

# A FIRST ORDER SYSTEM LEAST SQUARES METHOD FOR THE HELMHOLTZ EQUATION

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ABSTRACT. We present a first order system least squares (FOSLS) method for the Helmholtz equation at high wave number  $k$ , which always leads to a Hermitian positive definite algebraic system. By utilizing a non-trivial solution decomposition to the dual FOSLS problem which is quite different from that of the standard finite element methods, we give an error analysis to the  $hp$ -version of the FOSLS method where the dependence on the mesh size  $h$ , the approximation order  $p$ , and the wave number  $k$  is given explicitly. In particular, under some assumption of the boundary of the domain, the  $L^2$  norm error estimate of the scalar solution from the FOSLS method is shown to be quasi optimal under the condition that  $kh/p$  is sufficiently small and the polynomial degree  $p$  is at least  $O(\log k)$ . Numerical experiments are given to verify the theoretical results.

## 1. INTRODUCTION

Lots of least squares methods have been extensively studied for the efficient and accurate numerical approximation of many partial differential equations such as the elliptic, elasticity and Stokes equations. As mentioned in [10], there are three kinds of least-squares methods: the inverse approach, the div approach, and the div-curl approach. The interest of this paper is to consider the div approach least squares method which applies a chosen  $L^2$  norm to a natural first order system for the Helmholtz equation with Robin boundary condition which is the first order approximation of the radiation condition:

$$-\Delta u - k^2 u = f \quad \text{in } \Omega, \quad (1.1a)$$

$$\frac{\partial u}{\partial \mathbf{n}} - \mathbf{i}ku = g \quad \text{on } \partial\Omega, \quad (1.1b)$$

where  $\Omega \subset \mathbb{R}^d$  ( $d = 2$  or  $3$ ) is a bounded, Lipschitz and connected domain, the wave number  $k$  is real and positive, and  $\mathbf{i}$  denotes the imaginary unit. We want to point out that if the sign before  $\mathbf{i}$  in (1.1b) is positive, the corresponding least squares method and theoretical analysis in this paper also hold. We impose further assumptions on the domain  $\Omega$  in the following:

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- (A1) There is a constant  $C > 0$  such that for any  $f \in L^2(\Omega)$  and  $g \in L^2(\partial\Omega)$ , the Helmholtz equation (1.1) has a unique solution  $u \in H^1(\Omega)$  satisfying

$$\|\nabla u\|_{L^2(\Omega)} + k\|u\|_{L^2(\Omega)} \leq C (\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}).$$

- (A2) The boundary of  $\Omega$  is analytic.

The above assumptions are intrinsic for the analysis in this paper, while the least squares method can be applied for more general cases. In fact, [35] shows the assumption (A1) holds if the domain  $\Omega$  is star-shaped; and [6, Theorem 1.8] obtains the same estimate without the star-shaped restriction.

Due to the well-known pollution effect for the numerical solution of the Helmholtz equation, the standard Galerkin finite element methods can maintain a desired accuracy only if the mesh resolution is appropriately increased. Numerous nonstandard methods have been proposed in the literature to obtain more stable and accurate approximation, which includes quasi-stabilized finite element methods [3], absolutely stable discontinuous Galerkin (DG) methods [28, 29, 30, 34, 18], continuous interior penalty finite element methods [52, 53], the partition of unity finite element methods [2, 42], the ultra weak variational formulation [13], plane wave DG methods [1, 38], spectral methods [50], generalized Galerkin/finite element methods [5, 41], meshless methods [4], and the geometrical optics approach [26].

Generally, the linear systems from most of the above nonstandard Galerkin finite element approximations of the Helmholtz equation with high wave number  $k$  are strongly indefinite. But the least-squares Galerkin method for the Helmholtz equation always yields a Hermitian positive definite system [16, 39]. Hence it attracts the design of an efficient solver. For instance, a div-curl approach least squares method was applied to the Helmholtz equation in [39], and an efficient solver based on wave-ray multigrid was proposed. Recently, numerical results in [33] show that a multiplicative Schwarz algorithm, without coarse solver, provides a  $p$ -preconditioner for solving the DPG system. The numerical observations suggest that the condition number of the preconditioned system is independent of the wavenumber  $k$  and the polynomial degree  $p$ . Since both DPG methods and FOSLS are residual minimization methods such that their linear systems are Hermitian positive definite, it is promising that the multiplicative Schwarz preconditioner in [33] will provide similar preconditioning for FOSLS. We will show the effect of the multiplicative Schwarz preconditioner for our FOSLS in a separate paper.

A key result revealed by J.M. Melenk and S. Sauter in [46] is that the polynomial degree  $p$  should be chosen in a wavenumber-dependent way to yield optimal convergent conditions. This important result was analyzed based on the standard Galerkin finite element method. It shows that, under the assumption that the solution operator for Helmholtz problems is polynomially bounded in  $k$ , quasi optimal convergence can be obtained under the conditions that  $kh/p$  is sufficiently small and the polynomial degree  $p$  is at least  $O(\log k)$ .

An objective of this paper is to extend the key result in [46] to the div approach FOSLS method, which will be called FOSLS method for brevity in the following. We use the standard Raviart-Thomas finite element space and continuous piece-wise polynomial finite element space for the discretization of the FOSLS method. The stability of the FOSLS solutions for the Helmholtz equation can be obtained by the property of FOSLS formulation and a Rellich-type identity approach. The main difficulty in the analysis lies in the establishment of quasi optimal convergence for the FOSLS method. We first mimic the technique proposed in [46] to decompose the Helmholtz solution into an oscillatory analytic part and a nonoscillatory

elliptic part. A key estimate for the oscillatory analytic part of the Helmholtz solution (cf. (4.5c) in Theorem 4.3) is further derived for the error analysis of the FOSLS method for the Helmholtz equation. Another crucial estimate lies in the derivation of the dependence of convergence on the polynomial degree  $p$ . A new  $H(\text{div})$  projection is designed to overcome this problem, and some important estimates, which reveal the dependence of the projection error on  $k, h, p$ , for this  $H(\text{div})$  projection are obtained. In Remark 5.2, we explain why it is necessary to use Raviart-Thomas space instead of vector valued continuous piece-wise polynomial space to approximate vector fields in  $H(\text{div}, \Omega)$ . In Remark 4.5, we give detailed explanation why the projection-based interpolation in [20] can *not* be applied for the quasi optimal convergent estimate for the Helmholtz equation. The most important part of the analysis lies in a modified duality argument for the FOSLS method which is motivated by the duality argument used in [10]. Roughly speaking, the corresponding dual FOSLS problem is to find  $(\boldsymbol{\psi}, v) \in \{\boldsymbol{\psi} \in H(\text{div}, \Omega) : \boldsymbol{\psi} \cdot \mathbf{n}|_{\partial\Omega} \in L^2(\partial\Omega)\} \times H^1(\Omega)$  satisfying

$$\begin{aligned} & \|u - u_h\|_{L^2(\Omega)}^2 \\ & = (\mathbf{i}k(\boldsymbol{\phi} - \boldsymbol{\phi}_h) + \nabla(u - u_h), \mathbf{i}k\boldsymbol{\psi} + \nabla v)_\Omega \\ & \quad + (\mathbf{i}k(u - u_h) + \nabla \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h), \mathbf{i}kv + \nabla \cdot \boldsymbol{\psi})_\Omega \\ & \quad + k\langle (\boldsymbol{\phi} - \boldsymbol{\phi}_h) \cdot \mathbf{n} + (u - u_h), \boldsymbol{\psi} \cdot \mathbf{n} + v \rangle_{\partial\Omega}. \end{aligned}$$

Here,  $\mathbf{i}k\boldsymbol{\phi} + \nabla u = 0$ , and  $(\boldsymbol{\phi}_h, u_h)$  is the numerical approximation to  $(\boldsymbol{\phi}, u)$ . Then, the regularity estimates for the oscillatory analytic part  $(\boldsymbol{\psi}_A, v_A)$  and the nonoscillatory elliptic part  $(\boldsymbol{\psi}_{H^2}, v_{H^2})$  of the solution of the above dual FOSLS problem are deduced. Since the above dual FOSLS problem is quite different from the dual problem used in [44, 46], these regularity estimates, especially the estimate of  $\|\nabla \cdot \boldsymbol{\psi}_{H^2}\|_{H^1(\Omega)}$  (cf. (5.1e) in Lemma 5.1), gets involved with non-trivial modification to the original proof of solution decomposition in [46]. Finally the quasi optimality of the  $L^2$  norm error estimate for the scalar solution of the FOSLS method for the Helmholtz equation can be finally obtained under the conditions that  $kh/p$  is sufficiently small and the polynomial degree  $p$  is at least  $O(\log k)$ .

We want to emphasize that FOSLS is closely related to the discontinuous Petrov-Galerkin (DPG) methods, see [7, 8, 9, 11, 12, 14, 15, 17, 21, 22, 23, 27, 32, 36, 37, 49]. Recently, the  $\text{DPG}_\varepsilon$  method, which is of the least-squares type, was proposed in [31]. The  $\text{DPG}_\varepsilon$  solution may yield less pollution error than the general FOSLS with fixed polynomial degree  $p$  and on the same mesh. The analysis for FOSLS in this paper can be useful to develop and analyze pollution free DPG methods. In addition, the implementation of DPG methods have been significantly simplified in [48].

The organization of the paper is as follows: We introduce some notation, the FOSLS method, and the main result in the next section. Section 3 is devoted to the proof of the stability estimate of the FOSLS method for the Helmholtz equation. In Section 4, we present some auxiliary results for the regularity estimates of the oscillatory analytic part and the nonoscillatory elliptic part of the Helmholtz solution, and the approximation properties of the finite element spaces. The regularity estimates of the solution to the dual FOSLS problem and the proof of a quasi optimal convergent result of this paper are stated in Section 5. In the final section, we give some numerical results to confirm our theoretical analysis.

## 2. THE FIRST ORDER SYSTEM LEAST SQUARES METHOD, AND MAIN RESULTS

**2.1. Geometry of the mesh.** We describe the meshes we are going to use. We first introduce the concept of generalized cell. Next, we define a  $C^0$ -compatible mesh and then the so-called quasi-uniform regular meshes which are the meshes we are going to work with. Finally, we propose a way to generate quasi-uniform regular meshes.

**2.1.1. Reference cells and curved cells.** We denote by  $\widehat{K}$  the reference cell in  $\mathbb{R}^d$ . This closed set is the standard unit tetrahedron when  $d = 3$ . It is the standard unit triangle when  $d = 2$ . We denote by  $\Delta_m(\widehat{K})$  the collection of all  $m$ -dimensional subcells of  $\widehat{K}$  for  $0 \leq m \leq d - 1$ . They are all faces of  $\widehat{K}$  when  $m = 2$ , are all edges of  $\widehat{K}$  when  $m = 1$ , and all vertexes of  $\widehat{K}$  when  $m = 0$ .

**Definition 2.1.** A closed subset  $K$  of  $\mathbb{R}^d$  is a generalized  $d$ -dimensional cell if there is a  $C^1$ -diffeomorphism  $G_K$  from the reference cell  $\widehat{K}$  to  $K$  such that  $G_K \in C^\infty(\widehat{K})$ .

We denote by  $h_K$  the diameter of  $K$ . We also denote by  $\Delta_m(K)$  the collection of all  $m$ -dimensional subcells of  $K$ , which are exactly  $G_K(\Delta_m(\widehat{K}))$ . Note that all points  $x$  in  $K$  are of the form  $x = G_K(\widehat{x})$  where  $\widehat{x}$  lies in  $\widehat{K}$ .

**2.1.2.  $C^0$ -compatible mesh.** We denote by  $\mathcal{T}_h$  the finite collection of generalized cells in  $\mathbb{R}^d$  such that for any two different generalized cells  $K, K' \in \mathcal{T}_h$ , either  $K \cap K' = \emptyset$  or  $K \cap K' \in \Delta_m(K) \cap \Delta_m(K')$  for some  $0 \leq m \leq d - 1$ . Here, the parameter  $h$  is the maximum of the diameters  $h_K$  of the cells  $K$  in  $\mathcal{T}_h$ .

We denote by  $\Delta_{d-1}(\mathcal{T}_h)$  the collection of  $\Delta_{d-1}(K)$  for all cells  $K$  in  $\mathcal{T}_h$ . Notice that for any  $F \in \Delta_{d-1}(\mathcal{T}_h)$ , either  $F = K \cap K'$  with  $K, K' \in \mathcal{T}_h$  or  $F \subset \partial\Omega$  where  $\Omega$  is an open subset in  $\mathbb{R}^d$  such that  $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$ .

**Definition 2.2.** We say that  $\mathcal{T}_h$  is a  $C^0$ -compatible mesh if, for any two subcells  $\widehat{F}, \widehat{F}' \in \Delta_{d-1}(\widehat{K})$ , where  $K, K' \in \mathcal{T}_h$ , such that  $G_K(\widehat{F}) = G_{K'}(\widehat{F}')$ , there is an affine mapping  $\mathcal{R} : \widehat{F} \rightarrow \widehat{F}'$  satisfying

$$G_K|_{\widehat{F}} = G_{K'}|_{\widehat{F}'} \circ \mathcal{R}. \quad (2.1)$$

We call  $K$  an element of  $\mathcal{T}_h$ . And, we call  $F \in \Delta_{d-1}(\mathcal{T}_h)$  a face in  $\mathcal{T}_h$ .

The  $C^0$ -compatible mesh is introduced in [19].

**2.1.3. The quasi-uniform regular meshes.** We use the symbol  $\nabla^n$  to denote derivatives of order  $n$ ; more precisely, for a function  $u : \Omega \rightarrow \mathbb{R}, \Omega \subset \mathbb{R}^d$ , we define

$$|\nabla^n u(x)|^2 = \sum_{\alpha \in \mathbb{N}_0^d, |\alpha|=n} \frac{n!}{\alpha!} |D^\alpha u(x)|^2. \quad (2.2)$$

Here,  $\mathbb{N}_0$  is the set of all non-negative integers. Now, we are ready to give the description of meshes we are going to use in this paper.

**Definition 2.3.** (quasi-uniform regular meshes) Let  $\{\mathcal{T}_h\}_{h \in I}$  be a family of  $C^0$ -compatible meshes. We call  $\{\mathcal{T}_h\}_{h \in I}$  a family of quasi-uniform regular meshes if for any  $h \in I$  and any  $K \in \mathcal{T}_h$ ,

$$\sup_{\widehat{x} \in \widehat{K}} \|(\nabla G_K(\widehat{x}))^{-1}\| \leq C_G h^{-1}, \quad \sup_{\widehat{x} \in \widehat{K}} \|\nabla^i G_K(\widehat{x})\| \leq C_G h^i \gamma^i i! \quad \forall i \geq 0,$$

where  $C_G, \gamma$  are a positive constants independent of  $h$  and of  $K$ ,  $\|\cdot\|$  is the Euclidean norm.

Throughout this paper, we assume that the domain  $\Omega$  admits a family of quasi-uniform regular meshes  $\{\mathcal{T}_h\}_{h \in \mathbf{I}}$  such that  $\bar{\Omega} = \cup_{K \in \mathcal{T}_h} K$  for any  $h \in \mathbf{I}$ . As usual, we can always pick an  $h$  in  $\mathbf{I}$  arbitrarily close to zero.

**2.1.4. Isoparametric refinement.** Next, we present a way of generating a family of quasi-uniform regular meshes for  $\Omega$ . We begin by obtaining a  $C^0$ -compatible mesh for  $\Omega$ ,  $\mathcal{T}_{h_0}$ , and by setting  $G^0 := \{G_K^0, \forall K \in \mathcal{T}_{h_0}\}$ . To obtain a finer mesh  $\mathcal{T}_{h_1}$ , we first divide the reference element  $\hat{K}$  uniformly into elements  $\hat{K}'$ . Then we refine the actual element  $K$  via the mapping  $G_K^0$ , that is  $K' = G_K^0(\hat{K}')$ . The remaining meshes are obtained by repeating this process. It is not difficult to verify that the family of meshes obtained in this manner is quasi-uniform regular if we have that

$$\sup_{\hat{x} \in \hat{K}} \|(\nabla G_K^0(\hat{x}))^{-1}\| \leq C_G h_K^{-1}, \quad \sup_{\hat{x} \in \hat{K}} \|\nabla^i G_K^0(\hat{x})\| \leq C_G h_K^i \gamma^i! \quad \forall i \geq 0,$$

for any  $K \in \mathcal{T}_{h_0}$ . We emphasize that the meshes satisfying [45, Assumption 5.2] are quasi-uniform regular.

**2.2. First order system least squares method.** We define complex valued vector field space and scalar function space

$$\mathbf{V} = \{\boldsymbol{\phi} \in H(\operatorname{div}, \Omega) : \boldsymbol{\phi}|_{\partial\Omega} \in L^2(\partial\Omega; \mathbb{R}^d)\}, \quad W = H^1(\Omega). \quad (2.3)$$

For any mesh  $\mathcal{T}_h$  and any  $p \geq 0$ , we denote by

$$\mathbf{V}_h = \{\boldsymbol{\phi} \in H(\operatorname{div}, \Omega) : \det(DG_K) DG_K^{-1}(\boldsymbol{\phi}|_K \circ G_K) \in \mathbf{RT}_{p+1}(\hat{K}) \text{ for any } K \in \mathcal{T}_h\}, \quad (2.4a)$$

$$W_h = \{v \in W : v|_K \circ G_K \in P_{p+1}(\hat{K}) \text{ for any } K \in \mathcal{T}_h\}, \quad (2.4b)$$

where  $\mathbf{RT}_{p+1}(\hat{K}) = P_{p+1}(\hat{K}; \mathbb{R}^d) + \mathbf{x}P_{p+1}(\hat{K})$  and  $P_{p+1}(\hat{K})$  are complex valued Raviart-Thomas space and complex valued polynomial with order up to  $p+1$ , respectively. Notice that  $\mathbf{V}_h \subset \mathbf{V}$  and the restriction of  $\mathbf{V}_h$  on each element  $K$  is exactly  $\mathbf{RT}_{p+1}(\hat{K})$  mapped onto  $K$  via the Piola transform corresponding to  $G_K$ .

The least squares functional is defined as

$$\begin{aligned} R((\boldsymbol{\phi}, u); (f, g)) &= \|\mathbf{i}k\boldsymbol{\phi} + \nabla u\|_{L^2(\Omega)}^2 + \|\mathbf{i}ku + \nabla \cdot \boldsymbol{\phi} + \mathbf{i}fk^{-1}\|_{L^2(\Omega)}^2 \\ &\quad + \|k^{1/2}(\boldsymbol{\phi} \cdot \mathbf{n} + u - k^{-1/2}g)\|_{L^2(\partial\Omega)}^2 \quad \forall (\boldsymbol{\phi}, u) \in \mathbf{V} \times W. \end{aligned}$$

The first order system least squares (FOSLS) method is to find  $(\boldsymbol{\phi}_h, u_h) \in \mathbf{V}_h \times W_h$  by

$$\begin{aligned} b((\boldsymbol{\phi}_h, u_h), (\boldsymbol{\psi}, v)) &= (-\mathbf{i}fk^{-1}, \mathbf{i}kv + \nabla \cdot \boldsymbol{\psi})_\Omega + (\mathbf{i}g, \boldsymbol{\psi} \cdot \mathbf{n} + v)_{\partial\Omega} \quad \forall (\boldsymbol{\psi}, v) \in \mathbf{V}_h \times W_h. \end{aligned} \quad (2.5)$$

Here, for any  $(\boldsymbol{\phi}, u), (\boldsymbol{\psi}, v) \in \mathbf{V} \times W$ ,

$$\begin{aligned} b((\boldsymbol{\phi}, u), (\boldsymbol{\psi}, v)) &= (\mathbf{i}k\boldsymbol{\phi} + \nabla u, \mathbf{i}k\boldsymbol{\psi} + \nabla v)_\Omega + (\mathbf{i}ku + \nabla \cdot \boldsymbol{\phi}, \mathbf{i}kv + \nabla \cdot \boldsymbol{\psi})_\Omega + k\langle \boldsymbol{\phi} \cdot \mathbf{n} + u, \boldsymbol{\psi} \cdot \mathbf{n} + v \rangle_{\partial\Omega}. \end{aligned} \quad (2.6)$$

For any complex valued functions  $u$  and  $v$ , we define

$$(u, v)_\Omega = \int_\Omega u \bar{v} \quad \langle u, v \rangle_{\partial\Omega} = \int_{\partial\Omega} u \bar{v}.$$

If  $u \in H^1(\Omega)$  is the solution of the Helmholtz equation (1.1), then  $(\boldsymbol{\phi} = \mathbf{i}k^{-1}\nabla u, u)$  satisfies

$$b((\boldsymbol{\phi}, u), (\boldsymbol{\psi}, v)) = (-\mathbf{i}fk^{-1}, \mathbf{i}kv + \nabla \cdot \boldsymbol{\psi})_{\Omega} + \langle \mathbf{i}g, \boldsymbol{\psi} \cdot \mathbf{n} + v \rangle_{\partial\Omega} \quad \forall (\boldsymbol{\psi}, v) \in \mathbf{V} \times W. \quad (2.7)$$

We notice that the FOSLS method (2.5) is very similar to the one in [39] except that we use the Raviart-Thomas space to approximate  $\boldsymbol{\phi}$  and the Robin boundary condition is imposed weakly. In Remark 5.2, we explain why it is necessary to use Raviart-Thomas space instead of vector valued continuous piece-wise polynomial space to approximate vector fields in  $H(\text{div}, \Omega)$ . In Remark 5.3, we explain why we weight with the factor  $k$  to the inner product  $\langle \boldsymbol{\phi} \cdot \mathbf{n} + u, \boldsymbol{\psi} \cdot \mathbf{n} + v \rangle_{\partial\Omega}$  on the boundary  $\partial\Omega$ .

**2.3. Main result.** We outline the main result in the following by showing the stability and the quasi optimality of the FOSLS method for the Helmholtz equation.

**Theorem 2.4.** (*Stability*) *We assume that the assumption (A1) holds. There is a constant  $C$ , which is independent of the wave number  $k \geq k_0 > 0$ , such that*

$$\|\boldsymbol{\phi}\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 + k\|\boldsymbol{\phi} \cdot \mathbf{n} + u\|_{L^2(\partial\Omega)}^2 \leq Cb((\boldsymbol{\phi}, u), (\boldsymbol{\phi}, u)) \quad \forall (\boldsymbol{\phi}, u) \in \mathbf{V} \times W.$$

**Theorem 2.5.** (*Quasi optimal convergence*) *We assume that the Assumptions (A1, A2) hold.  $(\boldsymbol{\phi}_h, u_h)$  is the solution of the FOSLS method (2.7). There are constants  $c_1, c_2, C > 0$  independent of  $h, p$  and  $k \geq k_0 > 0$  such that if*

$$\frac{kh}{p} < c_1 \text{ together with } p \geq c_2(\log k + 1), \quad (2.8)$$

then for any  $(\boldsymbol{\psi}_h, v_h) \in \mathbf{V}_h \times W_h$ , we have

$$\begin{aligned} & \|u - u_h\|_{L^2(\Omega)} \\ & \leq Ch(k\|\boldsymbol{\phi} - \boldsymbol{\psi}_h\|_{L^2(\Omega)} + \|\nabla \cdot (\boldsymbol{\phi} - \boldsymbol{\psi}_h)\|_{L^2(\Omega)} + \|\nabla(u - v_h)\|_{L^2(\Omega)} + k\|u - v_h\|_{L^2(\Omega)}) \\ & \quad + Ch^{1/2}\|(\boldsymbol{\phi} - \boldsymbol{\psi}_h) \cdot \mathbf{n}\|_{L^2(\partial\Omega)}. \end{aligned}$$

### 3. STABILITY

We give the proof of the stability estimate (Theorem 2.4) for the solution of the FOSLS method in the above section.

*Proof.* (Theorem 2.4) We define

$$\boldsymbol{\eta} = \mathbf{i}k\boldsymbol{\phi} + \nabla u \quad w = \mathbf{i}ku + \nabla \cdot \boldsymbol{\phi} \text{ in } \Omega, \quad \mu = \boldsymbol{\phi} \cdot \mathbf{n} + u \text{ on } \partial\Omega.$$

We consider two problems

$$\begin{aligned} \mathbf{i}k\boldsymbol{\phi}_1 + \nabla u_1 &= 0 & \text{in } \Omega, \\ \mathbf{i}ku_1 + \nabla \cdot \boldsymbol{\phi}_1 &= w & \text{in } \Omega, \\ \boldsymbol{\phi}_1 \cdot \mathbf{n} + u_1 &= \mu & \text{on } \partial\Omega, \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} \mathbf{i}k\boldsymbol{\phi}_2 + \nabla u_2 &= \boldsymbol{\eta} & \text{in } \Omega, \\ \mathbf{i}ku_2 + \nabla \cdot \boldsymbol{\phi}_2 &= 0 & \text{in } \Omega, \\ \boldsymbol{\phi}_2 \cdot \mathbf{n} + u_2 &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (3.2)$$

According to the assumption (A1), there is a unique solution  $u_1 \in H^1(\Omega)$  of the following problem

$$\begin{aligned} -\Delta u_1 - k^2 u_1 &= \mathbf{i}k w & \text{in } \Omega, \\ \frac{\partial u_1}{\partial \mathbf{n}} - \mathbf{i}k u_1 &= -\mathbf{i}k \mu & \text{on } \partial\Omega, \end{aligned}$$

and

$$\|\nabla u_1\|_{L^2(\Omega)} + k\|u_1\|_{L^2(\Omega)} \leq Ck (\|w\|_{L^2(\Omega)} + \|\mu\|_{L^2(\partial\Omega)}).$$

We define  $\phi_1$  by  $\mathbf{i}k\phi_1 + \nabla u_1 = 0$  in  $\Omega$ . Then, we have that  $(\phi_1, u_1)$  is a solution of the problem (3.1) such that  $\phi_1 \in \{\psi \in H(\text{div}, \Omega) : \psi \cdot \mathbf{n}|_{\partial\Omega} \in L^2(\partial\Omega)\}$  and

$$\|\phi_1\|_{L^2(\Omega)} + k^{-1}\|\nabla u_1\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)} \leq C (\|w\|_{L^2(\Omega)} + \|\mu\|_{L^2(\partial\Omega)}). \quad (3.3)$$

According to [24, Lemma 4.3] and the assumption (A1) again, there is a solution  $(\phi_2, u_2) \in H(\text{div}, \Omega) \times H^1(\Omega)$  of the problem (3.2) satisfying

$$\|\phi_2\|_{L^2(\Omega)} + k^{-1}\|\nabla u_2\|_{L^2(\Omega)} + \|u_2\|_{L^2(\Omega)} \leq C\|\boldsymbol{\eta}\|_{L^2(\Omega)}. \quad (3.4)$$

It is easy to see that

$$\begin{aligned} \mathbf{i}k(\phi_1 + \phi_2) + \nabla(u_1 + u_2) &= \boldsymbol{\eta} & \text{in } \Omega, \\ \mathbf{i}k(u_1 + u_2) + \nabla \cdot (\phi_1 + \phi_2) &= w & \text{in } \Omega, \\ (\phi_1 + \phi_2) \cdot \mathbf{n} + (u_1 + u_2) &= \mu & \text{on } \partial\Omega. \end{aligned}$$

By the assumption (A1),  $u_1 + u_2 = u$  and  $\phi_1 + \phi_2 = \phi$ . Then, by (3.3) and (3.4), we can conclude that the proof is complete.  $\square$

*Remark 3.1.* By the same argument in the above proof, for any  $(\phi, u) \in \mathbf{V} \times W$ ,

$$\begin{aligned} &C_0 \left( \|\phi\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 + \|\phi \cdot \mathbf{n} + u\|_{L^2(\partial\Omega)}^2 \right) \\ &\leq (\mathbf{i}k\phi + \nabla u, \mathbf{i}k\phi + \nabla u)_\Omega + (\mathbf{i}ku + \nabla \cdot \phi, \mathbf{i}ku + \nabla \cdot \phi)_\Omega + \langle \phi \cdot \mathbf{n} + u, \phi \cdot \mathbf{n} + u \rangle_{\partial\Omega}. \end{aligned}$$

#### 4. AUXILIARY RESULTS

In this section, we provide some auxiliary results.

**4.1. Decomposition of the Helmholtz solution.** The main results of this section are Theorem 4.3 and Lemma 4.4. Theorem 4.3 is the same as [46, Theorem 4.10] except (4.5c). We emphasize that (4.5c) is essential in the proof of duality argument (cf. (5.1e) in Lemma 5.1) of the first order system least squares method for the Helmholtz equation. We require the Assumption (A2) holds throughout section 4.1.

We need to recall several notations in [46]. We denote by  $\mathcal{F}$  the Fourier transform for functions in  $L^2(\mathbb{R}^d)$ . For functions  $f \in L^2(\mathbb{R}^d)$ , the high frequency filter  $H_{\mathbb{R}^d}$  and the low frequency filter  $L_{\mathbb{R}^d}$  are defined by

$$\mathcal{F}(L_{\mathbb{R}^d} f) = \chi_{\eta k} \mathcal{F}(f), \quad \mathcal{F}(H_{\mathbb{R}^d} f) = (1 - \chi_{\eta k}) \mathcal{F}(f), \quad (4.1)$$

where  $\chi_{\eta k}$  is the characteristic function of the ball  $B_{\eta k}(0)$  and  $\eta$  is a positive parameter which will be determined later. Let  $E_\Omega : L^2(\Omega) \rightarrow L^2(\mathbb{R}^d)$  be the Stein extension operator [51, Chapter VI]. Then, for  $f \in L^2(\Omega)$ , we define

$$L_\Omega f = (L_{\mathbb{R}^d}(E_\Omega f))|_\Omega, \quad H_\Omega f = (H_{\mathbb{R}^d}(E_\Omega f))|_\Omega. \quad (4.2)$$

We denote by  $G^N$  a lifting operator with the mapping property  $G^N : H^s(\partial\Omega) \rightarrow H^{3/2+s}(\Omega)$  for any  $s > 0$  and  $\partial_{\mathbf{n}}G^N g = g$ . As mentioned in [46, Remark 4.1], we can choose  $G^N$  independent of  $k$ . We then define  $H_{\partial\Omega}^N$  and  $L_{\partial\Omega}^N$  by

$$H_{\partial\Omega}^N(g) = \partial_{\mathbf{n}}H_{\Omega}(G^N(g)), \quad L_{\partial\Omega}^N(g) = \partial_{\mathbf{n}}L_{\Omega}(G^N(g)). \quad (4.3)$$

We denote by  $N_k$  the Newton potential operator defined in [46, (4.11)]. We define  $S_k : (f, g) \rightarrow u$  to be the solution operator of the Helmholtz equation (1.1), and  $S_k^{\Delta} : g \rightarrow u$  to be the solution operator of the modified Helmholtz equation with Robin boundary conditions; i.e.,  $u = S_k^{\Delta}(g)$  solves

$$-\Delta u + k^2 u = 0 \text{ in } \Omega, \quad \partial_{\mathbf{n}} u - \mathbf{i}k u = g \text{ on } \partial\Omega. \quad (4.4)$$

**Lemma 4.1.** *We assume that the Assumptions (A1, A2) hold. Let  $q \in (0, 1)$ . Then, there are constants  $C, \gamma > 0$  independent of  $k$  such that for any  $f \in L^2(\Omega)$ , the function  $u = S_k(f, 0)$  can be written as  $u = u_A + u_{H^2} + \tilde{u}$ , where*

$$\begin{aligned} k\|u_A\|_{L^2(\Omega)} + \|\nabla u_A\|_{L^2(\Omega)} &\leq C\|f\|_{L^2(\Omega)}, \\ \|\nabla^{p+2} u_A\|_{L^2(\Omega)} &\leq Ck^{-1}\gamma^p \max(p+2, k)^{p+2}\|f\|_{L^2(\Omega)} \quad \forall p \geq 0, \\ \|\Delta u_A + k^2 u_A\|_{H^1(\Omega)} &\leq Ck\|f\|_{L^2(\Omega)}, \\ k\|u_{H^2}\|_{L^2(\Omega)} + \|\nabla u_{H^2}\|_{L^2(\Omega)} &\leq qk^{-1}\|f\|_{L^2(\Omega)}, \\ \|u_{H^2}\|_{H^2(\Omega)} &\leq C\|f\|_{L^2(\Omega)}, \end{aligned}$$

and the remainder  $\tilde{u} = S_k(\tilde{f}, 0)$  satisfies

$$-\Delta \tilde{u} - k^2 \tilde{u} = \tilde{f}, \quad \partial_{\mathbf{n}} \tilde{u} - \mathbf{i}k \tilde{u}|_{\partial\Omega} = 0,$$

where

$$\|\tilde{f}\|_{L^2(\Omega)} \leq q\|f\|_{L^2(\Omega)}.$$

*Proof.* According to Theorem 2.4 and [46, Lemma 4.15], it is easy to see that except  $\|\Delta u_A + k^2 u_A\|_{H^1(\Omega)} \leq Ck\|f\|_{L^2(\Omega)}$ , all other statements hold.

In order to prove  $\|\Delta u_A + k^2 u_A\|_{H^1(\Omega)} \leq Ck\|f\|_{L^2(\Omega)}$ , we need to go through the proof of [46, Lemma 4.15]. It is shown that

$$u_A = u_A^I + u_A^{II}, \quad u_A^I = S_k(L_{\Omega}f, 0), \quad u_{H^2}^I = N_k(H_{\Omega}f), \quad u_A^{II} = S_k(0, L_{\partial\Omega}^N(\mathbf{i}k u_{H^2}^I - \partial_{\mathbf{n}} u_{H^2}^I)).$$

So, it is sufficient to show that

$$\|\Delta u_A^I + k^2 u_A^I\|_{H^1(\Omega)} \leq Ck\|f\|_{L^2(\Omega)}, \quad \|\Delta u_A^{II} + k^2 u_A^{II}\|_{H^1(\Omega)} \leq Ck\|f\|_{L^2(\Omega)}.$$

Since  $u_A^I = S_k(L_{\Omega}f, 0)$ , we have

$$\begin{aligned} \|\Delta u_A^I + k^2 u_A^I\|_{H^1(\Omega)}^2 &= \|L_{\Omega}f\|_{H^1(\Omega)}^2 \leq C \int_{\mathbb{R}^d} (1 + |\xi|^2) |\widehat{L_{\Omega}f}|^2 \\ &= C \int_{\mathbb{R}^d} (1 + |\xi|^2) \chi_{\eta k} |\widehat{E_{\Omega}f}|^2 \leq Ck^2 \|E_{\Omega}f\|_{L^2(\Omega)}^2 \leq Ck^2 \|f\|_{L^2(\Omega)}^2. \end{aligned}$$

On the other hand,  $u_A^{II} = S_k(0, L_{\partial\Omega}^N(\mathbf{i}k u_{H^2}^I - \partial_{\mathbf{n}} u_{H^2}^I))$  implies that  $\Delta u_A^{II} + k^2 u_A^{II} = 0$ . So, we can conclude that  $\|\Delta u_A + k^2 u_A\|_{H^1(\Omega)} \leq Ck\|f\|_{L^2(\Omega)}$ .  $\square$



**Lemma 4.2.** *We assume that the Assumptions (A1, A2) hold. Let  $q \in (0, 1)$ . Then, there are constants  $C, \gamma > 0$  independent of  $k$  such that for any  $g \in H^{1/2}(\partial\Omega)$ , the function  $u = S_k(0, g)$  can be written as  $u = u_A + u_{H^2} + \tilde{u}$ , where*

$$\begin{aligned} k\|u_A\|_{L^2(\Omega)} + \|\nabla u_A\|_{L^2(\Omega)} &\leq C\|g\|_{H^{1/2}(\partial\Omega)}, \\ \|\nabla^{p+2}u_A\|_{L^2(\Omega)} &\leq Ck^{-1}\gamma^p \max(p+2, k)^{p+2}\|g\|_{H^{1/2}(\partial\Omega)} \quad \forall p \geq 0, \\ \|\Delta u_A + k^2u_A\|_{H^1(\Omega)} &\leq Ck\|g\|_{H^{1/2}(\partial\Omega)}, \\ k\|u_{H^2}\|_{L^2(\Omega)} + \|\nabla u_{H^2}\|_{L^2(\Omega)} &\leq qk^{-1}\|g\|_{H^{1/2}(\partial\Omega)}, \\ \|u_{H^2}\|_{H^2(\Omega)} &\leq C\|g\|_{H^{1/2}(\partial\Omega)}, \end{aligned}$$

and the remainder  $\tilde{u} = S_k(0, \tilde{g})$  satisfies

$$-\Delta\tilde{u} - k^2\tilde{u} = 0, \quad \partial_{\mathbf{n}}\tilde{u} - \mathbf{i}k\tilde{u}|_{\partial\Omega} = \tilde{g},$$

where

$$\|\tilde{g}\|_{H^{1/2}(\partial\Omega)} \leq q\|g\|_{H^{1/2}(\partial\Omega)}.$$

*Proof.* According to Theorem 2.4 and [46, Lemma 4.16], it is easy to see that except  $\|\Delta u_A + k^2u_A\|_{H^1(\Omega)} \leq Ck\|g\|_{H^{1/2}(\partial\Omega)}$ , all other statements hold.

In order to prove  $\|\Delta u_A + k^2u_A\|_{H^1(\Omega)} \leq Ck\|g\|_{H^{1/2}(\partial\Omega)}$ , we need to go through the proof of [46, Lemma 4.16]. It is shown that

$$u_A = u_A^I + u_A^{II}, \quad u_A^I = S_k(0, L_{\partial\Omega}^N g), \quad u_{H^2}^I = S_k^\Delta(H_{\partial\Omega}^N g), \quad u_A^{II} = S_k(L_\Omega(2k^2u_{H^2}^I), 0).$$

So, it is sufficient to show that

$$\|\Delta u_A^I + k^2u_A^I\|_{H^1(\Omega)} \leq Ck\|g\|_{H^{1/2}(\partial\Omega)}, \quad \|\Delta u_A^{II} + k^2u_A^{II}\|_{H^1(\Omega)} \leq Ck\|g\|_{H^{1/2}(\partial\Omega)}.$$

Since  $u_A^{II} = S_k(L_\Omega(2k^2u_{H^2}^I), 0)$ , we have

$$\begin{aligned} \|\Delta u_A^{II} + k^2u_A^{II}\|_{H^1(\Omega)}^2 &= \|L_\Omega(2k^2u_{H^2}^I)\|_{H^1(\Omega)}^2 = 2k^2 \int_{\mathbb{R}^d} (1 + |\xi|^2) |L_\Omega(\widehat{u_{H^2}^I})|^2 \\ &= 2k^2 \int_{\mathbb{R}^d} (1 + |\xi|^2) \chi_{\eta k} |\widehat{E_\Omega u_{H^2}^I}|^2 \leq Ck^4 \|E_\Omega u_{H^2}^I\|_{L^2(\Omega)}^2 \leq Ck^4 \|u_{H^2}^I\|_{L^2(\Omega)}^2 \\ &\leq Ck^2 \|g\|_{H^{1/2}(\partial\Omega)}^2. \end{aligned}$$

The last inequality above is obtained by [46, (4.31)]. On the other hand,  $u_A^I = S_k(0, L_{\partial\Omega}^N g)$  implies that  $\Delta u_A^I + k^2u_A^I = 0$ . So, we can conclude that  $\|\Delta u_A + k^2u_A\|_{H^1(\Omega)} \leq Ck\|g\|_{H^{1/2}(\partial\Omega)}$ .  $\square$

**Theorem 4.3.** *We assume that the Assumptions (A1, A2) hold. Then, there are constants  $C, \gamma > 0$  independent of  $k \geq k_0$  such that for any  $(f, g) \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$ , the function  $u = S_k(f, g)$  can be written as  $u = u_A + u_{H^2}$ , where*

$$k\|u_A\|_{L^2(\Omega)} + \|\nabla u_A\|_{L^2(\Omega)} \leq C (\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}), \quad (4.5a)$$

$$\|\nabla^{p+2}u_A\|_{L^2(\Omega)} \leq Ck^{-1}\gamma^p \max(p+2, k)^{p+2} (\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}) \quad \forall p \geq 0, \quad (4.5b)$$

$$\|\Delta u_A + k^2u_A\|_{H^1(\Omega)} \leq Ck (\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}), \quad (4.5c)$$

$$\|u_{H^2}\|_{H^2(\Omega)} + k\|u_{H^2}\|_{H^1(\Omega)} + k^2\|u\|_{L^2(\Omega)} \leq C (\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}). \quad (4.5d)$$

*Proof.* By proceeding in the same way as the proof of [46, Theorem 4.10] with Lemma 4.1 and Lemma 4.2, we can conclude that the proof is complete.  $\square$

**Lemma 4.4.** *We assume that the Assumptions (A1, A2) hold. Then, there are constants  $C, \gamma > 0$  independent of  $k \geq k_0$  such that for any analytic functions  $\tilde{f}$  and  $\tilde{g}$  in  $\Omega$ ,*

$$\|\nabla^{p+2} S_k(\tilde{f}, \tilde{g})\|_{L^2(\Omega)} \leq C \gamma^{p+2} k^{-1} \max(p+2, k)^{p+2} \left( \|\tilde{f}\|_{L^2(\Omega)} + \|\tilde{g}\|_{H^1(\Omega)} \right),$$

$$k \|S_k(\tilde{f}, \tilde{g})\|_{L^2(\Omega)} + \|\nabla S_k(\tilde{f}, \tilde{g})\|_{L^2(\Omega)} \leq C \left( \|\tilde{f}\|_{L^2(\Omega)} + \|\tilde{g}\|_{H^1(\Omega)} \right).$$

*Proof.* We denote by  $v = S_k(\tilde{f}, \tilde{g})$ . By the Assumption (A1), we have immediately

$$k \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)} \leq C \left( \|\tilde{f}\|_{L^2(\Omega)} + \|\tilde{g}\|_{H^1(\Omega)} \right).$$

By the Assumption (A2) and [40, Theorem 4.18(ii)], it is easy to see  $v \in C^\infty(\Omega)$ .

In order to show the other estimate, we follow the main steps in the proof of [46, Lemma 4.13]. We take  $\epsilon = k^{-1}$ . It is easy to see that  $v$  satisfies

$$-\epsilon^2 \Delta v - v = \epsilon^2 \tilde{f} \text{ in } \Omega, \quad \epsilon^2 \partial_{\mathbf{n}} v = \epsilon (\epsilon \tilde{g} + \mathbf{i}v) \text{ on } \partial\Omega.$$

Then, by applying [43, Proposition 5.4.5 and Remark 5.4.6] to the above equation, we can conclude that the proof is complete.  $\square$

**4.2. Approximation properties of finite element spaces.** We would like to show approximation properties of some projection operators for finite element spaces (2.4).

We define a projection  $\widehat{\Pi}_{\mathbf{V}} : H^1(\widehat{K}; \mathbb{R}^d) \rightarrow \mathbf{RT}_{p+1}(\widehat{K})$  by

$$\langle (\widehat{\Pi}_{\mathbf{V}} \widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\psi}}) \cdot \widehat{\mathbf{n}}, \widehat{\boldsymbol{\mu}} \rangle_{\widehat{F}} = 0, \quad \forall \widehat{\boldsymbol{\mu}} \in P_{p+1}(\widehat{F}), \widehat{F} \in \Delta_{d-1}(\widehat{K}), \quad (4.6a)$$

$$(\widehat{\Pi}_{\mathbf{V}} \widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\psi}}, \widehat{\nabla} \times \widehat{\boldsymbol{\phi}})_{\widehat{K}} = 0, \quad \forall \widehat{\boldsymbol{\phi}} \in \mathbf{Q}_{p+1,0}(\widehat{K}), \quad (4.6b)$$

$$\|\widehat{\nabla} \cdot (\widehat{\Pi}_{\mathbf{V}} \widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\psi}})\|_{L^2(\widehat{K})} \rightarrow \min. \quad (4.6c)$$

Here,  $\mathbf{Q}_{p+1,0}(\widehat{K}) = \{\widehat{\boldsymbol{\phi}} \in P_{p+2}^- \Lambda^{d-2}(\widehat{K}) : \text{tr} \widehat{\boldsymbol{\phi}}|_{\partial\widehat{K}} = 0\}$ . When  $d = 3$ ,  $P_{p+2}^- \Lambda^{d-2}(\widehat{K})$  is the Nédélec1st-kind  $H(\text{curl})$  element of degree  $\leq p+1$ . When  $d = 2$ ,  $P_{p+2}^- \Lambda^{d-2}(\widehat{K})$  is the Lagrange element of degree  $\leq p+2$ .

We emphasize that the projection (4.6) is the same as the projection [20, (201)] except the way to impose normal component on the boundary of  $\widehat{K}$  (see the difference between (4.6a) and the first condition in [20, (201)]).

*Remark 4.5.* Since we use the Raviart-Thomas space for the approximation to functions in  $\mathbf{V}$ , the natural idea is to utilize the projection-based interpolation  $\Pi^{\text{div}}$  in [20, (201)]. We notice that in [20, Theorem 5.3], the estimate of approximation error  $\|\Pi^{\text{div}} \boldsymbol{\psi} - \boldsymbol{\psi}\|_{H(\text{div}, \Omega)}$  gets involved with  $\|\boldsymbol{\psi}\|_{H^r(\text{div}, \Omega)}$  where  $r > 0$  and the Sobolev norm  $\|\cdot\|_{H^r(\text{div}, \Omega)}$  is defined in Hörmander's style. When  $r$  is a non-negative integer, the norm  $\|\cdot\|_{H^r(\text{div}, \Omega)}$  in Hörmander's style is equivalent to  $\left( \sum_{0 \leq i \leq r} \|\nabla^i \cdot\|_{L^2(\Omega)}^2 \right)^{1/2}$  which is provided in Lemma 5.1. However, it is *not* obvious to see how the equivalent constants depend on  $r$ . So, we introduce projection (4.6) and give the following Lemma 4.6.

**Lemma 4.6.** *There is a constant  $C > 0$  such that for any  $\widehat{\boldsymbol{\psi}} \in H^1(\widehat{K}; \mathbb{R}^d)$ ,*

$$\begin{aligned} & \|\widehat{\Pi}_{\mathbf{V}} \widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\psi}}\|_{H(\text{div}, \widehat{K})} \\ & \leq C \left( \inf_{\widehat{\boldsymbol{\phi}} \in \mathbf{RT}_{p+1}(\widehat{K})} \|\widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\phi}}\|_{H(\text{div}, \widehat{K})} + \inf_{\widehat{\boldsymbol{\varphi}} \in \mathbf{RT}_{p+1}(\widehat{K})} \|(\widehat{\boldsymbol{\varphi}} - \widehat{\boldsymbol{\psi}}) \cdot \widehat{\mathbf{n}}\|_{L^2(\partial\widehat{K})} \right). \end{aligned}$$

In addition, we have

$$\widehat{\nabla} \cdot \widehat{\Pi}_{\mathbf{V}} \widehat{\psi} = \widehat{P}^{\text{div}} \widehat{\psi}, \quad (\widehat{\Pi}_{\mathbf{V}} \widehat{\psi}) \cdot \widehat{\mathbf{n}}|_{\widehat{F}} = \widehat{P}_{\widehat{F}}(\widehat{\psi} \cdot \widehat{\mathbf{n}}|_{\widehat{F}}) \quad \forall \widehat{F} \in \Delta_{d-1}(\widehat{K}). \quad (4.7)$$

Here,  $\widehat{P}$  and  $\widehat{P}_{\widehat{F}}$  are the standard  $L^2$ -orthogonal projections onto  $P_{p+1}(\widehat{K})$  and  $P_{p+1}(\widehat{F})$ , respectively.

*Proof.* (4.7) can be verified straightforwardly by the definition of  $\widehat{\Pi}_{\mathbf{V}}$ . In the following, we give the proof of the inequality for the case  $d = 3$ , which is similar to that of [20, Theorem 5.3].

We define  $\widehat{P}^{\text{div}} : H^1(\widehat{K}; \mathbb{R}^3) \rightarrow \mathbf{RT}_{p+1}(\widehat{K})$  by

$$\begin{aligned} (\widehat{P}^{\text{div}} \widehat{\psi} - \widehat{\psi}, \widehat{\nabla} \times \widehat{\phi})_{\widehat{K}} &= 0, \quad \forall \widehat{\phi} \in P_{p+2}^- \Lambda^1(\widehat{K}), \\ \|\widehat{\nabla} \cdot (\widehat{P}^{\text{div}} \widehat{\psi} - \widehat{\psi})\|_{L^2(\widehat{K})} &\rightarrow \min. \end{aligned}$$

$\widehat{P}^{\text{div}}$  is introduced in [20, (198)].

We denote by  $\mathbf{q} = \widehat{\Pi}_{\mathbf{V}} \widehat{\psi} - \widehat{P}^{\text{div}} \widehat{\psi}$ . Then,

$$\begin{aligned} \mathbf{q} \cdot \widehat{\mathbf{n}} &= (\widehat{\Pi}_{\mathbf{V}} \widehat{\psi} - \widehat{P}^{\text{div}} \widehat{\psi}) \cdot \widehat{\mathbf{n}} \text{ on } \partial \widehat{K}, \\ (\mathbf{q}, \widehat{\nabla} \times \widehat{\phi})_{\widehat{K}} &= 0, \quad \forall \widehat{\phi} \in \mathbf{Q}_{p+1,0}(\widehat{K}), \\ \|\widehat{\nabla} \cdot \mathbf{q}\|_{L^2(\widehat{K})} &\rightarrow \min. \end{aligned}$$

We define  $\mathbf{p} \in \mathbf{RT}_{p+1}(\widehat{K})$  by

$$\begin{aligned} \mathbf{p} \cdot \widehat{\mathbf{n}} &= (\widehat{\Pi}_{\mathbf{V}} \widehat{\psi} - \widehat{P}^{\text{div}} \widehat{\psi}) \cdot \widehat{\mathbf{n}} \text{ on } \partial \widehat{K}, \\ (\mathbf{p}, \widehat{\nabla} \times \widehat{\phi})_{\widehat{K}} &= 0, \quad \forall \widehat{\phi} \in \mathbf{Q}_{p+1,0}(\widehat{K}), \\ \|\mathbf{p}\|_{H(\text{div}, \widehat{K})} &\rightarrow \min. \end{aligned}$$

We claim that  $\mathbf{p}$  satisfies

$$\mathbf{p} \cdot \widehat{\mathbf{n}} = (\widehat{\Pi}_{\mathbf{V}} \widehat{\psi} - \widehat{P}^{\text{div}} \widehat{\psi}) \cdot \widehat{\mathbf{n}} \text{ on } \partial \widehat{K}, \quad (4.8a)$$

$$\|\mathbf{p}\|_{H(\text{div}, \widehat{K})} \rightarrow \min, \quad (4.8b)$$

$$\|\mathbf{q}\|_{H(\text{div}, \widehat{K})} \leq C \|\mathbf{p}\|_{H(\text{div}, \widehat{K})}. \quad (4.8c)$$

Let  $\varepsilon_{\widehat{K}}^{\text{div}}$  be the polynomial extension operator in [25, Theorem 7.1]. Then, by (4.8), we have

$$\begin{aligned} \|\widehat{\Pi}_{\mathbf{V}} \widehat{\psi} - \widehat{P}^{\text{div}} \widehat{\psi}\|_{H(\text{div}, \widehat{K})} &= \|\mathbf{q}\|_{H(\text{div}, \widehat{K})} \leq C \|\mathbf{p}\|_{H(\text{div}, \widehat{K})} \\ &\leq C \|\varepsilon_{\widehat{K}}^{\text{div}}((\widehat{\Pi}_{\mathbf{V}} \widehat{\psi} - \widehat{P}^{\text{div}} \widehat{\psi}) \cdot \widehat{\mathbf{n}}|_{\partial \widehat{K}})\|_{H(\text{div}, \widehat{K})} \\ &\leq C \|(\widehat{\Pi}_{\mathbf{V}} \widehat{\psi} - \widehat{P}^{\text{div}} \widehat{\psi}) \cdot \widehat{\mathbf{n}}\|_{H^{-1/2}(\partial \widehat{K})} \\ &\leq C (\|(\widehat{P}_{\mathbf{V}}^{\text{div}} \widehat{\psi} - \widehat{\psi}) \cdot \widehat{\mathbf{n}}\|_{H^{-1/2}(\partial \widehat{K})} + \|(\widehat{\Pi}_{\mathbf{V}} \widehat{\psi} - \widehat{\psi}) \cdot \widehat{\mathbf{n}}\|_{H^{-1/2}(\partial \widehat{K})}) \\ &\leq C (\|\widehat{P}_{\mathbf{V}}^{\text{div}} \widehat{\psi} - \widehat{\psi}\|_{H(\text{div}, \widehat{K})} + \|(\widehat{\Pi}_{\mathbf{V}} \widehat{\psi} - \widehat{\psi}) \cdot \widehat{\mathbf{n}}\|_{H^{-1/2}(\partial \widehat{K})}) \\ &\leq C (\|\widehat{P}_{\mathbf{V}}^{\text{div}} \widehat{\psi} - \widehat{\psi}\|_{H(\text{div}, \widehat{K})} + \|(\widehat{\Pi}_{\mathbf{V}} \widehat{\psi} - \widehat{\psi}) \cdot \widehat{\mathbf{n}}\|_{L^2(\partial \widehat{K})}). \end{aligned}$$

Then, by combining the above inequality with [20, Theorem 5.2], we can conclude that the proof is complete. So, we only need to show that the claims (4.8) are true.

It is easy to see that

$$\begin{aligned}
& \|\mathbf{q}\|_{H(\operatorname{div}, \widehat{K})} \\
& \leq C(\|\mathbf{q} - \mathbf{p}\|_{L^2(\widehat{K})} + \|\mathbf{p}\|_{L^2(\widehat{K})} + \|\widehat{\nabla} \cdot \mathbf{q}\|_{L^2(\widehat{K})}) \\
& \leq C(\|\widehat{\nabla} \cdot (\mathbf{q} - \mathbf{p})\|_{L^2(\widehat{K})} + \|\mathbf{p}\|_{L^2(\widehat{K})} + \|\widehat{\nabla} \cdot \mathbf{q}\|_{L^2(\widehat{K})}) \text{ by [20, Lemma 5.2 case 2]} \\
& \leq C(\|\widehat{\nabla} \cdot (\mathbf{q} - \mathbf{p})\|_{L^2(\widehat{K})} + \|\mathbf{p}\|_{L^2(\widehat{K})} + \|\widehat{\nabla} \cdot \mathbf{p}\|_{L^2(\widehat{K})}) \text{ by the definition of } \mathbf{q} \\
& \leq C(\|\widehat{\nabla} \cdot \mathbf{q}\|_{L^2(\widehat{K})} + \|\widehat{\nabla} \cdot \mathbf{p}\|_{L^2(\widehat{K})} + \|\mathbf{p}\|_{L^2(\widehat{K})}) \\
& \leq C(\|\widehat{\nabla} \cdot \mathbf{p}\|_{L^2(\widehat{K})} + \|\mathbf{p}\|_{L^2(\widehat{K})}) \text{ by the definition of } \mathbf{q} \\
& = C\|\mathbf{p}\|_{H(\operatorname{div}, \widehat{K})}.
\end{aligned}$$

Notice that for any  $\widehat{\phi} \in \mathbf{Q}_{p+1,0}(\widehat{K})$ , we have

$$(\mathbf{p} + \widehat{\nabla} \times \widehat{\phi}) \cdot \widehat{\mathbf{n}} = \mathbf{p} \cdot \widehat{\mathbf{n}} \text{ on } \partial \widehat{K}, \quad \widehat{\nabla} \times \widehat{\phi} \in \mathbf{RT}_{p+1}(\widehat{K}).$$

So, we have

$$\|\mathbf{p} + \widehat{\nabla} \times \widehat{\phi}\|_{H(\operatorname{div})(\widehat{K})}^2 = \|\mathbf{p}\|_{H(\operatorname{div})(\widehat{K})}^2 + \|\widehat{\nabla} \times \widehat{\phi}\|_{H(\operatorname{div})(\widehat{K})}^2.$$

Thus, we can conclude that  $\mathbf{p}$  satisfies the claims (4.8).  $\square$

For any  $\psi \in H^1(\Omega; \mathbb{R}^d)$ , we define  $\Pi_{\mathbf{V}}\psi$  by

$$\begin{aligned}
\Pi_{\mathbf{V}}\psi(G_K(\widehat{x})) &= (\det DG_K(\widehat{x}))^{-1} DG_K(\widehat{x})(\widehat{\Pi}_{\mathbf{V}}\widehat{\psi})(\widehat{x}) \quad \forall \widehat{x} \in \widehat{K}, K \in \mathcal{T}_h, \\
\text{where } \widehat{\psi}(G_K(\widehat{x})) &= (\det DG_K(\widehat{x}))^{-1} DG_K(\widehat{x})\widehat{\psi}(\widehat{x}).
\end{aligned} \tag{4.9}$$

**Lemma 4.7.** *Let  $\psi \in C^\infty(\Omega; \mathbb{R}^d)$  satisfy*

$$\|\nabla^p \psi\|_{L^2(\Omega)} \leq \gamma^p \max(p, k)^p C_\psi \quad \forall p \in \mathbb{N}_0.$$

Here,  $C_\psi$  and  $\gamma$  are independent of  $k$ ,  $h$  and  $p$ . Then, there are  $C, \sigma > 0$  which are also independent of  $k$ ,  $h$  and  $p$ , such that

$$\begin{aligned}
k\|\Pi_{\mathbf{V}}\psi - \psi\|_{L^2(\Omega)} &\leq Ck(h(p+2))^d C_\psi \left[ \left(\frac{h}{h+\sigma}\right)^{p+2} + \left(\frac{kh}{\sigma p}\right)^{p+2} \right], \\
\|\nabla \cdot (\Pi_{\mathbf{V}}\psi - \psi)\|_{L^2(\Omega)} &\leq Ch^{-1}(h(p+2))^d C_\psi \left[ \left(\frac{h}{h+\sigma}\right)^{p+2} + \left(\frac{kh}{\sigma p}\right)^{p+2} \right], \\
k^{1/2}\|(\Pi_{\mathbf{V}}\psi - \psi) \cdot \mathbf{n}\|_{L^2(\partial\Omega)} &\leq Ck^{1/2}h^{-1/2}(h(p+2))^d C_\psi \left[ \left(\frac{h}{h+\sigma}\right)^{p+2} + \left(\frac{kh}{\sigma p}\right)^{p+2} \right].
\end{aligned}$$

*Proof.* We follow the proof of [45, Theorem 5.5]. We start by defining for each  $K \in \mathcal{T}_h$  the constant  $C_K$  by

$$C_K^2 = \sum_{p \in \mathbb{N}_0} \frac{\|\nabla^p \psi\|_{L^2(K)}^2}{(2\gamma \max(p, k))^{2p}}. \tag{4.10}$$

It is easy to see that

$$\|\nabla^p \boldsymbol{\psi}\|_{L^2(K)} \leq (2\gamma \max(p, k))^p C_K \quad \forall p \in \mathbb{N}_0, \quad (4.11a)$$

$$\Sigma_{K \in \mathcal{T}_h} C_K^2 \leq \frac{4C}{3} C_\psi^2. \quad (4.11b)$$

We choose  $K \in \mathcal{T}_h$  arbitrarily. We define

$$\begin{aligned} A(\widehat{x}) &= DG_K(\widehat{x}) \quad \forall \widehat{x} \in \widehat{K}, \\ (\det A)^{-1}(\widehat{x}) A(\widehat{x}) \widehat{\boldsymbol{\psi}}(\widehat{x}) &= \boldsymbol{\psi}(G_K(\widehat{x})). \end{aligned}$$

Let  $\Pi_{\mathbf{V}}$  be the projection defined in (4.9). Then by standard change of variable, we have

$$\begin{aligned} \|\Pi_{\mathbf{V}} \boldsymbol{\psi} - \boldsymbol{\psi}\|_{L^2(K)} &\leq C(\|\det A^{-1}\|_{L^\infty(\widehat{K})})^{1/2} \|A\|_{L^\infty(\widehat{K})} \|\widehat{\Pi}_{\mathbf{V}} \widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\psi}}\|_{L^2(\widehat{K})}, \\ \|\nabla \cdot \Pi_{\mathbf{V}} \boldsymbol{\psi} - \nabla \cdot \boldsymbol{\psi}\|_{L^2(K)} &\leq C(\|\det A^{-1}\|_{L^\infty(\widehat{K})})^{1/2} \|\widehat{P}(\widehat{\nabla} \cdot \widehat{\boldsymbol{\psi}}) - \widehat{\nabla} \cdot \widehat{\boldsymbol{\psi}}\|_{L^2(\widehat{K})}, \\ \|(\Pi_{\mathbf{V}} \boldsymbol{\psi} - \boldsymbol{\psi}) \cdot \mathbf{n}\|_{L^2(\partial K)} &\leq C \|A\|_{L^\infty(\widehat{K})}^{(d-1)/2} \Sigma_{\widehat{F} \in \Delta_{d-1}(\widehat{K})} \|\widehat{P}_{\widehat{F}}(\widehat{\boldsymbol{\psi}} \cdot \widehat{\mathbf{n}}) - \widehat{\boldsymbol{\psi}} \cdot \widehat{\mathbf{n}}\|_{L^2(\widehat{F})}. \end{aligned}$$

$\widehat{P}$  and  $\widehat{P}_{\widehat{F}}$  are the standard  $L^2$ -orthogonal projections onto  $P_{p+1}(\widehat{K})$  and  $P_{p+1}(\widehat{F})$ , respectively. The last two inequalities above is due to (4.7) and the fact that Piola transform commutes with both divergence operator and trace of normal component. Then, by the properties of matrix  $A$  in Definition 2.3 and Lemma 4.6, we have

$$\begin{aligned} \|\Pi_{\mathbf{V}} \boldsymbol{\psi} - \boldsymbol{\psi}\|_{L^2(K)} &\leq Ch^{1-d/2} \|\widehat{\Pi}_{\mathbf{V}} \widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\psi}}\|_{L^2(\widehat{K})} \quad (4.12a) \\ &\leq Ch^{1-d/2} \left( \inf_{\widehat{\boldsymbol{\phi}} \in \mathbf{RT}_{p+1}(\widehat{K})} \|\widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\phi}}\|_{H(\operatorname{div}, \widehat{K})} + \inf_{\widehat{\boldsymbol{\varphi}} \in \mathbf{RT}_{p+1}(\widehat{K})} \|(\widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\varphi}}) \cdot \widehat{\mathbf{n}}\|_{L^2(\partial \widehat{K})} \right) \\ \|\nabla \cdot \Pi_{\mathbf{V}} \boldsymbol{\psi} - \nabla \cdot \boldsymbol{\psi}\|_{L^2(K)} &\leq Ch^{-d/2} \|\widehat{P}(\widehat{\nabla} \cdot \widehat{\boldsymbol{\psi}}) - \widehat{\nabla} \cdot \widehat{\boldsymbol{\psi}}\|_{L^2(\widehat{K})} \quad (4.12b) \\ &\leq Ch^{-d/2} \inf_{\widehat{\boldsymbol{\phi}} \in \mathbf{RT}_{p+1}(\widehat{K})} \|\widehat{\nabla} \cdot (\widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\phi}})\|_{L^2(\widehat{K})} = Ch^{-d/2} \inf_{\widehat{\mathbf{v}} \in \mathbf{P}_{p+1}(\widehat{K})} \|\widehat{\nabla} \cdot \widehat{\boldsymbol{\psi}} - \widehat{\mathbf{v}}\|_{L^2(\widehat{K})}, \\ \|(\Pi_{\mathbf{V}} \boldsymbol{\psi} - \boldsymbol{\psi}) \cdot \mathbf{n}\|_{L^2(\partial K)} &\leq Ch^{(1-d)/2} \Sigma_{\widehat{F} \in \Delta_{d-1}(\widehat{K})} \|\widehat{P}_{\widehat{F}}(\widehat{\boldsymbol{\psi}} \cdot \widehat{\mathbf{n}}) - \widehat{\boldsymbol{\psi}} \cdot \widehat{\mathbf{n}}\|_{L^2(\widehat{F})} \quad (4.12c) \\ &\leq Ch^{(1-d)/2} \Sigma_{\widehat{F} \in \Delta_{d-1}(\widehat{K})} \inf_{\widehat{\mathbf{v}} \in \mathbf{P}_{p+1}(\widehat{F})} \|\widehat{\boldsymbol{\psi}} \cdot \widehat{\mathbf{n}} - \widehat{\mathbf{v}}\|_{L^2(\widehat{F})}. \end{aligned}$$

By the definition of  $\widehat{\boldsymbol{\psi}}$ , we have

$$\widehat{\boldsymbol{\psi}}(\widehat{x}) = \operatorname{adj} A(\widehat{x}) \boldsymbol{\psi}(G_K(\widehat{x})) \quad \forall \widehat{x} \in \widehat{K}.$$

Here,  $\operatorname{adj} A$  is the adjoint matrix of  $A$ . By the properties of matrix  $A$  in Definition 2.3 and [43, Lemma A.1.3], we have

$$\|\widehat{\nabla}^p \operatorname{adj} A\|_{L^\infty(\widehat{K})} \leq Ch^{p+d-1} \gamma^{p+d-1} (p+d-1)! \quad \forall p \in \mathbb{N}_0. \quad (4.13)$$

By (4.11a), the properties of matrix  $A$  in Definition 2.3 and [45, Lemma C.1], we have

$$\|\widehat{\nabla}^p (\boldsymbol{\psi} \circ G_K)\|_{L^2(\widehat{K})} \leq Ch^{p+d/2} \gamma_1^p \max(p, k)^p C_K \quad \forall p \in \mathbb{N}_0.$$

Here,  $\gamma_1 > 0$  is independent of  $h$ ,  $k$  and  $p$ . Then, by combining the above inequality with (4.13) and applying [43, Lemma A.1.3] again, we have

$$\|\widehat{\nabla}^p \widehat{\boldsymbol{\psi}}\|_{L^2(\widehat{K})} \leq Ch^{3d/2-1} (p \cdots (p+d-1)) (h\gamma)^p \max(p, k)^p C_K. \quad (4.14)$$

Here, the constant  $C$  is independent of  $h$ ,  $k$  and  $p$ . Now, we can apply [45, Lemma C.2] with  $R = 1$  and  $C_u = Ch^{3d/2-1}(p \cdots (p+d-1))C_K$ , such that

$$\begin{aligned} & \inf_{\widehat{\boldsymbol{\psi}} \in P_{p+1}(\widehat{K}; \mathbb{R}^d)} \|\widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\phi}}\|_{W^{1,\infty}(\widehat{K})} \\ & \leq Ch^{3d/2-1}(p \cdots (p+d-1))C_K \left[ \left( \frac{h}{h+\sigma} \right)^{p+2} + \left( \frac{kh}{\sigma p} \right)^{p+2} \right] \\ & \leq Ch^{3d/2-1}(p+2)^d C_K \left[ \left( \frac{h}{h+\sigma} \right)^{p+2} + \left( \frac{kh}{\sigma p} \right)^{p+2} \right]. \end{aligned} \quad (4.15)$$

According to (4.12a) and (4.15), we have

$$\|\Pi_{\mathbf{V}}\boldsymbol{\psi} - \boldsymbol{\psi}\|_{L^2(K)} \leq C(h(p+2))^d C_K \left[ \left( \frac{h}{h+\sigma} \right)^{p+2} + \left( \frac{kh}{\sigma p} \right)^{p+2} \right].$$

Then, by (4.11b), we have

$$\|\Pi_{\mathbf{V}}\boldsymbol{\psi} - \boldsymbol{\psi}\|_{L^2(\Omega)} \leq C(h(p+2))^d C_{\boldsymbol{\psi}} \left[ \left( \frac{h}{h+\sigma} \right)^{p+2} + \left( \frac{kh}{\sigma p} \right)^{p+2} \right].$$

We notice that  $\widehat{\nabla} \cdot \widehat{\boldsymbol{\psi}} = \det A \nabla \cdot \boldsymbol{\psi}$ . Then, by similar argument, we have the estimate of  $\|\nabla \cdot (\Pi_{\mathbf{V}}\boldsymbol{\psi} - \boldsymbol{\psi})\|_{L^2(\Omega)}$ . By (4.12c) and (4.15), we have the estimate of  $\|(\Pi_{\mathbf{V}}\boldsymbol{\psi} - \boldsymbol{\psi}) \cdot \mathbf{n}\|_{L^2(\partial\Omega)}$ . So, we can conclude that the proof is complete.  $\square$

**Lemma 4.8.** *Let  $\boldsymbol{\psi} \in \{\boldsymbol{\varphi} \in H^1(\Omega; \mathbb{R}^d) : \nabla \cdot \boldsymbol{\varphi} \in H^1(\Omega)\}$  satisfy*

$$k^2 \|\boldsymbol{\psi}\|_{L^2(\Omega)} + k \|\boldsymbol{\psi}\|_{H^1(\Omega)} + \|\nabla \cdot \boldsymbol{\psi}\|_{H^1(\Omega)} \leq CC_{\boldsymbol{\psi}}.$$

*Then, there exists  $\boldsymbol{\psi}_h \in \mathbf{V}_h$  such that*

$$\begin{aligned} k \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{L^2(\Omega)} + \|\nabla \cdot (\boldsymbol{\psi} - \boldsymbol{\psi}_h)\|_{L^2(\Omega)} & \leq ChC_{\boldsymbol{\psi}}, \\ k^{1/2} \|(\boldsymbol{\psi} - \boldsymbol{\psi}_h) \cdot \mathbf{n}\|_{L^2(\partial\Omega)} & \leq Ch^{1/2}k^{-1/2}C_{\boldsymbol{\psi}}. \end{aligned}$$

*Proof.* Let  $\widehat{\Pi}_{RT}^0$  be the lowest order standard Raviart-Thomas projection on  $\widehat{K}$ . We notice that for any  $\widehat{\boldsymbol{\psi}} \in H^1(\widehat{K}; \mathbb{R}^d)$

$$\widehat{\nabla} \cdot \widehat{\Pi}_{RT}^0 \widehat{\boldsymbol{\psi}} = \widehat{P}_0 \widehat{\nabla} \cdot \widehat{\boldsymbol{\psi}}, \quad (\widehat{\Pi}_{RT}^0 \widehat{\boldsymbol{\psi}}) \cdot \widehat{\mathbf{n}} = \widehat{P}_{0,\widehat{F}}(\widehat{\boldsymbol{\psi}} \cdot \mathbf{n}) \quad \forall \widehat{F} \in \Delta_{d-1}(\widehat{K}).$$

Here,  $\widehat{P}_0$  and  $\widehat{P}_{0,\widehat{F}}$  are standard  $L^2$ -orthogonal projection onto  $P_0(\widehat{K})$  and  $P_0(\widehat{F})$ , respectively. Then, we define  $\Pi_{RT}^0$  in the same way as  $\Pi_{\mathbf{V}}$  in (4.9) except that we replace  $\widehat{\Pi}_{\mathbf{V}}$  by  $\widehat{\Pi}_{RT}^0$ . According to the fact that the Piola transform commutes with both divergence operator and trace of normal component, we immediately have

$$\begin{aligned} k \|\Pi_{RT}^0 \boldsymbol{\psi} - \boldsymbol{\psi}\|_{L^2(\Omega)} + \|\nabla \cdot (\Pi_{RT}^0 \boldsymbol{\psi} - \boldsymbol{\psi})\|_{L^2(\Omega)} & \leq ChC_{\boldsymbol{\psi}}, \\ k^{1/2} \|(\Pi_{RT}^0 \boldsymbol{\psi} - \boldsymbol{\psi}) \cdot \mathbf{n}\|_{L^2(\partial\Omega)} & \leq Ch^{1/2}k^{-1/2}C_{\boldsymbol{\psi}}. \end{aligned}$$

So, we can conclude that the proof is complete.  $\square$

*Remark 4.9.* We notice that in [47, Theorem 3.5], it is shown that on a reference cube, the standard  $H(\text{div})$ -conforming Nédélec projection  $\widehat{\pi}_D$  has the following approximation property

$$\|\widehat{\pi}_D \boldsymbol{\psi} - \boldsymbol{\psi}\|_{L^2(\widehat{K})} \leq Cp^{-1/2} \|\boldsymbol{\psi}\|_{H^1(\widehat{K})}. \quad (4.16)$$

However, there is no  $H(\text{div})$ -conforming projection on triangle or tetrahedron elements satisfying (4.16). This is the reason why we use the lowest order standard Raviart-Thomas projection on  $\widehat{K}$ . If there is a  $H(\text{div})$ -conforming projection on triangular meshes satisfying (4.16), then the convergent result of Theorem 2.5 can be improved as

$$\begin{aligned} & \|u - u_h\|_{L^2(\Omega)} \\ & \leq Chp^{-1/2} (k\|\boldsymbol{\phi} - \boldsymbol{\psi}_h\|_{L^2(\Omega)} + \|\nabla \cdot (\boldsymbol{\phi} - \boldsymbol{\psi}_h)\|_{L^2(\Omega)} + \|\nabla(u - v_h)\|_{L^2(\Omega)} + k\|u - v_h\|_{L^2(\Omega)}) \\ & \quad + Ch^{1/2}p^{-1/2}\|(\boldsymbol{\phi} - \boldsymbol{\psi}_h) \cdot \mathbf{n}\|_{L^2(\partial\Omega)}, \end{aligned}$$

for any  $(\boldsymbol{\psi}_h, v_h) \in \mathbf{V}_h \times W_h$ .

We define two projections  $\Pi_W, \tilde{\Pi}_W : H^2(\Omega) \rightarrow W_h$  by

$$(\Pi_W v)|_K = \pi(v \circ G_K^{-1}) \quad (\tilde{\Pi}_W v)|_K = \tilde{\pi}(v \circ G_K^{-1}) \quad \forall K \in \mathcal{T}_h, \quad (4.17)$$

where  $\pi$  and  $\tilde{\pi}$  are the projections from  $H^2(\widehat{K})$  onto  $P_{p+1}(\widehat{K})$  introduced in [45, Theorem B.4, Lemma C.3], respectively.

**Lemma 4.10.** *Let  $v_{H^2} \in H^2(\Omega)$  and  $v_A$  is an analytic function in  $\Omega$ . We assume that there are constants  $C_v, \gamma > 0$  such that*

$$\|v_{H^2}\|_{H^2(\Omega)} \leq C_v \quad \|\nabla^p v_A\|_{L^2(\Omega)} \leq \gamma^p \max(p, k)^p C_v.$$

*Then, there are constants  $C, \sigma > 0$  independent of  $h, k$  and  $p$  such that*

$$\begin{aligned} & k\|\tilde{\Pi}_W v_A - v_A\|_{L^2(\Omega)} + \|\tilde{\Pi}_W v_A - v_A\|_{H^1(\Omega)} + k^{1/2}\|\tilde{\Pi}_W v_A - v_A\|_{H^{1/2}(\partial\Omega)} \\ & \leq C \left[ \left( \frac{h}{h+\sigma} \right)^p \left( 1 + \frac{hk}{h+\sigma} \right) + k \left( \frac{kh}{\sigma p} \right)^p \left( \frac{1}{p} + \frac{kh}{\sigma p} \right) \right] C_v, \\ & k\|\Pi_W v_{H^2} - v_{H^2}\|_{L^2(\Omega)} + \|\Pi_W v_{H^2} - v_{H^2}\|_{H^1(\Omega)} + k^{1/2}\|\Pi_W v_{H^2} - v_{H^2}\|_{H^{1/2}(\partial\Omega)} \\ & \leq Ck^{-1} \left( \frac{kh}{p} + \left( \frac{kh}{p} \right)^2 \right) C_v. \end{aligned}$$

*Proof.* The proof is a simple consequence of the procedure in [45, Theorem 5.5]. We have used the fact that for any  $k \geq 1$ ,

$$k^{1/2}\|v\|_{L^2(\partial\Omega)} \leq C(k\|v\|_{L^2(\Omega)} + \|v\|_{H^1(\Omega)}) \quad \forall v \in H^1(\Omega).$$

□

## 5. DUALITY ARGUMENT

We recall (2.2) that  $|\nabla^n u(x)|^2 = \sum_{\alpha \in \mathbb{N}_0^d: |\alpha|=n} \frac{n!}{\alpha!} |D^\alpha u(x)|^2$ .

**Lemma 5.1.** *We assume that the Assumptions (A1, A2) hold. Then, for any  $(\boldsymbol{\varphi}, w) \in \mathbf{V} \times W$ , there is  $(\boldsymbol{\psi}, v) \in \mathbf{V} \times W$  such that  $\|w\|_{L^2(\Omega)}^2 = b((\boldsymbol{\varphi}, w), (\boldsymbol{\psi}, v))$ .*

*$(\boldsymbol{\psi}, v)$  can be written as  $(\boldsymbol{\psi}, v) = (\boldsymbol{\psi}_A, v_A) + (\boldsymbol{\psi}_{H^2}, v_{H^2})$ . Here, both  $\boldsymbol{\psi}_A$  and  $v_A$  are analytic functions in  $\Omega$ ,  $\boldsymbol{\psi}_{H^2} \in H^1(\Omega; \mathbb{R}^d)$ , and  $v_{H^2} \in H^2(\Omega)$ . There are constants  $C, \gamma > 0$*

independent of  $k \geq k_0$  such that

$$k\|\boldsymbol{\psi}_A\|_{L^2(\Omega)} + \|\boldsymbol{\psi}_A\|_{H^1(\Omega)} \leq Ck\|w\|_{L^2(\Omega)}, \quad (5.1a)$$

$$k\|v_A\|_{L^2(\Omega)} + \|v_A\|_{H^1(\Omega)} \leq Ck\|w\|_{L^2(\Omega)}, \quad (5.1b)$$

$$\|\nabla^{p+2}\boldsymbol{\psi}_A\|_{L^2(\Omega)} + \|\nabla^{p+2}v_A\|_{L^2(\Omega)} \leq C\gamma^p \max(p, k)^{p+2}\|w\|_{L^2(\Omega)}, \quad (5.1c)$$

$$k^2\|\boldsymbol{\psi}_{H^2}\|_{L^2(\Omega)} + k\|\boldsymbol{\psi}_{H^2}\|_{H^1(\Omega)} \leq C\|w\|_{L^2(\Omega)}, \quad (5.1d)$$

$$\|\nabla \cdot \boldsymbol{\psi}_{H^2}\|_{H^1(\Omega)} \leq C\|w\|_{L^2(\Omega)}, \quad (5.1e)$$

$$k^2\|v_{H^2}\|_{L^2(\Omega)} + k\|v_{H^2}\|_{H^1(\Omega)} + \|v_{H^2}\|_{H^2(\Omega)} \leq C\|w\|_{L^2(\Omega)}. \quad (5.1f)$$

*Remark 5.2.* Since we can only have  $\|\nabla \cdot \boldsymbol{\psi}_{H^2}\|_{H^1(\Omega)} \leq C\|w\|_{L^2(\Omega)}$  instead of  $\|\boldsymbol{\psi}_{H^2}\|_{H^2(\Omega)} \leq C\|w\|_{L^2(\Omega)}$ , it is necessary to use Raviart-Thomas space to approximate  $\boldsymbol{\phi}$  and  $\boldsymbol{\psi}$  instead of vector valued continuous piece-wise polynomial space, in order to show quasi optimal convergence.

*Remark 5.3.* For any  $(\boldsymbol{\phi}, u), (\boldsymbol{\psi}, v) \in \mathbf{V} \times W$ , we define

$$\begin{aligned} & b_\tau((\boldsymbol{\phi}, u), (\boldsymbol{\psi}, v)) \\ &= (\mathbf{i}k\boldsymbol{\phi} + \nabla u, \mathbf{i}k\boldsymbol{\psi} + \nabla v)_\Omega + (\mathbf{i}ku + \nabla \cdot \boldsymbol{\phi}, \mathbf{i}kv + \nabla \cdot \boldsymbol{\psi})_\Omega + \tau \langle \boldsymbol{\phi} \cdot \mathbf{n} + u, \boldsymbol{\psi} \cdot \mathbf{n} + v \rangle_{\partial\Omega}, \end{aligned}$$

where  $\tau$  is a positive constant. It is easy to see that the exact solution  $(\boldsymbol{\phi} = \mathbf{i}k^{-1}\nabla u, u)$  satisfies

$$b_\tau((\boldsymbol{\phi}, u), (\boldsymbol{\psi}, v)) = (-\mathbf{i}fk^{-1}, \mathbf{i}kv + \nabla \cdot \boldsymbol{\psi})_\Omega + \tau k^{-1} \langle \mathbf{i}g, \boldsymbol{\psi} \cdot \mathbf{n} + v \rangle_{\partial\Omega} \quad \forall (\boldsymbol{\psi}, v) \in \mathbf{V} \times W.$$

As we mentioned in Remark 3.1, the variational form  $b_\tau$  is uniformly coercive with respect to the wave number  $k$  if  $\tau \geq 1$ . However, if we choose  $\tau$  to be 1, then the boundary condition (5.3c) in the proof of Lemma 5.1 will be

$$\boldsymbol{\phi} \cdot \mathbf{n} + v = \mathbf{i}kz \quad \text{on } \partial\Omega.$$

The consequence is that all right hand sides of regularity estimates (5.1) have to be multiplied by an extra factor  $k^{1/2}$ , such that the quasi optimal convergent result in Theorem 2.5 can *not* be obtained.

*Proof.* Let  $z$  be the solution of the Helmholtz equation

$$-\Delta z - k^2 z = w \text{ in } \Omega, \quad \frac{\partial z}{\partial \mathbf{n}} + \mathbf{i}kz = 0 \text{ on } \partial\Omega.$$

It is easy to check that

$$\begin{aligned} & \|w\|_{L^2(\Omega)}^2 \\ &= (\nabla w, \nabla z)_\Omega - k^2(w, z)_\Omega - \mathbf{i}k \langle w, z \rangle_{\partial\Omega} \\ &= (\mathbf{i}k\boldsymbol{\varphi} + \nabla w, \nabla z)_\Omega - (\mathbf{i}k\boldsymbol{\varphi}, \nabla z)_\Omega - k^2(w, z)_\Omega - \mathbf{i}k \langle w, z \rangle_{\partial\Omega} \\ &= (\mathbf{i}k\boldsymbol{\varphi} + \nabla w, \nabla z)_\Omega + (\nabla \cdot \boldsymbol{\varphi} + \mathbf{i}kw, -\mathbf{i}kz)_\Omega + \langle \boldsymbol{\varphi} \cdot \mathbf{n} + w, \mathbf{i}kz \rangle_{\partial\Omega}. \end{aligned}$$

According to Theorem 4.3 and Assumption (A2),  $z$  can be written as  $z = z_A + z_{H^2}$ .  $z_A$  is an analytic function and  $z_{H^2} \in H^2(\Omega)$ . In addition,

$$k\|z_A\|_{L^2(\Omega)} + \|z_A\|_{H^1(\Omega)} \leq C\|w\|_{L^2(\Omega)}, \quad (5.2a)$$

$$\|\nabla^{p+2}z_A\|_{L^2(\Omega)} \leq C\gamma^p k^{-1} \max(p, k)^{p+2}\|w\|_{L^2(\Omega)}, \quad (5.2b)$$

$$\|z_{H^2}\|_{H^2(\Omega)} + k\|z_{H^2}\|_{H^1(\Omega)} + k^2\|z_{H^2}\|_{L^2(\Omega)} \leq C\|w\|_{L^2(\Omega)}. \quad (5.2c)$$



According to Theorem 2.4, we can define  $(\boldsymbol{\psi}, v) \in \mathbf{V} \times W$  by

$$\mathbf{i}k\boldsymbol{\psi} + \nabla v = \nabla z \quad \text{in } \Omega, \quad (5.3a)$$

$$\mathbf{i}kv + \nabla \cdot \boldsymbol{\psi} = -\mathbf{i}kz \quad \text{in } \Omega, \quad (5.3b)$$

$$k^{1/2}(\boldsymbol{\psi} \cdot \mathbf{n} + v) = \mathbf{i}k^{1/2}z \quad \text{on } \partial\Omega. \quad (5.3c)$$

Obviously, we can write  $(\boldsymbol{\psi}, v)$  as  $(\tilde{\boldsymbol{\psi}}_A, \tilde{v}_A) + (\tilde{\boldsymbol{\psi}}_{H^2}, \tilde{v}_{H^2})$  where

$$\begin{aligned} \mathbf{i}k\tilde{\boldsymbol{\psi}}_A + \nabla\tilde{v}_A &= \nabla z_A & \mathbf{i}k\tilde{\boldsymbol{\psi}}_{H^2} + \nabla\tilde{v}_{H^2} &= \nabla z_{H^2} & \text{in } \Omega, \\ \mathbf{i}k\tilde{v}_A + \nabla \cdot \tilde{\boldsymbol{\psi}}_A &= -\mathbf{i}kz_A & \mathbf{i}k\tilde{v}_{H^2} + \nabla \cdot \tilde{\boldsymbol{\psi}}_{H^2} &= -\mathbf{i}kz_{H^2} & \text{in } \Omega, \\ k^{1/2}(\tilde{\boldsymbol{\psi}}_A \cdot \mathbf{n} + \tilde{v}_A) &= \mathbf{i}k^{1/2}z_A & k^{1/2}(\tilde{\boldsymbol{\psi}}_{H^2} \cdot \mathbf{n} + \tilde{v}_{H^2}) &= \mathbf{i}k^{1/2}z_{H^2} & \text{on } \partial\Omega. \end{aligned}$$

It is easy to see that

$$\begin{aligned} \Delta\tilde{v}_A + k^2\tilde{v}_A &= \Delta z_A - k^2 z_A & \Delta\tilde{v}_{H^2} + k^2\tilde{v}_{H^2} &= \Delta z_{H^2} - k^2 z_{H^2} & \text{in } \Omega, \\ \frac{\partial\tilde{v}_A}{\partial\mathbf{n}} - \mathbf{i}k\tilde{v}_A &= kz_A + \frac{\partial z_A}{\partial\mathbf{n}} & \frac{\partial\tilde{v}_{H^2}}{\partial\mathbf{n}} - \mathbf{i}k\tilde{v}_{H^2} &= kz_{H^2} + \frac{\partial z_{H^2}}{\partial\mathbf{n}} & \text{on } \partial\Omega. \end{aligned}$$

It is easy to see that  $\tilde{v}_A - z_A = S_k(-2k^2 z_A, (1 + \mathbf{i})kz_A)$ . By Lemma 4.4, we have

$$\begin{aligned} \|\nabla^{p+2}\tilde{v}_A\|_{L^2(\Omega)} &\leq \|\nabla^{p+2}(\tilde{v}_A - z_A)\|_{L^2(\Omega)} + \|\nabla^{p+2}z_A\|_{L^2(\Omega)} \\ &\leq C\gamma^{p+2} \max(p+2, k)^{p+2} (k\|z_A\|_{L^2(\Omega)} + \|\nabla z_A\|_{L^2(\Omega)}) + \|\nabla^{p+2}z_A\|_{L^2(\Omega)} \\ &\leq C\gamma^p \max(p, k)^{p+2} \|w\|_{L^2(\Omega)}, \\ k\|\tilde{v}_A\|_{L^2(\Omega)} + \|\tilde{v}_A\|_{H^1(\Omega)} &\leq Ck\|w\|_{L^2(\Omega)}. \end{aligned}$$

Since  $\mathbf{i}k\tilde{\boldsymbol{\psi}}_A + \nabla\tilde{v}_A = \nabla z_A$ , we have

$$\begin{aligned} \|\nabla^{p+2}\tilde{\boldsymbol{\psi}}_A\|_{L^2(\Omega)} &\leq C\gamma^p \max(p, k)^{p+2} \|w\|_{L^2(\Omega)}, \\ k\|\tilde{\boldsymbol{\psi}}_A\|_{L^2(\Omega)} + \|\tilde{\boldsymbol{\psi}}_A\|_{H^1(\Omega)} &\leq Ck\|w\|_{L^2(\Omega)}. \end{aligned}$$

By (5.2c), we have

$$\|\Delta z_{H^2} - k^2 z_{H^2}\|_{L^2(\Omega)} + \|kz_{H^2} + \frac{\partial z_{H^2}}{\partial\mathbf{n}}\|_{H^{1/2}(\partial\Omega)} \leq C\|w\|_{L^2(\Omega)}.$$

By Theorem 4.3 and the Assumption (A2) again,  $\tilde{v}_{H^2}$  can be written as  $\tilde{v}_{H^2} = \tilde{v}_{A,H^2} + \tilde{v}_{H^2,H^2}$  where  $\tilde{v}_{A,H^2}$  and  $\tilde{v}_{H^2,H^2}$  are analytic and in  $H^2(\Omega)$ , respectively. In addition, we have

$$k^2\|\tilde{v}_{H^2,H^2}\|_{L^2(\Omega)} + k\|\tilde{v}_{H^2,H^2}\|_{H^1(\Omega)} + \|\tilde{v}_{H^2,H^2}\|_{H^2(\Omega)} \leq C\|w\|_{L^2(\Omega)}.$$

We define  $v_A = \tilde{v}_A + \tilde{v}_{A,H^2}$  and  $v_{H^2} = \tilde{v}_{H^2,H^2}$ . Then, (5.1b, 5.1f) hold. We write  $\tilde{\boldsymbol{\psi}}_{H^2}$  as  $\tilde{\boldsymbol{\psi}}_{H^2} = \tilde{\boldsymbol{\psi}}_{A,H^2} + \tilde{\boldsymbol{\psi}}_{H^2,H^2}$  such that  $\mathbf{i}k\tilde{\boldsymbol{\psi}}_{A,H^2} + \nabla\tilde{v}_{A,H^2} = 0$ . We define  $\boldsymbol{\psi}_A = \tilde{\boldsymbol{\psi}}_A + \tilde{\boldsymbol{\psi}}_{A,H^2}$ . Then, (5.1a, 5.1c, 5.1d) hold.

Now, we only need to prove (5.1e). Since  $\mathbf{i}k\tilde{\boldsymbol{\psi}}_{A,H^2} + \nabla\tilde{v}_{A,H^2} = 0$  and (4.5c), we have

$$\begin{aligned} \|\mathbf{i}k\tilde{v}_{A,H^2} + \nabla \cdot \tilde{\boldsymbol{\psi}}_{A,H^2}\|_{H^1(\Omega)} &= k^{-1}\|k^2\tilde{v}_{A,H^2} + \Delta\tilde{v}_{A,H^2}\|_{H^1(\Omega)} \\ &\leq C \left( \|\Delta z_{H^2} - k^2 z_{H^2}\|_{L^2(\Omega)} + \|kz_{H^2} + \frac{\partial z_{H^2}}{\partial\mathbf{n}}\|_{H^{1/2}(\partial\Omega)} \right) \leq C\|w\|_{L^2(\Omega)}. \end{aligned}$$

Notice that

$$\begin{aligned} & (\mathbf{i}k\tilde{v}_{H^2,H^2} + \nabla \cdot \tilde{\boldsymbol{\psi}}_{H^2,H^2}) + (\mathbf{i}k\tilde{v}_{A,H^2} + \nabla \cdot \tilde{\boldsymbol{\psi}}_{A,H^2}) \\ &= \mathbf{i}k\tilde{v}_{H^2} + \nabla \cdot \tilde{\boldsymbol{\psi}}_{H^2} = -\mathbf{i}kz_{H^2}. \end{aligned}$$

We define  $\boldsymbol{\psi}_{H^2} = \tilde{\boldsymbol{\psi}}_{H^2,H^2}$ . Then, we can conclude that (5.1e) is true.  $\square$

Now we can provide the proof for the quasi optimal convergent result in Theorem 2.5.

*Proof.* (Theorem 2.5) We denote by  $e^\phi = \boldsymbol{\phi} - \boldsymbol{\phi}_h$  and  $e^u = u - u_h$ . Applying Lemma 5.1 with  $w = e^u$ , we have

$$\begin{aligned} \|e^u\|_{L^2(\Omega)}^2 &= b((e^\phi, e^u), (\boldsymbol{\psi}, v)) \\ &= b((e^\phi, e^u), (\boldsymbol{\psi}_A, v_A)) + b((e^\phi, e^u), (\boldsymbol{\psi}_{H^2}, v_{H^2})), \end{aligned}$$

such that  $\boldsymbol{\psi}_A, v_A, \boldsymbol{\psi}_{H^2}, v_{H^2}$  satisfy (5.1). Then, by the standard Galerkin orthogonality of the first order system least squares method (2.5), we have that for any  $\tilde{\boldsymbol{\psi}}_A, \tilde{\boldsymbol{\psi}}_{H^2} \in \mathbf{V}_h$  and any  $\tilde{v}_A, \tilde{v}_{H^2} \in W_h$

$$\|e^u\|_{L^2(\Omega)}^2 = b((e^\phi, e^u), (\boldsymbol{\psi}_A - \tilde{\boldsymbol{\psi}}_A, v_A - \tilde{v}_A)) + b((e^\phi, e^u), (\boldsymbol{\psi}_{H^2} - \tilde{\boldsymbol{\psi}}_{H^2}, v_{H^2} - \tilde{v}_{H^2})).$$

We choose  $\tilde{\boldsymbol{\psi}}_A = \Pi_{\mathbf{V}}\boldsymbol{\psi}_A$  ( $\Pi_{\mathbf{V}}$  defined in (4.9)),  $\tilde{\boldsymbol{\psi}}_{H^2}$  to be  $\boldsymbol{\psi}_h$  in Lemma 4.8,  $\tilde{v}_A = \tilde{\Pi}_W v_A$  and  $\tilde{v}_{H^2} = \Pi_W v_{H^2}$  ( $\tilde{\Pi}_W$  and  $\Pi_W$  defined in (4.17)). Then by Lemma 4.7, Lemma 4.8 and Lemma 4.10, it is straightforward to see that if (2.8) holds,

$$\begin{aligned} & \|e^u\|_{L^2(\Omega)} \\ & \leq Ch \left( \|\mathbf{i}ke^\phi + \nabla e^u\|_{L^2(\Omega)} + \|\mathbf{i}ke^u + \nabla \cdot e^\phi\|_{L^2(\Omega)} \right) \\ & \quad + Ch^{1/2} \|e^\phi \cdot \mathbf{n} + e^u\|_{L^2(\partial\Omega)}. \end{aligned}$$

Finally, by Theorem 2.4 (the stability estimate) and the standard Galerkin orthogonality argument for the first order system least squares method (2.5) again, we can conclude that the proof is complete.  $\square$

## 6. NUMERICAL RESULTS

In this section, we present some numerical results of the FOSLS method for the following 2-d Helmholtz problem:

$$\begin{aligned} -\Delta u - \kappa^2 u &= f := \frac{\sin \kappa r}{r} && \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} + \mathbf{i}\kappa u &= g && \text{on } \partial\Omega. \end{aligned}$$

Here  $\Omega$  is unit square  $[-0.5, 0.5] \times [-0.5, 0.5]$ , and  $g$  is chosen such that the exact solution is given by

$$u = \frac{\cos \kappa r}{\kappa} - \frac{\cos \kappa + \mathbf{i} \sin \kappa}{\kappa(J_0(\kappa) + \mathbf{i}J_1(\kappa))} J_0(\kappa r)$$

in polar coordinates, where  $J_\nu(z)$  are Bessel functions of the first kind.

In the following experiments, the FOSLS method is implemented for the pair of finite element spaces  $\mathbf{V}_h \times W_h$  with  $p+1 = 1, 2, 3, 4$ , which are denoted by  $RT1P1$ ,  $RT2P2$ ,  $RT3P3$  and  $RT4P4$ . For the fixed wave number  $k$ , we first show the dependence of the relative error  $\|u - u_h\|_{L^2(\Omega)} / \|u\|_{L^2(\Omega)}$  and  $\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{L^2(\Omega)} / \|\boldsymbol{\phi}\|_{L^2(\Omega)}$  on polynomial degree  $p$  and mesh size  $h$ . The Figure 1 displaying the relative error is plotted in log-log coordinates. The dotted lines

in the two graphs are denoted for the convergence rate  $O(kh^2/p^2)$  (left) and  $O(kh/p)$  (right) respectively. Since the dotted lines in each graph of Figure 1 are parallel for different  $p$ , we only plot a single dotted line in each graph to reveal its dependence on  $h$ . The left graph of Figure 1 displays the relative error  $\|u - u_h\|_{L^2(\Omega)} / \|u\|_{L^2(\Omega)}$  for the case  $k = 200$  by the FOSLS method based on  $RT1P1$ ,  $RT2P2$ ,  $RT3P3$  and  $RT4P4$  approximations, while the right graph displays the corresponding relative error  $\|\phi - \phi_h\|_{L^2(\Omega)} / \|\phi\|_{L^2(\Omega)}$ . We find that the relative error for  $u$  converges slower than  $O(kh^2/p^2)$  when  $p = 1$  or 2 on the underlying meshes, however, it converges almost or faster than  $O(kh^2/p^2)$  for higher polynomial degree  $p = 3$  or 4. The similar phenomenon can also be observed from the right graph of Figure 1 for the relative error for  $\phi$  compared with the convergence rate  $O(kh/p)$ .

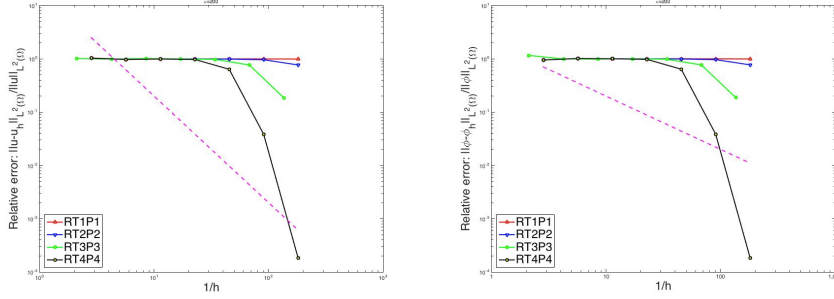


FIGURE 1. The relative errors  $\|u - u_h\|_{L^2(\Omega)} / \|u\|_{L^2(\Omega)}$  (left) and  $\|\phi - \phi_h\|_{L^2(\Omega)} / \|\phi\|_{L^2(\Omega)}$  (right) for the case  $k = 200$  based on the FOSLS method with different polynomial degree.

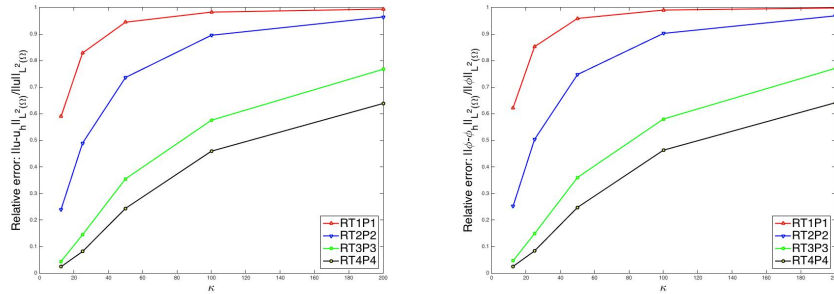


FIGURE 2. The relative errors  $\|u - u_h\|_{L^2(\Omega)} / \|u\|_{L^2(\Omega)}$  (left) and  $\|\phi - \phi_h\|_{L^2(\Omega)} / \|\phi\|_{L^2(\Omega)}$  (right) under the mesh condition  $kh/p \approx 1$ .

Figure 2 displays the relative errors for  $u$  and  $\phi$  under the mesh condition  $kh/p \approx 1$  respectively. It shows that for the FOSLS method based on different polynomial degree approximations ( $p=1,2,3,4$ ), both two types of relative errors cannot be controlled under the mesh condition  $kh/p \approx 1$  and increase with the wave number  $k$ , which indicates the existence of the pollution error. Figure 3 displays the same relative errors under the mesh condition  $kh/p \approx 0.5$ . We observe that under this mesh condition, although the relative errors still increase with the wave number  $k$  for the FOSLS method based on lower order polynomial approximations, the relative errors are quite small for different wave number  $k$  when the polynomial degree  $p = 4$ . The results support the theoretical analysis.

For more detailed comparison between FOSLS methods with different polynomial degree approximations, we consider the Helmholtz problem with wave number  $k = 200$ . Figure 4

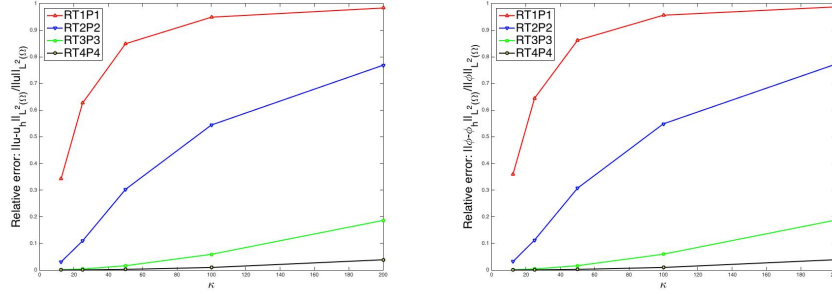


FIGURE 3. The relative errors  $\|u - u_h\|_{L^2(\Omega)} / \|u\|_{L^2(\Omega)}$  (left) and  $\|\phi - \phi_h\|_{L^2(\Omega)} / \|\phi\|_{L^2(\Omega)}$  (right) under the mesh condition  $kh/p \approx 0.5$ .

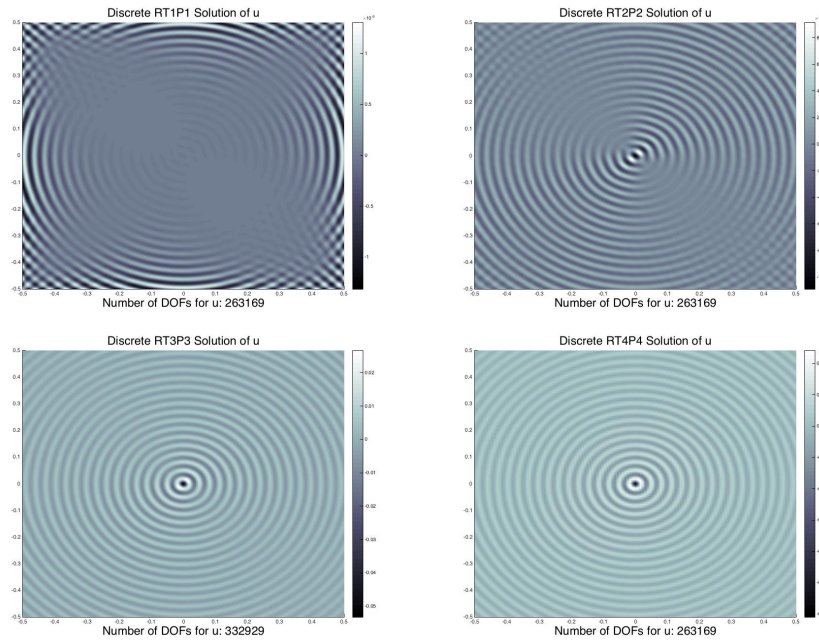


FIGURE 4. Surface plot of the imaginary parts of the FOSLS solution  $u_h$  based on  $RT1P1$  (top-left),  $RT2P2$  (top-right),  $RT3P3$  (bottom-left),  $RT4P4$  (bottom-right) approximations for the case  $k = 200$  under the mesh condition  $kh/p \approx 0.5$ .

displays the surface plots of the imaginary parts of the FOSLS solutions of  $u_h$  based on the  $RT1P1$ ,  $RT2P2$ ,  $RT3P3$  and  $RT4P4$  approximations under mesh condition  $kh/p \approx 0.5$ . The traces of imaginary part of the FOSLS solution  $u_h$  based on the  $RT1P1$ ,  $RT2P2$ ,  $RT3P3$  and  $RT4P4$  approximations in the  $xz$ -plane under mesh condition  $kh/p \approx 0.5$ , and the trace of imaginary part of the exact solution, are both shown in Figure 5. It is shown that the FOSLS solutions  $u_h$  based on  $RT3P3$  and  $RT4P4$  approximations have almost correct shapes and amplitudes as the exact solution, while the FOSLS solution  $u_h$  based on low order polynomial approximations does not match the exact solution well. Thus we can observe that although the phase error appears in the case of low order polynomial approximation, it can be reduced by high order polynomial approximation.

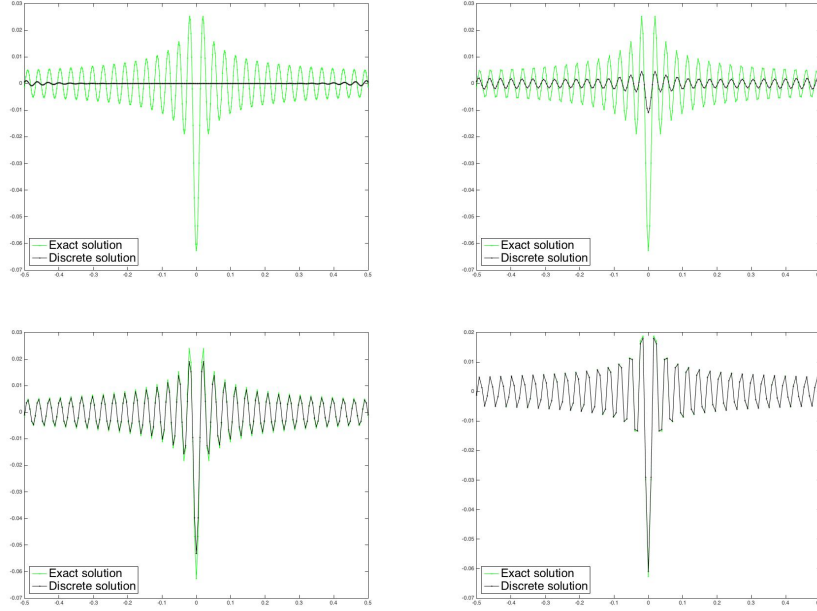


FIGURE 5. The traces of imaginary part of the FOSLS solution  $u_h$  based on  $RT1P1$  (top-left),  $RT2P2$  (top-right),  $RT3P3$  (bottom-left),  $RT4P4$  (bottom-right) approximations for the case  $k = 200$  under the mesh condition  $kh/p \approx 0.5$ . The trace of imaginary part of the exact solution is plotted in the green lines.

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