The dual Jacobian of a generalised hyperbolic tetrahedron, and volumes of prisms

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Abstract

We derive an analytic formula for the dual Jacobian matrix of a generalised hyperbolic tetrahedron. Two cases are considered: a mildly truncated and a prism truncated tetrahedron. The Jacobian for the latter arises as an analytic continuation of the former, that falls in line with a similar behaviour of the corresponding volume formulae.

Also, we obtain a volume formula for a hyperbolic n-gonal prism: the proof requires the above mentioned Jacobian, employed in the analysis of the edge lengths behaviour of such a prism, needed later for the Schläfli formula.

Key words: hyperbolic polyhedron, Gram matrix, volume.

1 Introduction

Let T be a generalised hyperbolic tetrahedron (in the sense of [19, 22]) depicted in Fig. 1. If the truncating planes associated with its ultra-ideal vertices do not intersect, we call such a tetrahedron *mildly truncated*, otherwise we call it *intensely truncated*. If only two of them intersect, we call such a tetrahedron *prism truncated* [12]. Let us note that a prism truncated orthoscheme is, in fact, a Lambert cube [11].

The volumes of the tetrahedron and its truncations are of particular interest, since they are the simplest representatives of hyperbolic polyhedra. Over the last decade an extensive study produced a number of volume formulae suitable for analytic and numerical exploration [3, 5, 11, 12, 20, 22]. A similar study was done for the spherical tetrahedron [14, 17], which can be viewed as a natural counterpart of the hyperbolic one. Many analytic properties of the volume formula for a hyperbolic tetrahedron came into view concerning the Volume Conjecture [10, 18].

However, other geometric characteristics of a generalised hyperbolic tetrahedron T are also important and bring some useful information. In particular,

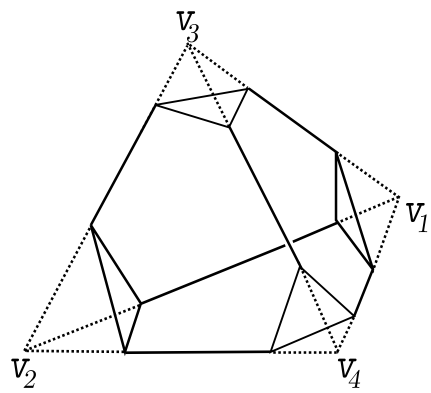

Figure 1: Generalised hyperbolic tetrahedron

Jac(T), the Jacobian of T, which is the Jacobian matrix of the edge length with respect to the dihedral angles, is such. This matrix enjoys many symmetries [15] and can be computed out of the Gram matrix of T [7].
In the present paper, we consider Jac$^\star(T)$, *the dual Jacobian* of a generalised hyperbolic tetrahedron T. By the dual Jacobian of T we mean the Jacobian matrix of the dihedral angles with respect to the edge length. Such an object behaves nicely when T undergoes both mild and intense truncation: the dual Jacobian of a prism truncated tetrahedron is an analytic continuation for that of a mildly doubly truncated one. Let us mention, that the respective volume formulae are also connected by an analytic continuation, in an analogous manner [12, 19].
As an application of our technique, we give a volume formula for a hyperbolic n-gonal prism, c.f. [4].

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2 Preliminaries

Let T be a mildly truncated hyperbolic tetrahedron with vertices v_k, $k \in \{1, 2, 3, 4\}$, edges e_{ij} (connecting the vertices v_i and v_j) with dihedral angles

a_{ij} and lengths ℓ_{ij}, $i, j \in \{1, 2, 3, 4\}$, $i < j$.
Depending on whether the vertex v_k is proper ($\mathrm{v}_k \in \mathbb{H}^3$), ideal ($\mathrm{v}_k \in \partial\mathbb{H}^3$) or ultra-ideal ($\mathrm{v}_k$ defines a polar hyperplane as described in [21, Section 3], c.f. Theorem 3.2.12), let us set the quantity ε_k to be $+1$, 0 or -1, respectively.
For each vertex v_i of T let us consider the face F_{jkl} opposite to it, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. The link $L(\mathrm{v}_l)$ of the vertex v_l is either a spherical triangle ($\varepsilon_l = +1$), a Euclidean triangle ($\varepsilon_l = 0$) or a hyperbolic triangle ($\varepsilon_l = -1$). Let us define the quantity b^i_{jk} as follows:

$$b^i_{jk} := \begin{cases} \text{the plane angle of } F_{jkl} \text{ opposite to the edge } e_{jk}, \text{ if } \varepsilon_l = +1; \\ \text{zero, if } \varepsilon_l = 0; \\ \text{the length of the common perpendicular to the edges } e_{jl} \\ \text{and } e_{kl} \text{ of } F_{jkl}, \text{ if } \varepsilon_l = -1. \end{cases}$$

Here, we consider the face F_{jkl} as a generalised hyperbolic triangle, for which the trigonometric laws hold as described in [2, 9].
Let us also define a quantity μ^i_{jk} by means of the formula

$$\mu^i_{jk} := \int_0^{b^i_{jk}} \cos(\sqrt{\varepsilon_l}s)\mathrm{d}s.$$

Let ${\mu'}^i_{jk}$ denote the derivative of μ^i_{jk} with respect to b^i_{jk}, which means that

$${\mu'}^i_{jk} = \cos(\sqrt{\varepsilon_l} b^i_{jk}).$$

Let σ_{kl} denote the following quantity associated with an edge e_{kl}, $k, l \in \{1, 2, 3, 4\}$, $k < l$,

$$\sigma_{kl} := \frac{1}{2}e^{\ell_{kl}} - \frac{1}{2}\varepsilon_k\varepsilon_l e^{-\ell_{kl}}.$$

Let σ'_{kl} denote the derivative of σ_{kl} with respect to ℓ_{kl}, so we have that

$$\sigma'_{kl} = \frac{1}{2}e^{\ell_{kl}} + \frac{1}{2}\varepsilon_k\varepsilon_l e^{-\ell_{kl}}.$$

Let us define the momentum M_i of the vertex v_i opposite to the face F_{jkl}, $\{i, j, k, l\} = \{1, 2, 3, 4\}$ by the following equality (c.f. [6, VII.6]):

$$M^i := \mu^i_{jk}\, \mu^i_{jl}\, \sigma_{kl}.$$

The quantity above is well defined grace to the following theorem.

Theorem 1 (The Sine Law for faces) *Let F_{jkl} be the face of T opposite to the vertex v_i, $\{i,j,k,l\}=\{1,2,3,4\}$. Then F_{jkl} is a generalised hyperbolic triangle and the following equalities hold:*

$$\frac{\mu^i_{jk}}{\sigma_{jk}} = \frac{\mu^i_{jl}}{\sigma_{jl}} = \frac{\mu^i_{kl}}{\sigma_{kl}}.$$

Let us also define the momentum M_{jkl} of the face F_{jkl} opposite to the vertex v_i, $\{i,j,k,l\}=\{1,2,3,4\}$ by setting (c.f. [6, VII.6])

$$M_{jkl} := \mu^j_{kl} \sin a_{ik} \sin a_{il}.$$

The quantity above is well defined, according to the following theorem.

Theorem 2 (The Sine Law for links) *Let v_i be the vertex of T opposite to the face F_{jkl}, $\{i,j,k,l\}=\{1,2,3,4\}$. Then $L(\mathrm{v}_i)$ is either a spherical, a Euclidean or a hyperbolic triangle and the following equalities hold:*

$$\frac{\sin a_{ij}}{\mu^j_{kl}} = \frac{\sin a_{ik}}{\mu^k_{jl}} = \frac{\sin a_{il}}{\mu^l_{jk}}.$$

Both Theorem 1 and Theorem 2 are paraphrases of the spherical, Euclidean or hyperbolic sine laws (for a generalised hyperbolic triangle, see [9]). The following theorems are the cosine laws for a generalised hyperbolic triangle adopted to the notation of the present paper.

Theorem 3 (The first Cosine Law for faces) *Let F_{jkl} be the face of T opposite to the vertex v_i, $\{i,j,k,l\}=\{1,2,3,4\}$. Then F_{jkl} is a generalised hyperbolic triangle and the following equality holds:*

$$\sigma'_{kl} = \frac{{\mu'}^i_{kl} + {\mu'}^i_{jk}\,{\mu'}^i_{jl}}{\mu^i_{jk}\,\mu^i_{jl}}.$$

Theorem 4 (The second Cosine Law for faces) *Let F_{jkl} be the face of T opposite to the vertex v_i, $\{i,j,k,l\}=\{1,2,3,4\}$. Then F_{jkl} is a generalised hyperbolic triangle and the following equality holds:*

$${\mu'}^i_{jk} = \frac{-\varepsilon_l \sigma'_{jk} + \sigma'_{jl}\sigma'_{kl}}{\sigma_{jl}\sigma_{kl}}.$$

Theorem 5 (The Cosine Law for links) *Let v_i be the vertex of T opposite to the face F_{jkl}, $\{i,j,k,l\}=\{1,2,3,4\}$. Then $L(\mathrm{v}_i)$ is either a spherical, a Euclidean or a generalised hyperbolic triangle and the following equality holds:*

$${\mu'}^j_{kl} = \frac{\cos a_{ij} + \cos a_{ik}\cos a_{il}}{\sin a_{ik}\sin a_{il}}.$$

3 Auxiliary lemmata

In the present section we shall consider various partial derivatives of certain geometric quantities associated with either the faces or the vertex links of a generalised hyperbolic tetrahedron T. These derivatives will be used later on in the computation of the entries of $\mathrm{Jac}^\star(T)$.

Lemma 1 *For* $\{i,j,k,l\} = \{1,2,3,4\}$ *we have*

$$\frac{\partial \ell_{kl}}{\partial b^i_{kl}} = -\varepsilon_j\, \frac{\mu^i_{kl}}{M^i},$$

$$\frac{\partial \ell_{kl}}{\partial b^i_{jk}} = -\sigma'_{jl}\, \frac{\mu^i_{kl}}{M^i},\quad \frac{\partial \ell_{kl}}{\partial b^i_{jl}} = -\sigma'_{jk}\, \frac{\mu^i_{kl}}{M^i}.$$

Proof. According to the definition of σ_{kl}, we have $\sigma_{kl} = 0$ only in the following two cases: $\varepsilon_k = \varepsilon_l = +1$ and $\ell_{kl} = 0$, or $\varepsilon_k = \varepsilon_l = -1$ and $\ell_{kl} = 0$. In the former case, we have a degenerate tetrahedron with two proper vertices collapsing to one point. In the latter case the tetrahedron has two ultra-ideal vertices, whose polar planes are tangent at a point on the ideal boundary $\partial\mathbb{H}^3$. This is a limiting case, since in a generalised (mildly truncated) tetrahedron two polar planes never intersect or become tangent. Thus, we suppose that $\sigma_{kl} \neq 0$.
By taking derivatives on both sides of the first Cosine Law for faces, we get the following formulae:

$$\sigma_{kl}\, \frac{\partial \ell_{kl}}{\partial b^i_{kl}} = \frac{\partial \sigma'_{kl}}{\partial b^i_{kl}} = \frac{1}{\mu^i_{jk}\mu^i_{jl}} \frac{\partial {\mu'}^i_{kl}}{\partial b^i_{kl}} = -\varepsilon_j\, \frac{\mu^i_{kl}}{\mu^i_{jk}\mu^i_{jl}},$$

since

$$\frac{\partial \sigma'_{kl}}{\partial b^i_{kl}} = \sigma_{kl}\, \frac{\partial \ell_{kl}}{\partial b^i_{kl}} \quad \text{and} \quad \frac{\partial {\mu'}^i_{kl}}{\partial b^i_{kl}} = -\varepsilon_j\, \mu^i_{kl}$$

by a direct computation. This implies the first identity of the lemma.
Now we compute

$$\sigma_{kl}\, \frac{\partial \ell_{kl}}{\partial b^i_{jk}} = \frac{\partial \sigma'_{kl}}{\partial b^i_{jk}} = -\frac{(({\mu'}^i_{jk})^2 + \varepsilon_l (\mu^i_{jk})^2)\mu^i_{jl}{\mu'}^i_{jl} + \mu^i_{jl}{\mu'}^i_{jk}{\mu'}^i_{kl}}{(\mu^i_{jk}\mu^i_{jl})^2} =$$

$$-\frac{{\mu'}^i_{jl} + {\mu'}^i_{jk}{\mu'}^i_{kl}}{\mu^i_{jk}\mu^i_{kl}} \cdot \frac{\mu^i_{kl}}{\mu^i_{jk}\mu^i_{jl}} = -\sigma'_{jl}\, \frac{\mu^i_{kl}}{\mu^i_{jk}\mu^i_{jl}}.$$

where we use the identity $({\mu'}^i_{jk})^2 + \varepsilon_l(\mu^i_{jk})^2 = 1$ and, as before, the fact that $\frac{\partial {\mu'}^i_{jk}}{\partial b^i_{jk}} = -\varepsilon_l \mu^i_{jk}$. Then the second identity follows. The third one is analogous to the second one under the permutation of the indices k and l. □

Lemma 2 *For* $\{i,j,k,l\} = \{1,2,3,4\}$ *we have*

$$\frac{\partial b^j_{kl}}{\partial a_{ij}} = \varepsilon_i \, \frac{\sin a_{ij}}{M_{jkl}}$$

$$\frac{\partial b^j_{kl}}{\partial a_{ik}} = \varepsilon_i \, \frac{\sin a_{ij}}{M_{jkl}} \, {\mu'}^l_{jk}, \quad \frac{\partial b^j_{kl}}{\partial a_{il}} = \varepsilon_i \, \frac{\sin a_{ij}}{M_{jkl}} \, {\mu'}^k_{jl}.$$

Proof. By taking derivatives on both sides of the Cosine Law for links, we get the following formulae:

$$-\varepsilon_i \, \mu^j_{kl} \, \frac{\partial b^j_{kl}}{\partial a_{ij}} = \frac{\partial {\mu'}^j_{kl}}{\partial a_{ij}} = -\frac{\sin a_{ij}}{\sin a_{ik} \sin a_{il}}.$$

The first identity of the lemma follows.
Then we subsequently compute

$$-\varepsilon_i \, \mu^j_{kl} \, \frac{\partial b^j_{kl}}{\partial a_{ik}} = \frac{\partial {\mu'}^j_{kl}}{\partial a_{ik}} = -\frac{\cos a_{il} + \cos a_{ij} \cos a_{ik}}{\sin a_{ij} \sin a_{ik}} \, \frac{\sin a_{ij}}{\sin a_{ik} \sin a_{il}} =$$

$${\mu'}^l_{jk} \, \frac{\sin a_{ij}}{\sin a_{ik} \sin a_{il}}.$$

The second identity follows. The third one is analogous under the permutation of the indices k and l. □
Now we shall prove several identities that relate the principal minors G_{ii}, $i \in \{1,2,3,4\}$ of the Gram matrix $G := G(T)$ of the tetrahedron T with its face or vertex momenta.

Lemma 3 *For* $\{i,j,k,l\} = \{1,2,3,4\}$, *we have that*

$$\det G_{ii} = \varepsilon_i \, M^2_{jkl}.$$

Proof. Let us perform the computation for G_{11} and other cases will follow by analogy. We have that

$$\det \begin{pmatrix} 1 & -\cos a_{14} & -\cos a_{13} \\ -\cos a_{14} & 1 & -\cos a_{12} \\ -\cos a_{13} & -\cos a_{12} & 1 \end{pmatrix} =$$

$$= \det \begin{pmatrix} 1 & -\cos a_{14} & -\cos a_{13} \\ 0 & \sin^2 a_{14} & -\mu'^2_{34} \sin a_{13} \sin a_{14} \\ 0 & -\mu'^2_{34} \sin a_{13} \sin a_{14} & \sin^2 a_{13} \end{pmatrix} =$$

$$= (1 - (\mu'^2_{34})^2) \sin^2 a_{13} \sin^2 a_{14} = \varepsilon_1 (\mu^2_{34})^2 \sin^2 a_{13} \sin^2 a_{14} = \varepsilon_1 M^2_{234}.$$

By permuting the set $\{i, j, k, l\} = \{1, 2, 3, 4\}$, one gets all other identities of the lemma. □

Lemma 4 *For* $\{i, j, k, l\} = \{1, 2, 3, 4\}$, *we have that*

$$-\det G = \sin^2 a_{jk} \, \sin^2 a_{jl} \, \sin^2 a_{kl} \, (M^i)^2.$$

Proof. Let us subsequently compute

$$\det G = \det \begin{pmatrix} 1 & -\cos a_{34} & -\cos a_{24} & -\cos a_{23} \\ -\cos a_{34} & 1 & -\cos a_{14} & -\cos a_{13} \\ -\cos a_{24} & -\cos a_{14} & 1 & -\cos a_{12} \\ -\cos a_{23} & -\cos a_{13} & -\cos a_{12} & 1 \end{pmatrix} =$$

$$\det \begin{pmatrix} 1 & -\cos a_{34} & -\cos a_{24} & -\cos a_{23} \\ 0 & \sin^2 a_{34} & -\mu'^1_{23} \sin a_{24} \sin a_{34} & -\mu'^1_{24} \sin a_{23} \sin a_{34} \\ 0 & -\mu'^1_{23} \sin a_{24} \sin a_{34} & \sin^2 a_{24} & -\mu'^1_{34} \sin a_{23} \sin a_{24} \\ 0 & -\mu'^1_{24} \sin a_{23} \sin a_{34} & -\mu'^1_{34} \sin a_{23} \sin a_{24} & \sin^2 a_{23} \end{pmatrix} =$$

$$\sin^2 a_{23} \, \sin^2 a_{24} \sin^2 a_{34} \quad \det \begin{pmatrix} 1 & -\mu'^1_{23} & -\mu'^1_{24} \\ -\mu'^1_{23} & 1 & -\mu'^1_{34} \\ -\mu'^1_{24} & -\mu'^1_{34} & 1 \end{pmatrix} =$$

$$\sin^2 a_{23} \, \sin^2 a_{24} \sin^2 a_{34} \quad \det \begin{pmatrix} 1 & -\mu'^1_{23} & -\mu'^1_{24} \\ 0 & \varepsilon_4 (\mu^1_{23})^2 & -\sigma'_{34} \mu^1_{23} \mu^1_{24} \\ 0 & -\sigma'_{34} \mu^1_{23} \mu^1_{24} & \varepsilon_3 (\mu^1_{24})^2 \end{pmatrix} =$$

$$\sin^2 a_{23} \, \sin^2 a_{24} \sin^2 a_{34} \quad (\varepsilon_3 \varepsilon_4 - (\sigma'_{34})^2)(\mu^1_{23} \mu^1_{24})^2 =$$

$$-\sin^2 a_{23} \, \sin^2 a_{24} \sin^2 a_{34} \quad (M^1)^2.$$

Here we used the Cosine Law for links in the second equality and the first Cosine Law for faces in the fourth equality. Also, we used the fact that for $\{i, j, k, l\} = \{1, 2, 3, 4\}$ one has $1 - \varepsilon_l (\mu^i_{jk})^2 = (\mu'^i_{jk})^2$ (in the third equality) and $\sigma^2_{ij} - (\sigma'_{ij})^2 = \varepsilon_i \varepsilon_j$ (in the sixth equality). All other identities of the lemma follow by permuting the set $\{i, j, k, l\} = \{1, 2, 3, 4\}$. □

4 Dual Jacobian of a generalised hyperbolic tetrahedron

In this section we shall compute the entries of the dual Jacobian matrix $\mathrm{Jac}^{\star}(T)$ of a generalised hyperbolic tetrahedron T.

Theorem 6 *Let T be a generalised hyperbolic tetrahedron. Then*

$$\mathrm{Jac}^{\star}(T) := \frac{\partial(\ell_{12}, \ell_{13}, \ell_{14}, \ell_{23}, \ell_{24}, \ell_{34})}{\partial(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34})} = -\eta\, \mathscr{D}\mathscr{S}\mathscr{D},$$

where

$$\eta := \left(\frac{\Pi_{i=1}^{4} \varepsilon_i \, \det G_{ii}}{(-\det G)^3} \right)^{1/2}, \quad \mathscr{D} := \begin{pmatrix} \sigma_{12} & & & & & \\ & \sigma_{13} & & & & \\ & & \sigma_{14} & & & \\ & & & \sigma_{23} & & \\ & & & & \sigma_{24} & \\ & & & & & \sigma_{34} \end{pmatrix}$$

and

$$\mathscr{S} := \begin{pmatrix} \omega_{12} & \varepsilon_1\sigma'_{14} & \varepsilon_1\sigma'_{13} & \varepsilon_2\sigma'_{24} & \varepsilon_2\sigma'_{23} & 1 \\ \varepsilon_1\sigma'_{14} & \omega_{13} & \varepsilon_1\sigma'_{12} & \varepsilon_3\sigma'_{34} & 1 & \varepsilon_3\sigma'_{23} \\ \varepsilon_1\sigma'_{13} & \varepsilon_1\sigma'_{12} & \omega_{14} & 1 & \varepsilon_4\sigma'_{34} & \varepsilon_4\sigma'_{24} \\ \varepsilon_2\sigma'_{24} & \varepsilon_3\sigma'_{34} & 1 & \omega_{23} & \varepsilon_2\sigma'_{12} & \varepsilon_3\sigma'_{13} \\ \varepsilon_2\sigma'_{23} & 1 & \varepsilon_4\sigma'_{34} & \varepsilon_2\sigma'_{12} & \omega_{24} & \varepsilon_4\sigma'_{14} \\ 1 & \varepsilon_3\sigma'_{23} & \varepsilon_4\sigma'_{24} & \varepsilon_3\sigma'_{13} & \varepsilon_4\sigma'_{14} & \omega_{34} \end{pmatrix},$$

where

$$\omega_{kl} := \frac{\sigma'_{ik}\sigma'_{jl} + \varepsilon_l \sigma'_{il}\sigma'_{jl}\sigma'_{kl} + \sigma'_{il}\sigma'_{jk} + \varepsilon_k \sigma'_{ik}\sigma'_{jk}\sigma'_{kl}}{\sigma_{kl}^2}.$$

Proof. We compute the respective derivatives, that constitute the entries of $\mathrm{Jac}^{\star}(T)$. Suppose that $\varepsilon_i \neq 0$, $i \in \{1,2,3,4\}$, since the cases when $\varepsilon_j = 0$ for some $j \in \{1,2,3,4\}$ can be dealt with in an analogous manner. Then for $\{i,j,k,l\} = \{1,2,3,4\}$, one has

$$\frac{\partial \ell_{kl}}{\partial a_{ij}} = \frac{\partial \ell_{kl}}{\partial b^j_{kl}} \frac{\partial b^j_{kl}}{\partial a_{ij}} = -\varepsilon_i \frac{\mu^j_{kl}}{M^j} \cdot \varepsilon_i \frac{\sin a_{ij}}{M_{jkl}} \overbrace{=}^{(1)} -\frac{1}{M^j} \frac{\sin a_{ij}}{\sin a_{ik} \sin a_{il}} =$$

$$-\frac{1}{M^j} \frac{\sin a_{ij}}{\sin a_{ik} \sin a_{il}} \frac{1}{\sigma_{ij}} \frac{1}{\sigma_{kl}} \sigma_{ij}\sigma_{kl} \overbrace{=}^{(2)} -\frac{1}{M^j} \frac{\sin a_{ij}}{\sin a_{ik} \sin a_{il}} \sigma_{ij}\sigma_{kl} \frac{\mu^i_{jk}\mu^i_{jl}}{M^i} \frac{\mu^l_{ik}\mu^l_{jk}}{M^l} \overbrace{=}^{(3)}$$

$$-\frac{M_{ijk}M_{ikl}M_{ijl}M_{jkl}}{\sqrt{(-\det G)^3}}\,\sigma_{ij}\sigma_{kl} \overbrace{=}^{(4)} -\sqrt{\frac{\Pi_{i=1}^4 \varepsilon_i \det G_{ii}}{(-\det G)^3}}\,\sigma_{ij}\sigma_{kl} = -\eta\,\sigma_{ij}\sigma_{kl}.$$

Here we used the definitions of vertex and face momenta, as well as Lemmata 1 and 4. Indeed, in (1) we have that $M_{ijk} = \mu^j_{kl}\,\sin a_{ik}\,\sin a_{il}$ and in (2) we use the fact that $\sigma_{ij} = \frac{\mu^l_{ik}\,\mu^l_{jk}}{M^l}$, $\sigma_{kl} = \frac{\mu^i_{jk}\,\mu^i_{jl}}{M^i}$. In (3) we use

$$\mu^i_{jk} = \frac{M_{ijk}}{\sin a_{jl}\,\sin a_{kl}},\quad \mu^l_{ik} = \frac{M_{ikl}}{\sin a_{ij}\,\sin a_{jk}},$$

$$\mu^i_{jl} = \frac{M_{ijl}}{\sin a_{jk}\,\sin a_{kl}},\quad \mu^l_{jk} = \frac{M_{jkl}}{\sin a_{ij}\,\sin a_{ik}},$$

together with the identities of Lemma 4. In (4) we use Lemma 3.
Analogous to the above, we compute for $\{i,j,k,l\} = \{1,2,3,4\}$,

$$\frac{\partial \ell_{kl}}{\partial a_{ik}} = \frac{\partial \ell_{kl}}{\partial b^i_{jk}}\,\frac{\partial b^i_{jk}}{\partial a_{ik}} = -\varepsilon_k \frac{\sin a_{ik}}{M_{ijl}}\,\frac{\mu^i_{kl}}{M^i}\,\sigma'_{jk} =$$

$$-\varepsilon_k \frac{\sqrt{\varepsilon_j \det G_{jj}}}{M^i \sin a_{jk} \sin a_{jl}}\,\frac{\sin a_{ik}}{\sqrt{\varepsilon_k \det G_{kk}}}\,\frac{1}{\sigma_{ik}\sigma_{kl}}\,\sigma_{ik}\sigma_{kl}\,\sigma'_{jk} =$$

$$-\varepsilon_k \frac{\sqrt{\varepsilon_j \det G_{jj}}}{M^i \sin a_{jk} \sin a_{jl}}\,\frac{\sin a_{ik}}{\sqrt{\varepsilon_k \det G_{kk}}}\,\frac{\mu^i_{jk}\mu^i_{jl}}{M^i}\,\frac{\mu^j_{il}\mu^j_{kl}}{M^j}\,\sigma_{ik}\sigma_{kl}\,\sigma'_{jk} =$$

$$-\varepsilon_k \frac{\sqrt{\varepsilon_j \det G_{jj}}}{\sqrt{\varepsilon_k \det G_{kk}}}\,\frac{\sqrt{\varepsilon_l \det G_{ll}}\sqrt{\varepsilon_k \det G_{kk}}\sqrt{\varepsilon_k \det G_{kk}}\sqrt{\varepsilon_i \det G_{ii}}}{\sqrt{(-\det G)^3}}\,\sigma_{ik}\sigma_{kl}\,\sigma'_{jk} =$$

$$-\varepsilon_k \sqrt{\frac{\Pi_{i=1}^4 \varepsilon_i \det G_{ii}}{(-\det G)^3}}\,\sigma_{ik}\sigma_{kl}\,\sigma'_{jk} = -\varepsilon_k\,\eta\,\sigma_{ik}\sigma_{kl}\,\sigma'_{jk}.$$

Finally, for $\{i,j,k,l\} = \{1,2,3,4\}$, we compute the derivative

$$\frac{\partial \ell_{kl}}{\partial a_{kl}} = \frac{\partial \ell_{kl}}{\partial b^i_{jk}}\,\frac{\partial b^i_{jk}}{\partial a_{kl}} + \frac{\partial \ell_{kl}}{\partial b^i_{jl}}\,\frac{\partial b^i_{jl}}{\partial a_{kl}}.$$

Since the two terms of the above sum are symmetric under the permutation of k and l, we may compute only the first one. The second one will be analogous. By Lemmata 1 and 2, we get

$$\frac{\partial \ell_{kl}}{\partial b^i_{jk}}\,\frac{\partial b^i_{jk}}{\partial a_{kl}} = -\varepsilon_l \frac{\mu^i_{kl}}{M^i}\,\frac{\sin a_{il}}{M_{ijk}}\,\sigma'_{jl}\,{\mu'}^j_{ik} \overbrace{=}^{(5)}$$

$$-(\sigma'_{ik}\sigma'_{jl} + \varepsilon_l \sigma'_{il}\sigma'_{jl}\sigma'_{kl})\, \frac{\mu^i_{kl}}{M^i}\, \frac{\sin a_{il}}{M_{ijk}}\, \frac{1}{\sigma_{il}\sigma_{kl}} \overbrace{=}^{(6)}$$

$$-(\sigma'_{ik}\sigma'_{jl} + \varepsilon_l \sigma'_{il}\sigma'_{jl}\sigma'_{kl})\, \frac{\sqrt{\varepsilon_j \det G_{jj}}}{M^i \sin a_{jk} \sin a_{jl}}\, \frac{\sin a_{il}}{\sqrt{\varepsilon_l \det G_{ll}}}\, \frac{\mu^i_{jk}\mu^i_{jl}}{M^i}\, \frac{\mu^j_{ik}\mu^j_{kl}}{M^j} \overbrace{=}^{(7)}$$

$$-(\sigma'_{ik}\sigma'_{jl} + \varepsilon_l \sigma'_{il}\sigma'_{jl}\sigma'_{kl}) \sqrt{\frac{\varepsilon_j \det G_{jj}}{(-\det G)^3}}\, \frac{\sqrt{\varepsilon_l \det G_{ll}}\sqrt{\varepsilon_k \det G_{kk}}\sqrt{\varepsilon_l \det G_{ll}}\sqrt{\varepsilon_i \det G_{ii}}}{\sqrt{\varepsilon_l \det G_{ll}}} =$$

$$-(\sigma'_{ik}\sigma'_{jl} + \varepsilon_l \sigma'_{il}\sigma'_{jl}\sigma'_{kl}) \sqrt{\frac{\Pi_{i=1}^4 \varepsilon_i \det G_{ii}}{(-\det G)^3}} = -\eta\, (\sigma'_{ik}\sigma'_{jl} + \varepsilon_l \sigma'_{il}\sigma'_{jl}\sigma'_{kl}).$$

Here, in (5) we used the second Cosine Law for faces and in (6) we used the equality $M_{ikl} = \mu^i_{kl} \sin a_{jk} \sin a_{jl}$ together with Lemma 3. In (7) we perform a computation analogous to (3).
Thus, we obtain

$$\frac{\partial \ell_{kl}}{\partial a_{kl}} = \frac{\partial \ell_{kl}}{\partial b^i_{jk}}\, \frac{\partial b^i_{jk}}{\partial a_{kl}} + \frac{\partial \ell_{kl}}{\partial b^i_{jl}}\, \frac{\partial b^i_{jl}}{\partial a_{kl}} =$$

$$-\eta\, (\sigma'_{ik}\sigma'_{jl} + \varepsilon_l \sigma'_{il}\sigma'_{jl}\sigma'_{kl}) - \eta\, (\sigma'_{il}\sigma'_{jk} + \varepsilon_k \sigma'_{ik}\sigma'_{jk}\sigma'_{kl}) = -\eta\, \omega_{kl}\, \sigma^2_{kl}.$$

The proof is completed. □

5 Dual Jacobian of a doubly truncated hyperbolic tetrahedron

Let us consider the case when T is a (mildly) doubly truncated tetrahedron depicted in Fig. 2 with dihedral angles θ_i and edge lengths ℓ_i, $i \in \{1, 2, 3, 4, 5, 6\}$. We suppose that the vertices cut off by the respective polar planes are v_1 and v_2.
If T is mildly truncated then the formula from Theorem 6 applies. If T is a prism truncated tetrahedron, as in Fig. 3, with dihedral angles μ, θ_i and edge lengths ℓ, ℓ_i, $i \in \{1, 2, 3, 5, 6\}$ then its Gram matrix is given by

$$G = \begin{pmatrix} 1 & -\cos\theta_1 & -\cos\theta_5 & -\cos\theta_3 \\ -\cos\theta_1 & 1 & -\cos\theta_6 & -\cos\theta_2 \\ -\cos\theta_5 & -\cos\theta_6 & 1 & -\cosh\ell \\ -\cos\theta_3 & -\cos\theta_2 & -\cosh\ell & 1 \end{pmatrix},$$

which is a slightly different notation compared to [12, 13].
Each link $L(\mathrm{v}_k)$, $k = 1, 2$, is a hyperbolic quadrilateral with two right same-side angles, which can be seen as a hyperbolic triangle with a single truncated

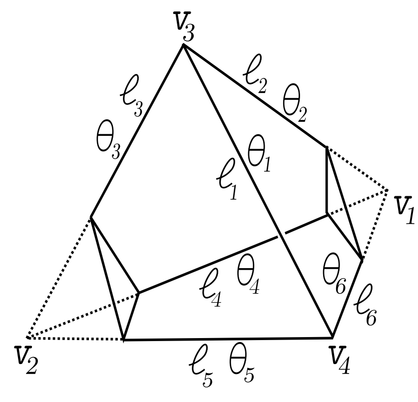


Figure 2: Doubly truncated tetrahedron (mild truncation)

vertex. Each link $L(\mathrm{v}_k)$, $k = 3, 4$, is a spherical triangle. In the definitions of Section 2 we change each b^i_{1j}, with $i, j \in \{2, 3, 4\}$, $i \neq j$, for $b^i_{1j} + \sqrt{-1}\frac{\pi}{2}$ and each b^i_{2j}, with $i, j \in \{1, 3, 4\}$, $i \neq j$, for $b^i_{2j} + \sqrt{-1}\frac{\pi}{2}$. Thus, some of the vertex and face momenta become complex numbers. All the trigonometric rules of Section 2 still hold grace to [2, Section 4.3]. Computing the respective derivatives in a complete analogy to the proof of Theorem 6, we obtain the following statement.

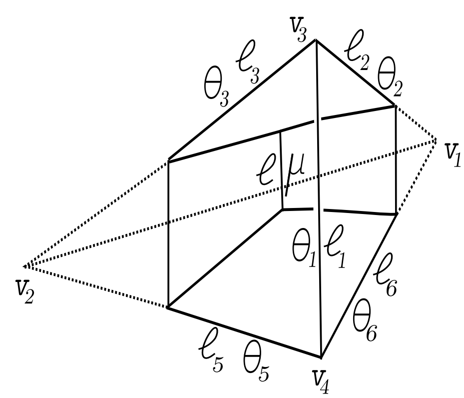


Figure 3: Doubly truncated tetrahedron (prism truncation)

Theorem 7 *Let T be a prism truncated tetrahedron depicted in Fig. 3. Then* *by means of the analytic continuation* $a_{12} := \sqrt{-1}\,\ell$, $\ell_{12} = \sqrt{-1}\,\mu$ *we have*

$$\mathrm{Jac}^\star(T) := \frac{\partial(\mu, \ell_1, \ell_2, \ell_3, \ell_5, \ell_6)}{\partial(\ell, \theta_1, \theta_2, \theta_3, \theta_5, \theta_6)} = \frac{\partial(\ell_{12}, \ell_{34}, \ell_{13}, \ell_{23}, \ell_{24}, \ell_{14})}{\partial(a_{12}, a_{34}, a_{13}, a_{23}, a_{24}, a_{14})}.$$

6 Volume of a hyperbolic prism

Let $\vec{\alpha}_n$ denote the n-tuple $(\alpha_1, \ldots, \alpha_n)$ with $0 < \alpha_k < \pi$, $k = 1, \ldots, n$. Let $\vec{\beta}_n$ and $\vec{\gamma}_n$ be analogous n-tuples. Let $\Pi_n := \Pi_n(\vec{\alpha}_n, \vec{\beta}_n, \vec{\gamma}_n)$ be the hyperbolic n-sided prism depicted in Fig. 4, with the respective dihedral angles, as shown in the picture.

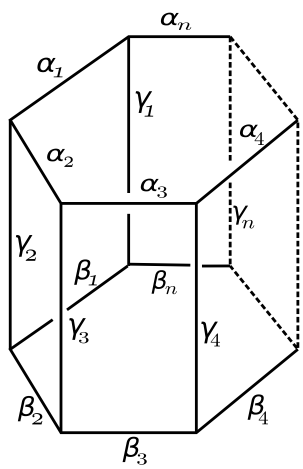


Figure 4: The prism $\Pi_n(\vec{\alpha}_n, \vec{\beta}_n, \vec{\gamma}_n)$

Let S_k, $k = 1, \ldots, n$, be the supporting hyperplane for the k-th side face of the prism Π_n (we start numbering the faces anti-clockwise from the side face adjacent to the angles α_1, β_1 and γ_1, γ_2), and let S_0 and S_{n+1} be those of the top and the bottom face, correspondingly. For each S_k, $k = 0, \ldots, n+1$, let S_k^+ be the respective half-space containing the unit outer normal to it. Let $S_k^- = \mathbb{H}^3 \setminus S_k^+$. Then $\Pi_n = \bigcap_{i=0}^{n+1} S_i^-$.
Let $T := T(\alpha, \alpha', \beta, \beta', \gamma; \ell)$ be the prism truncated tetrahedron depicted in Fig. 5. Here α, α', β, β' and γ are the respective dihedral angles, ℓ is the length of the respective edge. The volume $\mathrm{Vol}\,T$ of the tetrahedron T is given by [12, Theorem 1]*. Let $v(\alpha, \alpha', \beta, \beta', \gamma; \ell) := \mathrm{Vol}\,T(\alpha, \alpha', \beta, \beta', \gamma; \ell)$ denote the respective volume function.

*in Section 7 we give a simplified formula for the volume of T.

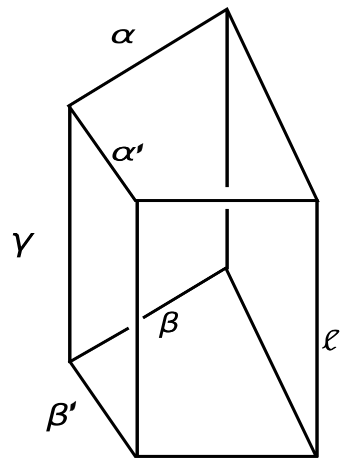


Figure 5: The prism truncated tetrahedron $T(\alpha, \alpha', \beta, \beta', \gamma; \ell)$

Let $p_0 p_{n+1}$ be the common perpendicular to S_0 and S_{n+1}. Let also define $k \oplus m := (k+m) \bmod n$, for $k, m \in \mathbb{N}$. Then we can state the main theorem of this section.

Theorem 8 *Let $\Pi_n = \Pi_n(\vec{\alpha}_n, \vec{\beta}_n, \vec{\gamma}_n)$ be a hyperbolic n-sided prism, as in Fig. 4. If $p_0 p_{n+1} \subset \Pi_n$, then the volume of Π_n is given by the formula*

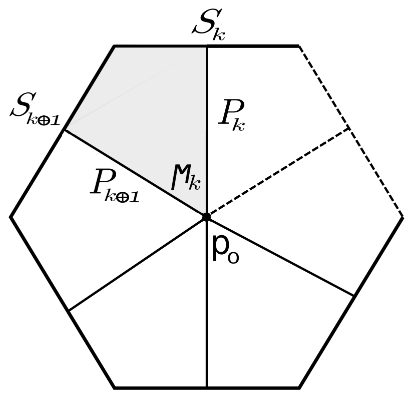
$$\operatorname{Vol} \Pi_n = \sum_{k=1}^{n} v(\alpha_k, \alpha_{k\oplus 1}, \beta_k, \beta_{k\oplus 1}, \gamma_{k\oplus 1}; \ell^\star),$$

where $\ell^\star$ is the unique solution to the equation $\frac{\partial \Phi}{\partial \ell}(\ell) = 0$, with

$$\Phi(\ell) := \pi \ell + \sum_{k=1}^{n} v(\alpha_k, \alpha_{k\oplus 1}, \beta_k, \beta_{k\oplus 1}, \gamma_{k\oplus 1}; \ell).$$

Let P_k, $k = 1, \ldots, n$, be the plane containing $p_0 p_{n+1}$ and orthogonal to S_k. First, we consider the case when $p_0 p_{n+1}$ lies inside the prism Π_n and the planes P_k, $k = 1, \ldots, n$, divide the prism Π_n into n prism truncated tetrahedra, as shown in Fig. 6.

Then each P_k meets the k-th side face of the prism Π_n. Thus, the planes S_0, S_k, $S_{k\oplus 1}$ and S_{n+1} together with P_k and $P_{k\oplus 1}$ become the supporting planes for the faces of a prism truncated tetrahedron, which we denote by T_k. Each P_k is orthogonal to S_k, S_0 and S_{n+1}. The dihedral angles of



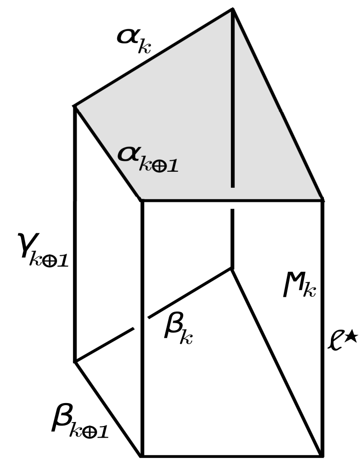


Figure 6: The decomposition of Π_n (top view, on the left) and the prism truncated tetrahedron T_k (on the right)

T_k inherited from the prism Π_n are easily identifiable. Let μ_k denote the dihedral angle along the edge p_0p_{n+1} and let $\ell^\star$ be its length. Then we have $T_k = T(\alpha_k, \alpha_{k\oplus 1}, \beta_k, \beta_{k\oplus 1}, \gamma_{k\oplus 1}; \ell^\star)$, $k = 1, \ldots, n$. Clearly,

$$\operatorname{Vol} \Pi_n = \sum_{k=1}^{n} \operatorname{Vol} T_k = \sum_{k=1}^{n} v(\alpha_k, \alpha_{k\oplus 1}, \beta_k, \beta_{k\oplus 1}, \gamma_{k\oplus 1}; \ell^\star).$$

Thus, we have to prove only the following statement.

Proposition 1 *If the common perpendicular p_0p_{n+1} is inside the prism Π_n and each P_k meets the respective side also inside Π_n, $k = 1, \ldots, n$, then the equation $\frac{\partial \Phi}{\partial \ell} = 0$ has a unique solution $\ell = \ell^\star$, the length of p_0p_{n+1}.*

Proof. Let us consider the collection of prism truncated tetrahedra $T_k = T(\alpha_k, \alpha_{k\oplus 1}, \beta_k, \beta_{k\oplus 1}, \gamma_{k\oplus 1}; \ell)$, $k = 1, \ldots, n$. Each pair $\{T_k, T_{k\oplus 1}\}$ of them has an isometric face corresponding to the plane $P_{k\oplus 1}$. Indeed, each such face is completely determined by the plane angles (two right angles at the side of length ℓ, the angles α_k and β_k at the opposite side) and one side length. We obtain the prism $\Pi_n(\vec{\alpha}_n, \vec{\beta}_n, \vec{\gamma}_n)$ by glueing the tetrahedra T_k together along the faces P_k, $k = 1, \ldots, n$, in the respective order. Their edges of length ℓ match together, and one obtains a prism if the angle sum of the dihedral angles μ_k, $k = 1, \ldots, n$, along them equals 2π. We have that

$$\frac{\partial \Phi}{\partial \ell} = \pi + \sum_{k=1}^{n} \frac{\partial v}{\partial \ell}(\alpha_k, \alpha_{k\oplus 1}, \beta_k, \beta_{k\oplus 1}, \gamma_{k\oplus 1}; \ell).$$

Since v is the volume function from [12, Theorem 1], then by applying the Schläfli formula [16, Equation 1] one obtains

$$\frac{\partial \Phi}{\partial \ell} = \pi - \frac{1}{2} \sum_{k=1}^{n} \mu_k.$$

Thus, whenever the tetrahedra T_k constitute a prism, we have $\sum_{k=1}^{n} \mu_k = 2\pi$ or, equivalently, $\frac{\partial \Phi}{\partial \ell} = 0$. The length ℓ in this case is exactly the length of the common perpendicular $p_0 p_{n+1}$ to the planes S_0 and S_{n+1}.
The rest is to prove that $\ell = \ell^\star$ is a unique solution. In order to do so, we shall show that $\frac{\partial \mu_k}{\partial \ell} > 0$, $k = 1, \ldots, n$. By using Theorem 7 we get the following formulae for a prism truncated tetrahedron (as depicted in Fig. 3):

$$\frac{\partial \ell_2}{\partial \ell} = -\eta \sin \mu_k \sinh \ell_6 \cosh \ell_2, \quad \frac{\partial \ell_3}{\partial \ell} = -\eta \sin \mu_k \sinh \ell_5 \cosh \ell_3,$$

$$\frac{\partial \ell_5}{\partial \ell} = -\eta \sin \mu_k \sinh \ell_3 \cosh \ell_5, \quad \frac{\partial \ell_6}{\partial \ell} = -\eta \sin \mu_k \sinh \ell_2 \cosh \ell_6.$$

Note that the above derivatives are all negative. In our present notation it means that for each prism truncated tetrahedron T_k, $k = 1, \ldots, n$, the edges of the top and bottom faces inherited from the prism Π_n diminish their length if we increase solely the parameter ℓ. Recall that $T_k = T(\alpha_k, \alpha_{k\oplus 1}, \beta_k, \beta_{k\oplus 1}, \gamma_{k\oplus 1}; \ell)$, and let us denote $T'_k := T(\alpha_k, \alpha_{k\oplus 1}, \beta_k, \beta_{k\oplus 1}, \gamma_{k\oplus 1}; \ell')$ with $\ell' > \ell$.
Let $ABCD$ be the top (equiv., bottom) face of T_k, as shown in Fig. 7, and $A'B'C'D'$ be the top (equiv., bottom) face of T'_k. Since the dihedral angles accept for μ_k and μ'_k remain the same, the plane angles of $ABCD$ at A, B, C and those of $A'B'C'D'$ at A', B' and C' are respectively equal. One sees easily that we can match then $ABCD$ and $A'B'C'D'$ such that B and B' coincide, the sides AB and $A'B'$, BC and $B'C'$ overlap and the point D' lies inside the quadrilateral $ABCD$. Then the area of $A'B'C'D'$ is less than that of $ABCD$. Equivalently, by the angle defect formula [1, Theorem 1.1.7], $\mu'_k > \mu_k$. Thus, $\frac{\partial \mu_k}{\partial \ell} > 0$, $k = 1, \ldots, n$, and the proposition follows. □
However, there is a possibility that, although the common perpendicular $p_0 p_{n+1}$ is entirely inside the prism Π_n, one (or several) of the planes P_k meets the respective S_k partially outside of the face S_k.

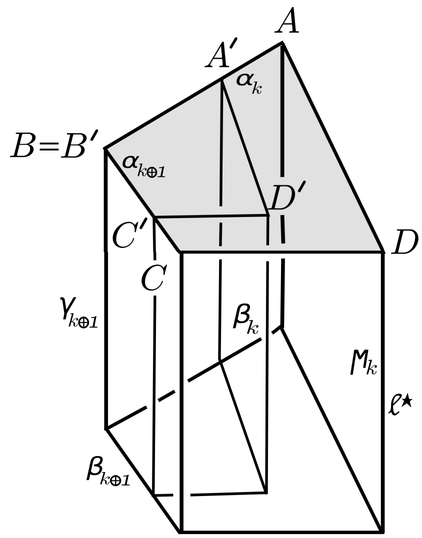


Figure 7: Prisms T_k and T'_k with top faces marked

First we consider the case when a single plane P_k meets S_k entirely outside, as depicted in Fig. 8. Like this, we obtain the figure shaded in grey, that consists of two triangular prisms sharing an edge.
Second we consider the case when a single plane P_k meets S_k partially outside, as depicted in Fig. 9. Like this, we obtain a more complicated figure that consists of two tetrahedra sharing an edge (one of which has two truncated vertices).
Thus the planes S_0, P_k, $P_{k\oplus 1}$, S_k, $S_{k\oplus 1}$ and S_{n+1} bound a “butterfly” prism. We put $k = 1$, for clarity. In the general case, $k \geq 2$, one uses induction on the number of planes P_k meeting S_k outside of Π_n. Here, some other cases of “butterfly” prisms are possible.

Proposition 2 *If the common perpendicular $p_0 p_{n+1}$ is completely inside the prism Π_n, the plane P_1 meets the plane S_1 outside of Π_n, and all other P_k, $k = 2, \ldots, n$, meet the respective side faces inside Π_n, then the volume of the prism equals*

$$\mathrm{Vol}\,\Pi_n = \sum_{k=1}^{n} v(\alpha_k, \alpha_{k\oplus 1}, \beta_k, \beta_{k\oplus 1}, \gamma_{k\oplus 1}; \ell^\star),$$

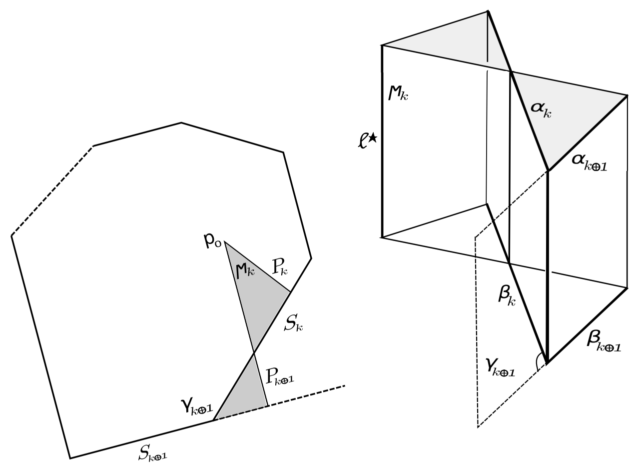


Figure 8: The decomposition of Π_n (top view, on the left) and the "butterfly" prism truncated tetrahedron T_k (on the right)

where $\ell^\star$ is the unique solution to the equation $\frac{\partial\Phi}{\partial\ell}(\ell) = 0$, with

$$\Phi(\ell) := \pi\ell + \sum_{k=1}^{n} v(\alpha_k, \alpha_{k\oplus 1}, \beta_k, \beta_{k\oplus 1}, \gamma_k; \ell).$$

Proof. We start with the case of a "butterfly" prism depicted in Fig. 8. Let us observe that the "butterfly" prism T_1 overlaps with the subsequent prism truncated tetrahedron T_2 exactly on its part $T_1^{(o)}$ outside of Π_n. The part of T_1 inside Π_n, called $T_1^{(i)}$, contributes to the total volume of the prism. The volume of $T_1^{(o)}$ is excessive in the respective volume formula and should be subtracted. In fact, we prove that

$$v(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_2; \ell^\star) = V := \mathrm{Vol}\, T_1^{(i)} - \mathrm{Vol}\, T_1^{(o)},$$

which implies that the excess in volume brought by T_2 is eliminated by the term "$-\mathrm{Vol}\, T_1^{(o)}$".

In order to do so, let us denote by θ the dihedral angle along the common edge of the triangular prisms $T_1^{(o)}$ and $T_1^{(i)}$. Let ℓ_θ be the length of this edge. Let $\gamma := \gamma_2$ and let ℓ_γ be the length of the vertical edge with dihedral angle γ. We know that $\frac{\partial V}{\partial \gamma} = -\frac{1}{2}\,\ell_\gamma$, by the structure of the volume formula for a prism truncated tetrahedron. Indeed, the function V does not correspond to the volume of a real prism truncated tetrahedron any more, however all the

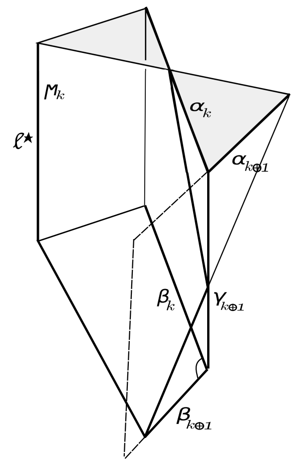


Figure 9: Another "butterfly" prism truncated tetrahedron T_k

metric relations defining the dihedral angles between the respective planes are preserved. Thus, after computing the derivative $\frac{\partial V}{\partial \ell}$ analogous to [12], we obtain the latter equality. Now we compute the respective derivatives for the parts of the "butterfly" prism T_1.

Observe that the parameter θ depends on γ, while we vary γ and keep all other dihedral angles fixed. Let us denote $\hat{\gamma} = \pi - \gamma$ for brevity. We have that

$$\frac{\partial \mathrm{Vol}\, T_1^{(o)}}{\partial \hat{\gamma}} = -\frac{\ell_\gamma}{2} - \frac{\ell_\theta}{2} \frac{\partial \theta}{\partial \hat{\gamma}}$$

and

$$\frac{\partial \mathrm{Vol}\, T_1^{(i)}}{\partial \gamma} = -\frac{\ell_\theta}{2} \frac{\partial \theta}{\partial \gamma},$$

by the Schläfli formula [16, Equation 1].

The above identities together with the fact that $\frac{\partial}{\partial \hat{\gamma}} = -\frac{\partial}{\partial \gamma}$ imply that

$$\frac{\partial}{\partial \gamma_2} v(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_2; \ell^\star) = \frac{\partial V}{\partial \gamma_2}.$$

By analogy, we can prove that

$$\frac{\partial}{\partial \xi} v(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_2; \ell^\star) = \frac{\partial V}{\partial \xi},$$

for any $\xi \in \{\alpha_1, \alpha_n, \beta_1, \beta_n, \mu_1\}$. The volume formula for a prism truncated tetrahedron implies that by setting $\alpha_1 = \alpha_n = \pi/2$ and $\beta_1 = \beta_n = \pi/2$ we

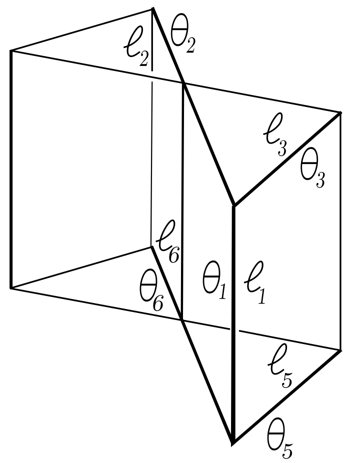


Figure 10: Parametrising the "butterfly" prism depicted in Fig. 8

get $v(\alpha_1, \alpha_n, \beta_1, \beta_n, \gamma_2; \ell^\star) = 0$. In the case of a "butterfly" prism T_1, under the same assignment of dihedral angles, we have that the bases of the two triangular prisms become orthogonal to their lateral sides. Thus $T_1^{(i)}$ and $T_1^{(o)}$ degenerate into Euclidean prisms, which means that their volumes tend to zero. Thus, we obtain the identity $v(\alpha_1, \alpha_n, \beta_1, \beta_n, \gamma_2; \ell^\star) = V$.
The proof of the monotonicity for the function $\frac{\partial \Phi}{\partial \ell}(\ell)$ is analogous to that in Proposition 1. However, since the part $T_1^{(o)}$ contributes to the function $v(\alpha_1, \alpha_n, \beta_1, \beta_n, \gamma_2; \ell)$ with the negative sign, we have to replace the edge lengths ℓ_3 and ℓ_5 with $-\ell_3$ and $-\ell_5$, respectively, as shown in Fig. 10. Then we recompute the respective derivatives of the lengths of the horizontal edges according to Theorem 7. We obtain that the lengths ℓ_2 and ℓ_6 diminish, as before, while the lengths ℓ_3 and ℓ_5 increase. This implies that the upper (resp., lower) triangular base of $T_1^{(i)\prime}$ can be placed entirely inside the upper (resp. lower) triangular base of $T_1^{(i)}$. By the area comparison argument, we have that $\mu_1' > \mu_1$. The inequality $\frac{\partial \mu_1}{\partial \ell} > 0$ follows.
All other cases of "butterfly prisms" (e.g. that in Fig. 9) can be considered by analogy. $\square$

Remark. In the general case, when the common perpendicular $p_0 p_{n+1}$ does not lie entirely inside the prism Π_n, we expect that an analogue to Theorem 8 holds with an exception that the equation $\frac{\partial \Phi}{\partial \ell}(\ell) = 0$ may have several solutions. However, one of these solutions is geometric and yields the volume of Π_n.

7 Modified volume formula

We modify the volume formula for a prism truncated tetrahedron from [12], in order to reduce it to a simpler form. Indeed, the formula in [12, Theorem 1] uses analytic continuation and accounts for possible branching with respect to any variable $a_j = e^{\ell}$, with some $j \in \{1, 2, \ldots, 6\}$, and $a_k = e^{i\,\theta_k}$, for any $k \in \{1, 2, \ldots, 6\}\backslash\{j\}$. Usually, we put $j = 4$ for simplicity. However, the formula allows for intense truncation at any edge, since it is invariant under a permutation of the variables a_l, $l \in \{1, \ldots, 6\}$.
In our case, given a prism Π_n and its decomposition into prism truncated tetrahedra T_i, $i \in \{1, \ldots, n\}$, we know that only the common perpendicular p_0p_{n+1} is produced by an intense truncation. Thus, we can always put $j = 4$ and, moreover, the variable a_4 will be the only one that might cause branching.
In this case, we suggest a simplified version of the formula from [12, Theorem 1]. This formula also has less numeric discrepancies and performs faster, if used for an actual computation.
Let us put $a_k := e^{i\,\theta_k}$, $k \in \{1, 2, 3, 5, 6\}$, $a_4 := e^{\ell}$, and let $\mathscr{U} = \mathscr{U}(a_1, a_2, a_3, a_4, a_5, a_6, z)$ denote

$$\begin{aligned}\mathscr{U} :=\ & \mathrm{Li}_2(z) + \mathrm{Li}_2(a_1a_2a_4a_5z) + \mathrm{Li}_2(a_1a_3a_4a_6z) + \mathrm{Li}_2(a_2a_3a_5a_6z)\\ & -\mathrm{Li}_2(-a_1a_2a_3z) - \mathrm{Li}_2(-a_1a_5a_6z) - \mathrm{Li}_2(-a_2a_4a_6z) - \mathrm{Li}_2(-a_3a_4a_5z),\end{aligned}$$

where $\mathrm{Li}_2(\circ)$ is the dilogarithm function.
Let z_- and z_+ be two solutions to the equation $e^{z\frac{\partial \mathscr{U}}{\partial z}} = 1$ in the variable z. According to [12, 20], these are

$$z_- := \frac{-q_1 - \sqrt{q_1^2 - 4q_0q_2}}{2q_2} \quad \text{and} \quad z_+ := \frac{-q_1 + \sqrt{q_1^2 - 4q_0q_2}}{2q_2},$$

where

$$q_0 := 1 + a_1a_2a_3 + a_1a_5a_6 + a_2a_4a_6 + a_3a_4a_5 + a_1a_2a_4a_5 + a_1a_3a_4a_6 + a_2a_3a_5a_6,$$

$$\begin{aligned}q_1 := -a_1a_2a_3a_4a_5a_6\Bigg(&\left(a_1 - \frac{1}{a_1}\right)\left(a_4 - \frac{1}{a_4}\right) + \left(a_2 - \frac{1}{a_2}\right)\left(a_5 - \frac{1}{a_5}\right)\\ &+ \left(a_3 - \frac{1}{a_3}\right)\left(a_6 - \frac{1}{a_6}\right)\Bigg),\end{aligned}$$

$$\begin{aligned}q_2 := a_1a_2a_3a_4a_5a_6(&a_1a_4 + a_2a_5 + a_3a_6 + a_1a_2a_6 + a_1a_3a_5 + a_2a_3a_4 +\\ & a_4a_5a_6 + a_1a_2a_3a_4a_5a_6).\end{aligned}$$

Given a function $f(x,y,\ldots,z)$, let $f(x,y,\ldots,z)\,|_{z=z_+}^{z=z_-}$ denote the difference $f(x,y,\ldots,z_-) - f(x,y,\ldots,z_+)$. Now we define the following function $\mathscr{V} = \mathscr{V}(a_1,a_2,a_3,a_4,a_5,a_6,z)$ by means of the equality

$$\mathscr{V} := \frac{i}{4}\left(\mathscr{U}(a_1,a_2,a_3,a_4,a_5,a_6,z) - z\,\frac{\partial \mathscr{U}}{\partial z}\,\log z\right)\Bigg|_{z=z_+}^{z=z_-}.$$

Proposition 3 *The volume of a prism truncated tetrahedron T is given by*

$$\operatorname{Vol} T = \Re\left(-\mathscr{V} + a_4\frac{\partial \mathscr{V}}{\partial a_4}\,\log a_4\right).$$

Proof. Let us denote

$$f(T) = \Re\left(-\mathscr{V} + a_4\frac{\partial \mathscr{V}}{\partial a_4}\,\log a_4\right),$$

and compute the derivative

$$\frac{\partial}{\partial \ell}\left(f(T) + \frac{\mu\,\ell}{2}\right) = a_4\,\frac{\partial}{\partial a_4}\left(f(T) + \frac{\mu\,\log a_4}{2}\right) =$$

$$= a_4\,\frac{\partial}{\partial a_4}\left(\Re\left(-\mathscr{V} + \left(a_4\,\frac{\partial \mathscr{V}}{\partial a_4} + \frac{\mu}{2}\right)\log a_4\right)\right).$$

The function $\Re\left(a_4\,\frac{\partial \mathscr{V}}{\partial a_4} + \frac{\mu}{2}\right)$ has an a.e. vanishing derivative, c.f. the note in [12] after Theorem 1 saying that $\mu \equiv -2\,\Re(a_4\,\frac{\partial \mathscr{V}}{\partial a_4}) \mod \pi$. Hence,

$$\frac{\partial}{\partial \ell}\left(f(T) + \frac{\mu\,\ell}{2}\right) = a_4\,\frac{\partial}{\partial a_4}\left(\Re\left(-\mathscr{V} + \left(a_4\,\frac{\partial \mathscr{V}}{\partial a_4} + \frac{\mu}{2}\right)\log a_4\right)\right)\overbrace{=}^{(1)}$$

$$\overbrace{=}^{(1)}\Re\left(-a_4\,\frac{\partial \mathscr{V}}{\partial a_4} + a_4\,\frac{\partial \mathscr{V}}{\partial a_4} + \frac{\mu}{2}\right) = \frac{\mu}{2}.$$

The equality (1) holds because of the commutativity of the operations $\Re$ and $\frac{\partial}{\partial a_4}$ for the function $-\mathscr{V} + \left(a_4\,\frac{\partial \mathscr{V}}{\partial a_4} + \frac{\mu}{2}\right)\log a_4$. The latter holds since $a_4 = e^{\ell}$ is a real parameter.
This implies that $\frac{\partial f(T)}{\partial \mu} = -\frac{\ell}{2}$. By analogy to the proof of [12, Theorem 1], we can show that $\frac{\partial f(T)}{\partial \theta_k} = -\frac{\ell_k}{2}$, and that if T degenerates into a right Euclidean prism, then $f(T) \to 0$. Thus, $\operatorname{Vol} T = f(T)$ and the proposition follows. □
Also, we have the following way to determine the dihedral angle μ along the length ℓ edge coming from the intense truncation.

Proposition 4 *The angle μ is given by*

$$\mu \equiv -\Re\left(\frac{i\,a_4}{2}\,\frac{\partial\mathscr{U}(a_1,\dots,a_6,z)}{\partial a_4}\Big|_{z=z_+}^{z=z_-}\right) \quad \mathrm{mod}\ \pi.$$

Proof. We have $\mu \equiv -2\Re\left(a_4\,\frac{\partial\mathscr{V}}{\partial a_4}\right)$ mod π, where $0<\mu<\pi$ and has an a.e. vanishing derivative.
Then we compute

$$\frac{\partial\mathscr{U}(a_1,\dots,a_6,z_\pm(a_1,\dots,a_6))}{\partial a_4} - \frac{\partial}{\partial a_4}\left(z_\pm\,\frac{\partial\mathscr{U}(a_1,\dots,a_6,z_\pm)}{\partial z}\,\log z_\pm\right) =$$

$$\frac{\partial\mathscr{U}(a_1,\dots,a_6,z_\pm)}{\partial a_4} + \frac{\partial z_\pm}{\partial a_4}\,\frac{\partial\mathscr{U}(a_1,\dots,a_6,z_\pm)}{\partial z} - \frac{\partial z_\pm}{\partial a_4}\,\frac{\partial\mathscr{U}(a_1,\dots,a_6,z_\pm)}{\partial z} = \frac{\partial\mathscr{U}(a_1,\dots,a_6,z_\pm)}{\partial a_4}, \qquad (1)$$

since, for some $m\in\mathbb{Z}$,

$$z_\pm\,\frac{\partial\mathscr{U}(a_1,\dots,a_6,z_\pm)}{\partial z} = 2\pi\,i\,m,$$

by the definition of z_- and z_+.
Therefore, we obtain

$$\mu \equiv -2\Re\left(a_4\,\frac{\partial\mathscr{V}}{\partial a_4}\right)\ \mathrm{mod}\ \pi \equiv -\Re\left(\frac{i\,a_4}{2}\,\frac{\partial\mathscr{U}(a_1,\dots,a_6,z)}{\partial a_4}\Big|_{z=z_+}^{z=z_-}\right)\ \mathrm{mod}\ \pi,$$

where $0<\mu<\pi$. □

8 Numerical examples

Finally, we produce some numerical examples concerning an n-gonal ($n\geq 5$) prism Π_n with the following distribution of dihedral angles: the angles along the vertical edges are $\frac{2\pi}{5}$, the angles adjacent to the bottom face are $\frac{\pi}{3}$, and those adjacent to the top face are $\frac{\pi}{2}$. Indeed, such a prism Π_n exists due to [8, Theorem 1.1]. Then we apply Theorem 8 for the cases $n=5,6,7$, and perform all necessary numeric computations with Wolfram Mathematica®.

In order to avoid excessive branching in numerical computations, we use the modified parameters

$$q_i' := \frac{q_i}{\Pi_{k=1}^6 a_k} \quad \text{and} \quad z_\pm := \frac{-q_1' - \sqrt{q_1'^2 \pm 4q_0'q_2'}}{2q_2'}.$$

in the formulae for $\mathscr{U}$ and $\mathscr{V}$ from Section 7.

It follows from the definition of q_i', $i = 1, 2, 3$, above that the quantity $q_1'^2 - 4q_0'q_2'$ is a real number, c.f. [20, Section 1.1, Lemma]. This fact prevents computational discrepancies and simplifies any further numerical analysis of the volume formula.

n	$(\ell^\star, \mu)$	$\mathrm{Vol}\,\Pi_n$
5	$(0.50672, 2\pi/5)$	2.63200
6	$(0.38360, \pi/3)$	3.43626
7	$(0.312595, 2\pi/7)$	4.19077

Table 1: Left: parameters $(\ell^\star, \mu)$ of T_n, right: volume of Π_n

Each of the above prisms Π_n can be subdivided into n isometric copies of a prism truncated tetrahedron T_n. Indeed, T_n is a prism truncated tetrahedron with angles $\theta_1 = \frac{2\pi}{5}$, $\theta_2 = \theta_3 = \frac{\pi}{2}$, $\theta_5 = \theta_6 = \frac{\pi}{3}$, and $\mu = \frac{2\pi}{n}$. By rotating it along the edge with dihedral angle μ, we compose the desired prism Π_n.

The graph of $\mathrm{Vol}\,T_n$, with $n = 5$, as a function of ℓ, is shown in Fig. 11 on the left. The graph of $\frac{\partial\Phi}{\partial\ell}(\ell)$ for the same prism truncated tetrahedron T_n is depicted in Fig. 11 on the right. We observe that the function $\frac{\partial\Phi}{\partial\ell}(\ell)$ is indeed monotone and has a single zero $\ell^\star \approx 0.50672....$

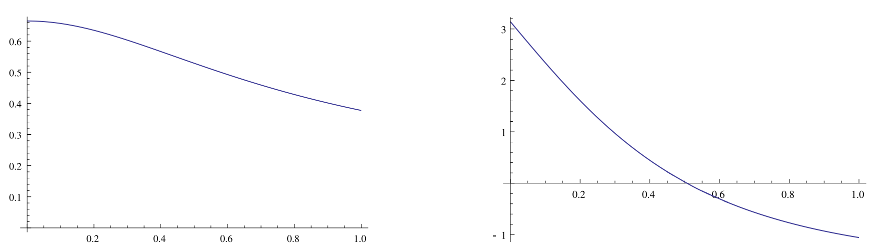


Figure 11: Left: $\mathrm{Vol}\,T_5$, right: $\frac{\partial\Phi}{\partial\ell}$, both as functions of ℓ

The volume of T_5 with $\theta_1 = \frac{2\pi}{5}$, $\theta_2 = \theta_3 = \frac{\pi}{2}$, $\theta_5 = \theta_6 = \frac{\pi}{3}$ and $\ell^\star \approx 0.50672...$ equals $\sim 0.52639...$ by Proposition 3. Thus, we can see that $\mathrm{Vol}\,\Pi_5 = 5 \cdot \mathrm{Vol}\,T_5$ in accordance with Theorem 8, and from Proposition 4 $\mu = 1.25664... \approx \frac{2\pi}{5}$.

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