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FINITE TIME STABILITY ANALYSIS OF NON-LINEAR FRACTIONAL ORDER WITH MULTI STATE TIME DELAY

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Abstract. Sufficient condition for the stability of a fractional order semi-linear system with multi-time delay is proposed.

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1. Introduction

Stability of linear and non-linear control system is an important area of research in control theory. Stability of finite dimensional system of ordinary and fractional differential equation is discussed in [6] and [3]. Gronwall Inequality was generalized in [7] and is used to discuss the finite time stability in [4]. In [2] Mittage-Leffler function and its properties are discussed in details.

The main result of this paper provides a condition for finite time stability of a fractional order system with multi-state time delay. It depends on the result of Generalized Gronwall' inequality for fractional order differential equation.

2. Fundamental concepts

In this section we provide an overviews on the fundamentals related to Riemann-Liouvill fractional integral and derivatives [5], fractional order differential equations and fractional order system, generalization of the Granwall' inequality for fractional order differential equation [7] and finite time stability.

2.1. **Fractional Integral and Differential.** Riemman Liouville Fractional integral of a function $f : \mathbb{R} \to \mathbb{R}$ of order ν is defined as [5] [1]

$$ {}_0\mathrm{D}_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} f(\xi) \mathrm{d}\xi \; ; \; Re(\nu) > 0 $$

The μ-th order Riemman Liouville fractional derivative is given by

$$ \mathrm{D}^\mu f(t) = \begin{cases} \frac{1}{\Gamma(\nu)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^t (t-\xi)^{\nu-1} f(\xi) \mathrm{d}\xi & \text{when } m-1 < \mu < m \\ \frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) & \text{when } \mu = m \in \mathbb{N} \end{cases} $$

Consider the semi-linear fractional order differential equations

$$\begin{aligned} \mathrm{D}^q x(t) &= A_0 x(t) + \sum_{i=1}^{p} A_i x(t-\tau_i) + B_0 u(t) + f(t,x);\ t \geq 0 \\ x(t) &= \psi_x(t);\ t \in [-\tau, 0] \end{aligned} \tag{2.1}$$

on the space $(\mathrm{C}[-\tau, T], \mathbb{R}^n)$ with the uniform norm defined as

$$\|x\|_{=} \max\{|x_1(t)|, |x_2(t)|, \ldots, |x_n(t)|\}.$$

Where

$\tau = max\{\tau_1, \tau_2, \ldots, \tau_p\};\ \tau_i > 0$ are contants.
$x(t) : \mathbb{R} \to \mathbb{R}^n;$ is a $n \times 1$ vector and $x \in C[[-\tau, T] : \mathbb{R}^n]$
$\mathrm{D}^q x(t) = (\mathrm{D}^q x_1(t), \mathrm{D}^q x_2(t), \ldots, \mathrm{D}^q x_n(t))^t;$ is an $n \times 1$ vector.
$(A_i)_{n\times n} = \{a_{j,k}\}_i;$ are constant matrix for $i = 0, 1, \ldots, p$.

$f(t,x)_{n\times 1} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$; satisfies Lipchitz condition on w.r.t x.

$$\therefore \|f(t,x) - f(t,y)\| \leq L\|x - y\|$$

$$\Rightarrow \|f(t,x)\| \leq L\|x\| + m,$$

where $m = \|f(t,\theta)\|, \theta$ is null vector.

2.2. **Generalized Gronwall Inequality.** [7] Suppose $y(t) : \mathbb{R} \to \mathbb{R}$ and $a(t) : \mathbb{R} \to \mathbb{R}$ are non-negative and integrable in every closed and bounded subinterval of $[0, T)$ and $g(t) : [0, T) \to \mathbb{R}$ is non-negative, non-decreasing, continuous and bounded function such that

$$y(t) \leq a(t) + g(t) \int_0^t (t-s)^{q-1} y(s) \mathrm{d}s$$

then for $0 \leq t \leq T$

$$y(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{g(t)\Gamma(q)}{\Gamma(nq)} (t-s)^{nq-1} a(s) \right] \mathrm{d}s$$

Moreover if $a(t)$ is nondecreasing then

$$y(t) \leq a(t) E_q(g(t).\Gamma(q) t^q).$$

Where E_q is the Mittag-Leffler function, defined by

$$E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kq+1)}.$$

2.3. **Finite Time Stability.** [4] The system given by (2.1) is finite time stable w.r.t. $\{\delta, \varepsilon, q_u, J\}$, if $\|\psi_x\| < \delta$ and $\|u(t)\| < q_u, \forall t \in J = [0, T]$ imply $\|x(t)\| < \varepsilon, \forall t \in J$.

The system (2.1) is called homogeneous if $u(t) = 0$. In this case the system will be finite time stable if $\|\psi_x\| < \delta$ imply $\|x(t)\| < \varepsilon, \forall t \in J$.

In [4] it has been shown that the linear system (2.1)is stable if the following condition is satisfied. [i]

$$\left[1 + \frac{(n+1)\sigma t^q}{\Gamma(q+1)} + \frac{q_u b_0 t^q}{\delta \Gamma(q+1)}\right] E_q((n+1)\sigma t^q) < \frac{\varepsilon}{\delta}$$

3. Main Result

Theorem 3.1. *The system given by* (2.1) *is finite time stable w.r.t.* $\{\delta, \varepsilon, q_u, J\}$, *if the following condition is satisfied*

$$\left[1 + \frac{(m + bq_u)T^q}{\delta \Gamma(q+1)}\right] E_q\{(L + \sigma(p+1))T^q\} < \frac{\varepsilon}{\delta}$$

Proof. The solution of (2.1) is given by

$$\begin{aligned} x(t) =& x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left\{ A_0 x(s) + \sum_{i=1}^{p} A_i x(s - \tau_i) + B_0 u(s) + f(s, x) \right\} \mathrm{d}s \\ \Rightarrow \|x(t)\| \leq& \|x(0)\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \{\|A_0\| \|x(s)\| + \sum_{i=1}^{p} \|A_i\| \|x(s - \tau_i)\| \\ &+ \|B_0\| \|u(s)\| + \|f(s, x)\|\} \mathrm{d}s \end{aligned}$$

Now let $\sigma_{max}(A)$ is the largest singular value of the matrix A and

$$\begin{aligned} \sigma &= \max_{0 \leq i \leq p} \{\sigma_{max}(A_i)\} \\ b_0 &= \sigma_{max}(B_0) \\ \therefore \|A_i\| &\leq \sigma;\ \forall i = 1, 2, \ldots, p. \end{aligned}$$

[i] The condition derived in this paper is stronger than that of [4]

$$
\begin{aligned}
\|x(t)\| \leq & \|x(0)\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \{\sigma \|x(s)\| + \sum_{i=1}^{p} \sigma \|x(s-\tau_i)\| \\
& + b_0 \|u(s)\| + \|f(s,x)\|\} \mathrm{d}s \\
\leq & \|\psi(0)\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \{\sigma \|x(s)\| + \sum_{i=1}^{p} \sigma \|x(s)\| \\
& + b_0 \|u(s)\| + m + L\|x(s)\|\} \mathrm{d}s \\
\leq & \|\psi(0)\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \{\sigma(p+1) \|x(s)\| \\
& + b_0 q_u + m + L\|x(s)\|\} \mathrm{d}s \\
\leq & \|\psi(0)\| + \frac{m + b_0 q_u}{\Gamma(q)} \int_0^t (t-s)^{q-1} \mathrm{d}s + \\
& \frac{L + \sigma(p+1)}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|x(s)\| \mathrm{d}s \\
\leq & \left\{ \delta + \frac{(m + b_0 q_u) t^q}{\Gamma(q+1)} \right\} + \frac{L + \sigma(p+1)}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|x(s)\| \mathrm{d}s \\
\|x(t)\| \leq & a(t) + g(t) \int_0^t (t-s)^{q-1} \|x(s)\| \mathrm{d}s \\
\text{Where } a(t) = & \delta + \frac{(m + b_0 q_u) t^q}{\Gamma(q+1)} \\
\text{and } g(t) = & \frac{L + \sigma(p+1)}{\Gamma(q)}
\end{aligned}
$$

Now $a(t)$ is nonnegative and locally integrable. $g(t)$ is non negative bounded and nondecreasing. Now by the Gronwall' inequality we shall say that

$$
\begin{aligned}
\|x(t)\| \leq & a(t) E_q(g(t)\Gamma(q) t^q) \\
= & \left\{ \delta + \frac{(m + b_0 q_u) t^q}{\Gamma(q+1)} \right\} E_q \left\{ \frac{L + \sigma(p+1)}{\Gamma(q)} \Gamma(q) t^q \right\} \\
\leq & \delta \left\{ 1 + \frac{(m + b_0 q_u) T^q}{\delta \Gamma(q+1)} \right\} E_q \left\{ (L + \sigma(p+1)) T^q \right\}
\end{aligned}
$$

So for the finite time stability we require $\|x(t)\| < \varepsilon$ if

$$
\left\{ 1 + \frac{(m + b_0 q_u) T^q}{\delta \Gamma(q+1)} \right\} E_q \{ (L + \sigma(p+1)) T^q \} < \frac{\varepsilon}{\delta}
$$

□

Special Cases

(1) If the control term $u(t) = 0$, then $q_u = 0$. Then theorem (3.1) will take the following form

$$\left\{1 + \frac{m}{\delta\Gamma(q+1)}T^q\right\} E_q\left\{(L + \sigma(p+1))T^q\right\} \leq \frac{\varepsilon}{\delta}$$

(2) If $f(t, \theta) = 0$ i.e. $m = 0$ them theorem (3.1) will take the following form

$$\left\{1 + \frac{b_0 q_u}{\delta\Gamma(q+1)}T^q\right\} E_q\left\{(L + \sigma(p+1))\right\} \leq \frac{\varepsilon}{\delta}$$

(3) Let there is no non-linear term i.e. $f(t) = 0$. So $L = 0$ and $m = 0$. Then the theorem (3.1) will take the following form

$$\left\{1 + \frac{b_0 q_u}{\delta\Gamma(q+1)}T^q\right\} E_q\left\{\sigma(p+1)T^q\right\} \leq \frac{\varepsilon}{\delta}$$

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