$  and thus at the level of the regular vectors  $(C_0(\mathcal{O}_{\mathbf{k}}, A)C_0^\gamma(\mathcal{O}_{\mathbf{k}}))_{\text{reg}}^\beta \subset C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A)_{\text{reg}}^\beta \subset C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta$ . But  $(C_0(\mathcal{O}_{\mathbf{k}}, A)C_0^\gamma(\mathcal{O}_{\mathbf{k}}))_{\text{reg}}^\beta = C_0(\mathcal{O}_{\mathbf{k}}, A)_{\text{reg}}^\beta C_0^\gamma(\mathcal{O}_{\mathbf{k}}) = C_0(\mathcal{O}_{\mathbf{k}}, A)_{\text{reg}}^\beta \star_\theta^\beta C_0^\gamma(\mathcal{O}_{\mathbf{k}})$  and thus  $\overline{C_0(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta C_0^\gamma(\mathcal{O}_{\mathbf{k}})} \subset C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta$ . On the other hand since  $\overline{C_0(\mathcal{O}_{\mathbf{k}}, A)C_0^\gamma(\mathcal{O}_{\mathbf{k}})} = C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A)$ , we have by the Cohen Factorization Theorem  $C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A) = C_0(\mathcal{O}_{\mathbf{k}}, A)C_0^\gamma(\mathcal{O}_{\mathbf{k}})$  (see for instance [13, Theorem 32.22]). Hence, any element  $\Phi \in C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A)_{\text{reg}}^\beta$  can be written as  $\Phi = \Xi\eta$  with  $\Xi \in C_0(\mathcal{O}_{\mathbf{k}}, A)$  and  $\eta \in C_0^\gamma(\mathcal{O}_{\mathbf{k}})$ . For  $\varphi \in \mathcal{S}(\mathbf{k}^{2d})$ , we have  $\beta_\varphi(\Phi) \in C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A)_{\text{reg}}^\beta$  and  $\beta_\varphi(\Phi) = \beta_\varphi(\Xi)\eta = \beta_\varphi(\Xi) \star_\theta^\beta \eta \in C_0(\mathcal{O}_{\mathbf{k}}, A)_{\text{reg}}^\beta \star_\theta^\beta C_0^\gamma(\mathcal{O}_{\mathbf{k}})$ . But by Proposition 5.1  $\Phi$  is approximated in  $C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta$  by elements of the form  $\beta_\varphi(\Phi)$ . Hence  $C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A)_{\text{reg}}^\beta \subset \overline{C_0(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta C_0^\gamma(\mathcal{O}_{\mathbf{k}})}$  and thus  $C_0^\gamma(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta \subset \overline{C_0(\mathcal{O}_{\mathbf{k}}, A)_\theta^\beta C_0^\gamma(\mathcal{O}_{\mathbf{k}})}$ , concluding the proof.  $\square$

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