

ON THE DISCRETE LOGARITHMIC MINKOWSKI PROBLEM

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ABSTRACT. A new sufficient condition for the existence of a solution for the logarithmic Minkowski problem is established. This new condition contains the one established by Zhu [69] and the discrete case established by Böröczky, Lutwak, Yang, Zhang [6] as two important special cases.

1. INTRODUCTION

The setting for this paper is n -dimensional Euclidean space \mathbb{R}^n . A *convex body* in \mathbb{R}^n is a compact convex set that has non-empty interior. If K is a convex body in \mathbb{R}^n , then the *surface area measure*, S_K , of K is a Borel measure on the unit sphere, S^{n-1} , defined for a Borel $\omega \subset S^{n-1}$ (see, e.g., Schneider [61]), by

$$S_K(\omega) = \int_{x \in \nu_K^{-1}(\omega)} d\mathcal{H}^{n-1}(x),$$

where $\nu_K : \partial'K \rightarrow S^{n-1}$ is the Gauss map of K , defined on $\partial'K$, the set of points of ∂K that have a unique outer unit normal, and \mathcal{H}^{n-1} is $(n-1)$ -dimensional Hausdorff measure.

As one of the cornerstones of the classical Brunn-Minkowski theory, the Minkowski's existence theorem can be stated as follows (see, e.g., Schneider [61]): If μ is not concentrated on a great subsphere of S^{n-1} , then μ is the surface area measure of a convex body if and only if

$$\int_{S^{n-1}} u d\mu(u) = 0.$$

The solution is unique up to translation, and even the regularity of the solution is well investigated, see e.g., Lewy [40], Nirenberg [57], Cheng and Yau [12], Pogorelov [60], and Caffarelli [9].

The surface area measure of a convex body has clear geometric significance. Another important measure that is associated with a convex body and that has clear geometric importance is the cone-volume measure. If K is a convex body in \mathbb{R}^n that contains the origin in its interior, then the *cone-volume measure*, V_K , of K is a Borel measure on S^{n-1} defined for each Borel $\omega \subset S^{n-1}$ by

$$V_K(\omega) = \frac{1}{n} \int_{x \in \nu_K^{-1}(\omega)} x \cdot \nu_K(x) d\mathcal{H}^{n-1}(x).$$

For references regarding cone-volume measure see, e.g., [5–8, 42–44, 55, 56, 58, 62–64, 69].

The Minkowski's existence theorem deals with the question of prescribing the surface area measure. The following problem is prescribing the cone-volume measure.

Logarithmic Minkowski problem: What are the necessary and sufficient conditions on a finite Borel measure μ on S^{n-1} so that μ is the cone-volume measure of a convex body in \mathbb{R}^n ?

In [45], Lutwak showed that there is an L_p analogue of the surface area measure (known as the L_p surface area measure). In recent years, the L_p surface area measure appeared in, e.g.,

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[1, 4, 10, 22, 23, 25, 26, 31, 42–44, 47–49, 52, 53, 55, 56, 58, 59, 64]. In [45], Lutwak posed the associated L_p Minkowski problem which extends the classical Minkowski problem for $p \geq 1$. In addition, the L_p Minkowski problem for $p < 1$ was publicized by a series of talks by Erwin Lutwak in the 1990's. The L_p Minkowski problem is the classical Minkowski problem when $p = 1$, while the L_p Minkowski problem is the logarithmic Minkowski problem when $p = 0$. The L_p Minkowski problem is interesting for all real p , and have been studied by, e.g., Lutwak [45], Lutwak and Oliker [46], Chou and Wang [14], Guan and Lin [21], Hug, et al. [35], Böröczky, et al. [6]. Additional references regarding the L_p Minkowski problem and Minkowski-type problems can be found in, e.g., [6, 11, 14, 20–24, 33–35, 38, 39, 41, 45, 46, 51, 54, 62, 63, 70, 71]. Applications of the solutions to the L_p Minkowski problem can be found in, e.g., [2, 3, 13, 15, 16, 27–29, 36, 37, 50, 66, 68].

A finite Borel measure μ on S^{n-1} is said to satisfy the *subspace concentration condition* if, for every subspace ξ of \mathbb{R}^n , such that $0 < \dim \xi < n$,

$$(1.2) \quad \mu(\xi \cap S^{n-1}) \leq \frac{\dim \xi}{n} \mu(S^{n-1}),$$

and if equality holds in (1.2) for some subspace ξ , then there exists a subspace ξ' , that is complementary to ξ in \mathbb{R}^n , so that also

$$\mu(\xi' \cap S^{n-1}) = \frac{\dim \xi'}{n} \mu(S^{n-1}).$$

The measure μ on S^{n-1} is said to satisfy the *strict subspace concentration inequality* if the inequality in (1.2) is strict for each subspace $\xi \subset \mathbb{R}^n$, such that $0 < \dim \xi < n$.

Very recently, Böröczky and Henk [5] proved that if the centroid of a convex body is the origin, then the cone-volume measure of this convex body satisfies the subspace concentration condition. For more references on the progress of the subspace concentration condition, see, e.g., Henk et al. [32], He et al. [30], Xiong [67], Böröczky et al. [8], and Henk and Linke [31].

In [6], Böröczky, et al. established the following necessary and sufficient conditions for the existence of solutions to the even logarithmic Minkowski problem.

Theorem 1.1 (Böröczky, Lutwak, Yang, Zhang). *A non-zero finite even Borel measure on S^{n-1} is the cone-volume measure of an origin-symmetric convex body in \mathbb{R}^n if and only if it satisfies the subspace concentration condition.*

The convex hull of a finite set is called a polytope provided that it has positive n -dimensional volume. The convex hull of a subset of these points is called a *facet* of the polytope if it lies entirely on the boundary of the polytope and has positive $(n - 1)$ -dimensional volume. If a polytope P contains the origin in its interior and has N facets whose outer unit normals are u_1, \dots, u_N , and such that if the facet with outer unit normal u_k has $(n - 1)$ -measure a_k and distance from the origin h_k for all $k \in \{1, \dots, N\}$, then

$$V_P = \frac{1}{n} \sum_{k=1}^N h_k a_k \delta_{u_k}.$$

where δ_{u_k} denotes the delta measure that is concentrated at the point u_k .

A finite subset U (with no less than n elements) of S^{n-1} is said to be *in general position* if any k elements of U , $1 \leq k \leq n$, are linearly independent.

For a long time, people believed that the data for a cone-volume measure can not be arbitrary. However, Zhu [69] proved that any discrete measure on S^{n-1} whose support is in general position is a cone-volume measure.

Theorem 1.2 (Zhu). *A discrete measure, μ , on the unit sphere S^{n-1} is the cone-volume measure of a polytope whose outer unit normals are in general position if and only if the support of μ is in general position and not concentrated on a closed hemisphere of S^{n-1} .*

A linear subspace ξ ($1 \leq \dim \xi \leq n - 1$) of \mathbb{R}^n is said to be essential with respect to a Borel measure μ on S^{n-1} if $\xi \cap \text{supp}(\mu)$ is not concentrated on any closed hemisphere of $\xi \cap S^{n-1}$.

Definition 1.3. *A finite Borel measure μ on S^{n-1} is said to satisfy the essential subspace concentration condition if, for every essential subspace ξ (with respect to μ) of \mathbb{R}^n , such that $0 < \dim \xi < n$,*

$$(1.3) \quad \mu(\xi \cap S^{n-1}) \leq \frac{\dim \xi}{n} \mu(S^{n-1}),$$

and if equality holds in (1.3) for some essential subspace ξ (with respect to μ), then there exists a subspace ξ' , that is complementary to ξ in \mathbb{R}^n , so that

$$(1.4) \quad \mu(\xi' \cap S^{n-1}) = \frac{\dim \xi'}{n} \mu(S^{n-1}).$$

Definition 1.4. *The measure μ on S^{n-1} is said to satisfy the strict essential subspace concentration inequality if the inequality in (1.3) is strict for each essential subspace ξ (with respect to μ) of \mathbb{R}^n , such that $0 < \dim \xi < n$.*

We would like to note that if μ is a Borel measure on the unit sphere that is not concentrated on a closed hemisphere and satisfies the essential subspace concentration condition, and ξ is an essential subspace (with respect to μ) that reaches the equality in (1.3), then by Lemma 5.2, ξ' (in (1.4)) is an essential subspace with respect to μ .

It is the aim of this paper to establish the following.

Theorem 1.5. *If μ is a discrete measure on S^{n-1} that is not concentrated on any closed hemisphere and satisfies the essential subspace concentration condition, then μ is the cone-volume measure of a polytope in \mathbb{R}^n containing the origin in its interior.*

If μ is a non-trivial even Borel measure on S^{n-1} , and ξ is a k -dimensional linear subspace of \mathbb{R}^n spanned by some vectors $v_1, \dots, v_k \in \text{supp}(\mu)$ for $1 \leq k \leq n - 1$, then $-v_1, \dots, -v_k \in \text{supp}(\mu)$, as well, and hence ξ is an essential subspace. In particular, for even discrete measures, Theorem 1.5 is equivalent to the sufficient condition of Theorem 1.1. However, there are non-even discrete measures that satisfy the essential subspace concentration condition, but not the subspace concentration condition. For example, if a k -dimensional subspace ξ , $1 \leq k \leq n - 1$, intersects the support of the measure in $k + 1$ unit vectors u_0, \dots, u_k such that u_1, \dots, u_k are independent, and $u_0 = \alpha_1 u_1 + \dots + \alpha_k u_k$ for $\alpha_1, \dots, \alpha_k > 0$, then there is no condition on the restriction of the measure to $\xi \cap S^{n-1}$. Therefore, for discrete measures, Theorem 1.5 is a generalization of the sufficient condition of Theorem 1.1.

We claim that if the support of a discrete measure μ is in general position, then the set of essential subspaces (with respect to μ) is empty. Otherwise, there exists a subspace ξ with $1 \leq \dim \xi \leq n - 1$ such that $\text{supp}(\mu) \cap \xi$ is not concentrated on a closed hemisphere of $S^{n-1} \cap \xi$. Then we can choose $\dim \xi + 1$ ($\leq n$) vectors from $\text{supp}(\mu) \cap \xi$ that are linearly dependent. But this contradicts the fact that $\text{supp}(\mu)$ is in general position. From our declaration, we have, Theorem 1.5 contains Theorem 1.2 as an important special case.

In \mathbb{R}^2 , Theorem 1.5 leads to the main result of Stancu ([62], pp. 162), where she applied a different method called the crystalline deformation.

New inequalities for cone-volume measures are established in section 6.

2. PRELIMINARIES

In this section, we collect some basic notations and facts about convex bodies. For general references regarding convex bodies see, e.g., [17–19, 61, 65].

The vectors of this paper are column vectors. For $x, y \in \mathbb{R}^n$, we will write $x \cdot y$ for the standard inner product of x and y , and write $|x|$ for the Euclidean norm of x . We write $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ for the boundary of the Euclidean unit ball B^n in \mathbb{R}^n , and write κ_n for the volume of the unit ball. Let $V_k(M)$ denote the k -dimensional Hausdorff measure of an at most k -dimensional convex set M . In addition, if $k = n - 1$, then we also use the notation $|M|$.

Suppose X_1, X_2 are subspaces of \mathbb{R}^n , we write $X_1 \perp X_2$ if $x_1 \cdot x_2 = 0$ for all $x_1 \in X_1$ and $x_2 \in X_2$. Suppose X is a subspace of \mathbb{R}^n and S is a subset of \mathbb{R}^n , we write $S|_X$ for the orthogonal projection of S on X .

Suppose C is a subset of \mathbb{R}^n , the positive hull, $\text{pos}(C)$, of C is the set of all positive combinations of any finitely many elements of C . Let $\text{lin}(C)$ be the smallest linear subspace of \mathbb{R}^n containing C . The diameter of C is defined by

$$d(C) = \sup\{|x - y| : x, y \in C\}.$$

For $K_1, K_2 \subset \mathbb{R}^n$ and $c_1, c_2 \geq 0$, the Minkowski combination, $c_1K_1 + c_2K_2$, is defined by

$$c_1K_1 + c_2K_2 = \{c_1x_1 + c_2x_2 : x_1 \in K_1, x_2 \in K_2\}.$$

The *support function* $h_K : \mathbb{R}^n \rightarrow \mathbb{R}$ of a compact convex set K is defined, for $x \in \mathbb{R}^n$, by

$$h(K, x) = \max\{x \cdot y : y \in K\}.$$

Obviously, for $c \geq 0$ and $x \in \mathbb{R}^n$, we have

$$h(cK, x) = h(K, cx) = ch(K, x).$$

The *convex hull* of two convex sets K, L in \mathbb{R}^n is defined by

$$[K, L] = \{z : z = \lambda x + (1 - \lambda)y, 0 \leq \lambda \leq 1 \text{ and } x, y \in K \cup L\}.$$

The *Hausdorff distance* of two compact sets K, L in \mathbb{R}^n is defined by

$$\delta(K, L) = \inf\{t \geq 0 : K \subset L + tB^n, L \subset K + tB^n\}.$$

It is known that the Hausdorff distance between two convex bodies, K and L , is

$$\delta(K, L) = \max_{u \in S^{n-1}} |h(K, u) - h(L, u)|.$$

We always consider the space of convex bodies as metric space equipped with the Hausdorff distance. It is known that if a sequence $\{K_m\}$ of convex bodies tends to a convex body K in \mathbb{R}^n containing the origin in its interior, then S_{K_m} tends weakly to S_K , and hence V_{K_m} tends weakly to V_K (see Schneider [61]).

For a convex body K in \mathbb{R}^n , and $u \in S^{n-1}$, the *support hyperplane* $H(K, u)$ in direction u is defined by

$$H(K, u) = \{x \in \mathbb{R}^n : x \cdot u = h(K, u)\},$$

the *face* $F(K, u)$ in direction u is defined by

$$F(K, u) = K \cap H(K, u).$$

Let \mathcal{P} be the set of all polytopes in \mathbb{R}^n . If the unit vectors u_1, \dots, u_N are not concentrated on a closed hemisphere, let $\mathcal{P}(u_1, \dots, u_N)$ be the set of all polytopes $P \in \mathcal{P}$ such that the set of outer unit normals of the facets of P is a subset of $\{u_1, \dots, u_N\}$, and let $\mathcal{P}_N(u_1, \dots, u_N)$ be the set of all polytopes $P \in \mathcal{P}$ such that the set of outer unit normals of the facets of P is $\{u_1, \dots, u_N\}$.

3. AN EXTREMAL PROBLEM RELATED TO THE LOGARITHMIC MINKOWSKI PROBLEM

Let us suppose $\gamma_1, \dots, \gamma_N \in (0, \infty)$, and the unit vectors u_1, \dots, u_N are not concentrated on a closed hemisphere. Let

$$(3.0) \quad \mu = \sum_{i=1}^N \gamma_i \delta_{u_i},$$

and for $P \in \mathcal{P}(u_1, \dots, u_N)$ define $\Phi_P : \text{Int}(P) \rightarrow \mathbb{R}$ by

$$(3.1) \quad \begin{aligned} \Phi_P(\xi) &= \int_{S^{n-1}} \log(h(P, u) - \xi \cdot u) d\mu(u) \\ &= \sum_{k=1}^N \gamma_k \log(h(P, u_k) - \xi \cdot u_k), \end{aligned}$$

where $\text{Int}(P)$ is the interior of P .

In this section, we study the following extremal problem:

$$(3.2) \quad \inf \left\{ \max_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = |\mu| \right\},$$

where $|\mu| = \sum_{k=1}^N \gamma_k$.

We will prove that the solution of problem (3.2) solves the corresponding logarithmic Minkowski problem.

For the case where u_1, \dots, u_N are in general position and $Q \in \mathcal{P}_N(u_1, \dots, u_N)$, problem (3.2) was studied in [69]. The results and proofs in this section are similar to [69]. However, for convenience of the readers, we give detailed proofs for these results.

Lemma 3.1. *Suppose $\mu = \sum_{k=1}^N \gamma_k \delta_{u_k}$ is a discrete measure on S^{n-1} that is not concentrated on a closed hemisphere, and $P \in \mathcal{P}(u_1, \dots, u_N)$, then there exists a unique point $\xi(P) \in \text{Int}(P)$ such that*

$$\Phi_P(\xi(P)) = \max_{\xi \in \text{Int}(P)} \Phi_P(\xi).$$

Proof. Let $0 < \lambda < 1$ and $\xi_1, \xi_2 \in \text{Int}(P)$. From the concavity of the logarithmic function,

$$\begin{aligned} \lambda \Phi_P(\xi_1) + (1 - \lambda) \Phi_P(\xi_2) &= \lambda \int_{S^{n-1}} \log(h(P, u) - \xi_1 \cdot u) d\mu(u) \\ &\quad + (1 - \lambda) \int_{S^{n-1}} \log(h(P, u) - \xi_2 \cdot u) d\mu(u) \\ &= \sum_{k=1}^N \gamma_k [\lambda \log(h(P, u_k) - \xi_1 \cdot u_k) + (1 - \lambda) \log(h(P, u_k) - \xi_2 \cdot u_k)] \\ &\leq \sum_{k=1}^N \gamma_k \log[h(P, u_k) - (\lambda \xi_1 + (1 - \lambda) \xi_2) \cdot u_k] \\ &= \Phi_P(\lambda \xi_1 + (1 - \lambda) \xi_2), \end{aligned}$$

with equality if and only if $\xi_1 \cdot u_k = \xi_2 \cdot u_k$ for all $k = 1, \dots, N$. Since the unit vectors u_1, \dots, u_N are not concentrated on a closed hemisphere, $\mathbb{R}^n = \text{lin}\{u_1, \dots, u_N\}$. Thus, $\xi_1 = \xi_2$. Therefore, Φ_P is strictly concave on $\text{Int}(P)$.

Since $P \in \mathcal{P}(u_1, \dots, u_N)$, for any $x \in \partial P$, there exists some $i_0 \in \{1, \dots, N\}$ such that

$$h(P, u_{i_0}) = x \cdot u_{i_0}.$$

Thus, $\Phi_P(\xi) \rightarrow -\infty$ whenever $\xi \in \text{Int}(P)$ and $\xi \rightarrow x$. Therefore, there exists a unique interior point $\xi(P)$ of P such that

$$\Phi_P(\xi(P)) = \max_{\xi \in \text{Int}(P)} \Phi_P(\xi).$$

□

Obviously, for $\lambda > 0$ and $P \in \mathcal{P}(u_1, \dots, u_N)$,

$$(3.3) \quad \xi(\lambda P) = \lambda \xi(P),$$

and if $P_i \in \mathcal{P}(u_1, \dots, u_N)$ and P_i converges to a polytope P , then $P \in \mathcal{P}(u_1, \dots, u_N)$.

For the case where u_1, \dots, u_N are in general position, the following lemma was proved in [69].

Lemma 3.2. *Suppose $\mu = \sum_{k=1}^N \gamma_k \delta_{u_k}$ is a discrete measure on S^{n-1} that is not concentrated on a closed hemisphere, $P_i \in \mathcal{P}(u_1, \dots, u_N)$ and P_i converges to a polytope P , then $\lim_{i \rightarrow \infty} \xi(P_i) = \xi(P)$ and*

$$\lim_{i \rightarrow \infty} \Phi_{P_i}(\xi(P_i)) = \Phi_P(\xi(P)).$$

Proof. Since $\xi(P) \in \text{Int}(P)$ by Lemma 3.1, we have

$$\liminf_{i \rightarrow \infty} \Phi_{P_i}(\xi(P_i)) \geq \liminf_{i \rightarrow \infty} \Phi_{P_i}(\xi(P)) = \Phi_P(\xi(P)).$$

Let z be any accumulation point of the sequence $\{\xi(P_i)\}$; namely, the limit of a subsequence $\{\xi(P_{i'})\}$. Since $\Phi_{P_i}(\xi(P_i))$ is bounded from below, and $h(P, u_k) - \xi(P_i) \cdot u_k$ is bounded from above for $k = 1, \dots, N$, it follows that

$$\liminf_{i \rightarrow \infty} (h(P, u_k) - \xi(P_i) \cdot u_k) = \liminf_{i \rightarrow \infty} (h(P_i, u_k) - \xi(P_i) \cdot u_k) > 0$$

for $k = 1, \dots, N$, and hence $z \in \text{Int}(P)$. We deduce that

$$\Phi_P(z) = \lim_{i' \rightarrow \infty} \Phi_P(\xi(P_{i'})) = \lim_{i' \rightarrow \infty} \Phi_{P_{i'}}(\xi(P_{i'})) \geq \liminf_{i \rightarrow \infty} \Phi_{P_i}(\xi(P_i)) \geq \Phi_P(\xi(P)).$$

Therefore Lemma 3.1 yields $z = \xi(P)$. □

The following lemma will be needed, as well.

Lemma 3.3. *Suppose $\mu = \sum_{k=1}^N \gamma_k \delta_{u_k}$ is a discrete measure on S^{n-1} that is not concentrated on a closed hemisphere, $P \in \mathcal{P}(u_1, \dots, u_N)$, then*

$$\sum_{k=1}^N \gamma_k \frac{u_k}{h(P, u_k) - \xi(P) \cdot u_k} = 0.$$

Proof. We may assume that $\xi(P)$ is the origin because for $x, \xi \in \text{Int} P$, we have $\Phi_{P-x}(\xi - x) = \Phi_P(\xi)$. Since $\Phi_P(\xi)$ attains its maximum at the origin that is an interior point of P , differentiation gives the desired equation. □

Lemma 3.4. *Suppose $\mu = \sum_{k=1}^N \gamma_k \delta_{u_k}$ is a discrete measure on S^{n-1} that is not concentrated on a closed hemisphere, and there exists a $P \in \mathcal{P}_N(u_1, \dots, u_N)$ with $\xi(P) = 0$, $V(P) = |\mu|$ such that*

$$\Phi_P(0) = \inf \left\{ \max_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = |\mu| \right\}.$$

Then,

$$V_P = \sum_{k=1}^N \gamma_k \delta_{u_k}.$$

Proof. According to Equation (3.3), it is sufficient to establish the lemma under the assumption that $|\mu| = 1$.

From the conditions, there exists a polytope $P \in \mathcal{P}_N(u_1, \dots, u_N)$ with $\xi(P)$ is the origin and $V(P) = 1$ such that

$$\Phi_P(o) = \inf \left\{ \max_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = 1 \right\}.$$

For $\tau_1, \dots, \tau_N \in \mathbb{R}$, choose $|t|$ small enough so that the polytope

$$P_t = \bigcap_{i=1}^N \{x : x \cdot u_i \leq h(P, u_i) + t\tau_i\} \in \mathcal{P}_N(u_1, \dots, u_N).$$

In particular, $h(P_t, u_i) = h(P, u_i) + t\tau_i$ for $i = 1, \dots, n$, and Lemma 7.5.3 in Schneider [61] yields that

$$\frac{\partial V(P_t)}{\partial t} = \sum_{i=1}^N \tau_i |F(P_t, u_i)|.$$

Let $\lambda(t) = V(P_t)^{-\frac{1}{n}}$. Then $\lambda(t)P_t \in \mathcal{P}_N(u_1, \dots, u_N)$, $V(\lambda(t)P_t) = 1$, $\lambda(t)$ is C^1 and

$$(3.5) \quad \lambda'(0) = -\frac{1}{n} \sum_{i=1}^N \tau_i |F(P, u_i)|.$$

Define $\xi(t) := \xi(\lambda(t)P_t)$, and

$$(3.6) \quad \begin{aligned} \Phi(t) &:= \max_{\xi \in \lambda(t)P_t} \int_{S^{n-1}} \log(h(\lambda(t)P_t, u) - \xi \cdot u) d\mu(u) \\ &= \sum_{k=1}^N \gamma_k \log(\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k). \end{aligned}$$

It follows from Lemma 3.3, that

$$(3.7) \quad \sum_{k=1}^N \gamma_k \frac{u_{k,i}}{\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k} = 0$$

for $i = 1, \dots, n$, where $u_k = (u_{k,1}, \dots, u_{k,n})^T$. In addition, since $\xi(P)$ is the origin, we have

$$(3.8) \quad \sum_{k=1}^N \gamma_k \frac{u_k}{h(P, u_k)} = 0.$$

Let $F = (F_1, \dots, F_n)$ be a function from a small neighbourhood of the origin in \mathbb{R}^{n+1} to \mathbb{R}^n such that

$$F_i(t, \xi_1, \dots, \xi_n) = \sum_{k=1}^N \gamma_k \frac{u_{k,i}}{\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + \dots + \xi_n u_{k,n})}$$

for $i = 1, \dots, n$. Then,

$$\begin{aligned} \frac{\partial F_i}{\partial t} \Big|_{(t, \xi_1, \dots, \xi_n)} &= \sum_{k=1}^N \gamma_k \frac{-u_{k,i}(\lambda'(t)h(P_t, u_k) + \lambda(t)\tau_k)}{[\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + \dots + \xi_n u_{k,n})]^2} \\ \frac{\partial F_i}{\partial \xi_j} \Big|_{(t, \xi_1, \dots, \xi_n)} &= \sum_{k=1}^N \gamma_k \frac{u_{k,i}u_{k,j}}{[\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + \dots + \xi_n u_{k,n})]^2} \end{aligned}$$

are continuous on a small neighborhood of $(0, 0, \dots, 0)$ with

$$\left(\frac{\partial F}{\partial \xi} \Big|_{(0, \dots, 0)} \right)_{n \times n} = \sum_{k=1}^N \frac{\gamma_k}{h(P, u_k)^2} u_k u_k^T,$$

where $u_k u_k^T$ is an $n \times n$ matrix. Since the unit vectors u_1, \dots, u_N are not concentrated on a closed hemisphere, $\mathbb{R}^n = \text{lin}\{u_1, \dots, u_N\}$. Thus, for any $x \in \mathbb{R}^n$ with $x \neq 0$, there exists a $u_{i_0} \in \{u_1, \dots, u_N\}$ such that $u_{i_0} \cdot x \neq 0$. Then,

$$\begin{aligned} x^T \left(\sum_{k=1}^N \frac{\gamma_k}{h(P, u_k)^2} u_k u_k^T \right) x &= \sum_{k=1}^N \frac{\gamma_k}{h(P, u_k)^2} (x \cdot u_k)^2 \\ &\geq \frac{\gamma_{i_0}}{h(P, u_{i_0})^2} (x \cdot u_{i_0})^2 > 0. \end{aligned}$$

Therefore, $(\frac{\partial F}{\partial \xi} \Big|_{(0, \dots, 0)})$ is positive definite. By this, the fact that $F_i(0, \dots, 0) = 0$ for $i = 1, \dots, n$, the fact that $\frac{\partial F_i}{\partial \xi_j}$ is continuous on a neighborhood of $(0, 0, \dots, 0)$ for all $1 \leq i, j \leq n$ and the implicit function theorem, we have

$$\xi'(0) = (\xi'_1(0), \dots, \xi'_n(0))$$

exists.

From the fact that $\Phi(0)$ is a minimizer of $\Phi(t)$ (in Equation (3.6)), Equation (3.5), the fact that $\sum_{k=1}^N \gamma_k = 1$ and Equation (3.8), we have

$$\begin{aligned} 0 &= \Phi'(0) \\ &= \sum_{k=1}^N \gamma_k \frac{\lambda'(0)h(P, u_k) + \lambda(0) \frac{dh(P, u_k)}{dt} \Big|_{t=0} - \xi'(0) \cdot u_k}{h(P, u_k)} \\ &= \sum_{k=1}^N \gamma_k \frac{-\frac{1}{n}(\sum_{i=1}^N \tau_i |F(P, u_i)|)h(P, u_k) + \tau_k - \xi'(0) \cdot u_k}{h(P, u_k)} \\ &= -\sum_{i=1}^N \frac{|F(P, u_i)|\tau_i}{n} + \sum_{k=1}^N \frac{\gamma_k \tau_k}{h(P, u_k)} - \xi'(0) \cdot \left[\sum_{k=1}^N \gamma_k \frac{u_k}{h(P, u_k)} \right] \\ &= \sum_{k=1}^N \left(\frac{\gamma_k}{h(P, u_k)} - \frac{|F(P, u_k)|}{n} \right) \tau_k. \end{aligned}$$

Since τ_1, \dots, τ_N are arbitrary, we deduce that $\gamma_k = \frac{1}{n}h(P, u_k)|F(P, u_k)|$ for $k = 1, \dots, N$. \square

4. EXISTENCE OF A SOLUTION OF THE EXTREMAL PROBLEM

In this section, we prove Lemma 4.7 about the existence of a solution of problem (3.2) for the case where the discrete measure is not concentrated on any closed hemisphere of S^{n-1} and satisfies the strict essential subspace concentration inequality. Having the results of the previous section, the essential new ingredient is the following statement (see Lemma 4.5).

If μ is a discrete measure on S^{n-1} that is not concentrated on any closed hemisphere of S^{n-1} and satisfies the strict essential subspace concentration inequality, and $\{P_m\}$ is a sequence of polytopes of unit volume such that the set of outer unit normals of P_m is a subset of the support of μ , and $\lim_{m \rightarrow \infty} d(P_m) = \infty$ then

$$\lim_{m \rightarrow \infty} \Phi_{P_m}(\xi(P_m)) = \infty.$$

It is equivalent to prove that any subsequence of $\{P_m\}$ has some subsequence $\{P_{m'}\}$ such that $\lim_{m \rightarrow \infty} \Phi_{P_{m'}}(\xi(P_{m'})) = \infty$.

To indicate the idea, we sketch the argument for $n = 2$. Let $\text{supp } \mu = \{u_1, \dots, u_N\}$, and let $w_m = \min\{h_{P_m}(u) + h_{P_m}(-u) : u \in S^1\}$ be the minimal width of P_m . Since $\lim_{m \rightarrow \infty} d(P_m) = \infty$ and $V(P_m) = 1$, we have $\lim_{m \rightarrow \infty} w_m = 0$. As P_m is a polygon, we may assume that $w_m = h_{P_m}(u_1) + h_{P_m}(-u_1)$ possibly after taking a subsequence and reindexing. If the angle of u_1 and u_i is $\alpha_i \in (0, \pi)$ then $V_1(F(P_m, u_i)) \leq w_m / \sin \alpha_i$, thus $\lim_{m \rightarrow \infty} d(P_m) = \infty$ implies that $-u_1 \in \text{supp } \mu$ for large m , say $u_2 = -u_1$. Let $v \in S^1$ be orthogonal to u_1 , and let $\gamma_i = \mu(\{u_i\})$ for $i = 1, \dots, N$. We may translate P_m in a way such that $o \in \text{Int } P_m$ in a way such that $h_{P_m}(u_1) = h_{P_m}(u_2) = w_m/2$, and $h_{P_m}(v) = h_{P_m}(-v)$ hold for large m . Thus $V(P_m) = 1$ yields the existence of a constant $c_1 > 0$ such that $h_{P_m}(u_i) > c_1/w_m$ for $i = 3, \dots, N$. Now $\text{lin } u_1$ is an essential subspace with respect to μ , and hence $\gamma_1 + \gamma_2 < \gamma_3 + \dots + \gamma_N$ according to the strict essential subspace concentration inequality. Therefore writing $c_2 = \min\{2, c_1\}$, we have

$$\begin{aligned} \liminf_{m \rightarrow \infty} \exp(\Phi_{P_m}(\xi(P_m))) &\geq \liminf_{m \rightarrow \infty} \exp(\Phi_{P_m}(o)) = \liminf_{m \rightarrow \infty} \prod_{i=1}^N h_{P_m}(u_i)^{\gamma_i} \\ &\geq \lim_{m \rightarrow \infty} \left(\frac{w_m}{2}\right)^{\gamma_1 + \gamma_2} \left(\frac{c_1}{w_m}\right)^{\gamma_3 + \dots + \gamma_N} \geq \lim_{m \rightarrow \infty} \left(\frac{c_2}{w_m}\right)^{\gamma_3 + \dots + \gamma_N - \gamma_1 - \gamma_2} = \infty. \end{aligned}$$

In the higher dimensional case, the idea is the very same. Only instead of one essential linear subspace like in the planar case, we will find essential subspaces $X_0 \subset \dots \subset X_{q-1}$ in a way such that for $j = 0, \dots, q-1$, $P_m|_{X_j^\perp}$ is "much larger" than $P_m|_{X_j}$ for large m after taking suitable subsequence. This is achieved in the preparatory statements Lemmas 4.1 to 4.4.

Given N sequences, the first two observations will help to do book keeping of how the limits of the sequences compare.

Lemma 4.1. *Let $\{h_{1j}\}_{j=1}^\infty, \dots, \{h_{Nj}\}_{j=1}^\infty$ be N ($N \geq 2$) sequences of real numbers. Then, there exists a subsequence, $\{j_n\}_{n=1}^\infty$, of \mathbb{N} and a rearrangement, i_1, \dots, i_N , of $1, \dots, N$ such that*

$$h_{i_1 j_n} \leq h_{i_2 j_n} \leq \dots \leq h_{i_N j_n},$$

for all $n \in \mathbb{N}$.

Proof. We prove it by induction on N . We first prove the case for $N = 2$. For $j \in \mathbb{N}$, consider the sequence

$$h_j = \max\{h_{1j}, h_{2j}\}.$$

Since $\{h_j\}_{j=1}^\infty$ is an infinite sequence and h_j either equals to h_{1j} or equals to h_{2j} for all $j \in \mathbb{N}$, there exists an $i_2 \in \{1, 2\}$ and a subsequence, $\{j_n\}_{n=1}^\infty$, of \mathbb{N} such that

$$h_{j_n} = h_{i_2 j_n}$$

for all $n \in \mathbb{N}$. Let $i_1 \in \{1, 2\}$ with $i_1 \neq i_2$. Then,

$$h_{i_1 j_n} \leq h_{i_2 j_n},$$

for all $n \in \mathbb{N}$.

Suppose the lemma is true for $N = k$ (with $k \geq 2$), we next prove that the lemma is true for $N = k + 1$. For $j \in \mathbb{N}$, consider the sequence

$$h_j = \max\{h_{1j}, h_{2j}, \dots, h_{k+1j}\}.$$

Since $\{h_j\}_{j=1}^\infty$ is an infinite sequence and h_j equals one of $h_{1j}, h_{2j}, \dots, h_{k+1j}$ for all $j \in \mathbb{N}$, there exists an $i_{k+1} \in \{1, 2, \dots, k+1\}$ and a subsequence, $\{j_n\}_{n=1}^\infty$, of \mathbb{N} such that

$$h_{j_n} = h_{i_{k+1} j_n}$$

for all $n \in \mathbb{N}$.

Consider the sequences $\{h_{ij_n}\}_{n=1}^\infty$ ($1 \leq i \leq k+1$ with $i \neq i_{k+1}$). By the inductive hypothesis, there exists a subsequence, j_{n_l} , of j_n and a rearrangement, i_1, \dots, i_k , of $1, \dots, \widehat{i_{k+1}}, \dots, k+1$ such that

$$h_{i_1 j_{n_l}} \leq h_{i_2 j_{n_l}} \leq \dots \leq h_{i_k j_{n_l}}$$

for all $l \in \mathbb{N}$. By this and the fact that $h_{j_{n_l}} = h_{i_{k+1} j_{n_l}}$ for all $l \in \mathbb{N}$, we have

$$h_{i_1 j_{n_l}} \leq h_{i_2 j_{n_l}} \leq \dots \leq h_{i_k j_{n_l}} \leq h_{i_{k+1} j_{n_l}}$$

for all $l \in \mathbb{N}$. □

Lemma 4.2. *Let $\{h_{1j}\}_{j=1}^\infty, \dots, \{h_{Nj}\}_{j=1}^\infty$ be N ($N \geq 2$) sequences of real numbers with*

$$h_{1j} \leq h_{2j} \leq \dots \leq h_{Nj}$$

for all $j \in \mathbb{N}$, $\lim_{j \rightarrow \infty} h_{1j} = 0$ and $\lim_{j \rightarrow \infty} h_{Nj} = \infty$. Then, there exist $q \geq 1$,

$$1 = \alpha_0 < \alpha_1 < \dots < \alpha_q \leq N < N+1 = \alpha_{q+1}$$

and a subsequence, $\{j_n\}_{n=1}^\infty$, of \mathbb{N} such that if $i = 1, \dots, q$, then

$$\lim_{n \rightarrow \infty} \frac{h_{\alpha_i j_n}}{h_{\alpha_{i-1} j_n}} = \infty,$$

if $i = 0, \dots, q$, and $\alpha_i \leq k \leq \alpha_{i+1} - 1$, then

$$\lim_{n \rightarrow \infty} \frac{h_{kj_n}}{h_{\alpha_i j_n}}$$

exists and equals to a positive number.

Proof. Let $\alpha_0 = 1$. By conditions,

$$\frac{h_{1j}}{h_{1j}} \leq \frac{h_{2j}}{h_{1j}} \leq \dots \leq \frac{h_{Nj}}{h_{1j}},$$

$\overline{\lim}_{j \rightarrow \infty} \frac{h_{ij}}{h_{1j}}$ either exists (equals to a positive number) or goes to ∞ , and $\overline{\lim}_{j \rightarrow \infty} \frac{h_{Nj}}{h_{1j}} = \infty$. Thus, there exists an α_1 ($1 < \alpha_1 \leq N$) such that for $1 \leq i \leq \alpha_1 - 1$,

$$\overline{\lim}_{j \rightarrow \infty} \frac{h_{ij}}{h_{1j}} < \infty$$

and

$$\overline{\lim}_{j \rightarrow \infty} \frac{h_{\alpha_1 j}}{h_{1j}} = \infty.$$

Hence, we can choose a subsequence, $\{j'_n\}_{n=1}^\infty$, of \mathbb{N} such that

$$\lim_{n \rightarrow \infty} \frac{h_{\alpha_1 j'_n}}{h_{1j'_n}} = \infty,$$

and for $1 \leq i \leq \alpha_1 - 1$,

$$\overline{\lim}_{n \rightarrow \infty} \frac{h_{ij'_n}}{h_{1j'_n}} \leq \overline{\lim}_{j \rightarrow \infty} \frac{h_{ij}}{h_{1j}} < \infty.$$

By choosing $\alpha_1 - 2$ times subsequences of j'_n , we can find a subsequence, $\{j''_n\}_{n=1}^\infty$, of $\{j'_n\}_{n=1}^\infty$ such that

$$\lim_{n \rightarrow \infty} \frac{h_{\alpha_1 j''_n}}{h_{1j''_n}} = \infty,$$

and for $1 \leq i \leq \alpha_1 - 1$,

$$\lim_{n \rightarrow \infty} \frac{h_{ij''_n}}{h_{1j''_n}}$$

exists and equals to a positive number.

By repeating (at most $N - \alpha_1$ times) similar arguments for the sequences $\{h_{ij_n}\}_{n=1}^\infty$ ($\alpha_1 \leq i \leq N$), we can find $q \geq 1$,

$$1 = \alpha_0 < \alpha_1 < \dots < \alpha_q \leq N < N + 1 = \alpha_{q+1}$$

and a subset, $\{j_n\}_{n=1}^\infty$, of \mathbb{N} that satisfy the conditions in the lemma. \square

The following lemma compares positive hull and linear hull.

Lemma 4.3. *Suppose $u_1, \dots, u_l \in S^{d-1}$ ($d \geq 2$), $\mathbb{R}^d = \text{lin}\{u_1, \dots, u_l\}$, and u_1, \dots, u_l are not concentrated on a closed hemisphere of S^{d-1} , then*

$$\mathbb{R}^d = \text{pos}\{u_1, \dots, u_l\}.$$

Moreover, there exists $\lambda > 0$ depending on u_1, \dots, u_l such that any $u \in S^{d-1}$ can be written in the form

$$u = a_{i_1}u_{i_1} + \dots + a_{i_d}u_{i_d}$$

where $\{u_{i_1}, \dots, u_{i_d}\} \subset \{u_1, \dots, u_l\}$ and $0 \leq a_{i_1}, \dots, a_{i_d} \leq \lambda$.

Proof. Let Q be the convex hull of $\{u_1, \dots, u_l\}$, which is a polytope. Since u_1, \dots, u_l are not concentrated on a closed hemisphere of S^{d-1} , the origin is an interior point of Q . In particular, $rB^d \subset Q$ for some $r > 0$.

For $u \in S^{d-1}$, there exists some $t \geq r$ such that $tu \in \partial Q$. It follows that $tu \in F$ for some facet F of Q . We deduce from the Charateodory theorem that there exists vertices u_{i_1}, \dots, u_{i_d} of F that tu lies in their convex hull. In other words,

$$tu = \alpha_{i_1}u_{i_1} + \dots + \alpha_{i_d}u_{i_d}$$

where $\alpha_{i_1}, \dots, \alpha_{i_d} \geq 0$ and $\alpha_{i_1} + \dots + \alpha_{i_d} = 1$. Therefore we choose $a_{i_j} = \alpha_{i_j}/t \leq 1/r$ for $j = 1, \dots, d$, which in turn satisfy $u = a_{i_1}u_{i_1} + \dots + a_{i_d}u_{i_d}$. In particular, we may take $\lambda = 1/r$. \square

The following lemma will be the last preparatory statement.

Lemma 4.4. *Suppose μ is a discrete measure on S^{n-1} that is not concentrated on any closed hemisphere of S^{n-1} with $\text{supp}(\mu) = \{u_1, \dots, u_N\}$ and $\mu(u_i) = \gamma_i$ for $i = 1, \dots, N$. If P_m is a sequence of polytopes with $V(P_m) = 1$, $\xi(P_m)$ is the origin, the set of outer unit normals of P_m is a subset of $\{u_1, \dots, u_N\}$, $\lim_{m \rightarrow \infty} d(P_m) = \infty$ and*

$$h(P_m, u_1) \leq h(P_m, u_2) \leq \dots \leq h(P_m, u_N)$$

for all $m \in \mathbb{N}$. Then, there exist $q \geq 1$, and $1 = \alpha_0 < \alpha_1 < \dots < \alpha_q \leq N < N + 1 = \alpha_{q+1}$ such that if $j = 1, \dots, q$, then

$$(4.0a) \quad \lim_{m \rightarrow \infty} \frac{h(P_m, u_{\alpha_j})}{h(P_m, u_{\alpha_{j-1}})} = \infty,$$

and if $j = 0, \dots, q$ and $\alpha_j \leq k \leq \alpha_{j+1} - 1$, then

$$(4.0b) \quad \lim_{m \rightarrow \infty} \frac{h(P_m, u_k)}{h(P_m, u_{\alpha_j})} = t_{kj} < \infty.$$

Moreover, $X_j = \text{pos}\{u_1, \dots, u_{\alpha_{j+1}-1}\}$ are subspaces of \mathbb{R}^n for all $0 \leq j \leq q$ and

$$1 \leq \dim(X_0) < \dim(X_1) < \dots < \dim(X_q) = n.$$

Proof. By the conditions that $\lim_{m \rightarrow \infty} d(P_m) = \infty$, $V(K) = 1$ and $h(P_m, u_1) \leq h(P_m, u_2) \leq \dots \leq h(P_m, u_N)$ for all $m \in \mathbb{N}$, we have,

$$\lim_{m \rightarrow \infty} h(P_m, u_1) = 0 \text{ and } \lim_{m \rightarrow \infty} h(P_m, u_N) = \infty.$$

From Lemma 4.2, we may assume that there exist $q \geq 1$, and

$$1 = \alpha_0 < \alpha_1 < \dots < \alpha_q \leq N < N + 1 = \alpha_{q+1}$$

that satisfy Equations (4.0a) and (4.0b).

For $j = 0, \dots, q - 1$, we consider the cone

$$\Sigma_j = \text{pos}\{u_1, \dots, u_{\alpha_{j+1}-1}\},$$

and its negative polar

$$\Sigma_j^* = \{v \in \mathbb{R}^n : v \cdot u_i \leq 0 \text{ for all } i = 1, \dots, \alpha_{j+1} - 1\}.$$

Let $0 \leq j \leq q - 1$, $1 \leq p \leq \alpha_{j+1} - 1$ and $v \in \Sigma_j^* \cap S^{n-1}$. From the condition that $\xi(P_m)$ is the origin and Lemma 3.3,

$$\sum_{i=1}^N \frac{\gamma_i(v \cdot u_i)}{h(P_m, u_i)} = 0.$$

By this and the fact that $v \in \Sigma_j^* \cap S^{n-1}$,

$$\begin{aligned} 0 &\geq \gamma_p(v \cdot u_p) = - \sum_{i \neq p} \frac{h(P_m, u_p)}{h(P_m, u_i)} \gamma_i(v \cdot u_i) \\ &\geq - \sum_{i \geq \alpha_{j+1}} \frac{h(P_m, u_p)}{h(P_m, u_i)} \gamma_i(v \cdot u_i) \\ &\geq - \sum_{i \geq \alpha_{j+1}} \frac{h(P_m, u_p)}{h(P_m, u_i)} \gamma_i. \end{aligned}$$

By this, (4.0a) and (4.0b), we have, $\gamma_p(v \cdot u_p)$ is no bigger than 0, and no less than any negative number. Thus,

$$v \cdot u_p = 0$$

for all $p = 1, \dots, \alpha_{j+1} - 1$ and $v \in \Sigma_j^* \cap S^{n-1}$. Then, for any $u \in \text{lin}\{u_1, \dots, u_{\alpha_{j+1}-1}\}$ and $v \in \Sigma_j^*$, $u \cdot v = 0$. Hence,

$$\Sigma_j^* \cap \text{lin}\{u_1, \dots, u_{\alpha_{j+1}-1}\} = \{0\}.$$

We claim that $\{u_1, \dots, u_{\alpha_{j+1}-1}\}$ is not concentrated on a closed hemisphere of $S^{n-1} \cap \text{lin}\{u_1, \dots, u_{\alpha_{j+1}-1}\}$. Otherwise, there exists a vector $u_0 \in \text{lin}\{u_1, \dots, u_{\alpha_{j+1}-1}\}$ such that $u_0 \neq 0$ and $u_0 \cdot u_p \leq 0$ for all $p = 1, \dots, \alpha_{j+1} - 1$. This contradicts the fact that $\Sigma_j^* \cap \text{lin}\{u_1, \dots, u_{\alpha_{j+1}-1}\} = \{0\}$. Hence, $\{u_1, \dots, u_{\alpha_{j+1}-1}\}$ is not concentrated on a closed hemisphere of $S^{n-1} \cap \text{lin}\{u_1, \dots, u_{\alpha_{j+1}-1}\}$. By Lemma 4.3,

$$\text{lin}\{u_1, \dots, u_{\alpha_{j+1}-1}\} = \text{pos}\{u_1, \dots, u_{\alpha_{j+1}-1}\}.$$

Let $X_j = \text{pos}\{u_1, \dots, u_{\alpha_{j+1}-1}\}$, $d_j = \dim X_j$ for $j = 0, \dots, q$, and $d_{-1} = 0$. Obviously, $d_0 \geq 1$ and $d_q = n$. We claim that $d_0 < d_1 < \dots < d_q$. Otherwise, there exist $0 \leq k < l \leq q$ such that $d_k = d_l$, and thus $X_k = X_l$. We write $\lambda > 0$ for the constant of Lemma 4.3 depending on u_1, \dots, u_N . By Lemma 4.3, there exist $u_{i_1}, \dots, u_{i_{d_k}} \in \{u_1, \dots, u_{\alpha_{k+1}-1}\}$ and $0 \leq a_{i_1}, \dots, a_{i_{d_k}} \leq \lambda$ such that

$$u_{\alpha_l} = a_{i_1} u_{i_1} + \dots + a_{i_{d_k}} u_{i_{d_k}}.$$

Hence,

$$\begin{aligned} h(P_m, u_{\alpha_l}) &= h(P_m, a_{i_1}u_{i_1} + \dots + a_{i_{d_k}}u_{i_{d_k}}) \\ &\leq a_{i_1}h(P_m, u_{i_1}) + \dots + a_{i_{d_k}}h(P_m, u_{i_{d_k}}), \end{aligned}$$

for all $m \in \mathbb{N}$. But this contradicts (4.0a) and (4.0b). Therefore,

$$1 \leq d_0 < d_1 < \dots < d_q = n.$$

□

Lemma 4.5. *Suppose μ is a discrete measure on S^{n-1} that is not concentrated on any closed hemisphere of S^{n-1} , and satisfies the strict essential subspace concentration inequality. If P_m is a sequence of polytopes with $V(P_m) = 1$, $\xi(P_m)$ is the origin, the set of outer unit normals of P_m is a subset of the support of μ and $\lim_{m \rightarrow \infty} d(P_m) = \infty$, then*

$$\int_{S^{n-1}} \log h(P_m, u) d\mu(u)$$

is not bounded from above.

Proof. Without loss of generality, we can suppose $|\mu| = 1$. Let $\text{supp}(\mu) = \{u_1, \dots, u_N\}$, and $\mu(\{u_i\}) = \gamma_i, i = 1, \dots, N$. From Lemma 4.1, we may assume that

$$(4.1) \quad h(P_m, u_1) \leq \dots \leq h(P_m, u_N),$$

for all $m \in \mathbb{N}$. Since $\lim_{m \rightarrow \infty} d(P_m) = \infty$ and $V(K) = 1$,

$$\lim_{m \rightarrow \infty} h(P_m, u_1) = 0 \quad \text{and} \quad \lim_{m \rightarrow \infty} h(P_m, u_N) = \infty.$$

By Lemma 4.4, there exist $q \geq 1$, and

$$1 = \alpha_0 < \alpha_1 < \dots < \alpha_q \leq N < N + 1 = \alpha_{q+1}$$

such that if $j = 1, \dots, q$, then

$$(4.2a) \quad \lim_{m \rightarrow \infty} \frac{h(P_m, u_{\alpha_j})}{h(P_m, u_{\alpha_{j-1}})} = \infty,$$

and if $j = 0, \dots, q$ and $\alpha_j \leq k \leq \alpha_{j+1} - 1$, then

$$(4.2b) \quad \lim_{m \rightarrow \infty} \frac{h(P_m, u_k)}{h(P_m, u_{\alpha_j})} = t_{k,j} < \infty.$$

Moreover, $X_j = \text{pos}\{u_1, \dots, u_{\alpha_{j+1}-1}\}$ are subspaces of \mathbb{R}^n with respect to μ for all $0 \leq j \leq q$ with

$$1 \leq d_0 < d_1 < \dots < d_q = n,$$

where $d_j = \dim(X_j)$. In particular, X_0, \dots, X_{q-1} are essential subspaces.

Let $\tilde{X}_0 = X_0$, and if $j = 1, \dots, q$, then let

$$\tilde{X}_j = X_{j-1}^\perp \cap X_j.$$

From the definition of X_j and \tilde{X}_j , we have, $\tilde{X}_{j_1} \perp \tilde{X}_{j_2}$ for $j_1 \neq j_2$, $\dim \tilde{X}_j = d_j - d_{j-1} > 0$ for $j = 0, \dots, q$, and \mathbb{R}^n is a direct sum of $\tilde{X}_0, \dots, \tilde{X}_q$.

Let $\lambda > 0$ be the constant of Lemma 4.3 for u_1, \dots, u_N . Suppose $0 \leq j \leq q$ and $u \in X_j \cap S^{n-1}$. By Lemma 4.3, there exists a subset, $\{u_{i_1}, \dots, u_{i_{d_j}}\}$, of $\{u_1, \dots, u_{\alpha_{j+1}-1}\}$ and $0 \leq a_{i_1}, \dots, a_{i_{d_j}} \leq \lambda$ such that

$$u = a_{i_1}u_{i_1} + \dots + a_{i_{d_j}}u_{i_{d_j}}.$$

Then,

$$\begin{aligned} h(P_m, u) &= h(P_m, a_{i_1} u_{i_1} + \dots + a_{i_{d_j}} u_{i_{d_j}}) \\ &\leq a_{i_1} h(P_m, u_{i_1}) + \dots + a_{i_{d_j}} h(P_m, u_{i_{d_j}}). \end{aligned}$$

By this, (4.2a) and (4.2b), if m is large, then

$$h(P_m, u) \leq t_j h(P_m, u_{\alpha_j}) \text{ for all } u \in X_j \cap S^{n-1}$$

where $t_j = d_j \lambda(t_{\alpha_{j+1}-1, j} + 1) > 0$. Hence, for $j = 0, \dots, q$,

$$P_m|_{\tilde{X}_j} \subset t_j h(P_m, u_{\alpha_j})(B^n \cap \tilde{X}_j).$$

By this and the fact that \mathbb{R}^n is a direct sum of $\tilde{X}_0, \dots, \tilde{X}_q$,

$$P_m \subset \sum_{j=0}^q t_j h(P_m, u_{\alpha_j})(B^n \cap \tilde{X}_j),$$

where the summation is Minkowski sum. Let

$$\omega = \max_{0 \leq j \leq q} t_j \kappa_{d_j - d_{j-1}}^{\frac{1}{d_j - d_{j-1}}},$$

where $\kappa_{d_j - d_{j-1}}$ is the volume of the $(d_j - d_{j-1})$ -dimensional unit ball. Then, for $j = 0, \dots, q$

$$V_{d_j - d_{j-1}} \left(t_j h(P_m, u_{\alpha_j})(B^n \cap \tilde{X}_j) \right) \leq (\omega h(P_m, u_{\alpha_j}))^{d_j - d_{j-1}}.$$

From this, the fact that \mathbb{R}^n is a direct sum of $\tilde{X}_0, \dots, \tilde{X}_q$, and Fubini's formula, we have

$$\begin{aligned} 1 &= V(P_m) \\ &\leq V \left(\sum_{j=0}^q t_j h(P_m, u_{\alpha_j})(B^n \cap \tilde{X}_j) \right) \\ &= \prod_{j=0}^q V_{d_j - d_{j-1}} \left(t_j h(P_m, u_{\alpha_j})(B^n \cap \tilde{X}_j) \right) \\ &\leq \prod_{j=0}^q (\omega h(P_m, u_{\alpha_j}))^{d_j - d_{j-1}}. \end{aligned}$$

It follows from $0 = d_{-1} < d_0 < \dots < d_q = n$ that if m is large, then

$$\sum_{j=0}^q \left(\frac{d_j}{n} - \frac{d_{j-1}}{n} \right) \log h(P_m, u_{\alpha_j}) \geq -\log \omega.$$

We rewrite the last inequality as

$$(4.3) \quad \log h(P_m, u_{\alpha_q}) \geq - \sum_{j=0}^{q-1} \frac{d_j}{n} \log \frac{h(P_m, u_{\alpha_j})}{h(P_m, u_{\alpha_{j+1}})} - \log \omega.$$

For $j = 0, \dots, q$, we set $\beta_j = \mu(X_j \cap S^{n-1}) = \sum_{i=1}^{\alpha_{j+1}-1} \gamma_i$, and $\beta_{-1} = 0$. We deduce from the facts that X_j is an essential subspace with $d_j = \dim(X_j)$, and from the condition that μ satisfies the strict essential subspace concentration condition that

$$(4.4) \quad \beta_j < \frac{d_j}{n} \quad \text{for } 0 \leq j \leq q-1.$$

By the fact that $h(P_m, u_1) \leq h(P_m, u_2) \leq \dots \leq h(P_m, u_N)$, the fact that $\beta_q = 1$ and (4.3),

$$\begin{aligned}
\sum_{i=1}^N \gamma_i \log h(P_m, u_i) &= \sum_{i=1}^{\alpha_1-1} \gamma_i \log h(P_m, u_i) + \sum_{i=\alpha_1}^{\alpha_2-1} \gamma_i \log h(P_m, u_i) + \dots + \sum_{i=\alpha_q}^N \gamma_i \log h(P_m, u_i) \\
&\geq \sum_{i=1}^{\alpha_1-1} \gamma_i \log h(P_m, u_{\alpha_0}) + \sum_{i=\alpha_1}^{\alpha_2-1} \gamma_i \log h(P_m, u_{\alpha_1}) + \dots + \sum_{i=\alpha_q}^N \gamma_i \log h(P_m, u_{\alpha_q}) \\
&= \sum_{j=0}^q (\beta_j - \beta_{j-1}) \log h(P_m, u_{\alpha_j}) \\
&= \log h(P_m, u_{\alpha_q}) + \sum_{j=0}^{q-1} \beta_j \log \frac{h(P_m, u_{\alpha_j})}{h(P_m, u_{\alpha_{j+1}})} \\
&\geq -\log \omega + \sum_{j=0}^{q-1} \left(\beta_j - \frac{d_j}{n} \right) \log \frac{h(P_m, u_{\alpha_j})}{h(P_m, u_{\alpha_{j+1}})}.
\end{aligned}$$

It follows from (4.1), (4.2a), (4.4) that for $j = 0, \dots, q-1$,

$$\lim_{m \rightarrow \infty} \left(\beta_j - \frac{d_j}{n} \right) \log \frac{h(P_m, u_{\alpha_j})}{h(P_m, u_{\alpha_{j+1}})} = \infty.$$

Therefore,

$$\lim_{m \rightarrow \infty} \sum_{i=1}^N \gamma_i \log h(P_m, u_i) = \infty.$$

□

The following lemma will be needed (see, [71], Lemma 3.5).

Lemma 4.6. *If P is a polytope in \mathbb{R}^n and $v_0 \in S^{n-1}$ with $V_{n-1}(F(P, v_0)) = 0$, then there exists a $\delta_0 > 0$ such that for $0 \leq \delta < \delta_0$*

$$V(P \cap \{x : x \cdot v_0 \geq h(P, v_0) - \delta\}) = c_n \delta^n + \dots + c_2 \delta^2,$$

where c_n, \dots, c_2 are constants that depend on P and v_0 .

Now, we have prepared enough to prove the main result of this section.

Lemma 4.7. *Suppose the discrete measure $\mu = \sum_{k=1}^N \gamma_k \delta_{u_k}$ is not concentrated on a closed hemisphere. If μ satisfies the strict essential subspace concentration inequality, then there exists a $P \in \mathcal{P}_N(u_1, \dots, u_N)$ such that $\xi(P) = 0$, $V(P) = |\mu|$ and*

$$\Phi_P(0) = \inf \left\{ \max_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = |\mu| \right\},$$

where $\Phi_Q(\xi) = \int_{S^{n-1}} \log(h(Q, u) - \xi \cdot u) d\mu(u)$.

Proof. It is easily seen that it is sufficient to establish the lemma under the assumption that $|\mu| = 1$.

Obviously, for $P, Q \in \mathcal{P}(u_1, \dots, u_N)$, if there exists an $x \in \mathbb{R}^n$ such that $P = Q + x$, then

$$\Phi_P(\xi(P)) = \Phi_Q(\xi(Q)).$$

Thus, we can choose a sequence $P_i \in \mathcal{P}(u_1, \dots, u_N)$ with $\xi(P_i) = 0$ and $V(P_i) = 1$ such that $\Phi_{P_i}(0)$ converges to

$$\inf \left\{ \max_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = 1 \right\}.$$

Choose a fixed $P_0 \in \mathcal{P}(u_1, \dots, u_N)$ with $V(P_0) = 1$, then

$$\inf \left\{ \max_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = 1 \right\} \leq \Phi_{P_0}(\xi(P_0)).$$

We claim that P_i is bounded. Otherwise, from Lemma 4.5, $\Phi_{P_i}(\xi(P_i))$ is not bounded from above. This contradicts the previous inequality. Therefore, P_i is bounded.

From Lemma 3.2 and the Blaschke selection theorem, there exists a subsequence of P_i that converges to a polytope P such that $P \in \mathcal{P}(u_1, \dots, u_N)$, $V(P) = 1$, $\xi(P) = 0$ and

$$(4.5) \quad \Phi_P(0) = \inf \left\{ \max_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = 1 \right\}.$$

We next prove that $F(P, u_i)$ are facets for all $i = 1, \dots, N$. Otherwise, there exists an $i_0 \in \{1, \dots, N\}$ such that

$$F(P, u_{i_0})$$

is not a facet of P .

Choose $\delta > 0$ small enough so that the polytope

$$P_\delta = P \cap \{x : x \cdot u_{i_0} \leq h(P, u_{i_0}) - \delta\} \in \mathcal{P}(u_1, \dots, u_N),$$

and (by Lemma 4.6)

$$V(P_\delta) = 1 - (c_n \delta^n + \dots + c_2 \delta^2),$$

where c_n, \dots, c_2 are constants that depend on P and direction u_{i_0} .

From Lemma 3.2, for any $\delta_i \rightarrow 0$ $\xi(P_{\delta_i}) \rightarrow 0$. We have,

$$\lim_{\delta \rightarrow 0} \xi(P_\delta) = 0.$$

Let δ be small enough so that $h(P, u_k) > \xi(P_\delta) \cdot u_k + \delta$ for all $k \in \{1, \dots, N\}$, and let

$$\lambda = V(P_\delta)^{-\frac{1}{n}} = (1 - (c_n \delta^n + \dots + c_2 \delta^2))^{-\frac{1}{n}}.$$

From this and Equation (3.3), we have

$$\begin{aligned} \prod_{k=1}^N (h(\lambda P_\delta, u_k) - \xi(\lambda P_\delta) \cdot u_k)^{\gamma_k} &= \lambda \prod_{k=1}^N (h(P_\delta, u_k) - \xi(P_\delta) \cdot u_k)^{\gamma_k} \\ &= \lambda \left[\prod_{k=1}^N (h(P, u_k) - \xi(P_\delta) \cdot u_k)^{\gamma_k} \right] \left[\frac{h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta}{h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0}} \right]^{\gamma_{i_0}} \\ &= \left[\prod_{k=1}^N (h(P, u_k) - \xi(P_\delta) \cdot u_k)^{\gamma_k} \right] \frac{(1 - \frac{\delta}{h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0}})^{\gamma_{i_0}}}{(1 - (c_n \delta^n + \dots + c_2 \delta^2))^{\frac{1}{n}}} \\ &\leq \left[\prod_{k=1}^N (h(P, u_k) - \xi(P_\delta) \cdot u_k)^{\gamma_k} \right] \frac{(1 - \frac{\delta}{d_0})^{\gamma_{i_0}}}{(1 - (c_n \delta^n + \dots + c_2 \delta^2))^{\frac{1}{n}}}, \end{aligned}$$

where $d_0 = d(P)$ is the diameter of P . Thus,

$$(4.6) \quad \Phi_{\lambda P_\delta}(\xi(\lambda P_\delta)) \leq \Phi_P(\xi(P_\delta)) + B(\delta),$$

where

$$(4.7) \quad B(\delta) = \gamma_{i_0} \log \left(1 - \frac{\delta}{d_0} \right) - \frac{1}{n} \log (1 - (c_n \delta^n + \dots + c_2 \delta^2)).$$

Obviously,

$$(4.8) \quad B'(\delta) = \gamma_{i_0} \frac{-1/d_0}{1 - \delta/d_0} + \frac{1}{n} \frac{nc_n \delta^{n-1} + \dots + 2c_2 \delta}{1 - (c_n \delta^n + \dots + c_2 \delta^2)} < 0,$$

when the positive δ is small enough. From this and the fact that $B_1(0) = 0$,

$$B(\delta) < 0$$

when the positive δ is small enough.

From this and Equations (4.6), (4.7), (4.8), there exists a $\delta_0 > 0$ such that $P_{\delta_0} \in \mathcal{P}(u_1, \dots, u_N)$ and

$$\Phi_{\lambda_0 P_{\delta_0}}(\xi(\lambda_0 P_{\delta_0})) < \Phi_P(\xi(P_{\delta_0})) \leq \Phi_P(\xi(P)) = \Phi_P(0),$$

where $\lambda_0 = V(P_{\delta_0})^{-\frac{1}{n}}$. Let $P_0 = \lambda_0 P_{\delta_0} - \xi(\lambda_0 P_{\delta_0})$, then $P_0 \in \mathcal{P}(u_1, \dots, u_N)$, $V(P_0) = 1$, $\xi(P_0) = 0$ and

$$\Phi_{P_0}(0) < \Phi_P(0).$$

This contradicts Equation (4.5). Therefore, $P \in \mathcal{P}_N(u_1, \dots, u_N)$. \square

5. EXISTENCE OF THE SOLUTION TO THE DISCRETE LOGARITHMIC MINKOWSKI PROBLEM

If μ is a Borel measure on S^{n-1} and ξ is a proper subspace of \mathbb{R}^n , it will be convenient to write μ_ξ for the restriction of μ to $S^{n-1} \cap \xi$. In this section, we prove the main result Theorem 1.5 of this paper based on the following idea. Let μ be discrete measure on S^{n-1} , $n \geq 2$, that is not concentrated on any closed hemisphere and satisfies the essential subspace concentration condition. If μ satisfies the strict essential subspace concentration inequality, then Lemma 4.7 yields that μ is a cone volume measure. Otherwise there exist complementary proper subspaces ξ and ξ' such that $\text{supp } \mu = S^{n-1} \cap (\xi \cup \xi')$, and μ_ξ and $\mu'_{\xi'}$ are not concentrated on any closed hemisphere of $\xi \cap S^{n-1}$ and $\xi' \cap S^{n-1}$, respectively, and satisfy the essential subspace concentration condition. Therefore μ_ξ and $\mu'_{\xi'}$ are cone volume measures on $\xi \cap S^{n-1}$ and $\xi' \cap S^{n-1}$, respectively, by induction on the dimension of the ambient space, which in turn imply that μ is a cone volume measure.

However, it is possible that $\dim \xi = 1$. Therefore in order to execute the plan, we extend the notions occurring in Theorem 1.5 to \mathbb{R}^1 . The role of a compact convex set containing the origin in its interior is played by some interval $K = [a, b]$ with $a < 0$ and $b > 0$, and closed hemispheres of $S^0 = \{-1, 1\}$ are $\{1\}$ and $\{-1\}$. The cone volume measure on S^0 associated to K satisfies $V_K(\{-1\}) = |a|$ and $V_K(\{1\}) = b$. In addition, we say that a non-trivial measure μ on S^0 satisfies the essential subspace concentration inequality if it is not concentrated on any closed hemisphere; namely, if $\mu(\{-1\}) > 0$ and $\mu(\{1\}) > 0$. These notions are in accordance with Definition 1.3 because if $n = 1$, then there is no subspace ξ such that $0 < \dim \xi < n$.

We note that the notion of strict essential subspace concentration inequality is defined and used only if the dimension $n \geq 2$.

The following lemma will be needed. The proof is the same that of Lemma 7.1 in [6].

Lemma 5.1. *Suppose $n \geq 2$, μ is a discrete measure on S^{n-1} that satisfies the essential subspace concentration condition. If ξ is an essential linear subspace with respect to μ for which*

$$\mu(\xi \cap S^{n-1}) = \frac{1}{n} \mu(S^{n-1}) \dim \xi,$$

then μ_ξ satisfies the essential subspace concentration condition.

For even measures, the following lemma was stated for even measures as Lemma 7.2 in [6]. However, the proof in [6] does not use the property that the measure is even.

Lemma 5.2. *Let ξ and ξ' be complementary subspaces in \mathbb{R}^n with $0 < \dim \xi < n$. Suppose μ is a Borel measure on S^{n-1} that is concentrated on $S^{n-1} \cap (\xi \cup \xi')$, and so that*

$$\mu(\xi \cap S^{n-1}) = \frac{1}{n} \mu(S^{n-1}) \dim \xi.$$

If μ_ξ and $\mu_{\xi'}$ are cone-volume measures of convex bodies in the subspaces ξ and ξ' , then μ is the cone-volume measure of a convex body in \mathbb{R}^n .

In addition, we also need the following lemma.

Lemma 5.3. *Suppose μ is a Borel measure on S^{n-1} , $n \geq 2$, that is not concentrated on any closed hemisphere, and μ concentrated on two complementary subspaces ξ and ξ' of \mathbb{R}^n . Then, μ_ξ is not concentrated on any closed hemisphere of $\xi \cap S^{n-1}$ and $\mu_{\xi'}$ is not concentrated on any closed hemisphere of $\xi' \cap S^{n-1}$.*

Proof. We only need prove that μ_ξ is not concentrated on any closed hemisphere of $\xi \cap S^{n-1}$.

Suppose μ_ξ is concentrated on a closed hemisphere, C , of $\xi \cap S^{n-1}$. Then, μ is concentrated on

$$S^{n-1} \cap \text{pos}\{C \cup \xi'\}.$$

However, $S^{n-1} \cap \text{pos}\{C \cup \xi'\}$ is a closed hemisphere of S^{n-1} . This contradicts the conditions of the lemma. Therefore, μ_ξ is not concentrated on any closed hemisphere of $\xi \cap S^{n-1}$. \square

Now, we have prepared enough to prove the main theorem of this paper.

Theorem 5.4. *If μ is a discrete measure on S^{n-1} , $n \geq 1$ that is not concentrated on any closed hemisphere and satisfies the essential subspace concentration condition, then μ is the cone-volume measure of a polytope in \mathbb{R}^n .*

Proof. We prove Theorem 5.4 by induction on the dimension $n \geq 1$. If $n = 1$, then the theorem trivially holds, therefore let $n \geq 2$.

If μ satisfies the strict essential subspace concentration inequality, then μ is the cone-volume measure of a polytope in \mathbb{R}^n according to Lemma 3.4 and Lemma 4.7.

Therefore we assume that there exists an essential subspace (with respect to μ), ξ , of \mathbb{R}^n , and a subspace, ξ' , of \mathbb{R}^n such that ξ, ξ' are complementary subspaces of \mathbb{R}^n , μ concentrated on $S^{n-1} \cap \{\xi \cup \xi'\}$ with

$$\mu(S^{n-1} \cap \xi) = \frac{\dim \xi}{n} \mu(S^{n-1}) \text{ and } \mu(S^{n-1} \cap \xi') = \frac{\dim \xi'}{n} \mu(S^{n-1}).$$

From the fact that μ is not concentrated on a closed hemisphere and Lemma 5.3, we have, μ_ξ is not concentrated on a closed hemisphere of $S^{n-1} \cap \xi$, and $\mu_{\xi'}$ is not concentrated on a closed hemisphere of $S^{n-1} \cap \xi'$. By Lemma 5.1, μ_ξ satisfies the essential subspace concentration condition on $\xi \cap S^{n-1}$, and $\mu_{\xi'}$ satisfies the essential subspace concentration condition on $\xi' \cap S^{n-1}$. From the induction hypothesis, μ_ξ is the cone-volume measure of a convex body in $\xi \cap \mathbb{R}^n$, and $\mu_{\xi'}$ is the cone-volume measure of a convex body in $\xi' \cap \mathbb{R}^n$. By Lemma 5.2, μ is the cone-volume measure of a convex body in \mathbb{R}^n . Since μ is discrete, μ is the cone-volume measure of a polytope in \mathbb{R}^n . \square

6. NEW INEQUALITIES FOR CONE-VOLUME MEASURES

In this section, we establish some inequalities for cone-volume measures.

The following example shows that the cone-volume measure of a convex body does not need to satisfy the essential subspace concentration condition with respect to essential linear subspace.

Example 6.1. Let u_1, \dots, u_n be an orthonormal basis of \mathbb{R}^n , and let $W = \{x \in u_1^\perp : |x \cdot u_i| \leq 1, i = 2, \dots, n\}$ be an $(n-1)$ -dimensional cube. For $r > 0$ and $i = 1, \dots, n-1$, $\xi_i = \text{lin}\{u_1, \dots, u_i\}$ is an essential subspace for the cone-volume measure of the truncated pyramid $P_r = [-ru_1 + rW, u_1 + W]$. If $r > 0$ is small, then P_r approximates $[o, u_1 + W]$, and thus

$$V_{P_r}(\xi_i \cap S^{n-1}) > V_{P_r}(\{u_1\}) = V([o, u_1 + W]) > \frac{i}{n} V(P_r).$$

We next establish new inequalities for the cone-volume measures.

Lemma 6.2. If K is a convex body in \mathbb{R}^n , $n \geq 3$, with $o \in \text{Int}(K)$, then for $u \in S^{n-1}$

$$(6.1) \quad V_K(\{u\}) + V_K(\{-u\}) + 2(n-1)\sqrt{V_K(\{u\})V_K(\{-u\})} \leq V(K),$$

with equality if and only if $F(K, -u)$ is a translate of $F(K, u)$, $K = [F(K, u), F(K, -u)]$, and $h(K, u) = h(K, -u)$.

In \mathbb{R}^2 , we have

Lemma 6.3. If K is a convex body containing the origin in its interior in \mathbb{R}^2 , and $u \in S^1$, then

$$(6.2) \quad \sqrt{V_K(\{u\})} + \sqrt{V_K(\{-u\})} \leq \sqrt{V(K)},$$

with equality if and only if K is a trapezoid with two sides parallel to u^\perp , and u^\perp contains the intersection of the diagonals.

We obtain the following estimate from Lemma 6.2 and Lemma 6.3.

Corollary 6.4. If K is a convex body in \mathbb{R}^n , $n \geq 2$ with $o \in \text{Int}(K)$ and $u \in S^{n-1}$, then

$$V_K(\{u\}) \cdot V_K(\{-u\}) \leq \frac{1}{4n^2} (V(K))^2,$$

with equality if and only if $F(K, -u)$ is a translate of $F(K, u)$, $K = [F(K, u), F(K, -u)]$, and $h(K, u) = h(K, -u)$.

We next prove Lemma 6.2 and Lemma 6.3 together.

Proof. For the case $|F(K, u)| \cdot |F(K, -u)| = 0$, Lemma 6.2 and Lemma 6.3 are trivially true. Thus we prove Lemma 6.2 and Lemma 6.3 under the condition that $|F(K, u)| \cdot |F(K, -u)| > 0$.

Let $V_K(\{u\}) = \alpha > 0$ and $V_K(\{-u\}) = \beta > 0$, let $h_K(u) = a$ and $h_K(-u) = b$, and for $0 \leq x \leq a + b$ let

$$K_x = ((a-x)u + u^\perp) \cap K.$$

Since K is a convex body,

$$\frac{x}{a+b} F(K, -u) + \frac{a+b-x}{a+b} F(K, u) \subset K_x.$$

From this and the Brunn-Minkowski inequality,

$$(6.3) \quad \begin{aligned} |K_x| &\geq \left| \frac{x}{a+b} F(K, -u) + \frac{a+b-x}{a+b} F(K, u) \right| \\ &= \left| \left(\frac{x}{a+b} F(K, -u) + \frac{a+b-x}{a+b} F(K, u) \right)_{u^\perp} \right| \\ &= \left| \frac{x}{a+b} F(K, -u)|_{u^\perp} + \frac{a+b-x}{a+b} F(K, u)|_{u^\perp} \right| \\ &\geq \left(\frac{x}{a+b} |F(K, -u)|_{u^\perp}^{\frac{1}{n-1}} + \frac{a+b-x}{a+b} |F(K, u)|_{u^\perp}^{\frac{1}{n-1}} \right)^{n-1} \\ &= \left(\frac{x}{a+b} |F(K, -u)|^{\frac{1}{n-1}} + \frac{a+b-x}{a+b} |F(K, u)|^{\frac{1}{n-1}} \right)^{n-1}, \end{aligned}$$

with equality if and only if $K_x = \frac{x}{a+b}F(u(K, -u) + \frac{a+b-x}{a+b}F(K, u))$, and $F(K, -u)|_{u^\perp}$ and $F(K, u)|_{u^\perp}$ are homothetic.

Let $t = \frac{a+b-x}{a+b}$. From (6.3) and Fubini's formula,

$$\begin{aligned}
(6.4) \quad V(K) &= \int_0^{a+b} |K_x| dx \\
&\geq \int_0^{a+b} \left(\frac{x}{a+b} |F(K, -u)|^{\frac{1}{n-1}} + \frac{a+b-x}{a+b} |F(K, u)|^{\frac{1}{n-1}} \right)^{n-1} dx \\
&= (a+b) \int_0^1 \left(t |F(K, u)|^{\frac{1}{n-1}} + (1-t) |F(K, -u)|^{\frac{1}{n-1}} \right)^{n-1} dt \\
&= (a+b) \sum_{i=0}^{n-1} |F(K, u)|^{\frac{i}{n-1}} |F(K, -u)|^{\frac{n-1-i}{n-1}} \binom{n-1}{i} \int_0^1 t^i (1-t)^{n-1-i} dt \\
&= \frac{a+b}{n} \sum_{i=0}^{n-1} |F(K, u)|^{\frac{i}{n-1}} |F(K, -u)|^{\frac{n-1-i}{n-1}}.
\end{aligned}$$

Let $S_1 = |F(K, u)|$ and $S_2 = |F(K, -u)|$. From (6.4) and the arithmetic-geometric inequality, we have

$$\begin{aligned}
(6.5) \quad V(K) &= \frac{a+b}{n} \sum_{i=0}^{n-1} S_1^{\frac{i}{n-1}} S_2^{\frac{n-1-i}{n-1}} \\
&= \frac{a}{n} S_1 + \frac{b}{n} S_2 + \frac{1}{n} \sum_{i=1}^{n-1} \left(a S_1^{\frac{n-1-i}{n-1}} S_2^{\frac{i}{n-1}} + b S_2^{\frac{n-1-i}{n-1}} S_1^{\frac{i}{n-1}} \right) \\
&\geq \alpha + \beta + 2(n-1) \sqrt{\alpha\beta}.
\end{aligned}$$

Thus, we get (6.1) and (6.2).

From the equality conditions for (6.3), (6.4) and the arithmetic-geometric inequality, we have, equality holds in (6.5) if and only if $F(K, u)|_{u^\perp}$ and $F(K, -u)|_{u^\perp}$ are homothetic, $K = [F(K, u), F(K, -u)]$, and

$$(6.6) \quad \frac{a}{b} = \left(\frac{S_1}{S_2} \right)^{\frac{2i-n+1}{n-1}},$$

for all $1 \leq i \leq n-1$.

Therefore, equality holds in (6.2) ($n=2$) if and only if K is a trapezoid with two sides parallel to u^\perp , and u^\perp contains the intersection of the diagonals.

When $n \geq 3$, (6.6) hold for $i=1, \dots, n-1$. Thus, $\frac{a}{b} = \frac{S_1}{S_2} = 1$. Therefore, equality holds in (6.1) if and only if $F(K, -u)$ is a translation of $F(K, u)$, $K = [F(K, u), F(K, -u)]$, and $h_K(u) = h_K(-u)$. \square

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REFERENCES

- [1] A.D. Alexandrov, Existence and uniqueness of a convex surface with a given integral curvature. C. R. (Doklady) Acad. Sci. USSR (N.S.) **35**, 131-134 (1942).
- [2] B. Andrews, Gauss curvature flow: the fate of the rolling stones. Invent. Math. **138**, 151-161 (1999).

- [3] B. Andrews, Classification of limiting shapes for isotropic curve flows. *J. Amer. Math. Soc.* **16**, 443-459 (2003).
- [4] F. Barthe, O. Guédon, S. Mendelson, A. Naor, A probabilistic approach to the geometry of the l_p^n -ball. *Ann. of Probability* **33**, 480-513 (2005).
- [5] K. Böröczky, M. Henk, Cone-volume measure and stability. submitted.
- [6] J. Böröczky, E. Lutwak, D. Yang, G. Zhang, The logarithmic Minkowski problem. *J. Amer. Math. Soc.* **26**, 831-852 (2013).
- [7] J. Böröczky, E. Lutwak, D. Yang, G. Zhang, The log-Brunn-Minkowski inequality. *Adv. Math.* **231**, 1974-1997 (2012).
- [8] Böröczky, J., Lutwak, E., Yang, D., Zhang, G.: Affine images of isotropic measures, to appear in *J. Diff. Geom.*
- [9] L. Caffarelli, Interior $W^{2,p}$ -estimates for solutions of the Monge-Ampère equation. *Ann. of Math. (2)* **131**, 135-150 (1990).
- [10] S. Campi, P. Gronchi, The L^p -Busemann-Petty centroid inequality. *Adv. Math.* **167**, 128-141 (2002).
- [11] W. Chen, L_p Minkowski problem with not necessarily positive data. *Adv. Math.* **201**, 77-89 (2006).
- [12] S.-Y. Cheng, S.-T. Yau, On the regularity of the solution of the n -dimensional Minkowski problem. *Comm. Pure Appl. Math.* **29**, 495-561 (1976).
- [13] K.-S. Chou, Deforming a hypersurface by its Gauss-Kronecker curvature. *Comm. Pure Appl. Math.* **38**, 867-882 (1985).
- [14] K.-S. Chou, X.-J. Wang, The L_p -Minkowski problem and the Minkowski problem in centroaffine geometry. *Adv. Math.* **205**, 33-83 (2006).
- [15] A. Cianchi, E. Lutwak, D. Yang, G. Zhang, Affine Moser-Trudinger and Morrey-Sobolev inequalities. *Calc. Var. Partial Differential Equations* **36**, 419-436 (2009).
- [16] M. Gage, R. Hamilton, The heat equation shrinking convex plane curves. *J. Differential Geom.* **23**, 69-96 (1986).
- [17] R.J. Gardner, *Geometric Tomography*, 2nd edition, *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2006.
- [18] P.M. Gruber, *Convex and discrete geometry*, *Grundlehren der Mathematischen Wissenschaften*, 336. Springer, Berlin, 2007.
- [19] B. Grünbaum, *convex polytope (graduate Texts in Mathematics)*, 2nd edition. Springer, 2003.
- [20] B. Guan, P. Guan, Convex hypersurfaces of prescribed curvatures. *Ann. of Math. (2)* **156**, 655-673 (2002).
- [21] P. Guan, C.-S. Lin, On equation $\det(u_{ij} + \delta_{ij}u) = u^p f$ on S^n . (preprint).
- [22] P. Guan, X. Ma, The Christoffel-Minkowski problem I: Convexity of solutions of a Hessian equation. *Invent. Math.* **151**, 553-577 (2003).
- [23] C. Haberl, Minkowski valuations intertwining with the special linear group. *J. Eur. Math. Soc.* **14**, 1565-1597 (2012).
- [24] C. Haberl, E. Lutwak, D. Yang, G. Zhang, The even Orlicz Minkowski problem. *Adv. Math.* **224**, 2485-2510 (2010).
- [25] C. Haberl, L. Parapatits, Valuations and surface area measures. *J. Reine Angew. Math.* **687**, 225-245 (2014).
- [26] C. Haberl, L. Parapatits, The centro-affine Hadwiger theorem. *J. Amer. Math. Soc.* **27**, 685-705 (2014).
- [27] C. Haberl, F. Schuster, General L_p affine isoperimetric inequalities. *J. Differential Geom.* **83**, 1-26 (2009).
- [28] C. Haberl, F. Schuster, Asymmetric affine L_p Sobolev inequalities. *J. Funct. Anal.* **257**, 641-658 (2009).
- [29] C. Haberl, F. Schuster, J. Xiao, An asymmetric affine Pólya-Szegő principle. *Math. Ann.* **352**, 517-542 (2012).
- [30] B. He, G. Leng & K. Li, Projection problems for symmetric polytopes. *Adv. Math.* **207**, 73-90 (2006).
- [31] M. Henk, E. Linke, Cone-volume measures of polytopes. *Adv. Math.* **253**, 50-62 (2014).
- [32] M. Henk, A. Schürmann & J.M. Wills, Ehrhart polynomials and successive minima. *Mathematika* **52(1-2)**:1-16 2005.
- [33] C. Hu, X. Ma, C. Shen, On the Christoffel-Minkowski problem of Firey's p -sum. *Calc. Var. Partial Differential Equations.* **21**, 137-155 (2004).
- [34] Y. Huang, Q. Lu, On the regularity of the L_p -Minkowski problem. *Adv. in Appl. Math.* **50**, 268-280 (2013).
- [35] D. Hug, E. Lutwak, D. Yang, G. Zhang, On the L_p Minkowski problem for polytopes. *Discrete Comput. Geom.* **33**, 699-715 (2005).
- [36] G. Huisken, Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.* **20**, 237-266 (1984).

- [37] M.N. Ivaki, A flow approach to the L_{-2} Minkowski problem. *Advances in Applied Mathematics*, 50 (2013), 445-464.
- [38] M.Y. Jiang, Remarks on the 2-dimensional L_p -Minkowski problem. *Adv. Nonlinear Stud.* **10**, 297-313 (2010).
- [39] D. Klain, The Minkowski problem for polytopes. *Adv. Math.* **185**, 270-288 (2004).
- [40] H. Lewy, On differential geometry in the large. I. Minkowski problem. *Trans. Amer. Math. Soc.* **43**, 258-270 (1938).
- [41] J. Lu, X.-J. Wang, Rotationally symmetric solution to the L_p -Minkowski problem. *J. Differential Equations.* **254**, 983-1005 (2013).
- [42] M. Ludwig, Ellipsoids and matrix-valued valuations. *Duke Math. J.* **119**, 159-188 (2003).
- [43] M. Ludwig, General affine surface areas. *Adv. Math.* **224**, 2346-2360 (2010).
- [44] M. Ludwig, M. Reitzner, A classification of $SL(n)$ invariant valuations. *Ann. of Math. (2)* **172**, 1219-1267 (2010).
- [45] E. Lutwak, The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem. *J. Differential Geom.* **38**, 131-150 (1993).
- [46] E. Lutwak, V. Oliker, On the regularity of solutions to a generalization of the Minkowski problem. *J. Differential Geom.* **41**, 227-246 (1995).
- [47] E. Lutwak, D. Yang, G. Zhang, L_p affine isoperimetric inequalities. *J. Differential Geom.* **56**, 111-132 (2000).
- [48] E. Lutwak, D. Yang, G. Zhang, A new ellipsoid associated with convex bodies. *Duke Math. J.* **104**, 375-390 (2000).
- [49] E. Lutwak, D. Yang, G. Zhang, The Cramer-Rao inequality for star bodies. *Duke Math. J.* **112**, 59-81 (2002).
- [50] E. Lutwak, D. Yang, G. Zhang, Sharp affine L_p Sobolev inequalities. *J. Differential Geom.* **62**, 17-38 (2002).
- [51] E. Lutwak, D. Yang, G. Zhang, On the L_p -Minkowski problem. *Trans. Amer. Math. Soc.* **356**, 4359-4370 (2004).
- [52] E. Lutwak, D. Yang, G. Zhang, Volume inequalities for subspaces of L_p . *J. Differential Geom.* **68**, 159-184 (2004).
- [53] E. Lutwak, G. Zhang, Blaschke-Santaló inequalities. *J. Differential Geom.* **47**, 1-16 (1997).
- [54] H. Minkowski, Allgemeine Lehrsätze über die konvexen Polyeder. *Gött. Nachr.* **1897**, 198-219 (1897).
- [55] A. Naor, The surface measure and cone measure on the sphere of l_p^n . *Trans. Amer. Math. Soc.* **359**, 1045-1079 (2007).
- [56] A. Naor, D. Romik, Projecting the surface measure of the sphere of l_p^n . *Ann. Inst. H. Poincaré Probab. Statist.* **39**, 241-261 (2003).
- [57] L. Nirenberg, The Weyl and Minkowski problems in differential geometry in the large. *Comm. Pure and Appl. Math.* **6**, 337-394 (1953).
- [58] G. Paouris, Concentration of mass on convex bodies. *Geom. Funct. Anal.* **16**, 1021-1049 (2006).
- [59] G. Paouris, E. Werner, Relative entropy of cone measures and L_p centroid bodies. *Proc. London Math. Soc.* **104**, 253-286 (2012).
- [60] A.V. Pogorelov, The Minkowski multidimensional problem. V.H. Winston & Sons, Washington, D.C, 1978.
- [61] R. Schneider, Convex bodies: the Brunn-Minkowski theory, *Encyclopedia of Mathematics and its Applications (Second Edition)*. Cambridge University Press, Cambridge, 2014.
- [62] A. Stancu, The discrete planar L_0 -Minkowski problem. *Adv. Math.* **167**, 160-174 (2002).
- [63] A. Stancu, On the number of solutions to the discrete two-dimensional L_0 -Minkowski problem. *Adv. Math.* **180**, 290-323 (2003).
- [64] A. Stancu, Centro-affine invariants for smooth convex bodies. *Int. Math. Res. Not.* **2012**, 2289-2320 (2012).
- [65] A. C. Thompson, Minkowski geometry, *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1996.
- [66] T. Wang, The affine Sobolev-Zhang inequality on $BV(\mathbb{R}^n)$. *Adv. Math.* **230**, 2457-2473 (2012).
- [67] G. Xiong, Extremum problems for the cone volume functional for convex polytopes. *Adv. Math.* **225**, 3214-3228 (2010).
- [68] G. Zhang, The affine Sobolev inequality. *J. Differential Geom.* **53**, 183-202 (1999).
- [69] G. Zhu, The logarithmic Minkowski problem for polytopes. *Adv. Math.* **262**, 909-931 (2014).
- [70] G. Zhu, The centro-affine Minkowski problem for polytopes. *J. Differential Geom.* (in press).
- [71] G. Zhu, The L_p Minkowski problem for polytopes for $0 < p < 1$. (submitted).

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