# Maxima of the *Q*-index: forbidden even cycles

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#### Abstract

Let G be a graph of order n and let q(G) be the largest eigenvalue of the signless Laplacian of G. Let  $S_{n,k}$  be the graph obtained by joining each vertex of a complete graph of order k to each vertex of an independent set of order n - k; and let  $S_{n,k}^+$ be the graph obtained by adding an edge to  $S_{n,k}$ .

It is shown that if  $k \ge 2$ ,  $n \ge 400k^2$ , and G is a graph of order n, with no cycle of length 2k + 2, then  $q(G) < q\left(S_{n,k}^+\right)$ , unless  $G = S_{n,k}^+$ . This result completes the proof of a conjecture of de Freitas, Nikiforov and Patuzzi.

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## 1 Introduction

Given a graph G, the Q-index of G is the largest eigenvalue q(G) of its signless Laplacian Q(G). In this paper we study how large can q(G) be if G is a graph of given order and contains no cycle of given even length.

Thus, let  $S_{n,k}$  be the graph obtained by joining each vertex of a complete graph of order k to each vertex of an independent set of order n-k, that is to say,  $S_{n,k} = K_k \vee \overline{K}_{n-k}$ . Also, let  $S_{n,k}^+$  be the graph obtained by adding an edge to  $S_{n,k}$ . In [7] the following conjecture has been raised:

**Conjecture 1** Let  $k \ge 2$  and let G be a graph of sufficiently large order n. If G has no cycle of length 2k + 1, then  $q(G) < q(S_{n,k})$ , unless  $G = S_{n,k}$ . If G has no cycle of length 2k + 2, then  $q(G) < q(S_{n,k}^+)$ , unless  $G = S_{n,k}^+$ .

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In [11] it was shown that Conjecture 1 is asymptotically true and some proof technique has been outlined. In [14] and [15] the second author developed this technique further and succeeded to prove Conjecture 1 for forbidden odd cycles. In this paper we shall prove the remaining case of the conjecture, which turns out to be by far more difficult than the odd case. Thus, our main result is the following theorem:

**Theorem 2** Let  $k \ge 2$ ,  $n \ge 400k^2$ , and let G be a graph of order n. If G has no cycle of length 2k + 2, then  $q(G) < q(S_{n,k}^+)$ , unless  $G = S_{n,k}^+$ .

In the next section we prepare the ground for the proof of Theorem 2 and in Section 3 we give the proof itself. At the end we give a sum up of our work.

## 2 Notation and supporting results

For graph notation and concepts undefined here, we refer the reader to [1]. For introductory material on the signless Laplacian see the survey of Cvetković [2] and its references. In particular, let G be a graph, and X and Y be disjoint sets of vertices of G. We write:

- V(G) for the set of vertices of G and E(G) for the set of edges of G;
- |G| for the number of vertices of G and e(G) for the number of edges of G;
- G[X] for the graph induced by X, and e(X) for e(G[X]);
- $G_u$  for the graph induced by the set  $V(G) \setminus \{u\}$ , where  $u \in V(G)$ ;
- e(X, Y) for the number of edges joining vertices in X to vertices in Y;

-  $\Gamma_G(u)$  (or simply  $\Gamma(u)$ ) for the set of neighbors of a vertex u, and  $d_G(u)$  (or simply d(u)) for  $|\Gamma(u)|$ .

We write  $P_k$ ,  $C_k$ , and  $K_k$  for the path, cycle, and complete graph of order k. We write  $F \subset G$  to indicate that F is a subgraph of G, and we say that a graph G is F-free if G contains no subgraphs isomorphic to F.

#### 2.1 Some auxiliary results

Here we state several known results, all of which are used in Section 3. We start with a result of Dirac [4].

**Theorem 3** If G is a graph with  $\delta(G) \ge 2$ , then G contains a cycle longer than  $\delta(G)$ .

Dirac's result has been further developed by Erdős and Gallai [5]. We shall need the following classical theorems from their paper.

**Theorem 4** Let  $k \ge 1$ . If G is a graph of order n, with no  $P_{k+2}$ , then  $e(G) \le kn/2$ , with equality holding if and only if G is a union of disjoint copies of  $K_{k+1}$ .

**Theorem 5** Let  $k \ge 2$ . If G is a graph of order n, with no cycle longer than k, then  $e(G) \le k(n-1)/2$ , with equality holding if and only if G is a union of copies of  $K_k$ , all sharing a single vertex.

The following structural extension of Theorem 4 has been established in [8].

**Lemma 6** Let  $k \ge 1$  and let the vertices of a graph G be partitioned into two sets A and B. If

$$2e(A) + e(A, B) > (2k - 1)|A| + k|B|,$$

then there exists a path of order 2k + 1 with both endvertices in A.

To state the next result define the graph  $L_{t,k}$  by  $L_{t,k} := K_1 \vee tK_k$ , i.e.,  $L_{t,k}$  consists of t complete graphs  $K_{k+1}$ , all sharing a single common vertex. In [10], the following stability result has been proved.

**Theorem 7** Let  $k \ge 2$ ,  $n \ge 2k + 3$ , and G be a graph of order n with  $\delta(G) \ge k$ . If G is connected, then  $P_{2k+3} \subset G$ , unless one of the following holds:

(i) 
$$G \subset S_{n,k}^+$$
;

(ii) n = tk + 1 and  $G = L_{t,k}$ ;

(iii) n = tk + 2 and  $G \subset K_1 \lor ((t - 1) K_k \cup K_{k+1});$ 

(iv) n = (s+t)k + 2 and G is obtained by joining the centers of two disjoint graphs  $L_{s,k}$  and  $L_{t,k}$ .

We shall need, in fact, a particular corollary of Theorem 7, which is easy to check directly.

**Corollary 8** Let  $k \ge 2$ ,  $n \ge 2k+3$ , and let G be a connected  $P_{2k+3}$ -free graph of order n with  $\delta(G) \ge k$ . Then  $e(G) \le (k+1)n/2$ , unless  $G \subset S_{n,k}^+$  and  $e(G) \le kn$ .

Another statement that we shall need is a variant of Theorem 7 for the case k = 1; we omit its easy proof.

**Lemma 9** If G is a connected graph of order  $n \ge 5$ . If G contains no  $P_5$ , then one of the following holds:

- (i)  $G \subset S_{n,1}^+$ ;
- (ii) G is obtained by joining the centers of two disjoint stars.

We finish this subsection with two known upper bounds on q(G). The first one can be traced back to Merris [12], while the case of equality has been established in [6].

**Theorem 10** For every graph G,

$$q(G) \le \max\left\{ d(u) + \frac{1}{d(u)} \sum_{v \in \Gamma(u)} d(v) : u \in V(G) \right\}.$$
(1)

If G is connected, equality holds if and only if G is regular or semiregular bipartite.

Finally, let us mention the following bound, due to Das [3].

**Theorem 11** If G is a graph with n vertices and m edges, then

$$q(G) \le \frac{2m}{n-1} + n - 2,$$
 (2)

with equality holding if and only if G is either complete, or is a star, or is a complete graph with one isolated vertex.

## 3 Proof of Theorem 2

Before going further, we shall make three remarks. First, recall an estimate of  $q(S_{n,k}^+)$  given in [7], where it was shown that if  $k \ge 2$  and  $n > 5k^2$ , then

$$n + 2k - 2 - \frac{2k(k-1)}{n+2k+2} > q\left(S_{n,k}^{+}\right) > n + 2k - 2 - \frac{2k(k-1)}{n+2k-3}.$$
(3)

Second, note that if G is a graph with  $q(G) \ge q(S_{n,k}^+)$ , then e(G) cannot be much smaller than  $e(S_{n,k}^+)$ . Indeed, in view of Das's bound (2) we have

$$q(S_{n,k}^+) \le q(G) \le \frac{2e(G)}{n-1} + n - 2,$$

and so

$$e(G) \ge kn - k^2 + 1 = e\left(S_{n,k}^+\right) - \frac{k(k-1)}{2}.$$
 (4)

Finally, given a vertex u of graph G, note that

$$\sum_{u,v\}\in E(G)} d(v) = 2e(\Gamma(u)) + e(\Gamma(u), V(G) \setminus \Gamma(u))$$

We shall use this equality with no explicit reference.

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Our proof of Theorem 2 is rather long and complicated. To improve the presentation we have extracted a large segment of it into Lemma 12 and Theorem 13 below. Since these assertions are subordinate to the proof of Theorem 2, their statements look somewhat technical.

**Lemma 12** Let  $k \ge 2$ ,  $n \ge 400k^2$ , let G be a  $C_{2k+2}$ -free graph of order n with

$$q\left(G\right) \ge q\left(S_{n,k}^{+}\right).$$

Let w be a dominating vertex of G, suppose that  $G_1, \ldots, G_p$  are the components of  $G_w$  of order at most  $3k^2$ , and let  $H := G - \bigcup_{i=1}^p G_i$ . Then  $|H| \ge 3n/10$  and

$$q\left(H\right) > q\left(S_{|H|,k}^{+}\right),$$

unless H = G and  $q(H) = q(G) = q(S_{n,k}^+)$ .

**Proof** If p = 0, the proof is completed, so let us assume that  $p \ge 1$ . We shall use induction on p. Let  $|G_i| := n_i$ , and  $H_s := G - \bigcup_{i=1}^s G_i$ . First we shall prove that  $q(H_1) > q(S_{n-n_1,k}^+)$ . For short, set q := q(G) and V := V(G). Letting  $(x_1, \ldots, x_n)$  be a positive unit eigenvector to q, from the eigenequation for q and the vertex w we see that

$$(q-n+1) x_w = \sum_{i \in V \setminus \{w\}} x_i \le \sqrt{(n-1)(1-x_w^2)}.$$

Now, in view of  $n \ge 400k^2 \ge 5k^2$  and (3), we see that

$$q \ge q\left(S_{n,k}^+\right) > n + 2k - 3,$$

and so,

$$x_w^2 \le \frac{n-1}{\left(q-n+1\right)^2 + n-1} \le \frac{n-1}{n-1+4\left(k-1\right)^2} < 1 - \frac{4\left(k-1\right)^2}{n+4k^2}.$$
(5)

Next, letting

$$x = \max\left\{x_i \mid i \in V(G_1)\right\},\tag{6}$$

we have

$$qx \le n_1 x + (n_1 - 1) x + x_w,$$

and so

$$x \le \frac{x_w}{q - 2n_1 + 1} \le \frac{x_w}{n + 2k - 2n_1 - 2} \le \frac{x_w}{n - 6k^2}.$$

Note that

$$q(H_1) \ge q(H_1) \left( 1 - \sum_{i \in V(G_1)} x_i^2 \right) \ge \sum_{\{i,j\} \in E(H_1)} (x_i + x_j)^2$$
  
=  $q(G) - \sum_{\{i,j\} \in E(G_1)} (x_i + x_j)^2 - \sum_{i \in V(G_1)} (x_i + x_w)^2.$ 

On the other hand, using (6) and (5), we see that

$$\sum_{\{i,j\}\in E(G_1)} (x_i + x_j)^2 + \sum_{i\in V(G_1)} (x_i + x_w)^2 \le 2n_1 (n_1 - 1) x^2 + n_1 (x + x_w)^2$$
$$\le n_1 \left( 1 + \frac{2}{n - 6k^2} + \frac{2n_1 - 1}{(n - 6k^2)^2} \right) x_w^2$$
$$\le n_1 \left( 1 + \frac{21}{10 (n - 6k^2)} \right) \left( 1 - \frac{4 (k - 1)^2}{n + 4k^2} \right)$$
$$\le n_1 \left( 1 - \frac{3}{2n} \right).$$

Hence, (3) implies that

$$q(H_1) \ge q(G) - n_1 + \frac{3n_1}{2n} \ge n - n_1 + 2k - 2 - \frac{2k(k-1)}{n+2k-3} + \frac{3n_1}{2n}$$
$$> n - n_1 + 2k - 2 - \frac{2k(k-1)}{n-n_1+2k+2} > q(S^+_{n-n_1,k}).$$

We can apply this argument repeatedly, as long as the remaining graph  $H_i$  is of order at least 100k<sup>2</sup>. Assume that there exists some j such that  $|H_{j-1}| \ge 100k^2$ , while  $|H_j| < 100k^2$ . Set for short  $t := n_1 + \cdots + n_j$ , and note that

$$q(H_j) \ge q(G) - t + \frac{3t}{2n}.$$

Since  $C_s \nsubseteq H_j$  for any  $s \ge 2k + 2$ , Theorem 5 implies that

$$e(H_j) \le (k+1/2)(n-t-1)$$

and Das's bound (2) implies that

$$q(H_j) \le \frac{2e(H_j)}{n-t-1} + n - t - 2 \le n - t + 2k - 1.$$

In view of

$$q(H_j) \ge q(G) - t + \frac{3t}{2n} > n - t + 2k - 2 - \frac{2k(k-1)}{n+2k-3} + \frac{3t}{2n},$$

we have

$$t < \frac{7}{10}n,$$

which implies that  $|H| \ge 3n/10 > 100k^2$ , and this contradiction completes the proof of Lemma 12. 

Lemma 12 is crucial for the next theorem, where it will be used to improve the structure of a graph at the price of a moderate reduction of its order.

**Theorem 13** Let  $k \ge 2$ ,  $n \ge 400k^2$ , and let G be a graph of order n. If  $q(G) \ge q(S_{n,k}^+)$ and  $\Delta(G) = n - 1$ , then  $C_{2k+2} \subset G$ , unless  $G = S_{n,k}^+$ .

**Proof** Assume that  $C_{2k+2} \not\subseteq G$ . To prove the theorem we need to show that  $G = S_{n,k}^+$ . Let w be a dominating vertex of G. Applying Lemma 12, we can find an induced subgraph  $H \subset G$  with the following properties:

- H is  $C_{2k+2}$ -free;
- w is a dominating vertex in H;

$$-|H| \ge 3n/10 > 100k^2;$$

- every component of  $H_w$  is of order greater than  $3k^2$ ; -  $q(H) > q\left(S^+_{|H|,k}\right)$ , unless H = G and  $q(H) = q(G) = q\left(S^+_{n,k}\right)$ .

Thus, to complete the proof of Theorem 13 it is enough to prove the following statement:

**Theorem A** Let  $k \geq 2$ ,  $n \geq 100k^2$ , let G be a  $C_{2k+2}$ -free graph of order n, and let w be a dominating vertex of G such that each component of  $G_w$  is greater than  $3k^2$ . If  $q(G) \ge q(S_{n,k}^+)$ , then  $G = S_{n,k}^+$ .

We proceed with the proof of Theorem A, keeping the notation for G and n, although now  $n > 100k^2$ . Thus, assume that G and n satisfy the premises of Theorem A and  $q(G) \ge q(S_{n,k}^+)$ . We shall prove that  $G = S_{n,k}^+$ . For convenience choose G so that it has maximum Q-index among all graphs satisfying the premises of Theorem A. Next observe that inequality (4) implies that

$$e(G) \ge kn - k^2 + 1,$$

and so

$$e(G_w) = e(G) - n + 1 \ge (k - 1)n - k^2 + 2.$$
(7)

We shall dispose of the case k = 2 before anything else, as most of our arguments work for  $k \ge 3$  and need changes to work for k = 2.

Claim 1. Theorem A holds for k = 2.

Proof. If k = 2, then  $G_w$  consist of components of order greater than 12. Since  $\delta(G_w) \geq 1$  and  $P_5 \not\subseteq G_w$ , Lemma 9 implies that the components of  $G_w$  are of the two types given in clauses (i) and (ii). If  $G_i \subset S^+_{|G_i|,1}$ , we see that  $G_i = S^+_{|G_i|,1}$ , as q(G) is maximal. Now assume that  $G_i$  is a component of  $G_w$  consisting of two stars whose centers are joined by an edge; let u and v be the centers of these stars; let  $u_1, \ldots, u_s$  be the neighbors of u and  $v_1, \ldots, v_t$  be the neighbors of v. Let  $(x_1, \ldots, x_n)$  be a unit positive eigenvector to q(G); by symmetry suppose that  $x_u \geq x_v$ . Remove all edges  $\{v_i, v\}$  and join  $v_1, \ldots, v_t$  to u. In this way  $G_i$  is transformed into a star  $S_{s+t+2,1}$ ; now make it an  $S^+_{s+t+2,1}$  by adding an edge to it. A brief calculation shows that the resulting graph G' satisfies q(G') > q(G), which contradicts the choice of G. Hence each component  $G_i$  of  $G_w$  satisfies  $G_i = S^+_{|G_i|,1}$ .

Finally, if  $G_w$  has two components  $G_1 = S_{|G_1|,1}^+$  and  $G_2 = S_{|G_2|,1}^+$ , replace them by a component  $S_{|G_1|+|G_2|,1}^+$  and write G'' for the resulting graph. Clearly  $C_6 \notin G''$  and we shall show that q(G'') > q(G), which contradicts our choice of G. Let u and v be the dominating vertices in  $G_1$  and  $G_2$ . Let  $\mathbf{x} = (x_1, \ldots, x_n)$  be a positive unit eigenvector to q(G). By symmetry we may suppose that  $x_u \geq x_v$ . Now remove all edges of  $G_2$  and join all vertices of  $G_2$  to u. In this way  $G_1$  and  $G_2$  are replaced by a single component  $S_{|G_1|+|G_2|,1}^+$ . For short, let  $n_1 := |G_1|$ ,  $n_2 := |G_2|$ , and q := q(G). Let  $W := V(S_{n_2,k-1}^+) \setminus \{v\}$  and let  $v', v'' \in W$  be the two exceptional vertices of  $G_2$  such that  $\{v', v''\} \in E(G)$ . By symmetry,  $x_{v'} = x_{v''}$  and from the eigenequations for q we see that

$$qx_{v'} = 3x_{v'} + x_{v''} + x_v + x_w = 4x_{v'} + x_v + x_w,$$
  

$$qx_v = n_2 x_v + x_{v'} + x_{v''} + x_w + \sum_{s \in W \setminus \{v', v''\}} x_s$$
  

$$> n_2 x_v + 2x_{v'} + x_w.$$

Excluding  $x_w$  from these relations, after some algebra we see that

$$(q - n_2 + 1) x_v > (q - 2) x_v$$

and so  $x_{v''} = x_{v'} < x_v$ . A brief calculation shows that the resulting graph G' satisfies q(G') > q(G), which contradicts the choice of G. This contradiction shows that  $G_w$  has only one component and so  $G = S_{n,2}^+$ , completing the proof of Claim 1.

To the end of the proof we shall assume that  $k \geq 3$ .

**Claim 2.** There exists an induced subgraph H of  $G_w$  such that  $\delta(H) \ge k - 1$  and  $|H| \ge n - k^2 + k$ .

*Proof.* Define a sequence of graphs,  $F_0 \supset F_1 \supset \cdots \supset F_r \supset \cdots$  using the following procedure:

$$\begin{split} F_0 &:= G_w;\\ i &:= 0;\\ \textbf{while } \delta(F_i) < k-1 \textbf{ do begin}\\ &\text{ select a vertex } v \in V(F_i) \text{ with } d(v) = \delta(F_i);\\ F_{i+1} &:= F_i - v;\\ &i &:= i+1; \end{split}$$

end.

Note that for each r = 0, 1, ..., we have  $|F_r| = n - r - 1$  and  $P_{2k+1} \nsubseteq F_r$ ; thus Theorem 4 implies that

$$e(F_r) \le \left(k - \frac{1}{2}\right)(n - r - 1)$$

On the other hand, in view of (7), we find that

$$e(F_r) = e(G_w) - \sum_{i=0}^{r-1} \delta(F_i) \ge e(G_w) - r(k-2)$$
  
$$\ge (k-1)n - k^2 + 2 - r(k-2).$$
(8)

Hence,

$$(k-1)n - k^{2} + 2 - r(k-2) \le \left(k - \frac{1}{2}\right)(n-r-1),$$

and after some algebra we find that

$$3r \le n + 2k^2 - 2k - 3 < 2n.$$

that is to say, the procedure stops before  $i \ge 2n/3$ . Next, with a more involved argument, we shall show that the procedure stops before  $i > k^2 - k - 1$ .

Let  $H = F_r$ , where r is the last value of the variable i. Let  $H_i$  be a component of H and set  $n_i := |H_i|$ . We claim that  $e(H_i) \leq (k-1)n_i$ .

Indeed if  $n_i \leq 2k - 1$ , then

$$e(H_i) \le \frac{n_i(n_i-1)}{2} \le (k-1)n_i$$

If  $n_i = 2k$  and  $H_i$  is Hamiltonian, then  $H_i$  is a component of  $G_w$  as otherwise  $P_{2k+1} \subset G_w$ . But all components of  $G_w$  are of order at least  $3k^2 > 2k$ , so  $H_i$  is not Hamiltonian. In this case, Ore's theorem [13] implies that

$$e(H_i) \le \frac{(n_i - 1)(n_i - 2)}{2} + 1 \le (k - 1)n_i.$$

If  $n \ge 2k+1$ , in view of  $P_{2k+1} \nsubseteq H_i$  and  $\delta(H_i) \ge k-1$ , it follows that  $H_i$  satisfies one of the clauses of Theorem 7, and so Corollary 8 implies that  $e(H_i) \le (k-1)n_i$ . Summing over all components of H, we find that

$$e(H) \le (k-1)(n-r-1).$$

On the other hand, in view of (8) we find that

$$(k-1)n - k^{2} + 2 - r(k-2) \le (k-1)(n-r-1)$$

and so,  $r \leq k^2 - k - 1$ . Therefore, H satisfies the requirements of Claim 2, which is thus proved.

Let H' be the subgraph of  $G_w$  induced by the vertex set  $V(G_w) \setminus V(H)$ , which may be empty. Let  $H_1, \ldots, H_p$  be the components of H and let  $n_1, \ldots, n_p$  be their orders.

**Claim 3.** Each component  $H_i$  of H satisfies  $H_i \subset S^+_{|H_i|,k=1}$ .

*Proof.* First note that each component of  $G_w$  contains at most one component of H. Indeed, since for each component  $H_i$  of H, we have  $\delta(H_i) \ge k - 1 \ge 2$ , Dirac's Theorem 3 implies that  $C_l \subset H_i$ , for some  $l \ge k$ ; hence each component of  $G_w$  contains at most one component of H, as otherwise  $P_{2k+1} \subset G_w$ .

Further, if  $H_i$  is a component of H and  $G_i$  is a component of  $G_w$  containing  $H_i$ , there are at most  $k^2 - k - 1$  vertices in  $V(G_i) \setminus V(H_i)$ ; since  $|G_i| > 3k^2$ , the order of  $H_i$  satisfies

$$|H_i| > 3k^2 - k^2 + k + 1 > 2k^2.$$

Assume for a contradiction that  $H_i$  is a component of H such that  $H_i \not\subseteq S^+_{n_i,k-1}$ . Note that  $n_i > 2k^2 > 2k + 1$ ,  $P_{2k+1} \not\subseteq H_i$ , and  $\delta(H_i) \ge k - 1$ . Using Theorem 7, we see that  $H_i$  is one of the graphs from clauses *(ii)*, *(iii)* or *(iv)*. Now Corollary 8 implies that

$$e\left(H_{i}\right) \leq \frac{kn_{i}}{2}.$$

Since *H* is  $P_{2k+1}$ -free, we have

$$e(H) = e(H_i) + e(V(H) \setminus V(H_i)) \le \frac{kn_i}{2} + (k-1)(n-1-|H'|-n_i).$$

and from (8) we know that

$$e(H) \ge (k-1)n - k^2 + 2 - (k-2)|H'|.$$

After some algebra we see that

$$k^{2} - k - 1 \ge \left(\frac{1}{2}k - 1\right)n_{i} + |H'| \ge k^{2},$$

a contradiction, showing that  $H_i \subset S^+_{n_i,k-1}$ , and completing the proof of Claim 3.

Our next goal is to prove that each component J of  $G_w$  is isomorphic to  $S^+_{|J|,k-1}$ . Since G has maximal q(G), it is enough to prove that each component J of  $G_w$  satisfies  $J \subset S^+_{|J|,k-1}$ .

 $J \subset S^+_{|J|,k-1}$ . Let J be a component of  $G_w$ . Note that J contains exactly one component F of H, as otherwise it would consist solely of vertices from H', which are at most  $k^2 - k - 1$ , while J has more than  $3k^2$  vertices. Set m := |F|; Claim 3 implies that  $F \subset S^+_{m,k-1}$ . Write A for the set of k-1 dominating vertices of  $S^+_{m,k-1}$ ; let  $B := V(F) \setminus A$  and

$$C := V(J) \setminus V(F).$$

Since  $\delta(F) \geq k-1$ , the bipartite subgraph of F induced by the vertex classes A and B contains at least |A| |B| - 2 edges. If G[B] contains an edge, then each vertex in B is endvertex of a path  $P_{2k} \subset F$ ; since  $P_{2k+1} \not\subseteq J$ , each vertex of C may be joined only to vertices from A. Therefore,  $J \subset S^+_{|J|,k-1}$  as long as G[B] contains an edge.

Assume therefore that the set B is independent. Together with  $\delta(F) \ge k - 1$ , this assumption implies that A and B induce a complete bipartite graph in G.

Claim 4. The set C is independent.

*Proof.* Let

$$C_A := \{ u : u \in C \text{ and } \Gamma(u) \cap A \neq \emptyset \},\$$
  

$$C_B := \{ u : u \in C \text{ and } \Gamma(u) \cap B \neq \emptyset \},\$$
  

$$C' := C \setminus C_B.$$

Our main goal is to prove that the set C' is independent, which easily implies that C is independent as well. Assume for a contradiction that G[C'] contains edges. This fact implies that  $C_B = \emptyset$ , as  $P_{2k+1} \nsubseteq J$ . For the same reason we see that G[C'] contains no  $P_4$  or cycles.

Further, G[C'] contains no isolated vertices. Indeed, if  $u \in C'$  and u is an isolated vertex in G[C'], then it has to be joined to all vertices of A as q(G) is maximal; we see that u has k-1 neighbors in H and so u cannot be removed by the procedure of Claim 2, a contradiction.

Hence G[C'] is a disjoint union of edges and stars. Note that if S is a star in G[C'] of order at least 3, then its center belongs to  $C_A$ , but no other vertex of S belongs to  $C_A$ , as  $P_{2k+1} \not\subseteq J$ .

Next, assume that G[C'] contains a star S of order  $t \ge 3$ , such that its center i is joined to exactly one vertex  $u \in A$ ; let  $u_1, \ldots, u_{t-1}$  be the peripheral vertices of S. Remove the edges  $\{u_1, i\}, \ldots, \{u_{t-1}, i\}$  and add the edges  $\{u_1, u\}, \ldots, \{u_{t-1}, u\}$ ; write G' for the

resulting graph, which obviously satisfies the hypothesis of Theorem A. We shall show that q(G') > q(G). First, by symmetry,

$$x_{u_1} = \dots = x_{u_{t-1}} = p.$$

Next we have

$$qp = 2p + x_i + x_w,$$
  

$$qx_u \ge (m+1)x_u + x_i + x_w,$$

and after some algebra we find that  $x_u > p$ . Also,

$$qx_{i} = (t+1)x_{i} + (t-1)p + x_{u} + x_{w} < (t+1)x_{i} + tx_{u} + x_{w},$$

and after some algebra we find that

$$(q - m - 1 + t) x_u > (q - t) x_i,$$

implying that  $x_u > x_i$ . Now a brief calculation shows that q(G') > q(G), contradicting the choice of G.

Hence, if  $S \subset G[C']$  is a star of order at least 3, then is center is joined to more than one vertex in A.

Next assume that G[C'] is connected. If G[C'] is just one edge, then  $J \subset S^+_{m,k-1}$ , and so  $J = S^+_{m,k-1}$  as q(G) is maximal. If G[C'] is a star S of order at least 3, then its center i is joined to more than one vertices A. Since q(G) is maximal, i must be joined to all vertices in A; thus i has k - 1 neighbors in H and so i cannot be removed by the procedure of Claim 2, a contradiction. So G[C'] has more than one component.

Finally, assume that u is a vertex of A having a neighbor in C'. If G[C'] contains a star S, then the center of S may be joined only to u, as otherwise we can find a  $P_{2k+1} \subset J$  using an additional component of G[C']. Hence G[C'] contains only disjoint edges. Clearly each edge of G[C'] contains a vertex of  $C_A$  and all such vertices must be joined exactly to u as  $P_{2k+1} \not\subseteq J$ . Since q(G) is maximal, we see that A induces a complete graph, and both ends of each disjoint edge in G[C'] are joined to u. We shall show that in this case q(G) is not maximal.

Indeed, let  $v \in A \setminus \{u\}$  and let  $\mathbf{x} = (x_1, \ldots, x_n)$  be a positive unit eigenvector to q(G). Suppose that  $\{i, j\}$  is an isolated edge in G[C']. Remove  $\{i, j\}$ , add the edges  $\{i, v\}$  and  $\{j, v\}$ ; write G' for the resulting graph, which obviously satisfies the hypothesis of Theorem A. By symmetry,  $x_i = x_j$ ; note that

$$\begin{split} qx_v &= mx_v + \sum_{s \in A \cup B \setminus \{v\}} x_s + x_w > mx_v + x_u + x_w, \\ qx_i &= 3x_i + x_u + x_j + x_w = 4x_i + x_u + x_w. \end{split}$$

After some algebra we find that

$$x_v > \frac{q-4}{q-m} x_i > \frac{q-4}{q-2k^2} x_i > x_i,$$

and a brief calculation shows that q(G') > q(G), contradicting the choice of G. This completes the proof that C' is independent. Therefore C is also independent, as no edge in C can be incident to a vertex in B as  $P_{2k+1} \not\subseteq J$ . This completes the proof of Claim 4.

Further, we can assume that  $C' = \emptyset$ , as if u is vertex in C', then, in  $G_w$ , u can be joined only to vertices of A; since q(G) is maximal, u is joined to each vertex in A; thus i has k - 1 neighbors in H and so i cannot be removed by the procedure of Claim 2, a contradiction.

Claim 5. Either  $J \subset S_{m+1,k-1}^+$  or the set C is empty.

*Proof.* Observe that if a vertex  $u \in C_B$  is joined to two or more vertices from B, then  $P_{2k+1} \subset J$ , so each vertex in  $C_B$  is joined to exactly one vertex in B. Now if  $C_B$  has two distinct vertices that are joined to two distinct vertices in B, then clearly  $P_{2k+1} \subset J$ . Therefore, all vertices in  $C_B$  are joined to the same vertex of B, say  $u \in B$ .

Suppose that  $C_A \neq \emptyset$  and let  $v \in C_A$ . Clearly  $C_B \setminus C_A = \emptyset$ , as  $P_{2k+1} \nsubseteq J$ ; therefore  $C_A = C_B = \{v\}$ , implying that  $J \subset S_{m+1,k-1}^+$ .

Hence we may assume that  $C_A = \emptyset$ , that is to say, all vertices in C are joined only to vertices in B. Now the graph J looks as follows: the set C is independent and all vertices of C are joined exactly to the vertex  $u \in B$ . We shall show that in this case q(G) is not maximal.

Indeed, choose a vertex  $v \in A$  and let  $C = \{u_1, \ldots, u_t\}$ ; remove the edges  $\{u_1, u\}, \ldots, \{u_t, u\}$ and add the edges  $\{u_1, v\}, \ldots, \{u_t, v\}$ ; write G' for the resulting graph, which satisfies the hypothesis of Theorem A. We shall show that  $x_v > x_u$ , which obviously implies that q(G') > q(G), contradicting the choice of G. Note that by symmetry,

$$x_{u_1} = \cdots = x_{u_t}$$
 and  $x_s = x_v$  for every  $s \in A$ .

Therefore, letting  $x_{u_1} = p$ , we see that

$$qx_{v} > mx_{v} + x_{u} + x_{w}, qx_{u} = (k+t) x_{u} + (k-1) x_{v} + tp + x_{w}, qp = 2p + x_{u} + x_{w}.$$

After some algebra we first find that  $x_v > p$ , and then  $x_v > x_u$ , as claimed. This completes the proof of Claim 5.

At this stage we see that each component J of  $G_w$  satisfies  $J = S^+_{|J|,k-1}$ ; to finish the proof we must show that  $G_w$  has only one component. Assume for a contradiction that  $G_w$  contains two components, say  $G_1 = S^+_{n_1,k-1}$  and  $G_2 = S^+_{n_2,k-1}$ . We shall show that in this case q(G) is not maximal, which contradicts the choice of G.

Let  $u_1, \ldots, u_{k-1}$  and  $v_1, \ldots, v_{k-1}$  be the dominating vertices in  $G_1$  and  $G_2$ . Let  $\mathbf{x} = (x_1, \ldots, x_n)$  be a positive unit eigenvector to q(G). By symmetry,

$$x_{u_1} = \dots = x_{u_{k-1}}, \quad x_{v_1} = \dots = x_{v_{k-1}}, \text{ and } x_{u_1} \ge x_{v_1}.$$

Now merge the components  $G_1$  and  $G_2$  into one component F by removing all edges of  $G_2$  and joining the vertices of  $G_2$  to each of the vertices  $u_1, \ldots, u_{k-1}$ . Note that F is

isomorphic to  $S_{n_1+n_2, k-1}^+$  and  $u_1, \ldots, u_{k-1}$  are its dominating vertices. Writing G' for the new graph, we shall show that q(G') > q(G). Indeed, let

$$W := V\left(S_{n_2,k-1}^+\right) \setminus \{v_1,\ldots,v_{k-1}\}$$

and let  $v', v'' \in W$  be the two exceptional vertices of  $G_2$  such that  $\{v', v''\} \in E(G)$ . By symmetry,  $x_{v'} = x_{v''}$  and from the eigenequations for q we see that

$$qx_{v'} = (k+1)x_{v'} + x_{v''} + (k-1)x_{v_1} + x_w = (k+2)x_{v'} + (k-1)x_{v_1} + x_w,$$
  

$$qx_{v_1} = n_2x_{v_1} + (k-2)x_{v_1} + x_{v'} + x_{v''} + x_w + \sum_{s \in W \setminus \{v', v''\}} x_s$$
  

$$> (n_2 + k - 2)x_{v_1} + 2x_{v'} + x_w.$$

Excluding  $x_w$  from these relations, after some algebra we see that

$$(q - n_2 + 1) x_{v_1} > (q - k) x_{v'}$$

and so  $x_{v''} = x_{v'} < x_{v_1}$ . Further,

$$q(G') - q(G) \ge \langle Q(G') \mathbf{x}, \mathbf{x} \rangle - \langle Q(G) \mathbf{x}, \mathbf{x} \rangle$$
  
=  $\sum_{i \in W} (k - 1) ((x_{u_1} + x_i)^2 - (x_{v_1} + x_i)^2)$   
+  $(k - 1)^2 (x_{u_1} + x_{v_1})^2 - 2 (k - 1) (k - 2) x_{v_1}^2 - 4x_{v'}^2$   
>  $4 (k - 1)^2 x_{v_1}^2 - 2 (k - 1) (k - 2) x_{v_1}^2 - 4x_{v_1}^2 \ge 0.$ 

This contradiction shows that indeed,  $G_w = S_{n-1,k-1}^+$  and so  $G = S_{n,k}^+$ . Theorem A is proved and so is Theorem 13.

### 3.1 Proof of Theorem 2

**Proof** Assume for a contradiction that G is a  $C_{2k+2}$ -free graph of order  $n > 400k^2$ , with  $q(G) \ge q(S_{n,k}^+)$ . To prove the theorem we shall show that  $G = S_{n,k}^+$ . For short, set q := q(G) and V := V(G).

Our proof of Theorem 2 will be based on a careful analysis of the Merris bound (1). Thus, let  $w \in V$  be a vertex for which the expression

$$d(w) + \frac{1}{d(w)} \sum_{\{w,i\} \in E(G)} d(i)$$

is maximal. First note that  $d(w) \ge 2k - 1$ , as otherwise, using (1), we obtain a contradiction

$$q(G) \le d(w) + \frac{1}{d(w)} \sum_{\{w,i\} \in E(G)} d(i) \le d(w) + \Delta(G) \le n + 2k - 3 < q(S_{n,k}^+).$$

We shall show that  $d(w) \ge n-2$ . Indeed, set  $A := \Gamma(w)$  and  $B := V(G) \setminus (\Gamma(w) \cup \{w\})$ . Obviously, |A| = d(w) and |B| = n - d(w) - 1. The assumption  $C_{2k+2} \nsubseteq G$  implies that the graph  $G_w$  contains no path  $P_{2k+1}$  with both endvertices in A. Therefore, using Lemma 6, we see that

$$d(w) + \frac{1}{d(w)} \sum_{\{w,i\} \in E(G)} d(i) = d(w) + 1 + \frac{2e(A) + e(A, B)}{d(w)}$$
$$\leq d(w) + 1 + \frac{(2k-1)d(w) + k(n-d(w)-1)}{d(w)}$$
$$= d(w) + k + \frac{k(n-1)}{d(w)}.$$

Note that the function x + [k(n-1) - 1]/x is convex for x > 0; hence, the maximum of the expression

$$d\left(w\right) + \frac{k\left(n-1\right)}{d\left(w\right)}$$

is attained for the minimum or the maximum admissible values for d(w). Thus, if

$$2k - 1 \le d(w) \le n - 3,$$

then

$$q(G) \le d(w) + \frac{k(n-1)}{d(w)} \le \max\left\{2k - 1 + \frac{k(n-1)}{2k-1}, n-3 + \frac{k(n-1)}{n-3}\right\}$$
  
$$< n + 2k - 2 - \frac{2(k^2 - k)}{n+2k-3} < q\left(S_{n,k}^+\right),$$

a contradiction, showing that  $d(w) \ge n-2$ .

At that stage we are left with two cases: d(w) = n - 1, covered by Theorem 13 and d(w) = n - 2, which will be disposed of in the rest of the proof.

Let v be the single vertex of G such that  $v \notin \Gamma(w)$ . Let G' be the graph obtained by adding the edge  $\{w, v\}$  to G. Since  $\Delta(G') = n - 1$ , and  $q(G') \ge q(G) \ge q(S_{n,k}^+)$ , Theorem 13 implies that G' contains a cycle  $C_{2k+2}$ , which obviously contains the edge  $\{w, v\}$ . Hence, G contains a path  $P_{2k+2}$  with endvertices w and v, and moreover, w is adjacent to all vertices of this path except v. This is a definite situation, and it is easy to see that if  $d(v) \ge 2$ , then  $C_{2k+2} \subset G$ ; hence, d(v) = 1.

Write u for the neighbor of v, and let  $\mathbf{x} = (x_1, ..., x_n)$  be the unit positive eigenvector to q. The eigenequation for the vertex v gives

$$qx_v = x_v + x_u$$

and so

$$x_v = \frac{1}{q-1}x_u.$$

Since  $d(u) \leq n-2$ , there is a vertex  $t \in \Gamma(w) \setminus (\Gamma(u) \cup \{u\})$ . Then, from the eigenequation for t we see that

$$qx_t \ge x_t + x_w$$

and so

$$x_t \ge \frac{1}{q-1} x_w$$

The eigenequation for the vertex w is

$$qx_w = (n-2)x_w + \sum_{i \in V \setminus \{w,v\}} x_i,$$
(9)

while the eigenequation for the vertex u implies that

$$qx_{u} = d(u)x_{u} + \sum_{i \in \Gamma(u)} x_{i} \le (n-2)x_{u} + \sum_{i \in V \setminus \{u,t\}} x_{i}.$$
 (10)

Subtracting (10) from (9), we find that

$$(q-n+3)(x_w - x_u) \ge x_t - x_v \ge \frac{1}{q-1}x_w - \frac{1}{q-1}x_u$$

and so,  $x_w \ge x_u$ .

Let G' be the graph obtained from G by removing the edge  $\{u, v\}$  and adding the edge  $\{w, v\}$ . Comparing the quadratic forms of Q(G) and Q(G'), we find that  $q(G') \ge q(G) \ge q(S_{n,k}^+)$ . However,  $G' \ne S_{n,k}^+$  and  $\Delta(G') = n - 1$ ; hence Theorem 13 implies that  $C_{2k+2} \subset G'$ , and consequently  $C_{2k+2} \subset G$ , as no cycle of G' contains v. This contradiction completes the proof of Theorem 2.

## 4 Concluding remarks

Theorem 2 and the main result of [14],[15] prove completely Conjecture 1. An important ingredient of our proof, Theorem 7, which is a nonspectral extremal result, has been obtained in [10]. We would like to reiterate a similar, but yet unproven conjecture for the spectral radius  $\mu(G)$ , raised in [9].

**Conjecture 14** Let  $k \ge 2$  and let G be a graph of sufficiently large order n. If G has no cycle of length 2k + 2, then  $\mu(G) < \mu(S_{n,k}^+)$ , unless  $G = S_{n,k}^+$ .

It is somewhat surprising that Conjecture 14 turned out to be more difficult than Conjecture 1, given that in general it is easier to work with the spectra radius than with the Q-index of a graph. Finally, let us note that the corresponding problem about the maximum number of edges in a  $C_{2k}$ -free graph of order n is notoriously difficult and is solved only for very few values of k.

## References

- B. Bollobás, Modern Graph Theory, Graduate Texts in Mathematics, 184, Springer-Verlag, New York (1998).
- [2] D. Cvetković. Spectral theory of graphs based on the Report, available signless Laplacian, Research (2010),at: http://www.mi.sanu.ac.rs/projects/signless\_L\_reportApr11.pdf.
- [3] K. Das, Maximizing the sum of the squares of the degrees of a graph, *Discrete Math* **285** (2004), 57-66.
- [4] G.A. Dirac, Some theorems on abstract graphs. Proc. London Math. Soc. 2 (1952) 69-81.
- [5] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hungar 10 (1959), 337–356.
- [6] L. Feng and G. Yu, On three conjectures involving the signless laplacian spectral radius of graphs, Publ. Inst. Math. (Beograd) (N.S.) 85 (2009), 35-38.
- [7] M.A.A. de Freitas, V. Nikiforov, and L. Patuzzi, Maxima of the Q-index: forbidden 4-cycle and 5-cycle, *Electronic J. Linear Algebra* 26 (2013), 905-916.
- [8] V. Nikiforov, Degree powers in graphs with forbidden even cycle, *Electronic J. Com*bin. 15 (2009), R107.
- [9] V. Nikiforov, The spectral radius of graphs without paths and cycles of specified length, *LinearAlgebra Appl.* **432** (2010), 2243-2256.
- [10] V. Nikiforov and X.Y. Yuan, Maxima of the Q-index: graphs without long paths, Electronic J. Linear Algebra 27 (2014), 504-514.
- [11] V. Nikiforov, An asymptotically tight bound on the Q-index of graphs with forbidden cycles, Publ. Inst. Math. (Beograd) (N.S.) 95(109) (2014), 189-199.
- [12] R. Merris, A note on Laplacian graph eigenvalues, *Linear Algebra Appl.* 295 (1998), 33-35.
- [13] O. Ore, Note on Hamilton circuits, Amer. Math. Monthly 67 (1960), 55.
- [14] X.Y. Yuan, Maxima of the Q-index: forbidden odd cycles, *Linear Algebra and Appl.* 458 (2014), 207-216.
- [15] X.Y. Yuan, Maxima of the Q-index: forbidden odd cycles, Preprint available at ArXiv:1401.4363v3