

Spectra of Abelian C^* -Subalgebra Sums

Christian Fleischhack*

*Institut für Mathematik
Universität Paderborn
Warburger Straße 100
33098 Paderborn
Germany*

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Abstract

Let $C_b(X)$ be the C^* -algebra of bounded continuous functions on some non-compact, but locally compact Hausdorff space X . Moreover, let \mathfrak{A}_0 be some ideal and \mathfrak{A}_1 be some unital C^* -subalgebra of $C_b(X)$. For \mathfrak{A}_0 and \mathfrak{A}_1 having trivial intersection, we show that the spectrum of their vector space sum equals the disjoint union of their individual spectra, whereas their topologies are nontrivially interwoven. Indeed, they form a so-called twisted-sum topology which we will investigate before. Within the whole framework, e.g., the one-point compactification of X and the spectrum of the algebra of asymptotically almost periodic functions can be described.

1 Introduction

It is well known that the spectrum of a direct sum of two abelian C^* -algebras equals the topological direct sum of the respective individual spectra. Sometimes, however, one is given only a *vector space* direct sum of two C^* -algebras. This applies, most prominently, to $\mathfrak{A} + \mathbb{C}\mathbf{1}$ to be considered when one adjoins a unit to the non-unital C^* -algebra \mathfrak{A} . Another example is the algebra $C_0(X) + C_{\text{AP}}(X)$ of asymptotically almost periodic functions [4] on a non-compact locally compact abelian group X , being our main motivation [3]. It is now natural to ask whether there are still general arguments on how to determine the spectrum in these cases. Or, to put it into a more abstract form: how does the spectrum of a sum of any two abelian C^* -algebras look like?

Of course, in this generality, the question makes no sense, as we do not know how to multiply elements of different addends. Even if both algebras are contained in a third algebra, their sum need not form an algebra again. Therefore, let us take a closer look at the situations above. In both cases, we are given two C^* -algebras \mathfrak{A}_0 and \mathfrak{A}_1 that fulfill $\mathfrak{A}_0\mathfrak{A}_1 \subseteq \mathfrak{A}_0$ and that trivially intersect. Here, their direct vector space sum $\mathfrak{A}_0 \boxplus \mathfrak{A}_1$ is, at least, a $*$ -algebra. However, in order to get a C^* -algebra structure, we need a norm. In both cases above, this does not cause a problem as \mathfrak{A}_0 and \mathfrak{A}_1 are contained in some C^* -algebra \mathfrak{C} . We shall assume this in the following, as then, by general arguments (Corollary 3.2), $\mathfrak{A}_0 + \mathfrak{A}_1$ is a C^* -algebra containing \mathfrak{A}_0 as an ideal. Note that, since C^* -norms are unique on $*$ -algebras, the whole construction is independent of the choice of \mathfrak{C} .

* e-mail: fleischh@math.upb.de

As we are going to calculate spectra of *abelian* C^* -algebras, we may assume that \mathfrak{C} equals the C^* -algebra $C_0(X)$ of continuous vanishing-at-infinity functions on some locally compact X . Assuming for the moment that \mathfrak{A}_0 is an ideal also in \mathfrak{C} , it is necessarily [1] of the form

$$\mathfrak{A}_0 \cong C_{0,Y}(X) := \{f \in C_0(X) \mid f \equiv 0 \text{ on } \mathbb{C}Y\} \cong C_0(Y)$$

for some open $Y \subseteq X$. This, however, is not the perfect framework for the case we are interested in most, namely the asymptotically almost periodic functions. Here, one is tempted to choose $Y = X = \mathbb{R}$, but then $C_{\text{AP}}(\mathbb{R})$ is not a subalgebra of $C_0(\mathbb{R})$. As we are aiming at unital \mathfrak{A}_1 anyway and since we are free to choose any \mathfrak{C} containing \mathfrak{A}_0 and \mathfrak{A}_1 , we will therefore prefer \mathfrak{C} to be the set $C_b(X)$ of all bounded continuous functions on X . To wrap up, we will let \mathfrak{A}_0 be an ideal and \mathfrak{A}_1 be a unital subalgebra of $C_b(X)$ with $\mathfrak{A}_0 \cap \mathfrak{A}_1 = \mathbf{0}$.

What can one say now about the spectrum of the sum $\mathfrak{A}_0 + \mathfrak{A}_1$ of these two algebras? Considering the unitization, i.e., $\mathfrak{A}_0 = C_0(X)$ and $\mathfrak{A}_1 = \mathbb{C}\mathbf{1}$ as subalgebras of $C_b(X)$, we see that, as a set, the spectrum of the sum is the disjoint union of the single spectra, namely X and $\{\infty\}$. However, there are certain matching conditions influencing the topology. In fact, the topology is *not* generated by the open sets in X and in $\{\infty\}$; it is given by the open sets in X and by complements of closed compacta in X together with ∞ . In other words, the spectrum of the sum is the disjoint union of the two spectra, but their topologies get nontrivially interwoven. This will indeed remain true for the general case. To see this, we will construct an appropriate isomorphism

$$\tau : \text{spec } \mathfrak{A}_0 \sqcup \text{spec } \mathfrak{A}_1 \longrightarrow \text{spec}(\mathfrak{A}_0 \boxplus \mathfrak{A}_1).$$

On $\text{spec } \mathfrak{A}_1$, the map τ should be given by $[\tau(\varphi)](a_0 + a_1) := \varphi(a_1)$. Indeed, $\tau(\varphi)$ is a character on \mathfrak{A}_1 since $\mathfrak{A}_0\mathfrak{A}_1 \subseteq \mathfrak{A}_0$ (check directly or see Theorem 4.2). On $\text{spec } \mathfrak{A}_0$, which we may assume to be an open subset Y of X , the situation is simpler: Here, we just set $[\tau(y)](a) := a(y)$. Taking an appropriate basis of the Gelfand topology on $\text{spec}(\mathfrak{A}_0 \boxplus \mathfrak{A}_1)$, we will get a simple description of the topology on $\text{spec } \mathfrak{A}_0 \sqcup \text{spec } \mathfrak{A}_1$ as mediated by τ . It explicitly shows how the topologies of the spectra of \mathfrak{A}_0 and \mathfrak{A}_1 are getting intertwined. This way, in particular, we generalize the results of Grigoryan and Tonev [4] on asymptotically almost periodic functions from \mathbb{R} to arbitrary non-compact, but locally compact abelian Hausdorff groups.

The paper is organized as follows: We will start in Section 2 with an abstract definition of the so-called twisted-sum topology on the disjoint union of topological spaces. This definition, of course, is directly motivated by the topology on $\text{spec}(\mathfrak{A}_0 \boxplus \mathfrak{A}_1)$ to be derived in Section 4. Before, in Section 3 we will summarize some general facts we need from the theory of (abelian) C^* -algebras. In Section 5, we study how X is contained in the spectrum of $\mathfrak{A}_0 \boxplus \mathfrak{A}_1$. We close in Section 6 with applications to the unitization and to asymptotically almost periodic functions. In Appendix A, we discuss measures on the spectra.

2 Twisted Sum

We are first going to describe a topology on the disjoint union of two topological spaces that is, in general, different from the standard direct sum.

Notation 2.1 Within this section, we let be

- \mathbf{Y}, \mathbf{Z} ... some disjoint topological spaces
- f ... some continuous map $f : \mathbf{Y} \longrightarrow \mathbf{Z}$
- \mathbb{C} ... the complement within \mathbf{Y} .

Definition 2.2 The f -twisted topology on the disjoint union $\mathbf{Y} \sqcup \mathbf{Z}$ is the topology generated by all sets of the following types:

- | | | |
|---------|---------------------------------|--|
| Type 1: | $V \sqcup \emptyset$ | with open $V \subseteq \mathbf{Y}$ |
| Type 2: | $\mathbb{C}K \sqcup \mathbf{Z}$ | with compact closed $K \subseteq \mathbf{Y}$ |
| Type 3: | $f^{-1}(W) \sqcup W$ | with open $W \subseteq \mathbf{Z}$. |

The sets above are called **standard sets**.

We denote $\mathbf{Y} \sqcup \mathbf{Z}$, equipped with the f -twisted topology, by

$$\mathbf{Y} \sqcup_f \mathbf{Z}$$

and call it f -twisted sum of \mathbf{Y} and \mathbf{Z} .

If we speak of **sum topologies** below, we will mean both the f -twisted sum and the direct sum on $\mathbf{Y} \sqcup \mathbf{Z}$. If f is clear from the context, we may drop it. Also note that $\emptyset \sqcup_f \mathbf{Z} = \mathbf{Z}$.

Lemma 2.1 A basis for the topology on $\mathbf{Y} \sqcup_f \mathbf{Z}$ is given by the following sets:

1. sets of type 1;
2. sets of type 23, i.e., intersections of a set of type 2 with a set of type 3.

Proof First note that the type is preserved under finite intersections of same-type sets. Next, any intersection of a type-1 set with any standard set is again of type 1; for this, simply observe that the \mathbf{Y} -part of any standard set is open in \mathbf{Y} . **qed**

Note that the total space $\mathbf{Y} \sqcup \mathbf{Z}$ is both a type-2 and a type-3 set.

Lemma 2.2 Let $\mathbf{Y} \sqcup \mathbf{Z}$ be given the f -twisted topology from Definition 2.2. Then we have:

- The relative topology on \mathbf{Y} coincides with the original topology on \mathbf{Y} .
- The relative topology on \mathbf{Z} coincides with the original topology on \mathbf{Z} .

Moreover, \mathbf{Y} is open and \mathbf{Z} is closed in $\mathbf{Y} \sqcup_f \mathbf{Z}$.

Proof Obvious. **qed**

If confusion is unlikely, we may write \mathbf{Y} and \mathbf{Z} instead of $\mathbf{Y} \sqcup \emptyset$ and $\emptyset \sqcup \mathbf{Z}$, respectively.

Lemma 2.3 $\mathbf{Y} \sqcup_f \mathbf{Z}$ is Hausdorff iff \mathbf{Y} is locally compact and both \mathbf{Y} and \mathbf{Z} are Hausdorff.

Proof \Leftarrow Obviously, any two distinct points in \mathbf{Y} can be separated by type-1 sets. Similarly, any two distinct points in \mathbf{Z} can be separated by type-3 sets. Let now $y \in \mathbf{Y}$ and $z \in \mathbf{Z}$. Choose some neighbourhood V of y contained in some compactum K . Then, $V \sqcup \emptyset$ and $\mathbb{C}K \sqcup \mathbf{Z}$ are disjoint open neighbourhoods of y and z , respectively.

\Rightarrow Lemma 2.2 implies that \mathbf{Y} and \mathbf{Z} are Hausdorff. To prove local compactness, let $y \in \mathbf{Y}$ be given. Note first, that y and $f(y)$ cannot be separated using a type-1 neighbourhood of $f(y)$, as this point is never contained in such a set. Note second, that both points can neither be separated by type-23 sets. In fact,

$$y \in (\mathbb{C}K_1 \cap f^{-1}(W_1)) \sqcup W_1 \quad \text{and} \quad f(y) \in (\mathbb{C}K_2 \cap f^{-1}(W_2)) \sqcup W_2$$

implies $f(y) \in W_2$, hence $y \in f^{-1}(W_2)$, hence $f^{-1}(W_1 \cap W_2)$ is non-empty; contradiction. As, on the other hand, two distinct elements in any Hausdorff space have to be separable by elements of any basis, y and $f(y)$ are now separated by a type-1 and a type-23 set, i.e., by

$$V \sqcup \emptyset \quad \text{and} \quad (\mathbb{C}K \cap f^{-1}(W)) \sqcup W,$$

respectively. As then V and $\mathbb{C}K \cap f^{-1}(W)$ are disjoint, we see that $V \cap f^{-1}(W)$ is contained in K . Now we are done, as the first set is an open neighbourhood of y and the latter one is compact. qed

Proposition 2.4 $\mathbf{Y} \sqcup_f \mathbf{Z}$ is compact iff \mathbf{Z} is compact.

Proof As \mathbf{Z} is always closed in $\mathbf{Y} \sqcup_f \mathbf{Z}$, the compactness of $\mathbf{Y} \sqcup_f \mathbf{Z}$ implies that of \mathbf{Z} . Let us now prove the other direction assuming \mathbf{Z} to be compact.

- Let \mathcal{O} be an open cover of $\mathbf{Y} \sqcup_f \mathbf{Z}$. We may assume that \mathcal{O} is contained in the basis of the topology.
- As \mathcal{O} covers, in particular, the compact set $\mathbf{Z} \subseteq \mathbf{Y} \sqcup_f \mathbf{Z}$, there is a finite $\mathcal{U} \subseteq \mathcal{O}$ still covering \mathbf{Z} . We may assume that none of the sets in \mathcal{U} is of first type, as their intersection with \mathbf{Z} is empty. Hence the elements of \mathcal{U} are type-23 sets

$$U_i = (\mathbb{C}K_i \cap f^{-1}(W_i)) \sqcup W_i.$$

As \mathcal{U} covers \mathbf{Z} , we have $\bigcup W_i = \mathbf{Z}$. Now, for $K := \bigcup K_i$, we have

$$\mathbf{Y} \cap \bigcup_i U_i = \bigcup_i (\mathbb{C}K_i \cap f^{-1}(W_i)) \supseteq \mathbb{C}K \cap f^{-1}(\bigcup_i W_i) = \mathbb{C}K.$$

This means that \mathcal{U} covers $\mathbb{C}K \sqcup \mathbf{Z}$.

- As K is compact, we may find some finite $\mathcal{U}' \subseteq \mathcal{O}$ covering K . Now, $\mathcal{U} \cup \mathcal{U}'$ covers all of $\mathbf{Y} \sqcup_f \mathbf{Z}$. qed

Let us finally compare the twisted-sum topology on $\mathbf{Y} \sqcup \mathbf{Z}$ with the standard direct-sum topology thereon. As any standard set is open in the direct-sum topology, we have

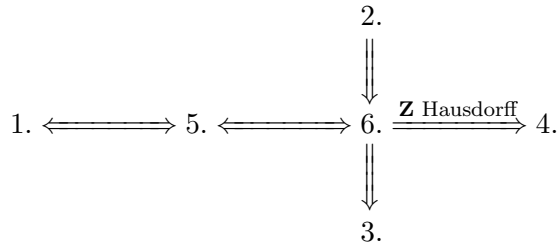
Lemma 2.5 The twisted-sum topology is always contained in the direct-sum topology.

Criteria for the equality of both sum topologies are summarized in

Proposition 2.6 Consider the following statements:

1. The sum topologies on $\mathbf{Y} \sqcup \mathbf{Z}$ coincide.
2. \mathbf{Y} is compact.
3. \mathbf{Y} is locally compact.
4. $f(\mathbf{Y})$ is closed in \mathbf{Z} .
5. $\emptyset \sqcup \mathbf{Z}$ is open in $\mathbf{Y} \sqcup_f \mathbf{Z}$.
6. \mathbf{Z} can be covered by open W_α , whose preimages $f^{-1}(W_\alpha)$ are contained in compacta in \mathbf{Y} .

These statements are correlated as follows:



Proof 1. \implies 5. Trivial.

5. \implies 1. Observe

$$V \sqcup W = (V \sqcup \emptyset) \cup [(\emptyset \sqcup \mathbf{Z}) \cap (f^{-1}(W) \sqcup W)]$$

for any $V \subseteq \mathbf{Y}$ and $W \subseteq \mathbf{Z}$.

5. \implies 6. As $\emptyset \sqcup \mathbf{Z} \subseteq \mathbf{Y} \sqcup_f \mathbf{Z}$ contains only the trivial type-1 set, it is a union of some type-23 sets

$$(\mathfrak{C}K_\alpha \cap f^{-1}(W_\alpha)) \sqcup W_\alpha.$$

By construction, each $\mathfrak{C}K_\alpha \cap f^{-1}(W_\alpha)$ is empty, i.e., $f^{-1}(W_\alpha) \subseteq K_\alpha$. As the W_α form an open cover of \mathbf{Z} , we get the proof from continuity of f .

6. \implies 5. This is clear from the proof of the reversed implication.
 2. \implies 6. Trivial.
 6. \implies 3. Given such W_α , there are compact K_α containing the open $f^{-1}(W_\alpha)$. Therefore, K_α is a compact neighbourhood for all the points in $f^{-1}(W_\alpha)$. As the latter sets form a cover of \mathbf{Y} , we get the claim.
 6. \implies 4. Assume that $f(\mathbf{Y})$ is not closed, i.e., there is some $z \in \overline{f(\mathbf{Y})} \setminus f(\mathbf{Y})$. Choose any open $W_\alpha =: W$ containing z . Because W is open, there is¹ a net $(f(y_\lambda))$ in $f(\mathbf{Y}) \cap W$ converging to z . Since $f^{-1}(W)$ is contained in some compactum K , there is a subnet of (y_λ) converging to some $y \in K$. Consequently, a subnet of $f(y_\lambda)$ converges to $f(y) \in f(\mathbf{Y})$. As \mathbf{Z} is Hausdorff, we get $f(y) = z$, hence a contradiction. **qed**

None of the implications in Proposition 2.6 above can be reversed, in general, nor can the Hausdorff property be removed there. Let us explain this by means of several examples.

Example 1 Let \mathbf{Z} carry the coarsest topology.

Then, $\{\mathbf{Z}\}$ is the only open cover for \mathbf{Z} . Proposition 2.6 now implies that both sum topologies coincide iff $\mathbf{Y} = f^{-1}(\mathbf{Z})$ is compact. In particular, there are locally compact \mathbf{Y} for which the two sum topologies differ, i.e., 3. \implies 6. is not given.

Moreover, taking any constant f , we see that the image $f(\mathbf{Y})$ is closed iff \mathbf{Z} consists of a single point only or, equivalently, \mathbf{Z} is Hausdorff. Thus, for any compact \mathbf{Y} (implying the desired equality of the twisted and the direct-sum topologies) and any non-Hausdorff \mathbf{Z} , 6. \implies 4. is not given. In other words, the Hausdorff property is necessary.

Example 2 Let $\mathbf{Y} = \mathbf{Z}$ be Hausdorff spaces and let f be the identity.

We are going to show that the two sum topologies coincide iff \mathbf{Y} is locally compact. The “only if”-part is already covered by the proposition above. To show the “if”-part, observe that, for each $y \in \mathbf{Y}$, we find open W_y and closed compact K_y with $y \in W_y \subseteq K_y$. The claim follows from $f^{-1}(W_y) \equiv W_y$ and Proposition 2.6. Altogether, 6. \implies 2. is not given.

On the other hand, $f(\mathbf{Y}) \equiv \mathbf{Z}$ is always closed in \mathbf{Z} , but \mathbf{Z} need not be locally compact, whence there is no need for the twisted and the direct sum topologies to coincide. In other words, 4. \implies 6. is not given. This remains true even if the Hausdorff assumption is dropped.

Example 3 Let $\mathbf{Y} = \mathbf{Z} \times \mathbf{Z}$ and $f : \mathbf{Z} \times \mathbf{Z} \longrightarrow \mathbf{Z}$ be the projection to the first component.

Of course, any $W \subseteq \mathbf{Z}$ has $W \times \mathbf{Z}$ as preimage, which surely is contained in some compactum iff \mathbf{Z} is compact (or W is empty). Therefore, also 4. and 3. together (even if \mathbf{Z} was Hausdorff) do not suffice to imply 6.

¹With $A := \overline{f(\mathbf{Y})}$, we have $z \notin A \cap W$. But, for each open U containing z , we get an open $W \cap U$ containing z , whence $z \in \overline{A \cap W}$ implies $z \in (A \cap W) \cap U$. Thus, $z \in \overline{A \cap W}$, whence there is a net in $A \cap W$ converging to z .

3 Preliminaries on C^* -algebras

Before going to the main statements, let us summarize the relevant prerequisites from C^* -algebras. Note that we assume any ideal to be closed and two-sided.

3.1 Closedness of Subalgebra Sums

For completeness, let us start with the well-known [7]

Proposition 3.1 Let \mathfrak{C} be a C^* -algebra with ideal \mathfrak{I} and C^* -subalgebra \mathfrak{A} .
Then $\mathfrak{I} + \mathfrak{A}$ is closed, hence a C^* -subalgebra of \mathfrak{C} .

Proof As \mathfrak{I} is an ideal, $\mathfrak{C}/\mathfrak{I}$ is a C^* -algebra and the canonical projection $\pi : \mathfrak{C} \rightarrow \mathfrak{C}/\mathfrak{I}$ is a $*$ -homomorphism. It restricts to a $*$ -homomorphism $\pi_{\mathfrak{A}} : \mathfrak{A} \rightarrow \mathfrak{C}/\mathfrak{I}$. Consequently [7], the range of $\pi_{\mathfrak{A}}$ is closed, hence $\pi^{-1}(\pi_{\mathfrak{A}}(\mathfrak{A})) = \mathfrak{I} + \mathfrak{A}$ as well. **qed**

Sometimes, it might not be clear a priori whether \mathfrak{I} is indeed an ideal in \mathfrak{C} – or even worse what \mathfrak{C} really is. This, however, does not destroy the closedness of $\mathfrak{I} + \mathfrak{A}$ as long as at least the relation between \mathfrak{I} and \mathfrak{A} resembles the ideal property:

Corollary 3.2 Let \mathfrak{I} and \mathfrak{A} be C^* -subalgebras of some C^* -algebra \mathfrak{C} with $\mathfrak{A}\mathfrak{I} \subseteq \mathfrak{I}$.
Then $\mathfrak{I} + \mathfrak{A}$ is a C^* -subalgebra of \mathfrak{C} , containing \mathfrak{I} as an ideal.

Proof Obviously, $\mathfrak{B} := \mathfrak{I} + \mathfrak{A}$ is a $*$ -subalgebra of \mathfrak{C} with $\mathfrak{B}\mathfrak{I} \subseteq \mathfrak{I}$. Consequently, its closure $\overline{\mathfrak{B}}$ is a C^* -subalgebra of \mathfrak{C} . Now, \mathfrak{I} and \mathfrak{A} are also C^* -subalgebras of $\overline{\mathfrak{B}}$. Even more, \mathfrak{I} is an ideal there. In fact, given $c \in \overline{\mathfrak{B}}$, there are b_i in \mathfrak{B} converging to c . For any $n \in \mathfrak{I}$, we have now $b_i n \in \mathfrak{I}$, whence $cn = \lim_i b_i n \in \mathfrak{I}$. By Proposition 3.1, $\mathfrak{I} + \mathfrak{A}$ is closed in $\overline{\mathfrak{B}}$, hence equal to $\overline{\mathfrak{B}}$ by denseness. **qed**

In other words, replacing \mathfrak{C} above by $\mathfrak{I} + \mathfrak{A}$ returns ourselves to the previous situation. Therefore, we will restrict ourselves to the case that \mathfrak{I} is an ideal.

3.2 Gelfand Transform

Let \mathfrak{A} be an abelian C^* -algebra with spectrum $\text{spec } \mathfrak{A}$. Recall [7]

Definition 3.1 1. The **Gelfand transform** \tilde{a} of any $a \in \mathfrak{A}$ is given by

$$\begin{aligned} \tilde{a} : \text{spec } \mathfrak{A} &\longrightarrow \mathbb{C}. \\ \chi &\longmapsto \chi(a) \end{aligned}$$

2. The topology on $\text{spec } \mathfrak{A}$ is the initial topology induced by all the Gelfand transforms. More precisely, it is generated by all the sets $\tilde{a}^{-1}(A)$ with $a \in \mathfrak{A}$ and open $A \subseteq \mathbb{C}$.

Proposition 3.3 Let $\mathfrak{B} \subseteq \mathfrak{A}$ be any subset that generates \mathfrak{A} as a C^* -algebra.
Then the topology on $\text{spec } \mathfrak{A}$ is already induced by $\{\tilde{a} \mid a \in \mathfrak{B}\}$.

Note that the Gelfand transform of $a \in \mathfrak{B}$ is taken w.r.t. \mathfrak{A} , not w.r.t. \mathfrak{B} .

For the proof of the proposition above, recall that the initial topology on some topological space N , induced by some functions h_α thereon, is characterized by the following condition: For any map $g : M \rightarrow N$ on any topological space M , we have

$$g \text{ continuous} \iff h_\alpha \circ g \text{ continuous for all } \alpha.$$

Now, the proposition above is an immediate consequence of

Lemma 3.4 Let M be a topological space and $g : M \rightarrow \text{spec } \mathfrak{A}$.

Then $\tilde{a} \circ g$ is continuous for all $a \in \mathfrak{A}$ iff $\tilde{b} \circ g$ is continuous for all $b \in \mathfrak{B}$.

Proof By continuity of the algebra operations (sum, product, conjugation), we may assume $\mathfrak{A} = \overline{\mathfrak{B}}$. Moreover, we only have to prove the “if”-part. For this, write $a \in \mathfrak{A}$ as $a = \lim b_k$ with $b_k \in \mathfrak{B}$. Then

$$\|\tilde{b}_k \circ g - \tilde{a} \circ g\|_\infty \leq \|\tilde{b}_k - \tilde{a}\|_\infty = \|b_k - a\| \rightarrow 0$$

by linearity and isometry of the Gelfand transform. Consequently, $\tilde{a} \circ g$ is continuous.

qed

3.3 Natural Mapping

Let X be some locally compact Hausdorff space and let $C_b(X)$ be the C^* -algebra of bounded continuous functions on it. Moreover, let \mathfrak{A} be a unital C^* -subalgebra of $C_b(X)$. Recall from [8, 3] the following definition and proposition.

Definition 3.2 The **natural mapping** $\iota : X \rightarrow \text{spec } \mathfrak{A}$ is given by

$$\begin{aligned} \iota(x) : \mathfrak{A} &\longrightarrow \mathbb{C}. \\ a &\longmapsto a(x) \end{aligned}$$

Proposition 3.5

- ι is well defined and continuous.
- ι is injective iff \mathfrak{A} separates the points in X .
- $\iota(X)$ is dense in $\text{spec } \mathfrak{A}$.
- $\tilde{a} \circ \iota = a$ on X for all $a \in \mathfrak{A}$.

4 Topology of $\text{spec } \mathfrak{A}$

From now on, we will use the following

Notation 4.1

- X ... some nonempty locally compact Hausdorff space
- \mathfrak{A}_0 ... some ideal in $C_0(X)$
- Y ... the open subset of X with

$$\mathfrak{A}_0 = C_{0,Y}(X) := \{a_0 \in C_0(X) \mid a_0 \equiv 0 \text{ on } \mathbb{C}Y\}$$

- \mathfrak{A}_1 ... some unital C^* -subalgebra of $C_b(X)$ with

$$\mathfrak{A}_0 \cap \mathfrak{A}_1 = \mathbf{0}$$

- \mathfrak{A} ... the direct vector space sum

$$\mathfrak{A} := \mathfrak{A}_0 \boxplus \mathfrak{A}_1$$

- f ... the restriction of the natural mapping $\iota_1 : X \rightarrow \text{spec } \mathfrak{A}_1$ to Y :

$$f := \iota_1|_Y : Y \rightarrow \text{spec } \mathfrak{A}_1$$

Remark

1. We assume the direct vector space sum of subspaces of a third vector space to be contained in this vector space again.
2. Note that Y is well defined as any ideal in $C_0(X)$ is of the form $C_{0,Y}(X)$. [1] Moreover, as $C_{0,Y}(X)$ is naturally isomorphic to $C_0(Y)$, we will identify its spectrum with Y .

- Lemma 4.1**
1. \mathfrak{A} is a unital abelian C^* -subalgebra of $C_b(X)$.
 2. \mathfrak{A}_0 is an ideal in \mathfrak{A} .

- Proof**
1. $\mathfrak{A}_0 \equiv C_{0,Y}(X)$ is an ideal in $C_b(X)$, whence the statement follows from $\mathfrak{A} \equiv \mathfrak{A}_0 + \mathfrak{A}_1$ and Proposition 3.1.
 2. This is trivial by the preceding argument. qed

We are now stating our main result.

Theorem 4.2 We have

$$\text{spec } \mathfrak{A} \cong Y \sqcup_f \text{spec } \mathfrak{A}_1.$$

Proof Let us define the mapping

$$\tau : Y \sqcup_f \text{spec } \mathfrak{A}_1 \longrightarrow \text{spec } \mathfrak{A}$$

for $y \in Y$ and $\varphi \in \text{spec } \mathfrak{A}_1$ as follows:

$$\begin{aligned} [\tau(y)](a_0 + a_1) &:= (a_0 + a_1)(y) \\ [\tau(\varphi)](a_0 + a_1) &:= \varphi(a_1). \end{aligned}$$

We may assume Y to be non-empty.

• **Well-definedness**

Of course, $\tau(y) \in \text{spec } \mathfrak{A}$. For the other part observe that $\tau(\varphi)$ is nonzero and multiplicative by

$$\begin{aligned} [\tau(\varphi)]((a_0 + a_1)(a'_0 + a'_1)) &= [\tau(\varphi)]((a_0a'_0 + a_0a'_1 + a_1a'_0) + a_1a'_1) \\ &= \varphi(a_1a'_1) \\ &= \varphi(a_1) \varphi(a'_1) \\ &= [\tau(\varphi)](a_0 + a_1) [\tau(\varphi)](a'_0 + a'_1). \end{aligned}$$

• **Surjectivity**

Let $\chi : \mathfrak{A} \longrightarrow \mathbb{C}$ be a character on \mathfrak{A} . Then there are two cases:

- If $\chi|_{\mathfrak{A}_0} = 0$, then, obviously, $\chi|_{\mathfrak{A}_1}$ is a character on \mathfrak{A}_1 , with $\tau(\chi|_{\mathfrak{A}_1}) = \chi$.
- If $\chi|_{\mathfrak{A}_0} \neq 0$, then it is a character on \mathfrak{A}_0 , whence, by Gelfand-Naimark theory, there is some $y \in Y$ with $\chi(a_0) = a_0(y)$ for all $a_0 \in \mathfrak{A}_0$. Given $a_1 \in \mathfrak{A}_1$, we have for some a_0 with $\chi(a_0) \neq 0$

$$a_0(y)a_1(y) = (a_0a_1)(y) = \chi(a_0a_1) = \chi(a_0)\chi(a_1) = a_0(y)\chi(a_1),$$

whence $\chi(a_1) = a_1(y)$. Here, we used $\mathfrak{A}_0\mathfrak{A}_1 \subseteq \mathfrak{A}_0$. Thus, we have $\chi(a) = a(y)$ for all $a = a_0 + a_1 \in \mathfrak{A}$, hence $\chi = \tau(y)$.

• **Injectivity**

There are three cases:

- Let $y, y' \in Y$ with $y \neq y'$. Taking some $a_0 \in \mathfrak{A}_0$ with $a_0(y) \neq a_0(y')$, we get

$$[\tau(y)](a_0) = a_0(y) \neq a_0(y') = [\tau(y')](a_0),$$

implying $\tau(y) \neq \tau(y')$.

- Let $y \in Y$ and $\varphi \in \text{spec } \mathfrak{A}_1$. Then, for any $a_0 \in \mathfrak{A}_0$ with $a_0(y) \neq 0$, we have

$$[\tau(y)](a_0) = a_0(y) \neq 0 = [\tau(\varphi)](a_0),$$

implying $\tau(y) \neq \tau(\varphi)$.

- Let $\varphi, \varphi' \in \text{spec } \mathfrak{A}_1$ with $\varphi \neq \varphi'$. Take $a_1 \in \mathfrak{A}_1$ with $\varphi(a_1) \neq \varphi'(a_1)$. Then

$$[\tau(\varphi)](a_1) = \varphi(a_1) \neq \varphi'(a_1) = [\tau(\varphi')](a_1),$$

implying $\tau(\varphi) \neq \tau(\varphi')$.

- **Continuity**

By Proposition 3.3, the topology of $\text{spec } \mathfrak{A}$ is generated by the sets $\tilde{a}_0^{-1}(U)$ and $\tilde{a}_1^{-1}(U)$ with $a_0 \in \mathfrak{A}_0$, $a_1 \in \mathfrak{A}_1$ and open $U \subseteq \mathbb{C}$. So, we only have to show that their preimages are open.

– Let $a_0 \in \mathfrak{A}_0$. Then

$$\begin{aligned} [\tilde{a}_0 \circ \tau](y) &= [\tau(y)](a_0) = a_0(y) \\ [\tilde{a}_0 \circ \tau](\varphi) &= [\tau(\varphi)](a_0) = 0 \end{aligned}$$

for $y \in Y$ and $\varphi \in \text{spec } \mathfrak{A}_1$. Let now $U \subseteq \mathbb{C}$ be open.

• If 0 is not contained in U , then

$$\tau^{-1}(\tilde{a}_0^{-1}(U)) \equiv (\tilde{a}_0 \circ \tau)^{-1}(U) = a_0^{-1}(U) \equiv a_0^{-1}(U) \sqcup \emptyset$$

is a type-1 element, as $a_0 \in \mathfrak{A}_0 \equiv C_{0,Y}(X)$, whence $a_0^{-1}(U)$ is open in Y .

• If 0 is contained in U , then

$$\tau^{-1}(\tilde{a}_0^{-1}(U)) \equiv (\tilde{a}_0 \circ \tau)^{-1}(U) = a_0^{-1}(U) \sqcup \text{spec } \mathfrak{A}_1$$

This is a type-2 element since the complement of $a_0^{-1}(U)$ in Y is compact, for $a_0 \in \mathfrak{A}_0$.

– Let $a_1 \in \mathfrak{A}_1$. Then, by Proposition 3.5,

$$\begin{aligned} [\tilde{a}_1 \circ \tau](y) &= [\tau(y)](a_1) = a_1(y) = \tilde{a}_1(\iota_1(y)) \equiv \tilde{a}_1(f(y)) \\ [\tilde{a}_1 \circ \tau](\varphi) &= [\tau(\varphi)](a_1) = \varphi(a_1) = \tilde{a}_1(\varphi) \end{aligned}$$

for $y \in Y$ and $\varphi \in \text{spec } \mathfrak{A}_1$. This means that

$$\tau^{-1}(\tilde{a}_1^{-1}(U)) \equiv [\tilde{a}_1 \circ \tau]^{-1}(U) = f^{-1}(\tilde{a}_1^{-1}(U)) \sqcup \tilde{a}_1^{-1}(U)$$

is a type-3 element as $\tilde{a}_1^{-1}(U)$ is open in $\text{spec } \mathfrak{A}_1$.

- **Homeomorphy**

As τ is a continuous bijection from a compact to a Hausdorff space, it is even a homeomorphism. **qed**

5 Embedding and Denseness

From the proof of Theorem 4.2, we get immediately

Corollary 5.1 The natural mapping $\iota : X \rightarrow \text{spec } \mathfrak{A}$ is given by

$$\iota = \begin{cases} \tau & \text{on } Y \\ \tau \circ \iota_1 & \text{on } X \setminus Y. \end{cases}$$

Here, $\iota_1 : X \rightarrow \text{spec } \mathfrak{A}_1$ is the natural mapping w.r.t. \mathfrak{A}_1 .

Ignoring the homeomorphism $\tau : Y \sqcup \text{spec } \mathfrak{A}_1 \rightarrow \text{spec } \mathfrak{A}$, the natural mapping for \mathfrak{A} equals the identity on Y and the natural mapping ι_1 for \mathfrak{A}_1 on $Y \setminus X$. In particular, for $Y = X$, the natural mapping is simply the identity on X .

Lemma 5.2 X is densely and continuously embedded into $\text{spec } \mathfrak{A}$, provided \mathfrak{A}_1 separates the points in $X \setminus Y$.

Proof First of all, $\mathfrak{A}_0 \equiv C_{0,Y}(X)$ separates any point in Y from any other point in X . Next, by assumption, \mathfrak{A}_1 separates any two points in $X \setminus Y$. Altogether, \mathfrak{A} separates any two points in X . The statement follows from Proposition 3.5. **qed**

Corollary 5.3 Let $X \setminus Y$ contain at most one point.

Then X is densely and continuously embedded into $\text{spec } \mathfrak{A} = Y \sqcup_f \text{spec } \mathfrak{A}_1$.

Note that, for $Y = X$, the space X might be embedded into $\text{spec } \mathfrak{A}$ in two different ways:

- Firstly, one uses the natural embedding ι of X into $X \sqcup \text{spec } \mathfrak{A}_1$, whose image is again X seen as a subset of $X \sqcup \text{spec } \mathfrak{A}_1$. That is the way we went above.
- Secondly, assuming that \mathfrak{A}_1 separates the points in X , one uses the natural embedding $\iota_1 : X \rightarrow \text{spec } \mathfrak{A}_1$ and then embeds $\text{spec } \mathfrak{A}_1$ into $X \sqcup \text{spec } \mathfrak{A}_1$. However, now the resulting embedding of X into $X \sqcup \text{spec } \mathfrak{A}_1$ is not dense anymore, unless X is empty. This is clear, as $\text{spec } \mathfrak{A}_1$ is closed in $\text{spec } \mathfrak{A}$.

A similar behaviour can be observed if Y equals X minus some point. Then the natural mapping ι is given by the identity on Y , but the “missing” point is taken from $\text{spec } \mathfrak{A}_1$. In other words, $\text{spec } \mathfrak{A}_1$ is attached to Y “filling” the gap.

6 Examples

Example 4 One-point compactification

Let $\mathfrak{A}_0 = C_0(X)$ and consider $\mathfrak{A}_1 := \mathbb{C} \mathbf{1}$ as a subset of $C_b(X)$. As $\text{spec } \mathfrak{A}_1$ consists of a single point, say ∞ , only, we have exactly two open sets: \emptyset and $\{\infty\}$. Moreover, the twisting map $f : X \rightarrow \text{spec } \mathfrak{A}_1$ is trivial. Consequently, the only type-3 sets are $\emptyset \sqcup \emptyset$ and $X \sqcup \{\infty\}$. This means that the topology of $\text{spec}(C_0(X) \boxplus \mathbb{C} \mathbf{1})$ is generated just by the open sets in X and by the complements of closed compact sets in X united with $\text{spec } \mathfrak{A}_1$. This is indeed nothing but the topology of the one-point compactification X^* of X . Of course, X is dense in X^* . Moreover, Lemma 2.3 generalizes the well-known fact [6], that X^* is Hausdorff iff X is locally compact Hausdorff. Finally, Example 1 comprises the fact [6] that ∞ is an isolated point of X^* iff X is compact.

Example 5 Asymptotically almost periodic functions

Let X be a non-compact, but locally compact abelian group, and let \mathfrak{A}_0 be full $C_0(X)$. If \mathfrak{A}_1 is the set of almost periodic functions on X , then \mathfrak{A} is the set of asymptotically almost periodic functions. (See [4] for $X = \mathbb{R}$.) Its spectrum is given by the twisted sum

$$X \sqcup_{\iota} X_{\text{Bohr}},$$

where $X_{\text{Bohr}} \equiv \text{spec } \mathfrak{A}_1$ is the Bohr compactification [9] of X , and ι is the canonical embedding of X into X_{Bohr} . Open sets are, in particular, the open sets in X and the type-3 sets $f^{-1}(U) \sqcup f^{-1}(U)$ with open $U \subseteq \mathbb{C}$ and with f running over the almost periodic functions. As $\iota(X)$ is dense in the compactum X_{Bohr} (being, e.g., a consequence of Proposition 3.5), but strictly smaller, it cannot be a closed subset. Now, Proposition 2.6 implies that $X \sqcup_{\iota} X_{\text{Bohr}}$ is not the direct-sum topology on $X \sqcup X_{\text{Bohr}}$.

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Appendix

A Borel algebra of $\mathbf{Y} \sqcup_f \mathbf{Z}$

Reusing the notation from Section 2, we will now describe the Borel algebra that corresponds to the twisted sum, as well as the (regular) measures on it. For the particular case of asymptotically almost periodic functions see also [5].

Proposition A.1 We have

$$\mathcal{B}(\mathbf{Y} \sqcup_f \mathbf{Z}) = \mathcal{B}(\mathbf{Y}) \oplus \mathcal{B}(\mathbf{Z}).$$

Here, $\mathcal{B}(X)$ denotes the Borel σ -algebra of a topological space X . The proposition above shows, in particular, that the direct sum and the twisted sum topologies yield one and the same Borel algebra. For brevity, we refrain from indicating f in $\mathbf{Y} \sqcup_f \mathbf{Z}$ in the following.

Proof • As \mathbf{Y} is open, any open set in \mathbf{Y} is open in $\mathbf{Y} \sqcup \mathbf{Z}$, whence

$$\mathcal{B}(\mathbf{Y}) \subseteq \mathcal{B}(\mathbf{Y} \sqcup \mathbf{Z}).$$

As \mathbf{Z} is closed, any closed set in \mathbf{Z} is closed in $\mathbf{Y} \sqcup \mathbf{Z}$, whence

$$\mathcal{B}(\mathbf{Z}) \subseteq \mathcal{B}(\mathbf{Y} \sqcup \mathbf{Z}).$$

- Obviously, $\mathcal{B}(\mathbf{Y}) \oplus \mathcal{B}(\mathbf{Z})$ is a Borel algebra and contains any standard open set in $\mathbf{Y} \sqcup \mathbf{Z}$. As these sets generate the twisted topology, hence the respective Borel algebra, we have

$$\mathcal{B}(\mathbf{Y}) \oplus \mathcal{B}(\mathbf{Z}) \supseteq \mathcal{B}(\mathbf{Y} \sqcup \mathbf{Z}).$$

qed

Recall that the direct sum $\mu_1 \oplus \mu_2$ of Borel measures μ_1 on \mathbf{Y} and μ_2 on \mathbf{Z} is given by

$$[\mu_1 \oplus \mu_2](Y \sqcup Z) := \mu_1(Y) + \mu_2(Z) \quad \text{for } Y \in \mathcal{B}(\mathbf{Y}) \text{ and } Z \in \mathcal{B}(\mathbf{Z}).$$

Of course, $\mu_1 \oplus \mu_2$ is a Borel measure on $\mathbf{Y} \sqcup \mathbf{Z}$ by Proposition A.1. Even more, we have

Corollary A.2 The finite Borel measures on $\mathbf{Y} \sqcup \mathbf{Z}$ are precisely the direct sums of finite Borel measures on \mathbf{Y} and on \mathbf{Z} . The respective statements hold for measures that are additionally inner or outer regular.

Proof • Obviously, any direct sum of finite Borel measures is finite Borel. The other way round, given a finite Borel measure μ on $\mathbf{Y} \sqcup \mathbf{Z}$, then $\mu_1 := \mu|_{\mathcal{B}(\mathbf{Y})}$ defines a finite Borel measure on \mathbf{Y} ; similarly, μ_2 is constructed on \mathbf{Z} . They fulfill $\mu = \mu_1 \oplus \mu_2$.

- As any finite inner regular Borel measure is outer regular [2], we may restrict ourselves to the inner regular case. First, on the one hand, let μ_1 and μ_2 be inner regular and let $Y \sqcup Z \in \mathcal{B}(\mathbf{Y} \sqcup \mathbf{Z})$. Then, for $\mu := \mu_1 \oplus \mu_2$,

$$\begin{aligned} \mu_1(Y) + \mu_2(Z) &\equiv \mu(Y \sqcup Z) \\ &\geq \sup\{\mu(K) \mid K \subseteq Y \sqcup Z \text{ compact}\} \\ &\geq \sup\{\mu(K_1 \sqcup K_2) \mid K_1 \subseteq Y, K_2 \subseteq Z \text{ compact}\} \\ &= \sup\{\mu_1(K_1) \mid K_1 \subseteq Y \text{ compact}\} \\ &\quad + \sup\{\mu_2(K_2) \mid K_2 \subseteq Z \text{ compact}\} \\ &= \mu_1(Y) + \mu_2(Z), \end{aligned}$$

whence $\mu_1 \oplus \mu_2$ is inner regular. Here, the first inequality comes from monotonicity and the second one as any two compact sets have compact union. Now, on

the other hand, let $\mu_1 \oplus \mu_2$ be inner regular. Then, however, both μ_1 and μ_2 have to be regular as well, since both spaces \mathbf{Y} and \mathbf{Z} carry the relative topology from $\mathbf{Y} \sqcup \mathbf{Z}$, whence a subset of, say, \mathbf{Y} is compact iff if it is so in $\mathbf{Y} \sqcup \mathbf{Z}$. **qed**

In particular, any measure on one of the subspaces \mathbf{Y} and \mathbf{Z} is a measure on the twisted sum with the respective additional properties. So, in the case of asymptotically almost periodic (AAP) functions, the Haar measure on \mathbb{R}_{Bohr} is also a measure on the AAP spectrum. Moreover, according to Hanusch [5], it is the only normalized regular Borel measure that is invariant w.r.t. the induced \mathbb{R} -action on $\mathbb{R} \sqcup \mathbb{R}_{\text{Bohr}}$.

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