

Coloring Grids ^{*}

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October 13, 2018

Abstract

A structure $\mathcal{A} = (A; E_i)_{i \in n}$ where each E_i is an equivalence relation on A is called an *n-grid* if any two equivalence classes coming from distinct E_i 's intersect in a finite set. A function $\chi : A \rightarrow n$ is an *acceptable coloring* if for all $i \in n$, the set $\chi^{-1}(i)$ intersects each E_i -equivalence class in a finite set. If B is a set, then the n -cube B^n may be seen as an n -grid, where the equivalence classes of E_i are the lines parallel to the i -th coordinate axis. We use elementary submodels of the universe to characterize those n -grids which admit an acceptable coloring. As an application we show that if an n -grid \mathcal{A} does not admit an acceptable coloring, then every finite n -cube is embeddable in \mathcal{A} .

1 Introduction

Following [3], for a natural number $n \geq 2$ we shall call an *n-grid* a structure of the form $\mathcal{A} = (A; E_i)_{i \in n}$ such that each E_i is an equivalence relation on the set A and $[a]_i \cap [a]_j$ is finite whenever $a \in A$ and $i < j < n$ (where $[a]_i$ denotes the equivalence class of a with respect to the relation E_i). An *n-cube* is a particular kind of n -grid where A is of the form $A = A_0 \times \cdots \times A_{n-1}$ and each E_i is the equivalence relation on A whose equivalence classes are the lines parallel to the i -th coordinate axis (i.e. two n -tuples are E_i -related if and only if all of their coordinates coincide except perhaps for the i -th one).

^{*}2000 Mathematics Subject Classification: Primary 03E50, Secondary 03E05, 51M05.
Key Words and Phrases: Continuum hypothesis, Sierpinski's theorem, n-grids.

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An *acceptable coloring* for an n -grid \mathcal{A} is a function $\chi : A \rightarrow n$ such that $[a]_i \cap \chi^{-1}(i)$ is finite for all $a \in A$ and $i \in n$.

In [3], J.H. Schmerl gives a really nice characterization of those semialgebraic n -grids which admit an acceptable coloring:

Theorem 1.1. (Schmerl) *Suppose that $2 \leq n < \omega$, \mathcal{A} is a semialgebraic n -grid and $2^{\aleph_0} \geq \aleph_{n-1}$. Then the following are equivalent:*

- (1) *some finite n -cube is not embeddable in \mathcal{A} .*
- (2) *\mathbb{R}^n is not embeddable in \mathcal{A} .*
- (3) *\mathcal{A} has an acceptable n -coloring.*

In this note, we present a characterization that works for any n -grid (see Definition 2.1 and Theorem 2.7). Then we use this characterization to show that (1) \Rightarrow (3) in the previous theorem holds for arbitrary n -grids (see Theorem 3.1). In fact, the size of the continuum turns out to be irrelevant for this implication. The implication (3) \Rightarrow (2) for arbitrary n -grids follows from a result of Kuratowski as it is mentioned in [3]. None of these implications can be reversed for arbitrary n -grids, regardless of the size of the continuum.

2 Twisted n -grids

In this section we use elementary submodels of the universe to obtain a characterization of those n -grids which admit an acceptable coloring. At first sight this characterization seems rather cumbersome, but it is the key to our results in the next section. The case $n = 3$ was already obtained in [1] with a bit different terminology and latter used in [2].

As it has become customary, whenever we say that M is an *elementary submodel of the universe*, we really mean that (M, \in) is an elementary submodel of $(H(\theta), \in)$ where $H(\theta)$ is the set of all sets of hereditary cardinality less than θ and θ is a large enough regular cardinal (e.g. when we are studying a fixed n -grid \mathcal{A} on a transitive set A , $\theta = (2^{|A|})^+$ is large enough).

Given an equivalence relation E on a set A , we say that $B \subseteq A$ is *E -small* if the E -equivalence classes restricted to B are all finite. Note that the E -small sets form an ideal in the power set of A . Using this terminology, an n -coloring $\chi : A \rightarrow n$ is acceptable for the n -grid $(A; E_i)_{i \in n}$ if and only if $\chi^{-1}(i)$ is E_i -small for each $i \in n$.

A *test set* for an n -grid \mathcal{A} is a set \mathcal{M} of elementary submodels of the universe such that $\mathcal{A} \in \bigcap \mathcal{M}$, $|\mathcal{M}| = n - 1$ and \mathcal{M} is linearly ordered by \in .

Definition 2.1. We say that an n -grid $\mathcal{A} = (A; E_i)_{i \in n}$ is *twisted* if for every test set \mathcal{M} for \mathcal{A} and every $k \in n$, the set

$$\{x \in A \setminus \bigcup \mathcal{M} : [x]_i \in \bigcup \mathcal{M} \text{ for all } i \neq k\}$$

is E_k -small.

The rest of this section is devoted to show that twisted n -grids are exactly the ones that admit acceptable colorings. For this, let us fix an arbitrary n -grid $\mathcal{A} = (A; E_i)_{i \in n}$; our first task is to cover A with countable elementary submodels in a way that allows us to define a suitable rank function for elements of A and for E_i -equivalence classes of elements of A .

We fix M_Λ an elementary submodel such that $A \cup \{\mathcal{A}\} \subseteq M_\Lambda$ and we let $\kappa = |M_\Lambda|$. Thinking of κ as an initial ordinal, we let $T = \bigcup_{m \in \omega} \kappa^m$ be the set of finite sequences of ordinals in κ . We have two natural orders on T , the tree (partial) order \subseteq and the lexicographic order \leq . In both orders we have the same minimum element Λ , the empty sequence. For $\sigma \in T$ and $\alpha \in \kappa$ we write $\sigma^\frown \alpha = \sigma \cup \{(|\sigma|, \alpha)\}$. Given $\sigma \in T \setminus \{\Lambda\}$ we write $\sigma + 1$ for the successor of σ in the lexicographic order of $\kappa^{|\sigma|}$; that is

$$\sigma + 1 = (\sigma \upharpoonright (|\sigma| - 1))^\frown (\sigma(|\sigma| - 1) + 1).$$

We shall write $\sigma \wedge \tau$ for the infimum of σ and τ with respect to the tree order; thus for $\sigma \neq \tau$ we have:

$$\sigma \wedge \tau = \sigma \upharpoonright |\sigma \wedge \tau| = \tau \upharpoonright |\sigma \wedge \tau| \text{ and}$$

$$\sigma(|\sigma \wedge \tau|) \neq \tau(|\sigma \wedge \tau|).$$

Now we can find inductively (on the length of $\sigma \in T$) elementary submodels M_σ such that:

- i) The sequence $\langle M_{\sigma^\frown \alpha} : \alpha \in \text{cof}(|M_\sigma|) \rangle$ is a continuous (increasing) elementary chain,
- ii) $M_\sigma \subseteq \bigcup \{M_{\sigma^\frown \alpha} : \alpha \in \text{cof}(|M_\sigma|)\}$,
- iii) $\{\mathcal{A}\} \cup \{M_\tau : \tau + 1 \subseteq \sigma\} \subseteq M_{\sigma^\frown 0}$, and

iv) If $\tau \subsetneq \sigma$ and M_τ is uncountable then $|M_\tau| > |M_\sigma|$.

We actually do not need to (and will not) define $M_{\sigma \frown \alpha}$ when M_σ is countable or if $\alpha \geq \text{cof}(|M_\sigma|)$.

Although the lexicographic order on T is not a well order, it is not hard to see that conditions *ii* and *iv* allow the following definition of rank to make sense:

Definition 2.2. For $x \in M_\Lambda$ we define $rk(x)$ as the minimum $\sigma \in T$ (in the lexicographic order) such that M_σ is countable and $x \in M_\tau$ for all $\tau \subseteq \sigma$.

Note that by the continuity of the elementary chains in condition *i*, we have that $rk(x)$ is always a finite sequence of ordinals which are either successor ordinals or 0. In particular, if $\sigma_x = rk(x)$, $\sigma_y = rk(y)$, $\sigma_x < \sigma_y$ and $m = |\sigma_x \wedge \sigma_y|$, then $\sigma_y(m)$ is a successor ordinal say $\alpha + 1$ and we can define

$$\Delta(x, y) = (\sigma_x \wedge \sigma_y)^\frown \alpha.$$

This last definition will only be used in the proof of Lemma 2.5. The following remark summarizes the basic properties of $\Delta(x, y)$ that we will be using; all of them follow rather easily from the definitions.

Remark 2.3. If $rk(x) < rk(y)$ then

- $x \in M_{\Delta(x, y)}$ and $y \notin M_{\Delta(x, y)}$,
- $\Delta(x, y) + 1 \subseteq rk(y)$,
- if $\sigma \supsetneq \Delta(x, y) + 1$ then $M_{\Delta(x, y)} \in M_\sigma$ (by conditions *i* and *iii*).

After assigning a rank to each member of M_Λ , we need a way to order in type ω all the elements of M_Λ of the same rank. This is easily done by fixing an injective enumeration

$$M_\sigma = \{t_m^\sigma : m \in \omega\}$$

for each σ for which M_σ is countable, and defining the degree of an element of M_Λ as follows:

Definition 2.4. For $x \in M_\Lambda$ we define $deg(x)$ as the unique natural number satisfying

$$x = t_{deg(x)}^{rk(x)}.$$

The following two lemmas will be used to construct an acceptable coloring for \mathcal{A} in the case that \mathcal{A} is twisted, although the second one does not make any assumptions on \mathcal{A} .

Lemma 2.5. *If \mathcal{A} is twisted then there is a set $B \subseteq A$ and a partition $B = \bigcup_{k \in n} B_k$ such that:*

- a) *Each B_k is E_k -small and*
- b) *$|\{i \in n : rk([x]_i) = rk(x)\}| \geq 2$ for any $x \in A \setminus B$.*

Proof. For each $k \in n$ we let B_k be the set of all $x \in A$ such that $rk([x]_k) > rk([x]_i)$ for all $i \neq k$. Let $B = \bigcup_{k \in n} B_k$.

Note that for any $x \in A$ and $i \in n$ we have that $rk([x]_i) \leq rk(x)$. On the other hand if $\sigma = rk([x]_k) = rk([x]_j)$ for some $k \neq j$, then by elementarity and the fact that $[x]_k \cap [x]_j$ is finite, it follows that $rk(x) \leq \sigma$ and hence $rk(x) = \sigma$. This observation easily implies that condition b) is met. It also implies that if $x \in B_k$ then

$$rk([x]_{k_0}) < \cdots < rk([x]_{k_{n-2}}) < rk([x]_k) \leq rk(x)$$

for some numbers k_0, \dots, k_{n-2} such that $\{k_0, \dots, k_{n-2}, k\} = n$.

Now we put $\mathcal{M} = \left\{ M_{\Delta([x]_{k_i}, x)} : i \in n - 1 \right\}$, and use \mathcal{M} as a test set for \mathcal{A} to conclude that, since \mathcal{A} is twisted, B_k is E_k -small.

To see that \mathcal{M} is indeed a test set, it is enough to show that $M_{\Delta([x]_{k_i}, x)} \in M_{\Delta([x]_{k_j}, x)}$ for $i < j$. So fix $i < j$ and note that since $[x]_{k_i} \cap [x]_{k_j}$ is finite we have $\Delta([x]_{k_i}, x) = \Delta([x]_{k_i}, [x]_{k_j})$ and therefore by Remark 2.3,

$$\Delta([x]_{k_i}, x) + 1 \subseteq rk(x) \wedge rk([x]_{k_j}).$$

But then $\Delta([x]_{k_i}, x) + 1 \subsetneq \Delta([x]_{k_j}, x)$ and again by Remark 2.3 we get $M_{\Delta([x]_{k_i}, x)} \in M_{\Delta([x]_{k_j}, x)}$. □

Lemma 2.6. *For all $i, k \in n$ with $i \neq k$, the set*

$$C_{i,k} = \{x \in A : rk([x]_i) = rk([x]_k) \text{ and } deg([x]_i) < deg([x]_k)\}$$

is E_k -small.

Proof. Fix $a \in A$ and let $\sigma = rk([a]_k)$ and $d = deg([a]_k)$. Note that if $x \in C_{i,k} \cap [a]_k$ then there is an $m < d$ (namely $m = deg([x]_i)$) such that $x \in t_m^\sigma \cap t_d^\sigma$ and $t_m^\sigma \cap t_d^\sigma$ is finite. Hence $C_{i,k} \cap [a]_k$ is contained in a finite union of finite sets. □

We are finally ready to prove the main result of this section.

Theorem 2.7. *The following are equivalent:*

- 1) \mathcal{A} is twisted.
- 2) \mathcal{A} admits an acceptable coloring.

Proof. Suppose first that \mathcal{A} is twisted. Let B and B_k for $k \in n$ be as in Lemma 2.5, and let $C_{i,k}$ for $i, k \in n$ be as in Lemma 2.6. For each $k \in n$ define C_k as the set of all $x \in A \setminus B$ such that:

- i) $rk(x) = rk([x]_k)$, and
- ii) for all $i \in n \setminus \{k\}$, if $rk([x]_i) = rk([x]_k)$ then $deg([x]_i) < deg([x]_k)$.

By condition b) in Lemma 2.5, we have that $C_k \subseteq \bigcup_{i \in n} C_{i,k}$ and therefore each C_k is E_k -small. It also follows that the C_k 's form a partition of $A \setminus B$ so that we can define an acceptable coloring for \mathcal{A} by:

$$\chi(x) = k \text{ if and only if } x \in B_k \cup C_k.$$

Now suppose that \mathcal{A} admits an acceptable coloring and fix a test set \mathcal{M} and $k \in n$. We want to show that the set

$$X = \{x \in A \setminus \cup \mathcal{M} : [x]_i \in \cup \mathcal{M} \text{ for all } i \neq k\}$$

is E_k -small. For this let $\chi : A \rightarrow n$ be an acceptable coloring such that (using elementarity and the fact that \mathcal{M} is linearly ordered by \in) χ belongs to each $M \in \mathcal{M}$. Now if $x \in X$ and $i \neq k$ then there is an $M \in \mathcal{M}$ such that $[x]_i \cap \chi^{-1}(i) \in M$ and hence $[x]_i \cap \chi^{-1}(i) \subset M$ (since χ is acceptable); this implies that $\chi(x) \neq i$. It follows that $X \subseteq \chi^{-1}(k)$ so that X is E_k -small. □

3 Embedding cubes into n -grids

Given an n -grid $\mathcal{A} = (A; E_i)_{i \in n}$ it will be convenient in this section to have a name $\rho_i : A \rightarrow A/E_i$ for the quotient maps ($\rho_i(\cdot) = [\cdot]_i$). Note that if $i \neq k$, $C \subseteq A$ is infinite and $\rho_k \upharpoonright C$ is constant, then there is an infinite $D \subseteq C$ such that $\rho_i \upharpoonright D$ is injective. We will make repeated use of this fact without explicitly saying so, in the proof of the following:

Theorem 3.1. *If \mathcal{A} is a non-twisted n -grid then any finite n -cube l^n (with $l \in \omega$) can be embedded in \mathcal{A} .*

Proof. By definition, since \mathcal{A} is not twisted, there is a test set \mathcal{M} and a $k \in n$ such that for some $a \in A$, the set

$$B = \{x \in [a]_k \setminus \cup \mathcal{M} : [x]_i \in \cup \mathcal{M} \text{ for all } i \neq k\}$$

is infinite. For each $x \in B$ and each $i \in n \setminus \{k\}$ there is an $M_i^x \in \mathcal{M}$ such that $[x]_i \in M_i^x$. Since \mathcal{M} is finite, there must be an infinite $C \subseteq B$ on which the map $x \mapsto \langle M_i^x : i \in n \setminus \{k\} \rangle$ is constant, say with value $\langle M_i : i \in n \setminus \{k\} \rangle$. Note that since C is disjoint from $\cup \mathcal{M}$, the map $i \mapsto M_i$ must be injective and hence $\mathcal{M} = \{M_i : i \in n \setminus \{k\}\}$, because $|\mathcal{M}| = n - 1$. Finally, we can find an infinite set $D \subseteq C$ such that $\rho_i \upharpoonright D$ is injective for all $i \neq k$.

Now taking $k_1 = k$ and letting φ be any injection from l into D , we easily see that the following statement is true for $j = 1$:

$P(j)$: *There are distinct $k_1, \dots, k_j \in n$ and an embedding $\varphi : l^j \rightarrow (A; E_{k_1}, \dots, E_{k_j})$ such that:*

- a) *for $i \in n \setminus \{k_1, \dots, k_j\}$, $\rho_i \circ \varphi$ is injective and belongs to M_i ,*
- b) *φ takes values in $A \setminus \cup \{M_i : i \in n \setminus \{k_1, \dots, k_j\}\}$.*

Note that when $j = n$, conditions a) and b) become trivially true, and $P(n)$ just says that there is an embedding (modulo an irrelevant permutation of coordinates) of the finite cube l^n into \mathcal{A} , which is exactly what we want to show. We already know that $P(1)$ is true, so we are done if we can show that $P(j)$ implies $P(j+1)$ for $1 \leq j < n$.

Assuming $P(j)$, let $\varphi : l^j \rightarrow (A; E_{k_1}, \dots, E_{k_j})$ be such an embedding, and let $k_{j+1} \in n \setminus \{k_1, \dots, k_j\}$ be such that $M_{k_{j+1}}$ is the \in -maximum element of $\{M_i : i \in n \setminus \{k_1, \dots, k_j\}\}$. Let us call

$$\delta := \rho_{k_{j+1}} \circ \varphi \in M_{k_{j+1}}.$$

Now note that $\varphi \notin M_{k_{j+1}}$ and at the same time φ satisfies the following properties (on the free variable Φ), all of which can be expressed using parameters from $M_{k_{j+1}}$:

- $\Phi : l^j \rightarrow (A; E_{k_1}, \dots, E_{k_j})$ is an embedding,
- $\rho_{k_{j+1}} \circ \Phi = \delta$,
- for $i \in n \setminus \{k_1, \dots, k_j, k_{j+1}\}$, $\rho_i \circ \Phi$ is injective and belongs to M_i ,
- Φ takes values in $A \setminus \bigcup \{M_i : i \in n \setminus \{k_1, \dots, k_j, k_{j+1}\}\}$.

This means that there must be an infinite set (in fact there must be an uncountable one, but we won't be using this) $\{\varphi_m : m \in \omega\}$ of distinct functions satisfying those properties. Going to a subsequence l^j -many times, we may assume without loss of generality that for each $t \in l^j$, the map $m \mapsto \varphi_m(t)$ is either constant or injective. Now since they cannot all be constant, it is not hard to see that in fact all these maps have to be injective: just note that if $t, t' \in l^j$ are in a line parallel to the $(r-1)$ -th coordinate axis then it cannot be the case that the map associated with t is constant while the one associated with t' is injective, since otherwise $\{\varphi_m(t') : m \in \omega\}$ would be an infinite set contained in $[\varphi_0(t)]_{k_r} \cap [\varphi_0(t')]_{k_{j+1}}$. To see this, just note that in that situation we would have $[\varphi_m(t')]_{k_r} = [\varphi_m(t)]_{k_r} = [\varphi_0(t)]_{k_r}$ and $[\varphi_m(t')]_{k_{j+1}} = (\rho_{k_{j+1}} \circ \varphi_m)(t') = \delta(t') = (\rho_{k_{j+1}} \circ \varphi_0)(t') = [\varphi_0(t')]_{k_{j+1}}$.

Next we can find an infinite $I \subseteq \omega$ such that for each $t \in l^j$ and each $i \in n \setminus \{k_{j+1}\}$ the map $m \mapsto [\varphi_m(t)]_i$ is injective when restricted to I . From here one can find (one at a time) l distinct elements m_0, \dots, m_{l-1} of I such that for all $t, t' \in l^j$, for all $r, r' \in l$ with $r \neq r'$ and for all $i \in n \setminus \{k_{j+1}\}$, we have that $[\varphi_{m_r}(t)]_i \neq [\varphi_{m_{r'}}(t')]_i$.

Finally we let $\psi : l^{j+1} \rightarrow (A; E_{k_1}, \dots, E_{k_{j+1}})$ be the function defined by $\psi(t, r) = \varphi_{m_r}(t)$. By the way that we constructed the m_r 's and using the fact that all the φ_m 's are embeddings and also using that δ is injective, one can see that ψ is in fact an embedding. From the fact that ψ is essentially a finite union of some φ_m 's and by the way we chose those φ_m 's, it follows that conditions a) and b) in $P(j+1)$ are satisfied. □

This last theorem only goes one way: for example, the n -cube ω^n is twisted for $n \geq 2$, but of course any finite n -cube can be embedded in it. I suspect that only for very "nice" classes of n -grids one can reverse this

theorem. Schmerl's theorem does it for semialgebraic n -grids; perhaps some form of o-minimality is what is required.

The question of when can an infinite cube be embedded in an arbitrary n -grid seems more subtle. For instance, let us consider the case $n = 2$. Using the same idea as for the proof of 3.1, one can easily show:

Theorem 3.2. *If \mathcal{A} is a non-twisted 2-grid then either $l \times \omega_1$ can be embedded in \mathcal{A} for all $l \in \omega$, or $\omega_1 \times l$ can be embedded in \mathcal{A} for all $l \in \omega$.*

However, it is not true that $\omega \times \omega$ embeds in any non-twisted 2-grid. For example, fix an uncountable family $\{A_\alpha : \alpha \in \omega_1\}$ of almost disjoint subsets of ω and let $A = \{(n, \alpha) \in \omega \times \omega_1 : n \in A_\alpha\}$. Think of A as a subgrid of the 2-cube $\omega \times \omega_1$. It is easy to see that this is a non-twisted grid, but not even $\omega \times 2$ can be embedded in it.

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