A CHARACTERIZATION OF BARYCENTRICALLY PREASSOCIATIVE FUNCTIONS

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ABSTRACT. We provide a characterization of the variadic functions which are barycentrically preassociative as compositions of length-preserving associative string functions with one-to-one unary maps. We also discuss some consequences of this characterization.

1. INTRODUCTION

Let X and Y be arbitrary nonempty sets. Throughout this paper we regard tuples $\mathbf{x}$ in X^n as n-strings over X. We let $X^* = \bigcup_{n\geqslant 0} X^n$ be the set of all strings over X, with the convention that $X^0 = \{\varepsilon\}$ (i.e., ε denotes the unique 0-string on X). We denote the elements of X^* by bold roman letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. If we want to stress that such an element is a letter of X, we use non-bold italic letters x, y, z, etc. The *length* of a string $\mathbf{x}$ is denoted by $|\mathbf{x}|$. For instance, $|\varepsilon| = 0$. We endow the set X^* with the concatenation operation, for which ε is the neutral element, i.e., $\varepsilon\mathbf{x} = \mathbf{x}\varepsilon = \mathbf{x}$. For instance, if $\mathbf{x} \in X^m$ and $y \in X$, then $\mathbf{x}y \in X^{m+1}$. Moreover, for every string $\mathbf{x}$ and every integer $n \geqslant 0$, the power $\mathbf{x}^n$ stands for the string obtained by concatenating n copies of $\mathbf{x}$. In particular we have $\mathbf{x}^0 = \varepsilon$.

As usual, a map $F\colon X^n \to Y$ is said to be an *n-ary function* (an *n-ary operation on X* if $Y = X$). Also, a map $F\colon X^* \to Y$ is said to be a *variadic function* (a *string function on X* if $Y = X^*$; see [5]). For every variadic function $F\colon X^* \to Y$ and every integer $n \geqslant 0$, we denote by F_n the *n-ary part* $F|_{X^n}$ of F.

Recall that a variadic function $F\colon X^* \to Y$ is said to be *preassociative* [6, 7] if, for any $\mathbf{x}, \mathbf{y}, \mathbf{y}', \mathbf{z} \in X^*$, we have

$$F(\mathbf{y}) = F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}\mathbf{y}'\mathbf{z}).$$

Also, a variadic function $F\colon X^* \to Y$ is said to be *barycentrically preassociative* (or *B-preassociative* for short) [8] if, for any $\mathbf{x}, \mathbf{y}, \mathbf{y}', \mathbf{z} \in X^*$, we have

$$|\mathbf{y}| = |\mathbf{y}'| \quad \text{and} \quad F(\mathbf{y}) = F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}\mathbf{y}'\mathbf{z}).$$

Contrary to preassociativity, B-preassociativity recalls the associativity-like property of the barycenter (just regard $F(\mathbf{x})$ as the barycenter of a set $\mathbf{x}$ of identical homogeneous balls in $X = \mathbb{R}^n$). In descriptive statistics and aggregation function theory, this condition says that the aggregated value of a series of numerical values remains unchanged when modifying a bundle of these values without changing their partial aggregation.

B-preassociativity has been recently utilized by the authors in the following characterization of the *quasi-arithmetic pre-mean functions*, thus generalizing the well-known Kolmogoroff-Nagumo's characterization of the quasi-arithmetic mean functions.

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Theorem 1 ([8]). *Let $\mathbb{I}$ be a nontrivial real interval, possibly unbounded. A function $F:\mathbb{I}^* \to \mathbb{R}$ is B-preassociative and, for every $n \geqslant 1$, the function F_n is symmetric, continuous, and strictly increasing in each argument if and only if there are continuous and strictly increasing functions $f:\mathbb{I} \to \mathbb{R}$ and $f_n:\mathbb{R} \to \mathbb{R}$ $(n \geqslant 1)$ such that*

$$F_n(\mathbf{x}) \;=\; f_n\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right), \qquad n \geqslant 1.$$

Remark 1. If we add the condition that every F_n is idempotent (i.e., $F_n(x^n) = x$ for every $x \in X$) in Theorem 1, then we necessarily have $f_n = f^{-1}$ for every $n \geqslant 1$, thus reducing this result to Kolmogoroff-Nagumo's characterization of the quasi-arithmetic mean functions [4, 9]. However, there are also many non-idempotent quasi-arithmetic pre-mean functions. Taking for instance $f_n(x) = nx$ and $f(x) = x$ over the reals $\mathbb{I} = \mathbb{R}$, we obtain the sum function. Taking $f_n(x) = \exp(nx)$ and $f(x) = \ln(x)$ over $\mathbb{I} =]0, \infty[$, we obtain the product function.

In this paper we show that B-preassociative functions can be factorized as compositions of length-preserving associative string functions with one-to-one unary maps. We also show how this factorization result generalizes a characterization of a noteworthy subclass of B-preassociative functions given by the authors in [8]. Finally, we mention some interesting consequences of this new characterization.

The terminology used throughout this paper is the following. The domain, range, and kernel of any function f are denoted by $\mathrm{dom}(f)$, $\mathrm{ran}(f)$, and $\ker(f)$, respectively. The identity function on any nonempty set is denoted by id. For every $n \geqslant 1$, the diagonal section $\delta_F: X \to Y$ of a function $F: X^n \to Y$ is defined as $\delta_F(x) = F(x^n)$.

Remark 2. Although B-preassociativity was recently defined by the authors [8], the basic idea behind this definition goes back to 1931 when de Finetti [1] introduced an associativity-like property for mean functions. Indeed, according to de Finetti, for a real function $F:\bigcup_{n\geqslant 1}\mathbb{R}^n \to \mathbb{R}$ to be considered as a mean, it is natural that it be "associative" in the following sense: for any $u \in X$ and any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*$ such that $|\mathbf{xz}| \geqslant 1$ and $|\mathbf{y}| \geqslant 1$, we have $F(\mathbf{xyz}) = F(\mathbf{x}u^{|\mathbf{y}|}\mathbf{z})$ whenever $F(\mathbf{y}) = F(u^{|\mathbf{y}|})$.

2. Main results

As mentioned in the introduction, in this section we mainly show that B-preassociative functions can be factorized as compositions of length-preserving associative string functions with one-to-one unary maps. This result is stated in Theorem 8.

Recall that a string function $F: X^* \to X^*$ is said to be *associative* [5] if it satisfies the equation $F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z})$ for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*$.

Definition 2. We say that a string function $F: X^* \to X^*$ is *length-preserving* if $|F(\mathbf{x})| = |\mathbf{x}|$ for every $\mathbf{x} \in X^*$, or equivalently, if $\mathrm{ran}(F_n) \subseteq X^n$ for every $n \geqslant 0$.

Clearly, the identity function on X^* is associative and length-preserving. The following example gives nontrivial instances of associative and length-preserving string functions. Further examples of associative string functions can be found in [5].

Example 3. Let $(h_n)_{n\geqslant 1}$ be a sequence of unary operations on X. One can easily see that the length-preserving function $F: X^* \to X^*$ defined by $F_0(\varepsilon) = \varepsilon$ and

$$F_n(x_1 \cdots x_n) \;=\; h_1(x_1) \cdots h_n(x_n), \qquad n \geqslant 1,$$

is associative if and only if $h_n \circ h_m = h_n$ for all $n, m \geqslant 1$ such that $m \leqslant n$. Using an elementary induction, one can also show that the latter condition is equivalent to $h_n \circ h_n =$

h_n and $h_{n+1} \circ h_n = h_{n+1}$ for every $n \geqslant 1$. To give an example, take any constant sequence $h_n = h$ such that $h \circ h = h$ (for instance, the positive part function $h(x) = x^+$ over $X = \mathbb{R}$). As a second example, consider the sequence h_n of unary operations on $X = \{1, 2, 3, \ldots\}$ defined by $h_n(k) = 1$ if $k \leqslant n+1$, and $h_n(k) = k$, otherwise.

Proposition 4. *Let $F\colon X^* \to X^*$ be a length-preserving function. Then F is associative if and only if it is B-preassociative and satisfies $F_n = F_n \circ F_n$ for every $n \geqslant 0$.*

Proof. To see that the necessity holds, we recall from [5] that any associative string function is preassociative and hence B-preassociative. The second part of the statement is immediate. For the sufficiency, we merely observe that we have $F(F(\mathbf{y})) = F(\mathbf{y})$ for every $\mathbf{y} \in X^*$ and therefore, by B-preassociativity, we also have $F(\mathbf{x}F(\mathbf{y})\mathbf{z}) = F(\mathbf{xyz})$ for every $\mathbf{xyz} \in X^*$, that is, F is associative. □

The following proposition, established in [8], shows how we can construct new B-preassociative functions from given B-preassociative functions.

Proposition 5 ([8])**.** *Let $F\colon X^* \to Y$ be a B-preassociative function and let $(g_n)_{n\geqslant 1}$ be a sequence of functions from Y to a nonempty set Y'. If $g_n|_{\mathrm{ran}(F_n)}$ is one-to-one for every $n \geqslant 1$, then any function $H\colon X^* \to Y'$ such that $H_n = g_n \circ F_n$ for every $n \geqslant 1$ is B-preassociative.*

Recall that a function g is a *quasi-inverse* [10, Sect. 2.1] of a function f if

$$f \circ g|_{\mathrm{ran}(f)} = \mathrm{id}|_{\mathrm{ran}(f)} \qquad \text{and} \qquad \mathrm{ran}(g|_{\mathrm{ran}(f)}) = \mathrm{ran}(g).$$

We denote the set of quasi-inverses of a function f by $Q(f)$. Under the assumption of the Axiom of Choice (AC), the set $Q(f)$ is nonempty for any function f. In fact, the Axiom of Choice is just another form of the statement "every function has a quasi-inverse". Note also that the relation of being quasi-inverse is symmetric: if $g \in Q(f)$ then $f \in Q(g)$; moreover, we have $\mathrm{ran}(g) \subseteq \mathrm{dom}(f)$ and $\mathrm{ran}(f) \subseteq \mathrm{dom}(g)$ and the functions $f|_{\mathrm{ran}(g)}$ and $g|_{\mathrm{ran}(f)}$ are one-to-one.

Lemma 6. *Assume AC and let $F\colon X^n \to Y$ be a function. For any $g \in Q(F)$, define the function $H\colon X^n \to X^n$ by $H = g \circ F$. Then we have $F = F \circ H$ and $H = H \circ H$. Moreover, the map $F|_{\mathrm{ran}(H)}$ is one-to-one.*

Proof. By definition of H we have $F \circ H = F \circ g \circ F = F$ and $H \circ H = g \circ F \circ g \circ F = g \circ F = H$. Also, the map $F|_{\mathrm{ran}(H)} = F|_{\mathrm{ran}(g)}$ is one-to-one. □

Lemma 7. *Assume AC and let $F\colon X^* \to Y$ be a function. The following assertions are equivalent.*

(i) *F is B-preassociative.*
(ii) *For every sequence $(g_n \in Q(F_n))_{n\geqslant 1}$, the function $H\colon X^* \to X^*$ defined by $H_0(\varepsilon) = \varepsilon$ and $H_n = g_n \circ F_n$ for every $n \geqslant 1$ is associative and length-preserving.*
(iii) *There exists a sequence $(g_n \in Q(F_n))_{n\geqslant 1}$ such that the function $H\colon X^* \to X^*$ defined by $H_0(\varepsilon) = \varepsilon$ and $H_n = g_n \circ F_n$ for every $n \geqslant 1$ is associative and length-preserving.*

Proof. (i) ⇒ (ii). Let $H\colon X^* \to X^*$ be defined as indicated in the statement. We know by Lemma 6 that $H \circ H = H$ and H is length-preserving. Since $g_n|_{\mathrm{ran}(F_n)}$ is one-to-one, we have that H is B-preassociative by Proposition 5. It follows from Proposition 4 that H is associative.

(ii) ⇒ (iii). Trivial.

(iii) ⇒ (i). By Proposition 4, H is B-preassociative. For every $n \geqslant 1$, since $g_n|_{\mathrm{ran}(F_n)}$ is a one-to-one map from $\mathrm{ran}(F_n)$ onto $\mathrm{ran}(g_n) = \mathrm{ran}(H_n)$, we have $F_n = (g_n|_{\mathrm{ran}(F_n)})^{-1} \circ H_n$. By Proposition 5 it follows that F is B-preassociative. □

We are now ready to present our main result, which gives a characterization of any B-preassociative function as a composition of a length-preserving associative string function with one-to-one unary maps.

Theorem 8. *Assume AC and let $F\colon X^* \to Y$ be a function. The following assertions are equivalent.*

(i) *F is B-preassociative.*
(ii) *There exist an associative and length-preserving function $H\colon X^* \to X^*$ and a sequence $(f_n)_{n\geqslant 1}$ of one-to-one functions $f_n\colon \mathrm{ran}(H_n) \to Y$ such that $F_n = f_n \circ H_n$ for every $n \geqslant 1$.*

If condition (ii) holds, then for every $n \geqslant 1$ we have $f_n = F|_{\mathrm{ran}(H_n)} = F_n|_{\mathrm{ran}(H_n)}$, $f_n^{-1} \in Q(F_n)$, and we may choose $H_n = g_n \circ F_n$ for any $g_n \in Q(F_n)$.

Proof. (i) ⇒ (ii). Let $H\colon X^* \to X^*$ be defined by $H_0(\varepsilon) = \varepsilon$ and $H_n = g_n \circ F_n$ for every $n \geqslant 1$, where $g_n \in Q(F_n)$. By Lemma 6 we have $F_n = f_n \circ H_n$ for every $n \geqslant 1$, where $f_n = F_n|_{\mathrm{ran}(H_n)}$ is one-to-one. By Lemma 7, H is associative and length-preserving.

(ii) ⇒ (i). H is B-preassociative by Proposition 4. By Proposition 5 it follows that also F is B-preassociative.

If condition (ii) holds, then for every $n \geqslant 1$ we have $F_n \circ H_n = f_n \circ H_n \circ H_n = f_n \circ H_n$ and hence $F_n|_{\mathrm{ran}(H_n)} = f_n$. Moreover, since f_n is one-to-one, we have $H_n = f_n^{-1} \circ F_n$ and hence $F_n \circ f_n^{-1} \circ F_n = F_n \circ H_n = f_n \circ H_n \circ H_n = f_n \circ H_n = F_n$, which shows that $f_n^{-1} \in Q(F_n)$. □

Remark 3. (a) It is clear that the trivial factorization $F_n = F_n \circ H_n$, where $H_n = \mathrm{id}$, holds for any function $F\colon X^* \to Y$. This observation could make us wrongly think that Theorem 8 is of no use. However, in our factorization $F_n = f_n \circ H_n$ the outer function f_n has the important feature that it is one-to-one.
(b) Similarly to Theorem 8, one can show [5] that any preassociative function $F\colon X^* \to Y$ can be factorized as a composition $F = f \circ H$, where $H\colon X^* \to X^*$ is associative and $f\colon \mathrm{ran}(H) \to Y$ is one-to-one.

In the rest of this section we show how Theorem 8 can be particularized to some nested subclasses of B-preassociative functions, including the subclass of B-preassociative functions $F\colon X^* \to Y$ for which the equality $\mathrm{ran}(F_n) = \mathrm{ran}(\delta_{F_n})$ holds for every $n \geqslant 1$ (see [8]).

For any integers $m, n \geqslant 1$, define $X_m^0 = X^0$ and

$$X_m^n = \{\mathbf{y}z^{n-\min\{n,m\}+1} : \mathbf{y}z \in X^{\min\{n,m\}}\}.$$

For instance $X_1^3 = \{z^3 : z \in X\}$, $X_2^3 = \{yz^2 : yz \in X^2\}$, and $X_m^3 = X^3$ for every $m \geqslant 3$.

Thus, we have $X_m^n = X^n$ if $m \geqslant n$ and $X_m^n = \{\mathbf{y}z^{n-m+1} : \mathbf{y}z \in X^m\}$ if $m \leqslant n$. It follows that for every $m \geqslant 1$ we have $X_m^n \subseteq X_{m+1}^n \subseteq X^n$.

Definition 9. Let $m \geqslant 1$ and $n \geqslant 0$ be integers. We say that a function $H\colon X^n \to X^n$ has an *m-generated range* if $\mathrm{ran}(H) \subseteq X_m^n$. We say that a function $H\colon X^* \to X^*$ has an *m-generated range* if H_n has an m-generated range for every $n \geqslant 0$.

Fact 10. *If a function $H: X^n \to X^n$ has an m-generated range, then it has an $(m+1)$-generated range. If a function $H: X^* \to X^*$ has an m-generated range, then it is length-preserving.*

Let $m \geqslant 1$ and $n \geqslant 0$ be integers. The *m-diagonal section* of a function $F: X^n \to Y$ is the map $\delta_F^m: X^{\min\{n,m\}} \to Y$ defined by $\delta_F^m = F$, if $n = 0$, and $\delta_F^m(\mathbf{y}z) = F(\mathbf{y}z^{n-\min\{n,m\}+1})$ for every $\mathbf{y}z \in X^{\min\{n,m\}}$, otherwise. We clearly have $\mathrm{ran}(\delta_F^m) \subseteq \mathrm{ran}(\delta_F^{m+1})$.

Definition 11. Let $m \geqslant 1$ and $n \geqslant 0$ be integers. We say that a function $F: X^n \to Y$ is *m-quasi-range-idempotent* if $\mathrm{ran}(F) = \mathrm{ran}(\delta_F^m)$.

By definition, any m-quasi-range-idempotent function $F: X^n \to Y$ is $(m+1)$-quasi-range-idempotent. We also observe that the property of being m-quasi-range-idempotent is preserved under left composition with unary maps: if $F: X^n \to Y$ is m-quasi-range-idempotent, then so is $g \circ F$ for any map $g: Y \to Y'$, where Y' is a nonempty set.

Proposition 12. *If $F: X^* \to X^*$ is associative and F_k has an m-generated range for some $k, m \geqslant 1$, then for any integer $p \geqslant 0$ the function F_{k+p} is $(m+p)$-quasi-range-idempotent. In particular, F_k is m-quasi-range-idempotent.*

Proof. Let $\mathbf{x} \in X^p$ and $\mathbf{x}' \in X^k$. Then, there exists $\mathbf{y}z \in X^{\min\{k,m\}}$ such that

$$\begin{aligned} F_{k+p}(\mathbf{x}\mathbf{x}') &= F_{k+p}(\mathbf{x}F_k(\mathbf{x}')) = F_{k+p}(\mathbf{x}\mathbf{y}z^{k-\min\{k,m\}+1}) \\ &= F_{k+p}(\mathbf{x}\mathbf{y}z^{(k+p)-\min\{k+p,m+p\}+1}) = \delta_{F_{k+p}}^{m+p}(\mathbf{x}\mathbf{y}z), \end{aligned}$$

which shows that $\mathrm{ran}(F_{k+p}) \subseteq \mathrm{ran}(\delta_{F_{k+p}}^{m+p})$. The converse inclusion is obvious. □

Lemma 13. *Let $m, n \geqslant 1$ be integers. Any map $F: X^n \to Y$ satisfying $F = F \circ H$, where $H: X^n \to X^n$ has an m-generated range, is m-quasi-range-idempotent.*

Proof. Since $\mathrm{ran}(H) \subseteq X_m^n$, we have $\mathrm{ran}(F) = \mathrm{ran}(F \circ H) \subseteq \mathrm{ran}(\delta_F^m)$. Since the converse inclusion $\mathrm{ran}(F) \supseteq \mathrm{ran}(\delta_F^m)$ holds for any map $F: X^n \to Y$, we have that F is m-quasi-range-idempotent. □

Lemma 14. *Under the assumptions of Lemma 6, if F is m-quasi-range-idempotent for some $m \geqslant 1$, then g can always be chosen so that $\mathrm{ran}(g) \subseteq X_m^n$ and therefore H has an m-generated range. Conversely, if H has an m-generated range for some $m \geqslant 1$, then F is m-quasi-range-idempotent.*

Proof. If F is m-quasi-range-idempotent for some $m \geqslant 1$, then there always exists $g \in Q(F)$ such that $\mathrm{ran}(g) \subseteq X_m^n$; indeed, if $y \in \mathrm{ran}(F) = \mathrm{ran}(\delta_F^m)$, then we can take $g(y) \in (\delta_F^m)^{-1}\{y\} \subseteq X_m^n$. Therefore $H = g \circ F$ has an m-generated range. Conversely, if H has an m-generated range for some $m \geqslant 1$, then F is m-quasi-range-idempotent by Lemma 13. □

Corollary 15. *For any $m \geqslant 1$, the equivalence in Lemma 7 holds if we add the condition that every F_n $(n \geqslant 1)$ is m-quasi-range-idempotent in assertion (i) and the conditions that $\mathrm{ran}(g_n) \subseteq X_m^n$ $(n \geqslant 1)$ and H has an m-generated range in assertions (ii) and (iii).*

Theorem 16. *For any $m \geqslant 1$, the equivalence between (i) and (ii) in Theorem 8 still holds if we add the condition that every F_n $(n \geqslant 1)$ is m-quasi-range-idempotent in assertion (i) and the condition that H has an m-generated range in assertion (ii). In this case the condition $\mathrm{ran}(g_n) \subseteq X_m^n$ $(n \geqslant 1)$ must be added in the last part of the statement.*

Proof. Follows from the results above. □

Setting $m = 1$ in Theorem 16, we immediately derive a factorization of any B-preassociative function whose n-ary part F_n is 1-quasi-range-idempotent for every $n \geqslant 1$. An alternative factorization for such functions is given in the following theorem, established in [8]. Recall that a function $F\colon X^* \to X \cup \{\varepsilon\}$ is *barycentrically associative* (or *B-associative* for short) [8] if it satisfies the equation $F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z})$ for any $\mathbf{x},\mathbf{y},\mathbf{z} \in X^*$. (B-associativity is also known as *decomposability*, see [2, 3]).

Theorem 17 ([8])**.** *Assume AC and let $F\colon X^* \to Y$ be a function. The following assertions are equivalent.*

(i) *F is B-preassociative and F_n is 1-quasi-range-idempotent for every $n \geqslant 1$.*
(ii) *There exists a B-associative function $H\colon X^* \to X \cup \{\varepsilon\}$ such that $H(\varepsilon) = \varepsilon$ and a sequence $(f_n)_{n\geqslant 1}$ of one-to-one functions $f_n\colon \mathrm{ran}(H_n) \to Y$ such that $F_n = f_n \circ H_n$ for every $n \geqslant 1$.*

If condition (ii) holds, then for every $n \geqslant 1$ we have $F_n = \delta_{F_n} \circ H_n$, $f_n = \delta_{F_n}|_{\mathrm{ran}(H_n)}$, $f_n^{-1} \in Q(\delta_{F_n})$, and we may choose $H_n = g_n \circ F_n$ for any $g_n \in Q(\delta_{F_n})$.

We now show how Theorem 17 can be easily derived from Theorem 16.

For every $m \geqslant 1$ and every $\mathbf{x} \in X^*$, denote by $\mathbf{x}_{[m]}$ the m-*prefix* of $\mathbf{x}$, that is the string in $\bigcup_{i=0}^m X^i$ defined as follows: if $|\mathbf{x}| \leqslant m$, then $\mathbf{x}_{[m]} = \mathbf{x}$; otherwise, if $\mathbf{x} = \mathbf{x}'\mathbf{x}''$, with $|\mathbf{x}'| = m$, then $\mathbf{x}_{[m]} = \mathbf{x}'$.

If $H\colon X^* \to X^*$ has an m-generated range, then by definition it can be assimilated with the function $H_{[m]}\colon X^* \to \bigcup_{i=0}^m X^i$ defined by $H_{[m]}(\mathbf{x}) = H(\mathbf{x})_{[m]}$. Indeed, H can be reconstructed from $H_{[m]}$ by setting

$$H(\mathbf{x}) = \begin{cases} H_{[m]}(\mathbf{x}), & \text{if } |\mathbf{x}| \leqslant m, \\ H_{[m]}(\mathbf{x})z^{n-m}, & \text{otherwise,} \end{cases}$$

where z is the last letter of $H_{[m]}(\mathbf{x})$.

Thus we can prove Theorem 17 from Theorem 16 as follows.

Proof of Theorem 17 as a corollary of Theorem 16. By setting $m = 1$ in Theorem 16, we see that H has a 1-generated range. By the observation above, H can then be assimilated with $H_{[1]}$ through the identity $H(\mathbf{x}) = H_{[1]}(\mathbf{x})^{|\mathbf{x}|}$ for every $\mathbf{x} \in X^*$. It is then clear that H is associative if and only if $H_{[1]}$ is B-associative. The other parts of Theorem 17 follow immediately. □

Remark 4. The question of generalizing Theorem 17 by dropping the 1-quasi-range-idempotent condition on every F_n was raised in [8]. Clearly, Theorem 8 answers this question.

3. Some consequences of the factorization result

Since any associative function $F\colon X^* \to X^*$ is preassociative and, in turn, B-preassociative, it can be factorized as indicated in Theorem 8. Therefore, up to one-to-one unary maps, the associative string functions can be completely described in terms of length-preserving associative string functions, and similarly for the preassociative and B-preassociative functions. This is an important observation which shows that in a sense any of these nested classes can be described in terms of the smallest one, namely the subclass of associative and length-preserving string functions (see Figure 1).

Example 18. Let $a \in X$ be fixed. Let the map $F\colon X^* \to X^*$ be defined inductively by $F(z) = z$ if $z \neq a$, $F(a) = \varepsilon$, and $F(\mathbf{x}z) = F(\mathbf{x})F(z)$ for every $\mathbf{x}z \in X^*$. Thus defined, $F(\mathbf{x})$ is obtained from $\mathbf{x}$ by removing all the 'a' letters (if any). Since F is

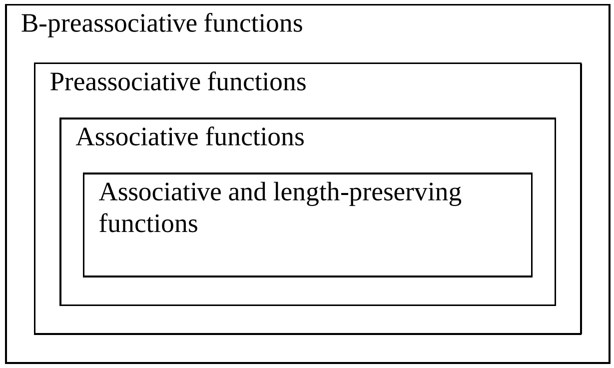

FIGURE 1. Nested subclasses of B-preassociative functions

associative (see [5] for more details), it is B-preassociative and therefore it can be factorized as indicated in Theorem 8. For every $n \geqslant 1$, define the function $g_n : \bigcup_{i=0}^{n}(X \smallsetminus \{a\})^i \to X^n$ by $g_n(\mathbf{x}) = \mathbf{x}a^{n-|\mathbf{x}|}$. Since $F_n \circ g_n \circ F_n = F_n$ for every $n \geqslant 1$, we see that $g_n \in Q(F_n)$. By Theorem 8, the function $H : X^* \to X^*$, defined by $H_0(\varepsilon) = \varepsilon$ and $H_n = g_n \circ F_n$ for every $n \geqslant 1$, is associative and length-preserving. Moreover, we have $F_n = f_n \circ H_n$ for every $n \geqslant 1$, where $f_n = F_n|_{\mathrm{ran}(H_n)}$. Thus defined, $H_n(\mathbf{x})$ is obtained from $\mathbf{x}$ by moving all the 'a' letters (if any) to the rightmost positions. For instance, $H_{11}(mathematics) = mthemticsaa$.

As observed in the previous section, setting $m = 1$ in Theorem 16, we can derive a factorization of any B-preassociative function whose n-ary part F_n is 1-quasi-range-idempotent for every $n \geqslant 1$ (Theorem 17). In the following example, we derive a similar factorization explicitly directly from Theorem 8 (without using Theorem 16).

Example 19. If we assume that F_n is 1-quasi-range-idempotent for every $n \geqslant 1$ in assertion (i) of Theorem 8, then the factorization given in assertion (ii) can be obtained by defining $H_n = g_n \circ F_n$, where $g_n(x) = h_n(x)^n$ and $h_n \in Q(\delta_{F_n})$. Indeed, since F_n is 1-quasi-range-idempotent, we have

$$(F_n \circ g_n \circ F_n)(\mathbf{x}) = (\delta_{F_n} \circ h_n \circ F_n)(\mathbf{x}) = F_n(\mathbf{x}),$$

which shows that $g_n \in Q(F_n)$.

It is clear that the B-associativity property, originally defined for functions $F : X^* \to X \cup \{\varepsilon\}$ can be immediately extended to string functions $F : X^* \to X^*$.

Definition 20. We say that a string function $F : X^* \to X^*$ is *barycentrically associative* (or *B-associative* for short) if it satisfies the equation $F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z})$ for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*$.

It is easy to see that any B-associative string function $F : X^* \to X^*$ is B-preassociative and hence can be factorized as indicated in Theorem 8. Moreover, any B-associative string function satisfying $\mathrm{ran}(F_n) \subseteq X$ for every $n \geqslant 1$ is also such that F_n is 1-quasi-range-idempotent for every $n \geqslant 1$ (see [8]) and therefore it can be factorized as described in Example 19. In this case we have $\delta_{F_n} \circ F_n = F_n$, which shows that $\mathrm{id}|_{\mathrm{ran}(F_n)} \in Q(\delta_{F_n})$ for every $n \geqslant 1$. Therefore, from Example 19 we immediately derive the following corollary.

Corollary 21. *Let $F\colon X^* \to X^*$ be a B-associative function satisfying $\mathrm{ran}(F_n) \subseteq X$ for every $n \geqslant 1$. Then, for every $n \geqslant 1$, we have $F_n = f_n \circ H_n$, where $H\colon X^* \to X^*$ is the length-preserving associative function defined by $H_n(\mathbf{x}) = F_n(\mathbf{x})^n$ for every $n \geqslant 1$ and $f_n\colon \mathrm{ran}(H_n) \to X$ is the one-to-one function defined by $f_n(x^n) = x$ for every $n \geqslant 1$.*

We end this section by an additional application of Theorem 8.

Definition 22. We say that a function $F\colon X^* \to Y$ has a *componentwise defined kernel* if there exists a family $\{E_n : n \geqslant 1\}$ of equivalence relations on X such that for any $n \geqslant 1$ and any $\mathbf{x}, \mathbf{y} \in X^n$, we have $F(\mathbf{x}) = F(\mathbf{y})$ if and only if $(x_i, y_i) \in E_i$ for $i = 1, \ldots, n$. In this case, we say that the family $\{E_n : n \geqslant 1\}$ *defines the kernel of F componentwise*.

This concept can be interpreted, e.g., in decision making, as follows. A function $F\colon X^* \to Y$ has a componentwise defined kernel if the equivalence between two n-profiles $\mathbf{x}, \mathbf{y} \in X^n$ can be defined attributewise.

The following proposition and corollary give characterizations of those B-preassociative functions which have a componentwise defined kernel.

Proposition 23. *Assume AC and let $F\colon X^* \to Y$ have a kernel defined componentwise by the family $\{E_n : n \geqslant 1\}$ of equivalence relations on X. Then F is B-preassociative if and only if $E_n \subseteq E_{n+1}$ for every $n \geqslant 1$.*

Proof. Let $F\colon X^* \to Y$ be defined as indicated in the statement. For the necessity, suppose that F is B-preassociative and let $(x, y) \in E_n$ for some $n \geqslant 1$. Then we have $F(x^n) = F(x^{n-1}y)$ and hence $F(x^{n+1}) = F(x^n y)$ by B-preassociativity. It follows that $(x, y) \in E_{n+1}$. For the sufficiency, for any $n \geqslant 1$ and any $\mathbf{x}, \mathbf{y} \in X^n$ such that $F(\mathbf{x}) = F(\mathbf{y})$, we have $F(\mathbf{xz}) = F(\mathbf{yz})$ for every $\mathbf{z} \in X^*$ by definition of F. Since $E_n \subseteq E_{n+1}$ for every $n \geqslant 1$, we also have $F(\mathbf{zx}) = F(\mathbf{zy})$ for every $\mathbf{z} \in X^*$. Therefore F is B-preassociative. □

Corollary 24. *Assume AC and let $F\colon X^* \to Y$ be a function. The following assertions are equivalent.*

(i) *F is B-preassociative and has a componentwise defined kernel.*
(ii) *There exists a sequence $(h_n)_{n \geqslant 1}$ of unary operations on X and a sequence $(f_n)_{n \geqslant 1}$ of one-to-one maps $f_n\colon \{h_1(x_1) \cdots h_n(x_n) : x_1 \cdots x_n \in X^n\} \to Y$ such that $h_n \circ h_n = h_n$, $h_{n+1} \circ h_n = h_{n+1}$, and $F_n(\mathbf{x}) = f_n(h_1(x_1) \cdots h_n(x_n))$ for every $n \geqslant 1$ and every $\mathbf{x} \in X^n$.*

Proof. (i) ⇒ (ii). By Proposition 23, the kernel of F is defined by some family of equivalence relations $\{E_n : n \geqslant 1\}$ on X satisfying $E_n \subseteq E_{n+1}$ for every $n \geqslant 1$. For every $c \in X/E_n$, let $s_n(c) \in c$ be a representative of c and define the map $h_n\colon X \to X$ by $h_n(x) = s_n(x/E_n)$. The map $g_n\colon \mathrm{ran}(F_n) \to X^n$ defined by $g_n(F(\mathbf{x})) = h_1(x_1) \cdots h_n(x_n)$ is a quasi-inverse of F_n. Indeed, since $(x_i, h_i(x_i)) \in E_i$ for every $\mathbf{x} \in X^n$ and every $i \in \{1, \ldots, n\}$, we have

$$(F_n \circ g_n \circ F_n)(x_1 \cdots x_n) = F_n(h_1(x_1) \cdots h_n(x_n)) = F_n(x_1 \cdots x_n).$$

By Theorem 8, setting $H_n = g_n \circ F_n$ for every $n \geqslant 1$, there is a one-to-one function $f_n\colon \mathrm{ran}(H_n) \to Y$ such that $F_n = f_n \circ H_n$ and such that the map $H\colon X^* \to X^*$ obtained by setting $H_0(\varepsilon) = \varepsilon$ is associative and length-preserving. The conclusion follows from Example 3.

(ii) ⇒ (i) By Example 3 and Proposition 4 we obtain that F is B-preassociative. Moreover, the kernel of F is defined by the family $\{\ker(h_i) : i \geqslant 1\}$ of equivalence relations on X. □

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