arXiv:1410.2181v1 [math.RA] 8 Oct 2014

NON-COMMUTATIVE TERNARY NAMBU-POISSON ALGEBRAS AND TERNARY HOM-NAMBU-POISSON ALGEBRAS

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Abstract. The main purpose of this paper is to study non-commutative ternary Nambu-Poisson algebras and their Hom-type version. We provide construction results dealing with tensor product and direct sums of two (non-commutative) ternary (Hom-)Nambu-Poisson algebras. Moreover, we explore twisting principle of (non-commutative) ternary Nambu-Poisson algebras along with an algebra morphism that lead to construct (non-commutative) ternary Hom-Nambu-Poisson algebras. Furthermore, we provide examples and a 3-dimensional classification of non-commutative ternary Nambu-Poisson algebras.

Introduction

In the 70's, Nambu proposed a generalized Hamiltonian system based on a ternary product, the Nambu-Poisson bracket, which allows to use more that one hamiltonian [19]. More recent motivation for ternary brackets appeared in string theory and M-branes, ternary Lie type structure was closely linked to the supersymmetry and gauge symmetry transformations of the world-volume theory of multiple coincident M2-branes and was applied to the study of Bagger-Lambert theory. Moreover ternary operations appeared in the study of some quarks models. In 1996, quantization of Nambu-Poisson brackets were investigated in [11], it was presented in a novel approach of Zariski, this quantization is based on the factorization on $\mathbb{R}$ of polynomials of several variables.

The algebraic formulation of Nambu mechanics was discussed in [23] and Nambu algebras was studied in [13] as a natural generalization of a Lie algebra for higher-order algebraic operations. By definition, Nambu algebra of order n over a field $\mathbb{K}$ of characteristic zero consists of a vector space V over $\mathbb{K}$ together with a $\mathbb{K}$-multilinear skew-symmetric operation $[.,\cdots,.] : \Lambda^n V \to V$, called the Nambu bracket, that satisfies the following generalization of the Jacobi identity. Namely, for any $x_1,...,x_{n-1} \in V$ define an adjoint action $ad(x_1,...,x_{n-1}) : V \to V$ by $ad(x_1,...,x_{n-1})x_n = [x_1,...,x_{n-1},x_n], x_n \in V$.

Then the fundamental identity is a condition saying that the adjoint action is a derivation with respect the Nambu bracket, i.e. for all $x_1,...,x_{n-1},y_1,...,y_n \in V$

$$ad(x_1,...,x_{n-1})[y_1,...,y_n] = \sum_{k=1}^{n}[y_1,...,ad(x_1,...,x_{n-1})y_k,...,y_n]. \tag{0.1}$$

2010 *Mathematics Subject Classification.* 17A42,17A30,17A40,17B63.

Key words and phrases. (non commutative) ternary Nambu-Poisson algebra, (non commutative) ternary Hom-Nambu-Poisson algebra, ternary Hom-Nambu-Poisson morphism.

When $n=2$, the fundamental identity becomes the Jacobi identity and we get a definition of a Lie algebra.

Different aspects of Nambu mechanics, including quantization, deformation and various algebraic constructions for Nambu algebras have recently been studied. Moreover a twisted generalization, called Hom-Nambu algebras, was introduced in [5]. This kind of algebras called Hom-algebras appeared as deformation of algebras of vector fields using σ-derivations. The first examples concerned q-deformations of Witt and Virasoro algebras. Then Hartwig, Larsson and Silvestrov introduced a general framework and studied Hom-Lie algebras [16], in which Jacobi identity is twisted by a homomorphism. The corresponding associative algebras, called Hom-associative algebras was introduced in [17]. Non-commutative Hom-Poisson algebras was discussed in [28]. Likewise, n-ary algebras of Hom-type was introduced in [5], see also [1, 2, 3, 26, 27].

We aim in this paper to explore and study non-commutative ternary Nambu-Poisson algebras and their Hom-type version. The paper includes five Sections. In the first one, we summarize basic definitions of (non-commutative) ternary Nambu-Poisson algebras and discuss examples. In the second Section, we recall some basics about Hom-algebra structures and introduce the notion of (non-commutative) ternary Hom-Nambu-Poisson algebra. Section 3 is dedicated to construction of (non-commutative) ternary Hom-Nambu-Poisson algebras using direct sums and tensor products. In Section 4, we extend twisting principle to ternary Hom-Nambu-Poisson algebras. It is used to build new structures with a given ternary (Hom-)Nambu-Poisson algebra and an algebra morphism. This process is used to construct ternary Hom-Nambu-Poisson algebras corresponding to the ternary algebra of polynomials where the bracket is defined by the Jacobian. We provide in the last section a classification of 3-dimensional ternary Nambu-Poisson algebras and corresponding Hom-Nambu-Poisson algebras using twisting principle.

1. Ternary (Non-commutative) Nambu-Poisson algebra

In the section we review some basic definitions and fix notations. In the sequel, A denotes a vector space over $\mathbb{K}$, where $\mathbb{K}$ is an algebraically closed field of characteristic zero. Let $\mu : A\times A \to A$ be a bilinear map, we denote by $\mu^{op} : A^{\times 2} \to A$ the opposite map, i.e., $\mu^{op} = \mu\circ\tau$ where $\tau : A^{\times 2} \to A^{\times 2}$ interchanges the two variables. A ternary algebra is given by a pair (A, m), where m is a ternary operation on A, that is a trilinear map $m : A\times A\times A \to A$, which is denoted sometimes by brackets.

Definition 1.1. A *ternary Nambu algebra* is a ternary algebra $(A, \{\ ,\ ,\ \})$ satisfying the fundamental identity defined as

$$\{x_1, x_2, \{x_3, x_4, x_5\}\} = \\ \{\{x_1, x_2, x_3\}, x_4, x_5\} + \{x_3, \{x_1, x_2, x_4\}, x_5\} + \{x_3, x_4, \{x_1, x_2, x_5\}\} \tag{1.1}$$

for all $x_1, x_2, x_3, x_4, x_5 \in A$.

This identity is sometimes called Filippov identity or Nambu identity, and it is equivalent to the identity (0.1) with $n=3$.

A *ternary Nambu-Lie algebra* or 3-Lie algebra is a ternary Nambu algebra for which the bracket is skew-symmetric, that is for all $\sigma \in S_3$, where S_3 is the permutation group,

$$[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = Sgn(\sigma)[x_1, x_2, x_3].$$

Let A and A' be two ternary Nambu algebras (resp. Nambu-Lie algebras). A linear map $f : A \to A'$ is a *morphism* of a ternary Nambu algebras (resp. ternary Nambu-Lie algebras) if it satisfies

$$f(\{x, y, z\}_A) = \{f(x), f(y), f(z)\}_{A'}.$$

Example 1.2. *The polynomials of variables x_1, x_2, x_3 with the ternary operation defined by the Jacobian function:*

$$\{f_1, f_2, f_3\} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{vmatrix}, \tag{1.2}$$

is a ternary Nambu-Lie algebra.

Example 1.3. *Let $V = \mathbb{R}^4$ be the 4-dimensional oriented Euclidian space over $\mathbb{R}$. The bracket of 3 vectors $\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}$ is given by*

$$[x, y, z] = \overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} = \begin{vmatrix} x_1 & y_1 & z_1 & e_1 \\ x_2 & y_2 & z_2 & e_2 \\ x_3 & y_3 & z_3 & e_3 \\ x_4 & y_4 & z_4 & e_4 \end{vmatrix},$$

where $\{e_1, e_2, e_3, e_4\}$ is a basis of $\mathbb{R}^4$ and $\overrightarrow{x} = \sum_{i=1}^{3} x_i \overrightarrow{e_i}$, $\overrightarrow{y} = \sum_{i=1}^{3} y_i \overrightarrow{e_i}$ and $\overrightarrow{z} = \sum_{i=1}^{3} z_i \overrightarrow{e_i}$. Then $(V, [.,.,.])$ is a ternary Nambu-Lie algebra.

Now, we introduce the notion of (non-commutative) ternary Nambu-Poisson algebra.

Definition 1.4. A *non-commutative ternary Nambu-Poisson algebra* is a triple $(A, \mu, \{.,.,.\})$ consisting of a $\mathbb{K}$-vector space A, a bilinear map $\mu : A \times A \to A$ and a trilinear map $\{.,.,.\} : A \otimes A \otimes A \to A$ such that

1. (A, μ) is a binary associative algebra,
2. $(A, \{.,.,.\})$ is a ternary Nambu-Lie algebra,
3. the following Leibniz rule

$$\{x_1, x_2, \mu(x_3, x_4)\} = \mu(x_3, \{x_1, x_2, x_4\}) + \mu(\{x_1, x_2, x_3\}, x_4)$$

holds for all $x_1, x_2, x_3 \in A$.

A ternary Nambu-Poisson algebra is a non-commutative ternary Nambu-Poisson algebra $(A, \mu, \{.,.,.\})$ for which μ is commutative, then μ is commutative unless otherwise stated.
In a (non-commutative) ternary Nambu-Poisson algebra, the ternary bracket $\{.,.,.\}$ is called Nambu-Poisson bracket.
Similarly, a non-commutative n-ary Nambu-Poisson algebra is a triple $(A, \mu, \{., \cdots, .\})$ where $(A, \{., \cdots, .\})$ defines an n-Lie algebra satisfying similar Leibniz rule with respect to μ.
A morphism of (non-commutative) ternary Nambu-Poisson algebras is a linear map that is a morphism of the underlying ternary Nambu-Lie algebras and associative algebras.

Example 1.5. *Let $C^\infty(\mathbb{R}^3)$ be the algebra of C^∞ functions on $\mathbb{R}^3$ and x_1, x_2, x_3 the coordinates on $\mathbb{R}^3$. We define the ternary brackets as in* (1.2)*, then $(C^\infty(\mathbb{R}^3), \{.,.,.\})$ is a ternary Nambu-Lie algebra. In addition the bracket satisfies the Leibniz rule: $\{fg, f_2, f_3\} = f\{g, f_2, f_3\} + \{f, f_2, f_3\}g$ where $f, g, f_2, f_3 \in C^\infty(\mathbb{R}^3)$ and the multiplication being the pointwise multiplication that is $fg(x) = f(x)g(x)$. Therefore, the algebra is a ternary Nambu-Poisson algebra.*
This algebra was considered already in 1973 by Nambu [19] *as a possibility of extending the Poisson bracket of standard hamiltonian mechanics to bracket of three functions defined by the Jacobian. Clearly, the Nambu bracket may be generalized further to a Nambu-Poisson allowing for an arbitrary number of entries.*
In particular, the algebra of polynomials of variables x_1, x_2, x_3 with the ternary operation defined by the Jacobian function in (1.2)*, is a ternary Nambu-Poisson algebra.*

Remark 1.6. The n-dimensional ternary Nambu-Lie algebra of Example 1.3 does not carry a non-commutative Nambu-Poisson algebra structure except that one given by a trivial multiplication.

2. Hom-type (non commutative) ternary Nambu-Poisson algebras

In this section, we present various Hom-algebra structures. The main feature of Hom-algebra structures is that usual identities are deformed by an endomorphism and when the structure map is the identity, we recover the usual algebra structure.

A Hom-algebra (resp. ternary Hom-algebra) is a triple (A, ν, α) consisting of a $\mathbb{K}$-vector space A, a bilinear map $\nu : A \times A \to A$ (resp. a trilinear map $\nu : A \times A \times A \to A$) and a linear map $\alpha : A \to A$. A binary Hom-algebra (A, μ, α) is said to be multiplicative if $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$ and it is called commutative if $\mu = \mu^{op}$. A ternary Hom-algebra (A, m, α) is said to be multiplicative if $\alpha \circ m = m \circ \alpha^{\otimes 3}$. Classical binary (resp. ternary) algebras are regarded as binary (resp. ternary) Hom-algebras with identity twisting map. We will often use the abbreviation xy for $\mu(x, y)$ when there is no ambiguity. For a linear map $\alpha : A \to A$, denote by α^n the n-fold composition of n-copies of α, with $\alpha^0 \equiv Id$.

Definition 2.1. A Hom-algebra (A, μ, α) is a *Hom-associative algebra* if it satisfies the Hom-associativity condition, that is
$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)) \text{ for all } x, y, z \in A.$$

Remark 2.2. When α is the identity map, we recover the classical associativity condition, then usual associative algebras.

Definition 2.3. A *ternary Hom-Nambu algebra* is a triple $(A, \{.,.,.\}, \widetilde{\alpha})$ consisting of a $\mathbb{K}$-vector space A, a ternary map $\{.,.,.\} : A \times A \times A \to A$ and a pair of linear maps $\widetilde{\alpha} = (\alpha_1, \alpha_2)$ where $\alpha_1, \alpha_2 : A \to A$ satisfying

$$\begin{aligned}\{\alpha_1(x_1), \alpha_2(x_2), \{x_3, x_4, x_5\}\} = \{\{x_1, x_2, x_3\}, \alpha_1(x_4), \alpha_2(x_5)\}+ \\ \{\alpha_1(x_3), \{x_1, x_2, x_4\}, \alpha_2(x_5)\} + \{\alpha_1(x_3), \alpha_2(x_4), \{x_1, x_2, x_5\}\}.\end{aligned} \tag{2.1}$$

We call the above condition the ternary Hom-Nambu identity.

Generally, the n-ary Hom-Nambu algebras are defined by the following Hom-Nambu identity

$$\{\alpha_1(x_1),...,\alpha_{n-1}(x_{n-1}),\{x_n,...,x_{2n-1}\}\}$$
$$=\sum_{i=n}^{2n-1}\{\alpha_1(x_n),...,\alpha_{i-n}(x_{i-1}),\{x_1,...,x_{n-1},x_i\},\alpha_{i-n+1}(x_{i+1})...,\alpha_{n-1}(x_{2n-1})\}$$

for all $(x_1,...,x_{2n-1})\in A^{2n-1}$.

Remark 2.4. A Hom-Nambu algebra is a *Hom-Nambu-Lie* algebra if the bracket is skew-symmetric.

Definition 2.5. A *non-commutative ternary Hom-Nambu-Poisson algebra* is a tuple $(A,\mu,\beta,\{.,.,.\},\widetilde{\alpha})$ consisting of a vector space A, a ternary operation $\{.,.,.\}$: $A\times A\times A\to A$, a binary operation $\mu: A\times A\to A$, a pair of linear maps $\widetilde{\alpha}=(\alpha_1,\alpha_2)$ where $\alpha_1,\alpha_2: A\to A$, and a linear map $\beta: A\to A$ such that:

(1) (A,μ,β) is a binary Hom-associative algebra,
(2) $(A,\{.,.,.\},\widetilde{\alpha})$ is a ternary Hom-Nambu-Lie algebra,
(3) $\{\mu(x_1,x_2),\alpha_1(x_3),\alpha_2(x_4)\}=\mu(\beta(x_1),\{x_2,x_3,x_4\})+\mu(\{x_1,x_3,x_4\},\beta(x_2))$.

The third condition is called Hom-Leibniz identity.

Remark 2.6. Notice that μ is not assumed to be commutative. When μ is a commutative multiplication, then $(A,\mu,\beta,\{.,.,.\},\widetilde{\alpha})$ is said to be a ternary Hom-Nambu-Poisson algebra.

We recover the classical (non-commutative) ternary Nambu-Poisson algebra when $\alpha_1=\alpha_2=\beta=Id$.

Similarly, a non-commutative n-ary Hom-Nambu-Poisson algebra is a tuple $(A,\mu,\beta,\{.,\cdots,.\},\widetilde{\alpha})$ where $(A,\{.,\cdots,.\},\widetilde{\alpha})$ with $\widetilde{\alpha}=(\alpha_1,\cdots,\alpha_{n-1})$ that defines an n-ary Hom-Nambu-Lie algebra satisfying similar Leibniz rule with respect to (A,μ,β).

In the sequel we will mainly interested in the class of non-commutative ternary Nambu-Poisson algebras where $\alpha=\alpha_1=\alpha_2=\beta$, for which we refer by a quadruple $(A,\mu,\{.,.,.\},\alpha)$.

Definition 2.7. Let $(A,\mu,\{.,.,.\},\alpha)$ be a (non-commutative) ternary Hom-Nambu-Poisson algebra. It is said to be *multiplicative* if

$$\alpha(\{x_1,x_2,x_3\}) = \{\alpha(x_1),\alpha(x_2),\alpha(x_3)\},$$
$$\alpha\circ\mu = \mu\circ\alpha^{\otimes 2}.$$

If in addition α is bijective then it is called *regular*.

Let $(A',\mu_{A'},\{.,.,.\}_{A'},\alpha_{A'})$ be another such quadruple. A *weak morphism* φ : $A\to A'$ is a linear map such that

- $\varphi\circ\{.,.,.\}=\{.,.,.\}_{A'}\circ\varphi^{\otimes 3}$,
- $\varphi\circ\mu=\mu_\beta\circ\varphi^{\otimes 2}$.

A morphism $\varphi: A\to A'$ is a weak morphism for which we have in addition $\varphi\circ\alpha=\alpha_{A'}\circ\varphi$.

Definition 2.8. Let $(A,\mu,\{.,.,.\},\alpha)$ and $(A',\mu',\{.,.,.\}',\alpha')$ be two (non-commutative) ternary Hom-Nambu-Poisson algebras. A linear map $f: A\to A'$ is a *morphism* of (non-commutative) ternary Hom-Nambu-Poisson algebras if it satisfies:

$$f(\{x_1, x_2, x_3\}) = \{f(x_1), f(x_2), f(x_3)\}', \tag{2.2}$$

$$f \circ \mu = \mu' \circ f^{\otimes 2}, \tag{2.3}$$

$$f \circ \alpha = \alpha' \circ f. \tag{2.4}$$

It said to be a *weak morphism* if hold only the two first conditions.

Proposition 2.9. *Let $(A_1, \mu_1, \{.,.,.\}_1, \alpha_1)$ and $(A_2, \mu_2, \{.,.,.\}_2, \alpha_2)$ be two ternary (non-commutative) Hom-Nambu-Poisson algebras. A linear map $\phi : A_1 \to A_2$ is a morphism of ternary (non-commutative) Hom-Nambu-Poisson algebras if and only if $\Gamma_\phi \subseteq A_1 \oplus A_2$ is a Hom-Nambu-Poisson subalgebra of*

$$(A_1 \oplus A_2, \mu_{A_1 \oplus A_2}, \{.,.,.\}_{A_1 \oplus A_2}, \alpha_{A_1 \oplus A_2})$$

where $\Gamma_\phi = \{(x, \phi(x)) : x \in A_1\} \subset A_1 \oplus A_2$.

Proof. Let $\phi : (A_1, \mu_1, \{.,.,.\}_1, \alpha_1) \to (A_2, \mu_2, \{.,.,.\}_2, \alpha_2)$ be a morphism of ternary Hom-Nambu-Poisson algebras.
We have

$$\begin{aligned}\{x_1 + \phi(x_1), x_2 + \phi(x_2), x_3 + \phi(x_3)\}_{A_1 \oplus A_2} &= \{x_1, x_2, x_3\}_1 + \{\phi(x_1), \phi(x_2), \phi(x_3)\}_2 \\ &= \{x_1, x_2, x_3\}_1 + \phi\{x_1, x_2, x_3\}_1.\end{aligned}$$

Then Γ_ϕ is closed under the bracket $\{.,.,.\}_{A_1 \oplus A_2}$.
We have also

$$(\alpha_1 + \alpha_2)(x_1 + \phi(x_1)) = \alpha_1(x_1) + \alpha_2 \circ \phi(x_1) = \alpha_1(x_1) + \phi \circ \alpha_1(x_1),$$

which implies that $(\alpha_1 + \alpha_2)\Gamma_\phi \subseteq \Gamma_\phi$.
Conversely, if the graph $\Gamma_\phi \subseteq A_1 \oplus A_2$ is a Hom-subalgebra of

$$(A_1 \oplus A_2, \mu_{A_1 \oplus A_2}, \{.,.,.\}_{A_1 \oplus A_2}, \alpha_{A_1 \oplus A_2}),$$

then we have

$$\{\phi(x_1), \phi(x_2), \phi(x_3)\}_2 = \phi\{x_1, x_2, x_3\}_1,$$

and

$$\begin{aligned}(\alpha_1 + \alpha_2)(x + \phi(x)) &= \alpha_1(x) + \alpha_2 \circ \phi(x) \in \Gamma_\phi \\ &= \alpha_1(x) + \phi \circ \alpha_1(x).\end{aligned}$$

Finally

$$\begin{aligned}\mu_{A_1 \oplus A_2}(x_1 + \phi(x_1), x_2 + \phi(x_2)) &= \mu_1(x_1, x_2) + \mu_2(\phi(x_1), \phi(x_2)) \\ &= \mu_1(x_1, x_2) + \phi \circ \mu_2(x_1, x_2) \subseteq \Gamma_\phi.\end{aligned}$$

Therefore ϕ is a morphism of ternary (non-commutative) Hom-Nambu-Poisson algebras.

□

3. Tensor product and direct sums

In the following, we define a direct sum of two ternary (non-commutative) Hom-Nambu-Poisson algebras.

Theorem 3.1. *Let* $(A_1,\mu_1,\{.,.,.\}_1,\alpha_1)$ *and* $(A_2,\mu_2,\{.,.,.\}_2,\alpha_2)$ *be two ternary (non-commutative) Hom-Nambu-Poisson algebras. Let* $\mu_{A_1\oplus A_2}$ *be a bilinear map on* $A_1\oplus A_2$ *defined for* $x_1,y_1,z_1\in A_1$ *and* $x_2,y_2,z_2\in A_1$ *by*

$$\mu(x_1+x_2,y_1+y_2)=\mu_1(x_1,y_1)+\mu_2(x_2,y_2),$$

$\{.,.,.\}_{A_1\oplus A_2}$ *a trilinear map defined by*

$$\{x_1+x_2,y_1+y_2,z_1+z_2\}_{A_1\oplus A_2}=\{x_1,y_1,z_1\}_1+\{x_2,y_2,z_2\}_2$$

and $\alpha_{A_1\oplus A_2}$ *a linear map defined by*

$$\alpha_{A_1\oplus A_2}(x_1+y_1)=\alpha_1(x_1)+\alpha_2(x_2).$$

Then

$$(A_1\oplus A_2,\mu_{A_1\oplus A_2},\{.,.,.\}_{A_1\oplus A_2},\alpha_{A_1\oplus A_2})$$

is a ternary (non-commutative) Hom-Nambu-Poisson algebra.

Proof. The commutativity of $\mu_{A_1\oplus A_2}$ is obvious since μ_1 and μ_2 are commutative. The skew-symmetry of the bracket follows from the skew-symmetry of $\{.,.,.\}_1$ and $\{.,.,.\}_2$. So it remains to check the Hom-associativity, the Hom-Nambu and the Hom-Leibniz identities. For Hom-associativity identity, we have

$$\begin{aligned}
&\mu_{A_1\oplus A_2}(\mu_{A_1\oplus A_2}(x_1+x_1',x_2+x_2'),\alpha_{A_1\oplus A_2}(x_3+x_3'))\\
&=\mu_{A_1\oplus A_2}(\mu_1(x_1,x_2)+\mu_2(x_1',x_2'),\alpha_1(x_3)+\alpha_2(x_3'))\\
&=\mu_1(\mu_1(x_1,x_2),\alpha_1(x_3))+\mu_2(\mu_2(x_1',x_2'),\alpha_2(x_3'))\\
&=\mu_1(\alpha_1(x_1),\mu_1(x_2,x_3))+\mu_2(\alpha_2(x_1'),\mu_2(x_2',x_3'))\\
&=\mu_{A_1\oplus A_2}(\alpha_1(x_1)+\alpha_2(x_1'),\mu_1(x_2,x_3)+\mu_2(x_2',x_3'))\\
&=\mu_{A_1\oplus A_2}(\alpha_{A_1\oplus A_2}(x_1,x_1'),\mu_{A_1\oplus A_2}(x_2+x_2',x_3+x_3')).
\end{aligned}$$

Now we prove the Hom-Nambu identity

$$\begin{aligned}
&\{\alpha_{A_1\oplus A_2}(x_1+x_1'),\alpha_{A_1\oplus A_2}(x_2+x_2'),\{x_3+x_3',x_4+x_4',x_5+x_5'\}_{A_1\oplus A_2}\}_{A_1\oplus A_2}\\
&=\{\alpha_1(x_1)+\alpha_2(x_1'),\alpha_1(x_2)+\alpha_2(x_2'),\{x_3,x_4,x_5\}_1+\{x_3',x_4',x_5'\}_2\}_{A_1\oplus A_2}\\
&=\{\alpha_1(x_1),\alpha_1(x_2),\{x_3,x_4,x_5\}_1\}_1+\{\alpha_2(x_1'),\alpha_2(x_2'),\{x_3',x_4',x_5'\}_2\}_2\\
&=\{\{x_1,x_2,x_3\}_1,\alpha_1(x_4),\alpha_1(x_5)\}_1+\{\alpha_1(x_3),\{x_1,x_2,x_4\}_1,\alpha_1(x_5)\}_1\\
&+\{\alpha_1(x_3),\alpha_1(x_4),\{x_1,x_2,x_5\}_1\}_1+\{\{x_1',x_2',x_3'\}_2,\alpha_2(x_4'),\alpha_2(x_5')\}_2\\
&+\{\alpha_2(x_3'),\{x_1',x_2',x_4'\}_2,\alpha_2(x_5')\}_2+\{\alpha_2(x_3'),\alpha_2(x_4'),\{x_1',x_2',x_5'\}_2\}_2\\
&=\{\{x_1,x_2,x_3\}_1+\{x_1',x_2',x_3'\}_2,\alpha_1(x_4)+\alpha_2(x_4'),\alpha_1(x_5)+\alpha_2(x_5')\}_{A_1\oplus A_2}\\
&+\{\alpha_1(x_3)+\alpha_2(x_3'),\{x_1,x_2,x_4\}_1+\{x_1',x_2',x_4'\}_2,\alpha_1(x_5)+\alpha_2(x_5')\}_{A_1\oplus A_2}\\
&+\{\alpha_1(x_3)+\alpha_2(x_3'),\alpha_1(x_3)+\alpha_2(x_3'),\{x_1,x_2,x_5\}_1+\{x_1',x_2',x_5'\}_2\}_{A_1\oplus A_2}\\
&=\{\{x_1+x_1',x_2+x_2',x_3+x_3'\}_{A_1\oplus A_2},\alpha_{A_1\oplus A_2}(x_4+x_4'),\alpha_{A_1\oplus A_2}(x_5+x_5')\}_{A1\oplus A_2}\\
&+\{\alpha_{A_1\oplus A_2}(x_3+x_3'),\{x_1+x_1',x_2+x_2',x_4+x_4'\}_{A_1\oplus A_2},\alpha_{A_1\oplus A_2}(x_5+x_5')\}_{A1\oplus A_2}\\
&+\{\alpha_{A_1\oplus A_2}(x_3+x_3'),\alpha_{A_1\oplus A_2}(x_4+x_4'),\{x_1+x_1',x_2+x_2',x_5+x_5'\}_{A_1\oplus A_2}\}_{A_1\oplus A_2}.
\end{aligned}$$

Finally, for Hom-Leibniz identity we have

$$\begin{aligned}
&\{\mu_{A_1\oplus A_2}(x_1+x'_1),\alpha_{A_1\oplus A_2}(x_3,x'_3),\alpha_{A_1+A_2}(x_4,x'_4)\}_{A_1\oplus A_2}\\
&=\{\mu_1(x_1,x_2)+\mu_2(x'_1,x'_2),\alpha_1(x_3)+\alpha_2(x'_3),\alpha_1(x_4)+\alpha_2(x'_4)\}_{A_1\oplus A_2}\\
&=\{\mu_1(x_1,x_2),\alpha_1(x_3),\alpha_1(x_4)\}_1+\{\mu_2(x'_1,x'_2),\alpha_2(x'_3),\alpha_2(x'_4)\}_2\\
&=\mu_1(\alpha_1(x_1),\{x_2,x_3,x_4\}_1)+\mu_1(\{x_1,x_3,x_4\}_1,\alpha_1(x_2))\\
&+\mu_2(\alpha_2(x'_1),\{x'_2,x'_3,x'_4\}_2)+\mu_2(\{x'_1,x'_3,x'_4\}_2,\alpha_2(x'_2))\\
&=\mu_{A_1\oplus A_2}(\alpha_{A_1\oplus A_2}(x_1,x'_1),\{x_2+x'_2,x_3+x'_3,x_4+x'_4\}_{A_1\oplus A_2})\\
&+\mu_{A_1\oplus A_2}(\{x_1+x'_1,x_3+x'_3,x_4+x'_4\}_{A_1\oplus A_2},\alpha_{A_1\oplus A_2}(x_2,x'_2)).
\end{aligned}$$

That ends the proof. □

Now, we define the tensor product of two ternary Hom-algebras. Moreover, we consider a tensor product of a ternary Hom-Nambu-Poisson algebra and a totally Hom-associative symmetric ternary algebra.

Let $A_1=(A,m,\alpha)$, where $\alpha=(\alpha_i)_{i=1,2}$ and $A_2=(A',m',\alpha')$ where $\alpha'=(\alpha'_i)_{i=1,2}$ be two ternary (non-commutative) Hom-algebras of a given type, the tensor product $A_1\otimes A_2$ is a ternary algebra defined by the triple $(A\otimes A',m\otimes m',\alpha\otimes\alpha')$ where $\alpha\otimes\alpha'=(\alpha_i\otimes\alpha'_i)_{i=1,2}$ with

$$m\otimes m'(x_1\otimes x'_1,x_2\otimes x'_2,x_3\otimes x'_3)=m(x_1,x_2,x_3)\otimes m'(x'_1,x'_2,x'_3), \tag{3.1}$$

$$\alpha_i\otimes\alpha'_i=\alpha_i(x_1)\otimes\alpha'_i(x'_1), \tag{3.2}$$

where $x_1,x_2,x_3\in A_1$ and $x'_1,x'_2,x'_3\in A_2$.

Recall that (A,m,α) is a totally Hom-associative ternary algebra if

$$\begin{aligned}
m(\alpha_1(x_1),\alpha_2(x_2),m(x_3,x_4,x_5))&=m(\alpha_1(x_1),m(x_2,x_3,x_4),\alpha_2(x_5))\\
&=m(m(x_1,x_2,x_3),\alpha_1(x_4),\alpha_2(x_5)).
\end{aligned}$$

for all $x_1\cdots,x_5\in A$, and the ternary multiplication m is symmetric if

$$m(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})=m(x_1,x_2,x_3). \tag{3.3}$$

for all $\sigma\in S_3$, $x_1,x_2,x_3\in A$.

Lemma 3.2. *Let $A_1=(A,m,\alpha)$ and $A_2=(A',m',\alpha')$ be two ternary Hom-algebras of given type (Hom-Nambu, totally Hom-associative). If m is symmetric and m' is skew-symmetric then $m\otimes m'$ is skew-symmetric.*

Proof. Straightforward. □

Theorem 3.3. *Let $(A,\mu,\beta,\{.,.,.\},(\alpha_1,\alpha_2))$ be a ternary (non-commutative) Hom-Nambu-Poisson algebra, $(B,\tau,(\alpha'_1,\alpha'_2))$ be a totally Hom-associative symmetric ternary algebra, and (B,μ',β') be a Hom-associative algebra, then*

$$(A\otimes B,\mu\otimes\mu',\beta\otimes\beta',\{.,.,.\}_{A\otimes B},(\alpha_1\otimes\alpha'_1,\alpha_2\otimes\alpha'_2))$$

is a (non-commutative) ternary Hom-Nambu-Poisson algebra if and only if

$$\tau(\mu'(b_1,b_2),b_3,b_4)=\mu'(b_1,\tau(b_2,b_3,b_4))=\mu'(\tau(b_1,b_3,b_4),b_2). \tag{3.4}$$

Proof. Since μ and μ' are both Hom-associative multiplication whence a tensor product $\mu \otimes \mu'$ is Hom-associative. Also the commutativity of $\mu \otimes \mu'$, the skew-symmetry of $\{.,.,.\}$ and the symmetry of τ imply the skew-symmetry of $\{.,.,.\}_{A\otimes B}$. Therefore, it remains to check Nambu identity and Leibniz identity.

We have

$$\begin{aligned} LHS =& \{\alpha_1 \otimes \alpha_1'(a_1 \otimes b_1), \alpha_2 \otimes \alpha_2'(a_2 \otimes b_2), \{a_3 \otimes b_3, a_4 \otimes b_4, a_5 \otimes b_5\}_{A\otimes B}\}_{A\otimes B} \\ &= \{\alpha_1(a_1) \otimes \alpha_1'(b_1), \alpha_2(a_2) \otimes \alpha_2'(b_2), \{a_3, a_4, a_5\}_A \otimes \tau(b_3, b_4, b_5)\}_{A\otimes B} \\ &= \underbrace{\{\alpha_1(a_1), \alpha_2(a_2), \{a_3, a_4, a_5\}\}}_{a} \otimes \underbrace{\tau(\alpha_1'(b_1), \alpha_2'(b_2), \tau(b_3, b_4, b_5))}_{b}, \end{aligned}$$

and

$$\begin{aligned} RHS =& \{\{a_1 \otimes b_1, a_2 \otimes b_2, a_3 \otimes b_3\}_{A\otimes B}, \alpha_1 \otimes \alpha_1'(a_4 \otimes b_4), \alpha_2 \otimes \alpha_2'(a_5 \otimes b_5)\}_{A\otimes B} \\ &+ \{\alpha_1 \otimes \alpha_1'(a_3 \otimes b_3), \{a_1 \otimes b_1, a_2 \otimes b_2, a_4 \otimes b_4\}_{A\otimes B}, \alpha_2 \otimes \alpha_2'(a_5 \otimes b_5)\}_{A\otimes B} \\ &+ \{\alpha_1 \otimes \alpha_1'(a_3 \otimes b_3), \alpha_2 \otimes \alpha_2'(a_4 \otimes b_4), \{a_1 \otimes b_1, a_2 \otimes b_2, a_5 \otimes b_5\}_{A\otimes B}\}_{A\otimes B} \\ &= \{\{a_1, a_2, a_3\}_A \otimes \tau(b_1, b_2, b_3), \alpha_1(a_4) \otimes \alpha_1'(b_4), \alpha_2(a_5) \otimes \alpha_2'(b_5)\}_{A\otimes B} \\ &+ \{\alpha_1(a_3) \otimes \alpha_1'(b_3), \{a_1, a_2, a_4\}_A \otimes \tau(b_1, b_2, b_4), \alpha_2(a_5) \otimes \alpha_2'(b_5)\}_{A\otimes B} \\ &+ \{\alpha_1(a_3) \otimes \alpha_1'(b_3), \alpha_2(a_4) \otimes \alpha_2'(b_4), \{a_1, a_2, a_5\}_A \otimes \tau(b_1, b_2, b_5)\}_{A\otimes B} \\ &= \underbrace{\{\{a_1, a_2, a_3\}, \alpha_1(a_4), \alpha_2(a_5)\}}_{c} \otimes \underbrace{\tau(\tau(b_1, b_2, b_3), \alpha_1'(b_4), \alpha_2'(b_5))}_{d} \\ &+ \underbrace{\{\alpha_1(a_3), \{a_1, a_2, a_4\}, \alpha_2(a_5)\}}_{e} \otimes \underbrace{\tau(\alpha_1'(b_3), \tau(b_1, b_2, b_4), \alpha_2'(b_5))}_{f} \\ &+ \underbrace{\{\alpha_1(a_3), \alpha_2(a_4), \{a_1, a_2, a_5\}\}}_{g} \otimes \underbrace{\tau(\alpha_1'(b_3), \alpha_2'(b_4), \tau(b_1, b_2, b_5))}_{h} \end{aligned}$$

Using ternary Nambu identity of $\{.,.,.\}$ we have $a = c + e + g$, and $b = d = f = h$ using the symmetry of τ and Hom-associativity of μ', then the left hand side is equal to the right hand side from where the ternary Hom-Nambu identity of bracket $\{.,.,.\}_{A\otimes B}$ is verified.

For the Hom-Leibniz identity, we have

$$\begin{aligned} LHS =& \{\mu \otimes \mu'(a_1 \otimes b_1, a_2 \otimes b_2), \alpha_1 \otimes \alpha_1'(a_3 \otimes b_3), \alpha_2 \otimes \alpha_2'(a_4 \otimes b_4)\}_{A\otimes B} \\ &= \{\mu(a_1, b_1) \otimes \mu'(a_2, b_2), \alpha_1(a_3) \otimes \alpha_1'(b_3), \alpha_2(a_4) \otimes \alpha_2'(b_4)\}_{A\otimes B} \\ &= \underbrace{\{\mu(a_1, b_1), \alpha_1(a_3), \alpha_2(a_4)\}_A}_{a'} \otimes \underbrace{\tau(\mu'(a_2, b_2), \alpha_1'(b_3), \alpha_2'(b_4))}_{b'} \end{aligned}$$

and

$$\begin{aligned}
RHS =& \mu\otimes\mu'(\beta\otimes\beta'(a_1\otimes b_1),\{a_2\otimes b_2,a_3\otimes b_3,a_4\otimes b_4\}_{A\otimes B})\\
&+\mu\otimes\mu'(\{a_1\otimes b_1,a_3\otimes b_3,a_4\otimes b_4\}_{A\otimes B},\beta\otimes\beta'(a_2\otimes b_2))\\
&=\mu\otimes\mu'(\beta(a_1)\otimes\beta'(b_1),\{a_2,a_3,a_4\}\otimes\tau(b_2,b_3,b_4))\\
&+\mu\otimes\mu'(\{a_1,a_3,a_4\}\otimes\tau(b_1,b_3,b_4),\beta(a_2)\otimes\beta'(b_2))\\
&=\underbrace{\mu(\beta(a_1),\{a_2,a_3,a_4\})}_{c'}\otimes\underbrace{\mu'(\beta'(b_1),\tau(b_2,b_3,b_4))}_{d'}\\
&+\underbrace{\mu(\{a_1,a_3,a_4\},\beta(a_2))}_{e'}\otimes\underbrace{\mu'(\tau(b_1,b_3,b_4),\beta'(b_2)}_{f'}
\end{aligned}$$

With Hom-Leibniz identity we have $a'=c'+e'$, and using condition (3.4) we have $b'=d'=f'$, for that the left hand side is equal to the right hand side and the Hom-Leibniz identity is proved. Then

$$(A\otimes B,\mu\otimes\mu',\beta\otimes\beta',\{.,.,.\}_{A\otimes B},(\alpha_1\otimes\alpha'_1,\alpha_2\otimes\alpha'_2))$$

is a (non-commutative) ternary Hom-Nambu-Poisson algebra.

□

4. Construction of ternary Hom-Nambu-Poisson algebras

In this section, we provide constructions of ternary Hom-Nambu-Poisson algebras using twisting principle.

Theorem 4.1. *Let $(A,\mu,\{.,.,.\},\alpha)$ be a (non-commutative) ternary Hom-Nambu-Poisson algebra and $\beta: A\to A$ be a weak morphism, then $A_\beta=(A,\{.,.,.\}_\beta=\beta\circ\{.,.,.\},\mu_\beta=\beta\circ\mu,\beta\alpha)$ is also a ternary (non-commutative) Hom-Nambu-Poisson algebra. Morever, if A is multiplicative and β is a algebra morphism, then A_β is a multiplicative (non-commutative) Hom-Nambu-Poisson algebra.*

Proof. If μ is commutative, then clearly so is μ_β. The rest of the proof applies whether μ is commutative or not. The skew-symmetry follows from the skew-symmetry of the bracket $\{.,.,.\}$. It remains to prove Hom-associativity condition, Hom-Nambu-identity and Hom-Leibniz identity. Indeed

$$\begin{aligned}
\mu_\beta(\mu_\beta(x,y),\beta\alpha(z))&=\mu_\beta(\beta(\mu(x,y),\beta\alpha(z)))=\beta^2(\mu(\mu(x,y),\alpha(z)))\\
&=\beta^2(\mu(\alpha(x),\mu(y,z)))=\mu_\beta(\beta\alpha(x),\mu_\beta(y,z)).
\end{aligned}$$

We check the Hom-Nambu identity

$$\begin{aligned}
&\{\beta\alpha(x_1),\beta\alpha(x_2),\{x_3,x_4,x_5\}_\beta\}_\beta=\beta^2\{\alpha(x_1),\alpha(x_2),\{x_3,x_4,x_5\}\}\\
&=\beta^2(\{\{x_1,x_2,x_3\},\alpha(x_4),\alpha(x_5)\}+\{\alpha(x_3),\{x_1,x_2,x_4\},\alpha(x_5)\}\\
&\quad+\{\alpha(x_3),\alpha(x_4),\{x_1,x_2,x_5\}\})\\
&=\{\{x_1,x_2,x_3\}_\beta,\beta\alpha(x_4),\beta\alpha(x_5)\}_\beta+\{\beta\alpha(x_3),\{x_1,x_2,x_4\}_\beta,\beta\alpha(x_5)\}_\beta\\
&\quad+\{\beta\alpha(x_3),\beta\alpha(x_4),\{x_1,x_2,x_5\}_\beta\}_\beta.
\end{aligned}$$

Then, it remains to show Hom-Leibniz identity

$$\{\mu_\beta(x_1,x_2),\beta\alpha(x_3),\beta\alpha(x_4)\}_\beta = \beta^2(\{\mu(x_1,x_2),\alpha(x_3),\alpha(x_4)\})$$
$$= \beta^2(\mu(\alpha(x_1),\{x_2,x_3,x_4\}) + \mu(\{x_1,x_3,x_4\},\alpha(x_2)))$$
$$= \mu_\beta(\beta\alpha(x_1),\{x_2,x_3,x_4\}_\beta) + \mu_\beta(\{x_1,x_3,x_4\}_\beta,\beta\alpha(x_2)).$$

Therefore $A_\beta = (A,\{.,.,.\}_\beta,\mu_\beta,\beta\alpha)$ is a ternary (non-commutative) Hom-Nambu-Poisson algebra. For the multiplicativity assertion, suppose that A is multiplicative and β is an algebra morphism. We have

$$(\beta\alpha)\circ(\mu_\beta) = \beta\alpha\circ\beta\circ\mu = \mu_\beta\circ\alpha^{\otimes 2}\beta^{\otimes 2} = \mu_\beta\circ(\beta\alpha)^{\otimes 2},$$

and

$$\beta\alpha\circ\{.,.,.\}_\beta = \beta\alpha\circ\beta\circ\{.,.,.\} = \{.,.,.\}_\beta\circ(\beta\alpha)^{\otimes 3}.$$

Then A_β is multiplicative.

□

Corollary 4.2. *Let $(A,\mu,\{.,.,.\},\alpha)$ be a multiplicative ternary (non-commutative) Hom-Nambu-Poisson algebra. Then*

$$A^n = (A,\mu^{(n)} = \alpha^n\circ\mu,\{.,.,.\}^{(n)} = \alpha^{(n)}\circ\{.,.,.\},\alpha^{n+1})$$

is a multiplicative (non-commutative) ternary Hom-Nambu-Poisson algebra for each integer $n\geq 0$.

Proof. The multiplicativity of A implies that $\alpha^n : A\to A$ is a Nambu-Poisson algebra morphism. By Theorem 4.2 $A_{\alpha^n} = A^n$ is a multiplicative ternary (non-commutative) Hom-Nambu-Poisson algebra. □

Corollary 4.3. *Let $(A,\mu,\{.,.,.\})$ be a ternary (non-commutative) Nambu-Poisson algebra and $\beta : A\to A$ be a Nambu-Poisson algebra morphism. Then*

$$A_\beta = (A,\mu_\beta = \beta\circ\mu,\{.,.,.\}_\beta = \beta\circ\{.,.,.\},\beta)$$

is a multiplicative (non-commutative) ternary Hom-Nambu-Poisson algebra.

Remark 4.4. Let $(A,\mu,\{.,.,.\},\alpha)$ and $(A',\mu',\{.,.,.\}',\alpha')$ be two (non-commutative) ternary Nambu-Poisson algebras and $\beta : A\to A$, $\beta' : A'\to A'$ be ternary Nambu-Poisson algebra endomorphisms. If $\varphi : A\to A'$ is a ternary Nambu-Poisson algebra morphism that satisfies $\varphi\circ\beta = \beta'\circ\varphi$, then

$$\varphi : (A,\mu_\beta,\{.,.,.\}_\beta,\beta\alpha)\to(A',\mu'_{\beta'},\{.,.,.\}'_{\beta'},\beta'\alpha')$$

is a (non-commutative) ternary Hom-Nambu-Poisson algebra morphism.

Indeed, we have

$$\varphi\circ\{.,.,.\}_\beta = \varphi\circ\beta\circ\{.,.,.\} = \beta'\circ\varphi\circ\{.,.,.\} = \beta'\circ\{.,.,.\}'\circ\varphi^{\times 3} = \{.,.,.\}'_{\beta'}\circ\varphi^{\times 3}$$

and

$$\varphi\circ\mu_\beta = \varphi\circ\beta\circ\mu = \beta'\circ\varphi\circ\mu = \beta'\circ\mu'\circ\varphi^{\times 2} = \mu'_{\beta'}\circ\varphi^{\times 2}.$$

In the sequel, we aim to construct Hom-type version of the ternary Nambu-Poisson algebra of polynomials of three variables $(\mathbb{R}[x,y,z],\cdot,\{.,.,.\})$, defined in Example 1.5. The Poisson bracket of three polynomials is defined in (1.2).
The twisted version is given by a structure of ternary Hom-Nambu-Poisson algebra $(\mathbb{R}[x,y,z],\cdot_\alpha = \alpha\circ\cdot,\{.,.,.\}_\alpha = \alpha\circ\{.,.,.\},\alpha)$ where $\alpha : \mathbb{R}[x,y,z]\to\mathbb{R}[x,y,z]$ is an algebra morphism satisfying for all $f,g\in\mathbb{R}[x,y,z]$

$$\alpha(f\cdot g)=\alpha(f)\cdot\alpha(g)$$
$$\alpha\{f,g,h\}=\{\alpha(f),\alpha(g),\alpha(h)\}.$$

Theorem 4.5. *A morphism $\alpha:\mathbb{R}[x,y,z]\to\mathbb{R}[x,y,z]$ which gives a structure of ternary Hom-Nambu-Poisson algebra $(\mathbb{R}[x,y,z],\cdot_\alpha=\alpha\circ\cdot,\{.,.,.\}_\alpha=\alpha\circ\{.,.,.\},\alpha)$ satisfies the following equation:*

$$1-\begin{vmatrix}\frac{\partial\alpha(x)}{\partial x} & \frac{\partial\alpha(x)}{\partial y} & \frac{\partial\alpha(x)}{\partial z}\\ \frac{\partial\alpha(y)}{\partial x} & \frac{\partial\alpha(y)}{\partial y} & \frac{\partial\alpha(y)}{\partial z}\\ \frac{\partial\alpha(z)}{\partial x} & \frac{\partial\alpha(z)}{\partial y} & \frac{\partial\alpha(z)}{\partial z}\end{vmatrix}=0, \tag{4.1}$$

Proof. let α be a Nambu-Poisson algebra morphism, then it satisfies for all $f,g\in\mathbb{R}[x,y,z]$

$$\alpha(f\cdot g)=\alpha(f)\cdot\alpha(g),$$
$$\alpha\{f,g,h\}=\{\alpha(f),\alpha(g),\alpha(h)\}.$$

The first equality shows that it is sufficient to just set α on x, y and z. For the second equality, we suppose by linearity that

$$f(x,y,z)=x^iy^jz^k,$$
$$g(x,y,z)=x^ly^mz^p,$$
$$f(x,y,z)=x^qy^rz^s.$$

Then we can write the second equation as follows

$$\alpha\begin{vmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z}\\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z}\end{vmatrix}=\begin{vmatrix}\frac{\partial\alpha(f)}{\partial x} & \frac{\partial\alpha(f)}{\partial y} & \frac{\partial\alpha(f)}{\partial z}\\ \frac{\partial\alpha(g)}{\partial x} & \frac{\partial\alpha(g)}{\partial y} & \frac{\partial\alpha(g)}{\partial z}\\ \frac{\partial\alpha(h)}{\partial x} & \frac{\partial\alpha(h)}{\partial y} & \frac{\partial\alpha(h)}{\partial z}\end{vmatrix},$$

which can be simplified to

$$1=\begin{vmatrix}\frac{\partial\alpha(x)}{\partial x} & \frac{\partial\alpha(x)}{\partial y} & \frac{\partial\alpha(x)}{\partial z}\\ \frac{\partial\alpha(y)}{\partial x} & \frac{\partial\alpha(y)}{\partial y} & \frac{\partial\alpha(y)}{\partial z}\\ \frac{\partial\alpha(z)}{\partial x} & \frac{\partial\alpha(z)}{\partial y} & \frac{\partial\alpha(z)}{\partial z}\end{vmatrix}. \tag{4.2}$$

□

Example 4.6. *We set polynomials:*

$$\alpha(x)=P_1(x,y,z)=\sum_{0\leq i,j,k\leq d}a_{ijk}x^iy^jz^k,$$
$$\alpha(y)=P_2(x,y,z)=\sum_{0\leq i,j,k\leq d}b_{ijk}x^iy^jz^k,$$
$$\alpha(z)=P_3(x,y,z)=\sum_{0\leq i,j,k\leq d}c_{ijk}x^iy^jz^k,$$

where $P_1,P_2,P_3\in\mathbb{R}[x,y,z]$, and d the largest degree for each variable.We assume that $a_0=b_0=c_0=0$.

Case of polynomials of degree one. *We take*

$$P_1(x,y,z) = a_1x + a_2y + a_3z,$$
$$P_2(x,y,z) = b_1x + b_2y + b_3z,$$
$$P_3(x,y,z) = c_1x + c_2y + c_3z.$$

Equation (2.5) *becomes*

$$1 - \begin{vmatrix} \frac{\partial P_1(x,y,z)}{\partial x} & \frac{\partial P_1(x,y,z))}{\partial y} & \frac{\partial P_1(x,y,z)}{\partial z} \\ \frac{\partial P_2(x,y,z)}{\partial x} & \frac{\partial P_2(x,y,z)}{\partial y} & \frac{\partial P_2(x,y,z)}{\partial z} \\ \frac{\partial P_3(x,y,z)}{\partial x} & \frac{\partial P_3(x,y,z)}{\partial y} & \frac{\partial P_3(x,y,z)}{\partial z} \end{vmatrix} = 0, \tag{4.3}$$

whence

$$1 - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0. \tag{4.4}$$

The polynomials P_1, P_2 and P_3 are of one of this form

(1) $P_1(x,y,z) = xa_1 + ya_2 + za_3,\ P_2(x,y,z) = b_2y - \frac{z}{a_1c_2},\ P_3(x,y,z) = c_2y.$

(2) $P_1(x,y,z) = a_1x + a_2y + a_3z,\ P_2(x,y,z) = \frac{1+a_1b_3c_2}{a_1c_3}y + b_3z,$
$P_3(x,y,z) = c_2y + c_3z.$

(3) $P_1(x,y,z) = a_1x + a_2y + a_3z,\ P_2(x,y,z) = b_1x + \frac{1}{a_2c_1}z,\ P_3(x,y,z) = c_1x.$

(4) $P_1(x,y,z) = a_1x + a_2y + a_3z,\ P_2(x,y,z) = \frac{-1+a_2b_3c_1}{a_2c_3}x + b_3z,$
$P_3(x,y,z) = c_1x + c_3z.$

(5) $P_1(x,y,z) = \frac{a_2b_1c_3+b_2}{c_3x} + a_2y + a_3z,\ P_2(x,y,z) = b_1x + b_2y + b_3z,$
$P_3(x,y,z) = c_3z.$

(6) $P_1(x,y,z) = \frac{1}{b_2c_3}x + a_2y + a_3z,\ P_2(x,y,z) = b_2y + b_3z,\ P_3(x,y,z) = c_3z.$

(7) $P_1(x,y,z) = a_1x + \frac{1}{b_1c_3}y + a_3z,\ P_2(x,y,z) = b_1x + b_3z,\ P_3(x,y,z) = c_3z.$

(8) $P_1(x,y,z) = a_1x + a_2y + \frac{1}{b_1c_2}z,\ P_2(x,y,z) = b_1x,\ P_3(x,y,z) = c_1x + c_2y.$

(9) $P_1(x,y,z) = a_1x + \frac{-1}{b_1c_3+a_3c_2c_3}y + a_3z,\ P_2(x,y,z) = b_1x,$
$P_3(x,y,z) = c_1x + c_2y + c_3z.$

(10) $P_1(x,y,z) = \frac{a_2b_1}{b_2} + \frac{1}{b_2c_3-b_3c_2}x + a_2y + a_3z,\ P_2(x,y,z) = b_1x + b_2y + b_3z,$
$P_3(x,y,z) = \frac{b_1c_2}{b_2}x + c_2y + c_3z.$

(11) $P_1(x,y,z) = \frac{-c_3+a_2c_1c_2}{b_3c_2^2}x + a_2y + a_3z,\ P_2(x,y,z) = b_3z,$
$P_3(x,y,z) = c_1x + c_2y + c_3z.$

(12) $P_1(x,y,z) = a_1x + a_2y + \frac{1}{b_1c_2-b_2c_1}z,\ P_2(x,y,z) = b_1x + b_2y,$
$P_3(x,y,z) = c_1x + c_2y.$

(13) $P_1(x,y,z)=\frac{1+a_2b_1c_3-a_3b_1c_2-a_2b_3c_1+a_3b_2c_1}{b_2c_3-b_3c_2}x+a_2y+a_3z$,
$P_2(x,y,z)=b_1x+b_2y+b_3z$, $P_3(x,y,z)=c_1x+c_2y+c_3z$.

(14) $P_1(x,y,z)=a_1x+\frac{b_2}{b_3}(a_3-\frac{1}{b_1c_2-b_2c_1})y+a_3z$, $P_2(x,y,z)=b_1x+b_2y+b_3z$,
$P_3(x,y,z)=c_1x+c_2y+\frac{b_3c_2}{b_2}z$.

Particular case of polynomials of degree two. *We take one of the polynomials of degree two*

$$P_1(x,y,z)=a_1x+a_2y+a_3z$$
$$P_2(x,y,z)=b_1x+b_2y+b_3z$$
$$P_3(x,y,z)=c_1x+c_2y+c_3z+c_4x^2$$

The polynomials P_1,P_2 and P_3 are of one of this form

(1) $P_1(x,y,z)=\frac{a_2b_1}{b_2}+\frac{1}{b_2c_3-b_3c_2}x+a_2y+\frac{a_2b_3}{b_2}z$, $P_2(x,y,z)=b_1x+b_2y+b_3z$,
$P_3(x,y,z)=c_4x^2+c_1x+c_2y+c_3z$.

(2) $P_1(x,y,z)=a_2x+\frac{a_3b_2}{b_3}y+a_3z$, $P_2(x,y,z)=b_2y+b_3z$,
$P_3(x,y,z)=c_4x^2+c_1x+c_2y+\frac{\frac{1}{a_1}+b_3c_2}{b_2}z$.

(3) $P_1(x,y,z)=a_2x+a_2y+a_3z$, $P_2(x,y,z)=b_2y$,
$P_3(x,y,z)=c_4x^2+c_1x+c_2y+\frac{1}{a_1b_2}z$.

(4) $P_1(x,y,z)=(\frac{a_2b_1}{b_3}-\frac{1}{c_2b_3})x+a_3z$, $P_2(x,y,z)=b_1x+b_3z$,
$P_3(x,y,z)=c_4x^2+c_1x+c_2y+c_3z$.

(5) $P_1(x,y,z)=-\frac{1}{b_3c_2}x+a_3z$, $P_2(x,y,z)=b_3z$,
$P_3(x,y,z)=c_4x^2+c_1x+c_2y+c_3z$.

(6) $P_1(x,y,z)=a_1x-\frac{1}{b_1c_3}y+a_3z$, $P_2(x,y,z)=b_1x$,
$P_3(x,y,z)=c_4x^2+c_1x+c_3z$.

(7) $P_1(x,y,z)=a_1x+\frac{-1}{b_1c_3}+\frac{a_3c_2}{c_3}y+a_3z$, $P_2(x,y,z)=b_1x$,
$P_3(x,y,z)=c_4x^2+c_1x+c_2y+c_3z$.

(8) $P_1(x,y,z)=a_1x+a_2y+\frac{1}{b_1c_2}z$, $P_2(x,y,z)=b_1x$,
$P_3(x,y,z)=c_4x^2+c_1x+c_2y$.

(9) $P_1(x,y,z)=a_1x+a_2y+a_3z$, $P_2(x,y,z)=\frac{(1+a_2b_3c_1)}{a_2c_3}x+b_3z$,
$P_3(x,y,z)=c_1x+c_3z$.

5. CLASSIFICATION

In this section, we provide the classification of 3-dimensional ternary non-commutative Nambu-Poisson algebras. By straightforward calculations and using a computer algebra system we obtain the following result.

Theorem 5.1. *Every 3-dimensional ternary Nambu-Lie algebra is isomorphic to the ternary algebra defined with respect to basis* $\{e_1, e_2, e_3\}$*, by the skew symmetric bracket defined as*

$$\{e_1, e_2, e_3\} = e_1$$

Moreover it define a 3-dimensional ternary non-commutative Nambu-Poisson algebra $(A, \{.,.,.\}, \mu)$ *if and only if* μ *is one of the following non-commutative associative algebra defined as*

(1)

$$\mu_1(e_2, e_1) = ae_1 \quad \mu_1(e_2, e_2) = e_2 \quad \mu_1(e_2, e_3) = e_3$$
$$\mu_1(e_3, e_1) = be_1 \quad \mu_1(e_3, e_2) = be_2 \quad \mu_1(e_3, e_3) = be_3,$$

where a*,* b *are parameters.*

(2)

$$\mu_2(e_1, e_2) = ae_1 \quad \mu_2(e_1, e_3) = be_1 \quad \mu_2(e_2, e_2) = ae_2$$
$$\mu_2(e_2, e_3) = be_2 \quad \mu_2(e_3, e_2) = ae_3 \quad \mu_2(e_3, e_3) = be_3,$$

where a*,* b *are parameters with* $a \neq 0$

(3)

$$\mu_3(e_1, e_3) = ae_1 \quad \mu_3(e_2, e_3) = ae_2 \quad \mu_3(e_3, e_3) = ae_3$$

where a *is a parameter with* $a \neq 0$

The multiplication not mentioned are equal to zero.

Remark 5.2. The 3-dimensional ternary Nambu-Lie algebra is endowed with a commutative Nambu-Poisson algebra structure only when the multiplication is trivial.

Using the twisting principle described in Theorem 4.1, we obtain the following 3-dimensional non-commutative ternary Hom-Nambu-Poisson algebras.

Proposition 5.3. *Any 3-dimensional ternary non-commutative Hom-Nambu-Poisson algebra* $(A, \{.,.,.\}_\alpha, \mu_\alpha, \alpha)$ *obtained by a twisting defined with respect to the basis* $\{e_1, e_2, e_3\}$ *by the ternary bracket* $\{e_1, e_2, e_3\}_\alpha = ce_1$*, where* c *is a parameter, and one of the following binary Hom-associative algebra defined by* μ_{α_i} *and a corresponding structure map*

(1)

$$\mu_{\alpha_1}(e_2, e_1) = ace_1, \qquad \mu_{\alpha_1}(e_3, e_1) = bce_1,$$
$$\mu_{\alpha_1}(e_2, e_2) = a(de_1 + e_2), \qquad \mu_{\alpha_1}(e_3, e_2) = b(de_1 + e_2),$$
$$\mu_{\alpha_1}(e_2, e_3) = a(he_1 + ge_2 + e_3), \qquad \mu_{\alpha_1}(e_3, e_3) = b(he_1 + ge_2 + e_3),$$

with

$$\alpha_1(e_1) = ce_1, \alpha_1(e_2) = de_1 + e_2, \alpha_1(e_3) = he_1 + ge_2 + e_3.$$

(2)

$$\mu_{\alpha_2}(e_1, e_2) = ace_1, \qquad \mu_{\alpha_2}(e_3, e_1) = bce_1,$$
$$\mu_{\alpha_2}(e_2, e_2) = a(de_1 + e_2 + le_3), \qquad \mu_{\alpha_2}(e_3, e_2) = b(de_1 + e_2 + le_3),$$
$$\mu_{\alpha_2}(e_2, e_3) = a(he_1 + e_3), \qquad \mu_{\alpha_2}(e_3, e_3) = b(he_1 + e_3),$$

with

$$\alpha_2(e_1) = ce_1, \alpha_2(e_2) = de_1 + e_2 + le_3, \alpha_2(e_3) = he_1 + e_3e_3.$$

(3)

$$\mu_{\alpha_3}(e_2, e_1) = ace_1, \qquad \mu_{\alpha_3}(e_3, e_1 = bce_1,$$
$$\mu_{\alpha_3}(e_2, e_2) = a(de_1 + fe_2 + \frac{a}{b}(1-f)e_3), \qquad \mu_{\alpha_3}(e_3, e_2) = b(de_1 + fe_2 + \frac{a}{b}(1-f)e_3),$$
$$\mu_{\alpha_3}(e_2, e_3) = a(he_1 + \frac{b}{a}(f-1)e_2 + \frac{(b-ga)}{b}e_3), \quad \mu_{\alpha_3}(e_3, e_3) = b(he_1 + \frac{b}{a}(f-1)e_2 + \frac{(b-ga)}{b}e_3),$$

with

$$\alpha_3(e_1) = ce_1, \alpha_3(e_2) = de_1 + fe_2 + \frac{a}{b}(1-f)e_3), \alpha_3(e_3) = he_1 + \frac{b}{a}(f-1)e_2 + \frac{(b-ga)}{b}e_3.$$

(4)

$$\mu_{\alpha_4}(e_1, e_2) = ace_1, \qquad \mu_{\alpha_4}(e_2, e_3) = b(de_1 + e_2),$$
$$\mu_{\alpha_4}(e_1, e_3) = bce_1, \qquad \mu_{\alpha_4}(e_3, e_2) = a(he_1 + ge_2 + e_3),$$
$$\mu_{\alpha_4}(e_2, e_2) = a(de_1 + e_2), \qquad \mu_{\alpha_4}(e_3, e_3) = b(he_1 + ge_2 + e_3),$$

with

$$\alpha_4(e_1) = ce_1, \alpha_4(e_2) = de_1 + e_2, \alpha_4(e_3) = he_1 + ge_2 + e_3.$$

(5)

$$\mu_{\alpha_5}(e_1, e_2) = ace_1, \qquad \mu_{\alpha_5}(e_2, e_3) = b(de_1 + e_2 + le_3),$$
$$\mu_{\alpha_5}(e_1, e_3) = bce_1, \qquad \mu_{\alpha_5}(e_3, e_2) = a(he_1 + e_3),$$
$$\mu_{\alpha_5}(e_2, e_2) = a(de_1 + e_2 + le_3), \qquad \mu_{\alpha_5}(e_3, e_3) = b(he_1 + e_3).$$

with

$$\alpha_5(e_1) = ce_1, \alpha_5(e_2) = de_1 + e_2 + le_3, \alpha_5(e_3) = he_1 + e_3.$$

(6)

$$\mu_{\alpha_6}(e_1, e_2) = ace_1, \qquad \mu_{\alpha_6}(e_2, e_3) = b(de_1 + fe_2 + \frac{a}{b}(1-f)e_3),$$
$$\mu_{\alpha_6}(e_1, e_3) = bce_1, \qquad \mu_{\alpha_6}(e_3, e_2) = a(he_1 + \frac{-b}{a}(f-1)e_2 + \frac{b-ag}{b}e_3),$$
$$\mu_{\alpha_6}(e_2, e_2) = a(de_1 + fe_2 + \frac{a}{b}(1-f)e_3), \quad \mu_{\alpha_6}(e_3, e_3) = b(he_1 + \frac{-b}{a}(f-1)e_2 + \frac{b-ag}{b}e_3),$$

with

$$\alpha_6(e_1) = ce_1, \alpha_6(e_2) = de_1 + fe_2 + \frac{a}{b}(1-f)e_3, \alpha_6(e_3) = he_1 + \frac{-b}{a}(f-1)e_2 + \frac{b-ag}{b}e_3.$$

(7)

$$\mu_{\alpha_7}(e_1, e_3) = ace_1,$$
$$\mu_{\alpha_7}(e_2, e_3) = a(de_1 + fe_2 + le_3),$$
$$\mu_{\alpha_7}(e_3, e_3) = a(he_1 + ge_2 + \frac{1+g+l}{f}e_3),$$

with

$$\alpha_7(e_1) = ce_1, \alpha_7(e_2) = de_1 + fe_2 + le_3, \alpha_7(e_3) = he_1 + ge_2 + \frac{1+g+l}{f}e_3.$$

(8)

$$\mu_{\alpha_8}(e_1, e_3) = ace_1,$$
$$\mu_{\alpha_8}(e_2, e_3) = a(de_1 + e_2),$$
$$\mu_{\alpha_8}(e_3, e_3) = a(he_1 + ge_2 + e_3),$$

with

$$\alpha_8(e_1) = ce_1, \alpha_8(e_2) = de_1 + e_2, \alpha_8(e_3) = he_1 + ge_2 + e_3.$$

(9)

$$\mu_{\alpha_9}(e_1, e_3) = ace_1,$$
$$\mu_{\alpha_9}(e_2, e_3) = a(de_1 - \frac{1}{g}e_3),$$
$$\mu_{\alpha_9}(e_3, e_3) = a(he_1 + ge_2 + re_3),$$

with

$$\alpha_9(e_1) = ce_1, \alpha_9(e_2) = de_1 - \frac{1}{g}e_3, \alpha_9(e_3) = he_1 + ge_2 + re_3.$$

References

[1] F. Ammar, A. Makhlouf and S. Silvestrov, Ternary q-Virasoro-Witt Hom-Nambu-Lie algebras, Journal of Physics A-Mathematical and Theoretical, **43**,no. 26, 265204, (2010).

[2] F. Ammar, S. Mabrouk, A. Makhlouf, Representations and cohomology of n-ary multiplicative Hom-Nambu-Lie algebras, - J. Geom. Physics , **61**, no. 10, 1898-1913, (2011).

[3] J. Arnlind,A. Makhlouf, S. Silvestrov, Ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras, J. Math. Phys. **51**, 043515, 11 pp. (2010).

[4] H. Ataguema and A. Makhlouf, Deformations of ternary algebras, J. Gen. Lie Theory and App. **1**, 41–55, (2007).

[5] H. Ataguema, A. Makhlouf, S. Silvestrov, Generalization of n-ary Nambu algebras and beyond. J. Math. Phys. **50**, 083501, (DOI:10.1063/1.3167801)(2009).

[6] J.A. de Azcarraga, A. M. Perelomov, J.C. Perez Bueno, New Generalized Poisson Structures. J. Phys. A **29**, L151–L157, (1996).

[7] J.A. de Azcarraga and J.M. Izquierdo, n-ary algebras: a review with applications J. Phys. A: Math. Theor.**43**, 293001, (2010).

[8] M. Bordemann, O. Elchinger and A. Makhlouf, Twisting Poisson algebras, coPoisson algebras and Quantization, Travaux Mathmatiques, Volume **20**, 83–119, (2012).

[9] J. M. Casas, T. Datuashvili, M. Ladra, Left-right noncommutative Poisson algebras, Central European Journal of Mathematics, Volume **12**, Issue 1, pp 57-78, January 2014.

[10] R. Chatterjee, L. Takhtajan, Aspects of classical and quantum Nambu mechanics, lett. Math. Phys. **37**, 475-482, (1996).

[11] G. Dito, M. Flato, D. Sternheimer, and L. Takhtajan. Deformation quantization and Nambu mechanics. Commun. Math. Phys., **183**, 1–22, (1997).

[12] G. Dito, M. Flato, and D. Sternheimer. Nambu mechanics, n-ary operations and their quantization.In Deformation Theory and Symplectic Geometry. Math. Phys. Stud. **20**, Kluwer Academic Publishers, Dordrecht, Boston, 43-66, (1997).

[13] V.T. Filippov, n-Lie algebras,(Russian), Sibirsk. Mat. Zh. **26**, no. 6, 126–140 (1985). (English translation: Siberian Math. J. **26**, no. 6, 879–891, (1985))

[14] N. Goze, and E. Remm, On n-ary algebras given by Gerstenhaber products, arXiv:0803.0553v2 [math.RA], (2008).

[15] H. J. Groenewold , On the principles of elementary quantum mechanics. Physics, **12**, 405–460, (1946).

[16] J. T. Hartwig, D. Larsson et S. D. Silvestrov, Deformations of Lie algebras using σ-derivations, Journal of Algebra **295** , no. 2, p. 314–361, (2006).
[17] A. Makhlouf and S. Silvestrov, Hom-algebra structures, Journal of Generalized Lie Theory and Applications, Volume **2**, No. 2, 51–64, (2008).
[18] A. Makhlouf. Associative algebras and computer algebra. Theoretical computer science **187**, 123-145, (1997).
[19] Y. Nambu , Generalized Hamiltonian mechanics, Phys. Rev. D (3), **7**, 2405–2412, (1973).
[20] S. Okubo Triple products and Yang-Baxter equation (I): Octonionic and quaternionic triple systems, J. Math.Phys. **34**, 3273–3291, (1993).
[21] D. Sahoo and M. C. Valsakumar, Nambu mechanics and its quantization, Phys. Rev. A**46**, 4410–4412, (1992).
[22] D. Sahoo , M. C. Valsakumar, Algebraic structure of Nambu mechanics, PARAMANA-Phys. **1**, 1–16, (1993).
[23] L. Takhtajan, On foundation of the generalized Nambu mechanics, Comm. Math. Phys. **160**, 295–315, (1994).
[24] D. Yau, Enveloping algebra of Hom-Lie algebras, Journal of Generalized Lie Theory and Applications. **2**, 95–108, (2008).
[25] D. Yau, Hom-algebras and homology Journal of Lie Theory 19 (2009) 409-421.
[26] D. Yau, On n-ary Hom-Nambu and Hom-Nambu-Lie algebras, J. Geom. Phys. 62 (2012), no. 2, 506–522..
[27] D. Yau, A Hom-associative analogue of n-ary Hom-Nambu algebras, arXiv:1005.2373v1, (2010).
[28] D. Yau, Non-commutative Hom-Poisson algebras, arXiv:1010.3408 v1 [math.RA], (2010).

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