VARIABLE EXPONENT SOBOLEV SPACES ASSOCIATED WITH JACOBI EXPANSIONS

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Abstract. In this paper we define variable exponent Sobolev spaces associated with Jacobi expansions. We prove that our generalized Sobolev spaces can be characterized as variable exponent potential spaces and as variable exponent Triebel-Lizorkin type spaces.

1. Introduction

Sobolev spaces associated with orthogonal systems have been studied in the last years. Bongioanni and Torrea ([8] and [9]) defined Sobolev spaces in the Hermite and Laguerre settings. Sobolev spaces associated with ultraspherical expansions were investigated by Betancor, Fariña, Rodríguez-Mesa, Testoni and Torrea [5]. The study in [5] was extended recently to Jacobi expansions by Langowski [24].

In this paper we define variable exponent Sobolev spaces in the Jacobi context. We now describe our main results.

Consider a measurable function $p : \Omega \subseteq \mathbb{R}^n \longrightarrow [1,\infty)$. By $L^{p(\cdot)}(\Omega)$ we denote the variable exponent Lebesgue space that consists of all those measurable functions on Ω such that for some $\lambda > 0$

$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

It is a Banach space with the Luxermburg norm defined by

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 \ : \ \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1\right\}, \quad f \in L^{p(\cdot)}(\Omega).$$

By $p'(\cdot)$ we represent the conjugate variable exponent. A complete study of $L^{p(\cdot)}$-spaces can be found in [17].

We define $\mathcal{P}(\Omega)$ as the set of measurable functions $p : \ \Omega \longrightarrow [1,\infty)$ such that

$$p_- = \text{ess inf}\{p(x) \ : \ x \in \Omega\} > 1 \qquad \text{and} \qquad p_+ = \text{ess sup}\{p(x) \ : \ x \in \Omega\} < \infty.$$

The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined as

$$\mathcal{M}f(x) = \sup_{B \ni x} \frac{1}{|B|}\int_B f(y)dy, \quad x \in \Omega.$$

The set B in the supremum represents a ball and $|B|$ denotes its Lebesgue measure.

We define $\mathcal{B}(\Omega)$ as the subset of $\mathcal{P}(\Omega)$ that consists of all those measurable functions p such that the maximal operator $\mathcal{M}$ is bounded from $L^{p(\cdot)}(\Omega)$ into itself. Diening [16, Theorem 3.5] proved that if Ω is a bounded subset of $\mathbb{R}^n$, $p \in \mathcal{P}(\Omega)$ and there exists $C > 0$ such that

$$(1) \qquad |p(x) - p(y)| \leq \frac{C}{-\log|x-y|}, \quad x, y \in \Omega, \quad |x-y| \leq 1/2,$$

then $p \in \mathcal{B}(\Omega)$.

Many classical operators in harmonic analysis (maximal operator, singular integrals, Fourier multipliers, commutators, fractional integrals, ...) have been studied in variable $L^{p(\cdot)}$-spaces (see, for instance, [15], [17], [18] and [39]).

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Let $k \in \mathbb{N}$, where by $\mathbb{N}$ we represent the set of positive integer with zero included, and $p \in \mathcal{P}(\Omega)$. A measurable function f on Ω is in the generalized Sobolev space $W^{k,p(\cdot)}(\Omega)$ if its weak partial derivatives $D^\alpha f \in L^{p(\cdot)}(\Omega)$, $\alpha \in \mathbb{N}^n$ and $0 \leq |\alpha| \leq k$. The norm in $W^{k,p(\cdot)}(\Omega)$ is defined by

$$\|f\|_{W^{k,p(\cdot)}(\Omega)} = \sum_{|\alpha|=0}^{k} \|D^\alpha f\|_{L^{p(\cdot)}(\Omega)}, \quad f \in W^{k,p(\cdot)}(\Omega).$$

It turns out that $W^{k,p(\cdot)}(\Omega)$ is a Banach space.

Variable exponent Sobolev spaces $W^{k,p(\cdot)}(\Omega)$ have been studied by a lot of authors in this century. Applications of these generalized Sobolev spaces can be seen in [17, Part III].

Now we turn to the Harmonic Analysis associated with the Jacobi differential operator $L_{\alpha,\beta}$ for $\alpha, \beta > -1$, which is defined as

$$L_{\alpha,\beta} = -\frac{d^2}{d\theta^2} - \frac{1-4\alpha^2}{16\sin^2\frac{\theta}{2}} - \frac{1-4\beta^2}{16\cos^2\frac{\theta}{2}}, \quad \text{on } (0,\pi).$$

This type of analysis has emerged as a prolific area of interest (see [1], [12], [13], [24], [25], [26], [34], [37] and [42], amongst others).

The Jacobi operator admits the following decomposition

$$L_{\alpha,\beta} = D_{\alpha,\beta}^* D_{\alpha,\beta} + \left(\frac{\alpha+\beta+1}{2}\right)^2,$$

where

$$\begin{aligned} D_{\alpha,\beta} &= \frac{d}{d\theta} - \frac{2\alpha+1}{4}\cot\frac{\theta}{2} + \frac{2\beta+1}{4}\tan\frac{\theta}{2} \\ &= \left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}\frac{d}{d\theta}\left[\left(\sin\frac{\theta}{2}\right)^{-\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{-\beta-1/2}\right], \end{aligned}$$

and $D_{\alpha,\beta}^*$ is the formal adjoint of $D_{\alpha,\beta}$ in $L^2(0,\pi)$. When $\alpha = \beta$ the Jacobi operator $L_{\alpha,\beta}$ reduces to the ultraspherical operator L_λ, $\lambda = \alpha + 1/2$, considered in [5]. According to [43, (4.24.2)] we have that, for every $n \in \mathbb{N}$,

$$L_{\alpha,\beta}\phi_n^{\alpha,\beta} = \lambda_n^{\alpha,\beta}\phi_n^{\alpha,\beta},$$

where $\lambda_n^{\alpha,\beta} = (n + \frac{\alpha+\beta+1}{2})^2$ and

$$\phi_n^{\alpha,\beta}(\theta) = \left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}\mathcal{P}_n^{\alpha,\beta}(\theta), \quad \theta \in (0,\pi).$$

If $p_n^{\alpha,\beta}$ denotes the n-th Jacobi polynomial considered in Szegö's monograph, then $\mathcal{P}_n^{\alpha,\beta} = d_n^{\alpha,\beta} p_n^{\alpha,\beta}$, where $d_n^{\alpha,\beta}$ is a normalization constant, for every $n \in \mathbb{N}$. The system $\{\phi_n^{\alpha,\beta}\}_{n\in\mathbb{N}}$ is orthonormal and complete in $L^2(0,\pi)$. We define the Jacobi operator $\mathcal{L}_{\alpha,\beta}$ by

$$\mathcal{L}_{\alpha,\beta} f = \sum_{n=0}^{\infty} \lambda_n^{\alpha,\beta} c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta}, \quad f \in D(\mathcal{L}_{\alpha,\beta}).$$

Here, for every $f \in L^2(0,\pi)$ and $n \in \mathbb{N}$,

$$c_n^{\alpha,\beta}(f) = \int_0^\pi \phi_n^{\alpha,\beta}(\theta) f(\theta) d\theta,$$

and by $D(\mathcal{L}_{\alpha,\beta})$ we denote the domain of $\mathcal{L}_{\alpha,\beta}$ given by

$$D(\mathcal{L}_{\alpha,\beta}) = \{f \in L^2(0,\pi) \ : \ \sum_{n=0}^{\infty} (\lambda_n^{\alpha,\beta})^2 |c_n^{\alpha,\beta}(f)|^2 < \infty\}.$$

Note that $C_c^\infty(0,\pi)$, the space of smooth function with compact support in $(0,\pi)$, is contained in $D(\mathcal{L}_{\alpha,\beta})$ and hence,

$$\mathcal{L}_{\alpha,\beta} f = L_{\alpha,\beta} f, \quad f \in C_c^\infty(0,\pi).$$

$\mathcal{L}_{\alpha,\beta}$ is a positive and selfadjoint operator in $L^2(0,\pi)$. Let us note that $-\mathcal{L}_{\alpha,\beta}$ generates a semigroup of operators $\{W_t^{\alpha,\beta}\}_{t>0}$ in $L^2(0,\pi)$ where, for every $t > 0$,

$$W_t^{\alpha,\beta} f = \sum_{n=0}^{\infty} e^{-t\lambda_n^{\alpha,\beta}} c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta}, \quad f \in L^2(0,\pi).$$

Moreover, for every $t > 0$ and $f \in L^2(0, \pi)$,

$$W_t^{\alpha,\beta} f(\theta) = \int_0^\pi W_t^{\alpha,\beta}(\theta, \varphi) f(\varphi) d\varphi, \quad \theta \in (0, \pi),$$

where

$$W_t^{\alpha,\beta}(\theta, \varphi) = \sum_{n=0}^\infty e^{-t\lambda_n^{\alpha,\beta}} \phi_n^{\alpha,\beta}(\theta) \phi_n^{\alpha,\beta}(\varphi), \quad \theta, \varphi \in (0, \pi) \text{ and } t > 0.$$

$\{W_t^{\alpha,\beta}\}_{t>0}$ is called the heat semigroup associated with the Jacobi operator $\mathcal{L}_{\alpha,\beta}$. By $\{P_t^{\alpha,\beta}\}_{t>0}$ we denote the Poisson semigroup defined by $\mathcal{L}_{\alpha,\beta}$. According to the subordination formula, we can write, for every $t > 0$ and $f \in L^2(0, \pi)$,

$$P_t^{\alpha,\beta} f(\theta) = \int_0^\pi P_t^{\alpha,\beta}(\theta, \varphi) f(\varphi) d\varphi, \quad \theta \in (0, \pi),$$

where

$$P_t^{\alpha,\beta}(\theta, \varphi) = \frac{t}{\sqrt{4\pi}} \int_0^\infty \frac{e^{-t^2/4u}}{u^{3/2}} W_u^{\alpha,\beta}(\theta, \varphi) du, \quad \theta, \varphi \in (0, \pi). \tag{2}$$

Jacobi Sobolev spaces were studied by Langowski [24]. We now introduce variable exponent Jacobi Sobolev spaces. Assume that $p \in \mathcal{P}(0, \pi)$ and $k \in \mathbb{N}$. We say that a measurable function $f \in L^{p(\cdot)}(0, \pi)$ is in the variable Jacobi Sobolev space $W_{\alpha,\beta}^{k,p(\cdot)}(0, \pi)$ if $\mathbb{D}_{\alpha,\beta}^\ell f \in L^{p(\cdot)}(0, \pi)$, for every $\ell \in \mathbb{N}$, $0 \le \ell \le k$, with $\mathbb{D}_{\alpha,\beta}^0 f = f$ and for $\ell \ge 1$,

$$\mathbb{D}_{\alpha,\beta}^\ell = D_{\alpha+l-1,\beta+l-1} \circ \ \dots \ \circ D_{\alpha+1,\beta+1} \circ D_{\alpha,\beta},$$

is understood in a weak sense. On $W_{\alpha,\beta}^{k,p(\cdot)}(0, \pi)$ we consider the norm defined by

$$\|f\|_{W_{\alpha,\beta}^{k,p(\cdot)}(0,\pi)} = \|f\|_{L^{p(\cdot)}(0,\pi)} + \sum_{l=1}^k \|\mathbb{D}_{\alpha,\beta}^\ell f\|_{L^{p(\cdot)}(0,\pi)}, \quad f \in W_{\alpha,\beta}^{k,p(\cdot)}(0, \pi).$$

Thus, $W_{\alpha,\beta}^{k,p(\cdot)}(0, \pi)$ becomes a Banach space. See the discussion in [24] (and also in [5]) for the use of the derivatives $\mathbb{D}_{\alpha,\beta}^\ell$, instead of the more natural choice $D_{\alpha,\beta}^\ell = D_{\alpha,\beta} \circ \ \dots \ \circ D_{\alpha,\beta}$.

Let $\gamma > 0$ and assume that $\alpha + \beta \neq -1$. The negative power $\mathcal{L}_{\alpha,\beta}^{-\gamma}$ of $\mathcal{L}_{\alpha,\beta}$ is given by

$$\mathcal{L}_{\alpha,\beta}^{-\gamma} f = \sum_{n=0}^\infty (\lambda_n^{\alpha,\beta})^{-\gamma} c_n^{\alpha,\beta}(f) \phi_n^{\alpha,\beta}, \quad f \in L^2(0, \pi). \tag{3}$$

$\mathcal{L}_{\alpha,\beta}^{-\gamma}$ defines a one to one and bounded operator from $L^{p(\cdot)}(0, \pi)$ into itself (see Propositions 3.3 and 3.4 below). The variable exponent Jacobi potential space $H_{\alpha,\beta}^{\gamma,p(\cdot)}(0, \pi)$ consists of all those functions $f \in L^{p(\cdot)}(0, \pi)$ such that $f = \mathcal{L}_{\alpha,\beta}^{-\gamma} g$ for some (unique) $g \in L^{p(\cdot)}(0, \pi)$. We considerer in $H_{\alpha,\beta}^{\gamma,p(\cdot)}(0, \pi)$ the following norm

$$\|f\|_{H_{\alpha,\beta}^{\gamma,p(\cdot)}(0,\pi)} = \|g\|_{L^{p(\cdot)}(0,\pi)}, \quad f = \mathcal{L}_{\alpha,\beta}^{-\gamma} g \in H_{\alpha,\beta}^{\gamma,p(\cdot)}(0, \pi).$$

Endowed with this norm $H_{\alpha,\beta}^{\gamma,p(\cdot)}(0, \pi)$ is a Banach space.

The variable exponent version of [24, Theorem A] is given in the following theorem.

Theorem 1.1. *Let $\alpha, \beta \ge -1/2$ such that $\alpha + \beta \neq -1$ and $k \in \mathbb{N}$, $k \ge 1$. Assume that $p \in \mathcal{B}(0, \pi)$. Then, $H_{\alpha,\beta}^{k/2,p(\cdot)}(0, \pi) = W_{\alpha,\beta}^{k,p(\cdot)}(0, \pi)$. Moreover, the norms $\|\cdot\|_{H_{\alpha,\beta}^{k/2,p(\cdot)}(0,\pi)}$ and $\|\cdot\|_{W_{\alpha,\beta}^{k,p(\cdot)}(0,\pi)}$ are equivalent.*

The proof of Theorem 1.1 is done in several steps. For a suitable function p we will prove.

(a) The linear subspace $S_{\alpha,\beta} = \text{span}\{\phi_n^{\alpha,\beta}\}$ is dense in both $W_{\alpha,\beta}^{k,p(\cdot)}(0, \pi)$ and $H_{\alpha,\beta}^{k/2,p(\cdot)}(0, \pi)$.
(b) The higher order Jacobi-Riesz transforms defined by

$$R_{\alpha,\beta}^k = \mathbb{D}_{\alpha,\beta}^k \mathcal{L}_{\alpha,\beta}^{-k/2} \text{ and } R_{\alpha,\beta}^{k,*} = \mathbb{D}_{\alpha,\beta}^{k,*} \mathcal{L}_{\alpha+k,\beta+k}^{-k/2}, \quad k \in \mathbb{N}. \tag{4}$$

are bounded operators on $L^{p(\cdot)}(0, \pi)$.

(c) We define a multiplier operator $m(\mathcal{L}_{\alpha,\beta})$ in such a way that

$$m(\mathcal{L}_{\alpha,\beta})R^{k,*}_{\alpha,\beta}R^k_{\alpha,\beta}f = f - \sum_{n=0}^{k-1} c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta}, \quad \text{for all } f \in S_{\alpha,\beta},$$

and prove its boundedness on $L^{p(\cdot)}(0,\pi)$.

(d) For every $\gamma > 0$, the potential operator $\mathcal{L}^{-\gamma}_{\alpha,\beta}$ is also bounded on $L^{p(\cdot)}(0,\pi)$.

According with [15] in order to get the boundedness of operators defined on $L^{p(\cdot)}(0,\pi)$ it is sufficient to prove boundedness of them on the weighted L^r-spaces, $L^r_\omega(0,\pi)$ for every $\omega \in A_r(0,\pi)$, the class of Muckenhoupt weights , and some $1 < r < \infty$. Let us note that, taking into account [15, Theorem 1.2], we can change the condition "$p \in \mathcal{P}(0,\pi)$ and for some $p_0 \in (1,p_-)$, $(p(\cdot)/p_0)' \in \mathcal{B}(0,\pi)$" used in [15, Theorem 1.3] by $p \in \mathcal{B}(0,\pi)$, because if $p \in \mathcal{B}(0,\pi)$ there exists an extension $\widetilde{p} \in \mathcal{B}(\mathbb{R})$ of p from $(0,\pi)$ to $\mathbb{R}$.

Once of all this has been proved, the proof of Theorem 1.1 is as follows:

From assertion (a) it is enough to prove the equivalence of norms for functions in $S_{\alpha,\beta}$. Let us take then $f, g \in S_{\alpha,\beta}$ such that $f = \mathcal{L}^{-k/2}_{\alpha,\beta}g$. From assertions (b) and (c) we get

$$\begin{aligned}\|g\|_{L^{p(\cdot)}(0,\pi)} \leq & C\Big(\|m(\mathcal{L}_{\alpha,\beta})R^{k,*}_{\alpha,\beta}\mathbb{D}^k_{\alpha,\beta}f\|_{L^{p(\cdot)}(0,\pi)} + \|f\|_{L^{p(\cdot)}(0,\pi)}\Big)\\ \leq & C\Big(\|\mathbb{D}^k_{\alpha,\beta}f\|_{L^{p(\cdot)}(0,\pi)} + \|f\|_{L^{p(\cdot)}(0,\pi)}\Big).\end{aligned}$$

Thus, we obtain

$$\|f\|_{H^{k/2,p(\cdot)}_{\alpha,\beta}(0,\pi)} \leq C\|f\|_{W^{k,p(\cdot)}_{\alpha,\beta}(0,\pi)}.$$

On the other hand, by using assertions (b) and (d), for every $m \in \mathbb{N}$ such that $0 \leq m \leq k$,

$$\|\mathbb{D}^m_{\alpha,\beta}f\|_{L^{p(\cdot)}(0,\pi)} = \|\mathbb{D}^m_{\alpha,\beta}\mathcal{L}^{-k/2}_{\alpha,\beta}g\|_{L^{p(\cdot)}(0,\pi)} = \|R^m_{\alpha,\beta}\mathcal{L}^{-(k-m)/2}_{\alpha,\beta}g\|_{L^{p(\cdot)}(0,\pi)} \leq C\|g\|_{L^{p(\cdot)}(0,\pi)}.$$

Hence,

$$\|f\|_{W^{k,p(\cdot)}_{\alpha,\beta}(0,\pi)} \leq C\|f\|_{H^{k/2,p(\cdot)}_{\alpha,\beta}(0,\pi)}.$$

We now define the positive power of the Jacobi operator $\mathcal{L}_{\alpha,\beta}$ according to the ideas of Lions and Peetre [27, Chapter VII, Section 2] and Berens, Butzer and Westphal [2]. Let $\gamma > 0$ and choose $r \in \mathbb{N}$ such that $\gamma < r \leq \gamma + 1$. For every $\varepsilon > 0$ and $f \in L^{p(\cdot)}(0,\pi)$, we define

$$I^{\gamma,r}_\varepsilon f = C_{\gamma,r}\int_\varepsilon^\infty \frac{\left(I - W^{\alpha,\beta}_u\right)^r f}{u^{\gamma+1}}du \ , \tag{5}$$

where the integral is understood in the $L^{p(\cdot)}$-Bochner sense and $C_{\gamma,r} = \left(\int_0^\infty \frac{(1-e^{-u})^r}{u^{\gamma+1}}du\right)^{-1}$. Note that, for every $f \in L^{p(\cdot)}(0,\pi)$,

$$\int_\varepsilon^\infty \frac{\|\left(I - W^{\alpha,\beta}_u\right)^r f\|_{L^{p(\cdot)}(0,\pi)}}{u^{\gamma+1}}du < \infty.$$

Moreover, the operator $I^{\gamma,r}_\varepsilon$ is bounded from $L^{p(\cdot)}(0,\pi)$ into itself (Proposition 5.1). We consider the domain of $\mathcal{L}^\gamma_{\alpha,\beta}$

$$D_{p(\cdot)}(\mathcal{L}^\gamma_{\alpha,\beta}) = \Big\{f \in L^{p(\cdot)}(0,\pi) \ : \ \lim_{\varepsilon\to 0^+} I^{\gamma,r}_\varepsilon f \text{ exists in } L^{p(\cdot)}(0,\pi)\Big\},$$

and we define

$$\mathcal{L}^\gamma_{\alpha,\beta}f = \lim_{\varepsilon\to 0^+} I^{\gamma,r}_\varepsilon f, \quad f \in D_{p(\cdot)}(\mathcal{L}^\gamma_{\alpha,\beta}). \tag{6}$$

As it will be shown in Section 5, in the definition of $\mathcal{L}^\gamma_{\alpha,\beta}$ we can take any $r \in \mathbb{N}$, $r > \gamma$. Next, we characterize the Jacobi potential space $H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi)$ as the domain of $\mathcal{L}^\gamma_{\alpha,\beta}$.

Theorem 1.2. *Let $\gamma > 0$ and $\alpha, \beta \geq -1/2$ such that $\alpha + \beta \neq -1$. Assume that $p \in \mathcal{B}(0,\pi)$. Then, $H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi) = D_{p(\cdot)}(\mathcal{L}^\gamma_{\alpha,\beta})$. Moreover, for every $f \in D_{p(\cdot)}(\mathcal{L}^\gamma_{\alpha,\beta})$,*

$$\mathcal{L}^{-\gamma}_{\alpha,\beta}\mathcal{L}^\gamma_{\alpha,\beta}f = f,$$

and, for every $f \in L^{p(\cdot)}(0,\pi)$,

$$\mathcal{L}^\gamma_{\alpha,\beta}\mathcal{L}^{-\gamma}_{\alpha,\beta}f = f.$$

Segovia and Wheeden [40] characterized potential spaces by using Littlewood-Paley square functions. In order to do this they introduced square functions involving fractional derivatives of the classical Poisson semigroup. Inspired by [40], Betancor, Fariña, Rodríguez-Mesa, Testoni and Torrea obtained characterizations using vertical and area Littlewood-Paley functions for the potential spaces associated with the Hermite and Ornstein-Uhlenbeck operators ([6]) and Schrödinger operators ([4]). We will characterize our variable exponent Jacobi potential spaces by using Littlewood-Paley function defined via derivatives of the Jacobi-Poisson semigroup.

Let $\gamma > 0$ and $k \in \mathbb{N}$ such that $0 < \gamma < k$. We consider the following Littlewood-Paley function

$$g_{\alpha,\beta}^{\gamma,k}(f)(\theta) = \left(\int_0^\infty \left|t^{k-\gamma}\partial_t^k P_t^{\alpha,\beta}(f)(\theta)\right|^2 \frac{dt}{t}\right)^{1/2}, \quad \theta \in (0,\pi).$$

We say that a measurable function $f \in L^{p(\cdot)}(0,\pi)$ is in $T_{\alpha,\beta}^{\gamma,k,p(\cdot)}(0,\pi)$ when $g_{\alpha,\beta}^{\gamma,k}(f) \in L^{p(\cdot)}(0,\pi)$. On $T_{\alpha,\beta}^{\gamma,k,p(\cdot)}(0,\pi)$ we define the norm

$$\|f\|_{T_{\alpha,\beta}^{\gamma,k,p(\cdot)}(0,\pi)} = \|f\|_{L^{p(\cdot)}(0,\pi)} + \|g_{\alpha,\beta}^{\gamma,k}(f)\|_{L^{p(\cdot)}(0,\pi)}, \quad f \in T_{\alpha,\beta}^{\gamma,k,p(\cdot)}(0,\pi).$$

Thus, $T_{\alpha,\beta}^{\gamma,k,p(\cdot)}(0,\pi)$ is a Banach space.

The space $T_{\alpha,\beta}^{\gamma,k,p(\cdot)}(0,\pi)$, which can be seen as a variable exponent Triebel-Lizorkin type space, coincides with the variable exponent potential space $H_{\alpha,\beta}^{\gamma/2,p(\cdot)}(0,\pi)$.

Theorem 1.3. *Let $\alpha,\beta \geq -1/2$ such that $\alpha+\beta \neq -1$ and $0 < \gamma < k$, $k \in \mathbb{N}$. Assume that $p \in \mathcal{B}(0,\pi)$. Then, $H_{\alpha,\beta}^{\gamma/2,p(\cdot)}(0,\pi) = T_{\alpha,\beta}^{\gamma,k,p(\cdot)}(0,\pi)$. Moreover, the norms $\|\cdot\|_{H_{\alpha,\beta}^{\gamma/2,p(\cdot)}(0,\pi)}$ and $\|\cdot\|_{T_{\alpha,\beta}^{\gamma,k,p(\cdot)}(0,\pi)}$ are equivalent.*

Note that from Theorem 1.3 we deduce that the space $T_{\alpha,\beta}^{\gamma,k,p(\cdot)}(0,\pi)$ does not depend on $k \in \mathbb{N}$ provided that $0 < \gamma < k$. The result in Theorem 1.3 is new even when $p \in \mathcal{P}(0,\pi)$ is constant and it gives a new characterization of the Jacobi Sobolev spaces introduced in [24].

In order to prove Theorem 1.3 we need to show that certain square function related to $g_{\alpha,\beta}^{\gamma,k}$, which involves fractional derivatives, is bounded on $L^{p(\cdot)}(0,\pi)$. In [40] fractional derivatives were introduced. Suppose that $\gamma > 0$ and F is a nice enough function defined in $(0,\pi)\times(0,\infty)$. The γ-th derivative $\partial_t^\gamma F$ is defined by

$$\partial_t^\gamma F(\theta,t) = \frac{e^{-i(m-\gamma)\pi}}{\Gamma(m-\gamma)} \int_0^\infty \partial_t^m F(\theta,t+s)s^{m-\gamma-1}ds, \quad \theta \in (0,\pi),\ t > 0,$$

where $m \in \mathbb{N}$ is such that $m-1 \leq \gamma < m$.

We consider the Littlewood-Paley function $g_{\alpha,\beta}^{\gamma}$ given by

$$g_{\alpha,\beta}^{\gamma}(f)(\theta) = \left(\int_0^\infty \left|t^{\gamma}\partial_t^\gamma P_t^{\alpha,\beta}(f)(\theta)\right|^2 \frac{dt}{t}\right)^{1/2}, \quad \theta \in (0,\pi).$$

The key relation between $g_{\alpha,\beta}^{\gamma,k}$ and $g_{\alpha,\beta}^{\gamma}$, $0 < \gamma < k$, which allows to connect the spaces $H_{\alpha,\beta}^{\gamma/2,p(\cdot)}(0,\pi)$ and $T_{\alpha,\beta}^{\gamma,k,p(\cdot)}(0,\pi)$, is the following

$$g_{\alpha,\beta}^{k-\gamma}(f) = g_{\alpha,\beta}^{\gamma,k}(\mathcal{L}_{\alpha,\beta}^{-\gamma/2}f), \quad f \in S_{\alpha,\beta}.$$

In [23] Kyriazis, Petrushev and Xu defined Besov and Triebel-Lizorkin spaces associated with Jacobi expansions with respect to $\big((-1,1),(1-x)^\alpha(1+x)^\beta dx\big)$. We now adapt the Triebel-Lizorkin definitions given in [23] to our Jacobi expansions in $\big((0,\pi),d\theta\big)$. We take a function $\mathfrak{a} \in C_c^\infty(0,\infty)$ such that $\operatorname{supp}\mathfrak{a} \subseteq [1/2,2]$ and $\inf_{t\in[3/5,5/3]}|\mathfrak{a}(t)| > 0$. The following construction is independent of the election of $\mathfrak{a}$ and, as it is said in [23], we can add the condition that $\mathfrak{a}(t)+\mathfrak{a}(2t) = 1$ for $t \in [1/2,1]$. We define the sequence $\{\Phi_j^{\alpha,\beta}\}_{j\in\mathbb{N}}$ of functions on $(0,\pi)^2$ as follows,

$$\Phi_0^{\alpha,\beta}(\theta,\varphi) = \phi_0^{\alpha,\beta}(\theta)\phi_0^{\alpha,\beta}(\varphi), \quad \theta,\varphi \in (0,\pi),$$

and, for every $j \in \mathbb{N}$, $j \geq 1$,

$$\Phi_j^{\alpha,\beta}(\theta,\varphi) = \sum_{n=0}^\infty \mathfrak{a}\Big(\frac{\lambda_n^{\alpha,\beta}}{2^{j-1}}\Big)\phi_n^{\alpha,\beta}(\theta)\phi_n^{\alpha,\beta}(\varphi), \quad \theta,\varphi \in (0,\pi).$$

If $\gamma \in \mathbb{R}$ and $0 < p, q < \infty$, a function $f \in L^1(0,\pi)$ is in the Jacobi-Triebel-Lizorkin space $F^{\gamma,q,p}_{\alpha,\beta}(0,\pi)$ provided that

$$\|f\|_{F^{\gamma,q,p}_{\alpha,\beta}(0,\pi)} = \Big\| \Big(\sum_{j=0}^{\infty} \big(2^{j\gamma} |\Phi_j^{\alpha,\beta}(f)(\cdot)|\big)^q \Big)^{1/q} \Big\|_{L^p(0,\pi)} < \infty.$$

Here, for every $j \in \mathbb{N}$,

$$\Phi_j^{\alpha,\beta}(f)(\theta) = \int_0^\pi \Phi_j^{\alpha,\beta}(\theta,\varphi) f(\varphi) d\varphi, \quad \theta \in (0,\pi).$$

It would be interesting to investigate Jacobi-Triebel-Lizorkin spaces with variable exponent in the $\big((-1,1),(1-x)^\alpha(1+x)^\beta dx\big)$ and $\big((0,\pi), d\theta\big)$ settings. This question will be considered on its whole generality in a forthcoming paper. Here we only introduce Jacobi-Triebel-Lizorkin spaces with $\gamma > 0$, $q = 2$ and variable exponent $p(\cdot)$. Assume that $p \in \mathcal{P}(0,\pi)$. A function $f \in L^{p(\cdot)}(0,\pi)$ is in $F^{\gamma,2,p(\cdot)}_{\alpha,\beta}(0,\pi)$ when

$$\|f\|_{F^{\gamma,2,p(\cdot)}_{\alpha,\beta}(0,\pi)} = \Big\| \Big(\sum_{j=0}^{\infty} \big(2^{j\gamma} |\Phi_j^{\alpha,\beta}(f)(\cdot)|\big)^2 \Big)^{1/2} \Big\|_{L^{p(\cdot)}(0,\pi)} < \infty.$$

In the following theorem we identify the variable exponent Jacobi-Triebel-Lizorkin space $F^{\gamma,2,p(\cdot)}_{\alpha,\beta}(0,\pi)$ with the potential space $H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi)$.

Theorem 1.4. *Let $\alpha, \beta \geq -1/2$ and $\gamma > 0$. Assume that $p \in \mathcal{B}(0,\pi)$. Then, $H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi) = F^{\gamma,2,p(\cdot)}_{\alpha,\beta}(0,\pi)$. Moreover, the norms $\|\cdot\|_{H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi)}$ and $\|\cdot\|_{F^{\gamma,2,p(\cdot)}_{\alpha,\beta}(0,\pi)}$ are equivalent.*

Note that as a special case of Theorem 1.4 we establish that the Jacobi potential space $H^{\gamma,p}_{\alpha,\beta}(0,\pi)$ considered by Langowski ([24]) coincides with the Jacobi-Triebel-Lizorkin space $F^{\gamma,2,p}_{\alpha,\beta}(0,\pi)$, for every $1 < p < \infty$.

The paper is organized as follows. In Sections 2, 3 and 4 we prove that assertions (a), (b), (c) and (d) are true. Theorems 1.2, 1.3 and 1.4 are proved in Sections 5, 6 and 7, respectively.

Throughout this paper by C and c we always denote positive constants that can change in each occurrence.

2. Dense subspaces

This section deals with the proof of the $W^{k,p(\cdot)}_{\alpha,\beta}$-density of $S_{\alpha,\beta}$ claimed in assertion (a) of Section 1.

Assume that $p \in \mathcal{P}(0,\pi)$. According to [17, Theorem 3.4.6] the space $L^{p'(\cdot)}(0,\pi)$ is isomorphic to the dual space $(L^{p(\cdot)}(0,\pi))^*$ of $L^{p(\cdot)}(0,\pi)$. On the other hand, for every $k \in \mathbb{N}$, $\phi_k^{\alpha,\beta} \in L^\infty(0,\pi)$. Then, $\phi_k^{\alpha,\beta} \in L^{p'(\cdot)}(0,\pi)$, $k \in \mathbb{N}$ ([17, Theorem 3.3.11]). We define, for every $f \in L^{p(\cdot)}(0,\pi)$ and $k \in \mathbb{N}$,

$$c_k^{\alpha,\beta}(f) = \int_0^\pi \phi_k^{\alpha,\beta}(\theta) f(\theta) d\theta.$$

By [17, Theorem 3.4.12] the space $C_c^\infty(0,\pi)$ is dense in $L^{p(\cdot)}(0,\pi)$.

Proposition 2.1. *Let $\alpha, \beta \geq -1/2$ and $p \in \mathcal{P}(0,\pi)$. The space $S_{\alpha,\beta} = \operatorname{span}\{\phi_k^{\alpha,\beta}\}_{k\in\mathbb{N}}$ is dense in $L^{p(\cdot)}(0,\pi)$.*

Proof. Since $C_c^\infty(0,\pi)$ is a dense subspace of $L^{p(\cdot)}(0,\pi)$, it is sufficient to see that $C_c^\infty(0,\pi)$ is contained in the closure of $S_{\alpha,\beta}$ in $L^{p(\cdot)}(0,\pi)$. Let $g \in C_c^\infty(0,\pi)$. By using integration by parts we deduce that, for every $m \in \mathbb{N}$, there exists $C_m > 0$ such that $|c_k^{\alpha,\beta}(g)| < C_m(k+1)^{-m}$, $k \in \mathbb{N}$. Hence,

$$S_n^{\alpha,\beta}(g) = \sum_{k=0}^{n} c_k^{\alpha,\beta}(g)\phi_k^{\alpha,\beta} \longrightarrow g\ , \quad \text{as } n \to \infty, \quad \text{in } L^\infty(0,\pi).$$

Hence, according to [17, Theorem 3.3.11], $S_n^{\alpha,\beta}(\phi) \to \phi$, as $n \to \infty$, in $L^{p(\cdot)}(0,\pi)$. □

Corollary 2.1. *Let $\alpha, \beta \geq -1/2$ and $p \in \mathcal{P}(0,\pi)$. If $f \in L^{p(\cdot)}(0,\pi)$ and $c_k^{\alpha,\beta}(f) = 0$, $k \in \mathbb{N}$, then $f = 0$.*

Proof. Since $p \in \mathcal{P}(0,\pi)$, p' is also in $\mathcal{P}(0,\pi)$. Then, by Proposition 2.1, $S_{\alpha,\beta}$ is dense in $L^{p'(\cdot)}(0,\pi)$. Assume that $f \in L^{p(\cdot)}(0,\pi)$ is such that $c_k^{\alpha,\beta}(f) = 0$, $k \in \mathbb{N}$. The norm conjugate formula ([17, Corollary 3.2.14]) leads to

$$\int_0^\pi f(\theta)g(\theta)d\theta = 0,$$

for every $g \in L^{p'(\cdot)}(0,\pi)$. By using again the norm conjugate formula (duality) we conclude that $f = 0$. $\square$

We can improve the result in Proposition 2.1 when the function $p(\cdot)$ satisfies additional conditions. According to [30, Theorem 1], if $1 < p < \infty$ and $f \in L^p(0,\pi)$, then

$$f = \lim_{n\to\infty} \sum_{k=0}^n c_k^{\alpha,\beta}(f)\phi_k^{\alpha,\beta},$$

where the convergence is understood in $L^p(0,\pi)$. We now establish this property in $L^p_w(0,\pi)$, $1 < p < \infty$ and $w \in A_p(0,\pi)$, and in $L^{p(\cdot)}(0,\pi)$ when the function $p(\cdot)$ is as in [15, Theorem 1.3].

Proposition 2.2. *Let $\alpha, \beta \geq -1/2$.*

(*i*) *If $1 < p < \infty$ and $w \in A_p(0,\pi)$, there exists $C > 0$ such that, for every $n \in \mathbb{N}$,*

$$\Big\| \sum_{k=0}^n c_k^{\alpha,\beta}(f)\phi_k^{\alpha,\beta} \Big\|_{L^p_w(0,\pi)} \leq C\|f\|_{L^p_w(0,\pi)}, \quad f \in L^p_w(0,\pi),$$

and

$$\lim_{n\to\infty} \sum_{k=0}^n c_k^{\alpha,\beta}(f)\phi_k^{\alpha,\beta} = f, \quad f \in L^p_w(0,\pi),$$

in the sense of convergence in $L^p_w(0,\pi)$.

(*ii*) *Assume that $p \in \mathcal{B}(0,\pi)$. Then, there exists $C > 0$ such that, for every $n \in \mathbb{N}$,*

$$\Big\| \sum_{k=0}^n c_k^{\alpha,\beta}(f)\phi_k^{\alpha,\beta} \Big\|_{L^{p(\cdot)}(0,\pi)} \leq C\|f\|_{L^{p(\cdot)}(0,\pi)}, \quad f \in L^{p(\cdot)}(0,\pi),$$

and

$$\lim_{n\to\infty} \sum_{k=0}^n c_k^{\alpha,\beta}(f)\phi_k^{\alpha,\beta} = f, \quad f \in L^{p(\cdot)}(0,\pi),$$

in the sense of convergence in $L^{p(\cdot)}(0,\pi)$.

Proof of Proposition 2.2, (*i*). In order to prove this property we proceed as in the proof of [22, Theorem 2]. Let $1 < p < \infty$ and $w \in A_p(0,\pi)$. Suppose that $f \in L^p_w(0,\pi)$ and $n \in \mathbb{N}$. We define

$$S_n f(\theta) = \sum_{k=0}^n c_k^{\alpha,\beta}(f)\phi_k^{\alpha,\beta}(\theta), \quad \theta \in (0,\pi).$$

As in [22, p. 13] we have that

$$|S_n f(\theta)| \leq C \sum_{\ell=1}^3 J_\ell^{\alpha,\beta,n} f(\theta), \quad \theta \in (0,\pi), \tag{7}$$

where the operators $J_\ell^{\alpha,\beta,n}$, $\ell = 1,2,3$ can be estimated as follows. Firstly, for $J_1^{\alpha,\beta,n}$ we get

$$\begin{aligned} J_1^{\alpha,\beta,n} f(\theta) \leq & C \frac{\big(\sin\frac{\theta}{2}\big)^{\alpha+1/2}\big(\cos\frac{\theta}{2}\big)^{\beta+1/2}}{\big(\sin\frac{\theta}{2} + \frac{1}{n+1}\big)^{\alpha+1/2}\big(\cos\frac{\theta}{2} + \frac{1}{n+1}\big)^{\beta+1/2}} \\ & \times \int_0^\pi \frac{\big(\sin\frac{\varphi}{2}\big)^{\alpha+1/2}\big(\cos\frac{\varphi}{2}\big)^{\beta+1/2}}{\big(\sin\frac{\varphi}{2} + \frac{1}{n+1}\big)^{\alpha+1/2}\big(\cos\frac{\varphi}{2} + \frac{1}{n+1}\big)^{\beta+1/2}} |f(\varphi)|d\varphi \\ \leq & C \int_0^\pi |f(\varphi)|d\varphi, \quad \theta \in (0,\pi). \end{aligned}$$

Then, Hölder's inequality implies that

$$\int_0^\pi |J_1^{\alpha,\beta,n} f(\theta)|^p w(\theta)d\theta \leq C \int_0^\pi |f(\theta)|^p w(\theta)d\theta, \tag{8}$$

because $L^p_w(0,\pi) \subseteq L^1(0,\pi)$.

For $J_2^{\alpha,\beta,n}$ the following estimate holds

$$J_2^{\alpha,\beta,n} f(\theta) \leq C \frac{\big(\sin\frac{\theta}{2}\big)^{\alpha+1/2}\big(\cos\frac{\theta}{2}\big)^{\beta+1/2}}{\big(\sin\frac{\theta}{2}+\frac{1}{n+1}\big)^{\alpha+1/2}\big(\cos\frac{\theta}{2}+\frac{1}{n+1}\big)^{\beta+1/2}} \times \Big|\int_0^\pi \frac{\sin\varphi}{\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}} \frac{\big(\sin\frac{\varphi}{2}\big)^{\alpha+3/2}\big(\cos\frac{\varphi}{2}\big)^{\beta+3/2}}{\big(\sin\frac{\varphi}{2}+\frac{1}{n}\big)^{\alpha+3/2}\big(\cos\frac{\varphi}{2}+\frac{1}{n}\big)^{\beta+3/2}} b_n(\varphi) f(\varphi) d\varphi\Big|,$$

where $\sup_{k\in\mathbb{N}} |b_k(\varphi)| \leq C$, $\varphi \in (0,\pi)$. We can write (see [22, p. 14])

$$\frac{\sin\varphi}{\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}} = \frac{1}{\sin\frac{\theta-\varphi}{2}} + R(\theta,\varphi), \quad \theta,\varphi\in(0,\pi),\ \theta\neq\varphi,$$

being

$$|R(\theta,\varphi)| \leq C \begin{cases} \dfrac{1}{\sin\frac{\theta}{2}+\sin\frac{\varphi}{2}}, & 0<\theta<\pi/2 \\ \dfrac{1}{\cos\frac{\theta}{2}+\cos\frac{\varphi}{2}}, & \pi/2<\theta<\pi, \end{cases} \qquad \varphi\in(0,\pi).$$

Thus, by defining

$$g(\varphi) = \frac{\big(\sin\frac{\varphi}{2}\big)^{\alpha+3/2}\big(\cos\frac{\varphi}{2}\big)^{\beta+3/2}}{\big(\sin\frac{\varphi}{2}+\frac{1}{n}\big)^{\alpha+3/2}\big(\cos\frac{\varphi}{2}+\frac{1}{n}\big)^{\beta+3/2}} b_n(\varphi) f(\varphi), \quad \varphi\in(0,\pi),$$

we obtain

$$J_2^{\alpha,\beta,n} f(\theta) \leq C\Big[|(Hg)(\theta)| + S^1(|g|)(\theta) + S^2(|g|)(\theta)\Big], \quad \theta\in(0,\pi), \tag{9}$$

where

$$(Hg)(\theta) = \text{P.V.}\int_0^\pi \frac{g(\varphi)}{\sin\frac{\theta-\varphi}{2}} d\varphi, \quad \text{a.e. } \theta\in(0,\pi),$$

$$(S^1 g)(\theta) = \int_0^\pi \frac{g(\varphi)}{\sin\frac{\theta}{2}+\sin\frac{\varphi}{2}} d\varphi, \quad \theta\in(0,\pi),$$

and

$$(S^2 g)(\theta) = \int_0^\pi \frac{g(\varphi)}{\cos\frac{\theta}{2}+\cos\frac{\varphi}{2}} d\varphi, \quad \theta\in(0,\pi).$$

The operator H is a singular integral operator related to the Hilbert transform and S^j, $j=1,2$, are Stieltjes type operators. It is well-known ([21]) that H is bounded from $L^p_w(0,\pi)$ into itself. In [22, Lemma 6] it was established that S^1 and S^2 are bounded from $L^p_w(0,\pi)$ into itself. Then, (9) implies that

$$\int_0^\pi |J_2^{\alpha,\beta,n} f(\theta)|^p w(\theta) d\theta \leq C\int_0^\pi |g(\theta)|^p w(\theta)d\theta \leq C\int_0^\pi |f(\theta)|^p w(\theta) d\theta. \tag{10}$$

In a similar way we can see

$$\int_0^\pi |J_3^{\alpha,\beta,n} f(\theta)|^p w(\theta) d\theta \leq C\int_0^\pi |f(\theta)|^p w(\theta) d\theta. \tag{11}$$

By putting together (7), (8), (10) and (11) we conclude that

$$\|S_n f\|_{L^p_w(0,\pi)} \leq C\|f\|_{L^p_w(0,\pi)}.$$

Note that the constant $C>0$ does not depend on $n\in\mathbb{N}$ and $f\in L^p_w(0,\pi)$.

Since $C^\infty_c(0,\pi)$ is a dense subspace of $L^p_w(0,\pi)$ and for every $h\in C^\infty_c(0,\pi)$,

$$\lim_{n\to\infty} S_n h = h, \quad \text{uniformly in}(0,\pi),$$

and hence in $L^p_w(0,\pi)$; standard arguments allow us to show that, for every $f\in L^p_w(0,\pi)$,

$$\lim_{n\to\infty} S_n f = f, \quad \text{in } L^p_w(0,\pi).$$

□

Proof of Proposition 2.2, (ii). From the property established in Proposition 2.2, (i), and according to [15, Theorem 1.3] we deduce that there exists $C > 0$ such that, for every $n \in \mathbb{N}$,

$$\|S_n f\|_{L^{p(\cdot)}(0,\pi)} \leq C\|f\|_{L^{p(\cdot)}(0,\pi)}, \quad f \in L^{p(\cdot)}(0,\pi). \tag{12}$$

By [17, Theorem 3.3.1], $C_c^\infty(0,\pi) \subseteq L^{p_+}(0,\pi) \subseteq L^{p(\cdot)}(0,\pi)$ and the inclusions are continuous. Hence, for every $h \in C_c^\infty(0,\pi)$,

$$\lim_{n\to\infty} S_n(h) = h, \quad \text{in } L^{p(\cdot)}(0,\pi).$$

Since $C_c^\infty(0,\pi)$ is dense in $L^{p(\cdot)}(0,\pi)$ we deduce from (12) that, for every $f \in L^{p(\cdot)}(0,\pi)$,

$$\lim_{n\to\infty} S_n f = f, \quad \text{in } L^{p(\cdot)}(0,\pi).$$

□

We are going to see that $S_{\alpha,\beta}$ is a dense subspace of $W^{k,p(\cdot)}_{\alpha,\beta}(0,\pi)$.

Proposition 2.3. *Let $\alpha, \beta \geq -1/2$, $k \in \mathbb{N}$ and $p \in \mathcal{B}(0,\pi)$. Then, $S_{\alpha,\beta}$ is a dense subspace of $W^{k,p(\cdot)}_{\alpha,\beta}(0,\pi)$.*

Proof. We proceed following the ideas in the proof of [5, Proposition 2] (see also [24, Proposition 3.2]). Note firstly that, since $L^{p(\cdot)}(0,\pi) \subseteq L^{p_-}(0,\pi)$ ([17, Theorem 3.3.1]), $W^{k,p(\cdot)}_{\alpha,\beta}(0,\pi) \subseteq W^{k,p_-}_{\alpha,\beta}(0,\pi)$, where the last Sobolev type space $W^{k,p_-}_{\alpha,\beta}(0,\pi)$ (with constant exponent p_-) was studied by Langowski [24].

Let $f \in W^{k,p(\cdot)}_{\alpha,\beta}(0,\pi)$. The maximal operator $W^{\alpha,\beta}_*$ associated with $\{W^{\alpha,\beta}_t\}_{t>0}$ is defined by

$$W^{\alpha,\beta}_*(f) = \sup_{t>0} |W^{\alpha,\beta}_t(f)|.$$

According to [36, Theorem A, and (3)] we have that

$$|W^{\alpha,\beta}_t(\theta,\varphi)| \leq C\frac{e^{-c(\theta-\varphi)^2/t}}{\sqrt{t}}, \quad \theta,\varphi \in (0,\pi) \text{ and } t > 0. \tag{13}$$

From (13) we deduce that

$$W^{\alpha,\beta}_t(f) \leq C\mathcal{M}_c(f),$$

where $\mathcal{M}_c$ denotes the centered Hardy-Littlewood maximal operator. Then, by [17, Theorem 4.3.8] $W^{\alpha,\beta}_*$ is a bounded (sublinear) operator from $L^{p(\cdot)}(0,\pi)$ into itself. It is clear that, for every $\phi \in S_{\alpha,\beta}$,

$$\lim_{t\to 0^+} W^{\alpha,\beta}_t(\phi) = \phi, \quad \text{in } L^{p(\cdot)}(0,\pi).$$

Then, since $S_{\alpha,\beta}$ is dense in $L^{p(\cdot)}(0,\pi)$ (Proposition 2.1), we obtain that,

$$\lim_{t\to 0^+} W^{\alpha,\beta}_t(f) = f, \quad \text{in } L^{p(\cdot)}(0,\pi).$$

By [24, Lemmas 3.1 and 3.3]

$$c^{\alpha+\ell,\beta+\ell}_m\left(\mathbb{D}^\ell_{\alpha,\beta} f\right) = (-1)^\ell\sqrt{(m+1)_\ell(m+\ell+\alpha+\beta+1)_\ell}\; c^{\alpha,\beta}_{m+\ell}(f), \quad \ell, m \in \mathbb{N}, \quad 0 \leq \ell \leq k. \tag{14}$$

Here and in the sequel we denote by $(z)_\ell$, $z > 0$, the ℓ-Pochhammer symbol, that is,

$$(z)_\ell = z(z+1)\cdots(z+\ell-1), \quad \ell \in \mathbb{N}, \quad \ell \geq 1 \quad \text{and} \quad (z)_0 = 1. \tag{15}$$

By taking into account [24, (1)] we can differentiate term by term inside the series and [24, Lemma 3.1] and (14) lead to

$$\begin{aligned}
\mathbb{D}^\ell_{\alpha,\beta} W^{\alpha,\beta}_t f &= \sum_{m=0}^\infty e^{-t\lambda^{\alpha,\beta}_m} c^{\alpha,\beta}_m(f)\; \mathbb{D}^\ell_{\alpha,\beta}\phi^{\alpha,\beta}_m \\
&= \sum_{m=\ell}^\infty e^{-t\lambda^{\alpha,\beta}_m} (-1)^m\sqrt{(m-\ell+1)_\ell(m+\alpha+\beta+1)_\ell}\; c^{\alpha,\beta}_m(f)\; \phi^{\alpha+\ell,\beta+\ell}_{m-\ell} \\
&= \sum_{m=\ell}^\infty e^{-t\lambda^{\alpha,\beta}_m} c^{\alpha+\ell,\beta+\ell}_{m-\ell}\left(\mathbb{D}^\ell_{\alpha,\beta} f\right) \phi^{\alpha+\ell,\beta+\ell}_{m-\ell} \\
&= \sum_{m=0}^\infty e^{-t\lambda^{\alpha+\ell,\beta+\ell}_m} c^{\alpha+\ell,\beta+\ell}_m\left(\mathbb{D}^\ell_{\alpha,\beta} f\right) \phi^{\alpha+\ell,\beta+\ell}_m, \quad \ell \in \mathbb{N},\ 0 \leq \ell \leq k.
\end{aligned}$$

Hence, for every $\ell \in \mathbb{N}$, $0 \leq \ell \leq k$,

$$\lim_{t\to 0^+} \mathbb{D}^\ell_{\alpha,\beta} W^{\alpha,\beta}_t f = \mathbb{D}^\ell_{\alpha,\beta} f, \quad \text{in } L^{p(\cdot)}(0,\pi).$$

Let $\varepsilon > 0$. There exists $t_0 > 0$ such that, for every $0 < t < t_0$,

$$\|\mathbb{D}^\ell_{\alpha,\beta} W^{\alpha,\beta}_t f - \mathbb{D}^\ell_{\alpha,\beta} f\|_{L^{p(\cdot)}(0,\pi)} < \varepsilon, \quad \ell \in \mathbb{N}, \quad 0 \leq l \leq k.$$

On the other hand, by using [24, (1)], [17, Theorem 3.3.11] and Hölder inequality we get, for every $\theta \in (0,\pi)$ and $\ell, m \in \mathbb{N}$,

$$\left|c^{\alpha+\ell,\beta+\ell}_m \left(\mathbb{D}^\ell_{\alpha,\beta} f\right)\right| \left|\phi^{\alpha+\ell,\beta+\ell}_m(\theta)\right| \leq C \|\mathbb{D}^\ell_{\alpha,\beta} f\|_{L^{p_-}(0,\pi)} (m+1)^{\alpha+\beta+2\ell+2}.$$

Hence, there exists $m_0 \in \mathbb{N}$, $m_0 \geq k$, such that

$$\left\| \sum_{m=M+1}^\infty e^{-t_0 \lambda^{\alpha+\ell,\beta+\ell}_m} c^{\alpha+\ell,\beta+\ell}_m \left(\mathbb{D}^\ell_{\alpha,\beta} f\right) \phi^{\alpha+\ell,\beta+\ell}_m \right\|_{L^{p(\cdot)}(0,\pi)}$$

$$\leq C \sum_{m=m_0+1}^\infty e^{-t_0 (m+\frac{\alpha+\beta+2\ell}{2})^2} (m+1)^{\alpha+\beta+2\ell+2} < \varepsilon, \quad \ell \in \mathbb{N},\ 0 \leq \ell \leq k,\ M \in \mathbb{N},\ M \geq m_0.$$

Then,

$$\left\| \sum_{m=0}^{m_0} e^{-t_0 \lambda^{\alpha,\beta}_m} c^{\alpha,\beta}_m(f) \phi^{\alpha,\beta}_m - f \right\|_{W^{k,p(\cdot)}_{\alpha,\beta}(0,\pi)} < 2\varepsilon.$$

Thus, we have proved that f is in the closure of $S_{\alpha,\beta}$ in $W^{k,p(\cdot)}_{\alpha,\beta}(0,\pi)$ and the proof is finished. □

3. Jacobi multipliers in weighted L^p-spaces

This section deals, among other things, with the proof of the $H^{k/2,p(\cdot)}_{\alpha,\beta}$-density of $S_{\alpha,\beta}$ claimed in assertions (a) and (d) of Section 1.

Let $m = (m_k)_{k=0}^\infty$ be a bounded sequence of real numbers. The Jacobi multiplier $T^{\alpha,\beta}_m$ associated with m is defined by

$$T^{\alpha,\beta}_m f = \sum_{k=0}^\infty m_k c^{\alpha,\beta}_k(f) \phi^{\alpha,\beta}_k, \quad f \in L^2(0,\pi).$$

Plancherel's equality implies that $T^{\alpha,\beta}_m$ is bounded on $L^2(0,\pi)$. Sufficient conditions which allow to extend $T^{\alpha,\beta}_m$ as a bounded operator to $L^p(0,\pi)$ and to certain weighted $L^p(0,\pi)$ spaces have been established by several authors (see [1], [7], [14], [20], [28], [31], [32] and [44], amongst others).

The goal of this section is to establish a multiplier theorem in $L^{p(\cdot)}(0,\pi)$. Previously we need to show a multiplier result for $L^p_w(0,\pi)$ when $w \in A_p(0,\pi)$. In order to achieve this we invoke a general multiplier theorem due to Meda [29] (see also [44]).

Let $-\infty < a < \left(\frac{\alpha+\beta+1}{2}\right)^2$. We consider the operator

$$\mathcal{L}_{\alpha,\beta;a} = \mathcal{L}_{\alpha,\beta} - a.$$

It is clear that, for every $k \in \mathbb{N}$, $\phi^{\alpha,\beta}_k$ is an eigenfunction for $\mathcal{L}_{\alpha,\beta;a}$ associated with the eigenvalue

$$\lambda^{\alpha,\beta;a}_k = \left(k + \frac{\alpha+\beta+1}{2}\right)^2 - a = k(k+\alpha+\beta+1) + \left(\frac{\alpha+\beta+1}{2}\right)^2 - a.$$

$\mathcal{L}_{\alpha,\beta;a}$ is a nonnegative and selfadjoint operator on $L^2(0,\pi)$. Moreover, $\mathcal{L}_{\alpha,\beta;a}$ generates a (heat) semigroup $\{W^{\alpha,\beta;a}_t\}_{t>0}$ on $L^2(0,\pi)$, given by

$$W^{\alpha,\beta;a}_t(f) = \int_0^\pi W^{\alpha,\beta;a}_t(\theta,\varphi) f(\varphi) d\varphi, \quad f \in L^2(0,\pi), \quad t > 0,$$

and

$$W^{\alpha,\beta;a}_t(\theta,\varphi) = \sum_{k=0}^\infty e^{-t\lambda^{\alpha,\beta;a}_k} \phi^{\alpha,\beta}_k(\theta) \phi^{\alpha,\beta}_k(\varphi), \qquad \theta,\varphi \in (0,\pi), \text{ and } t > 0.$$

According to [36, Theorem A, (3) and (9)] we have that

$$\left|W^{\alpha,\beta;a}_t(\theta,\varphi)\right| \leq C e^{-((\frac{\alpha+\beta+1}{2})^2 - a)t} \, \frac{e^{-c(\theta-\varphi)^2/t}}{\sqrt{t}}, \quad \theta,\varphi \in (0,\pi) \text{ and } t > 0.$$

Let $\gamma \in \mathbb{R}\backslash\{0\}$. The imaginary power $\mathcal{L}^{i\gamma}_{\alpha,\beta;a}$ of $\mathcal{L}_{\alpha,\beta;a}$ is the spectral multiplier $g(\mathcal{L}_{\alpha,\beta;a})$ where $g(x) = x^{i\gamma}$, $x > 0$, that is,

$$\mathcal{L}^{i\gamma}_{\alpha,\beta;a}(f) = \sum_{k=0}^{\infty} (\lambda_k^{\alpha,\beta;a})^{i\gamma} c_k^{\alpha,\beta}(f)\phi_k^{\alpha,\beta}, \quad f \in L^2(0,\pi).$$

The operator $\mathcal{L}^{i\gamma}_{\alpha,\beta;a}$ can be seen as a Laplace transform type multiplier for $\mathcal{L}_{\alpha,\beta;a}$. Then, a general result due to Stein [41, Corollary 3, p. 121] applies to deduce that $\mathcal{L}^{i\gamma}_{\alpha,\beta;a}$ can be extended from $L^2(0,\pi) \cap L^p(0,\pi)$ to $L^p(0,\pi)$ as a bounded operator on $L^p(0,\pi)$, for every $1 < p < \infty$. Also, by proceeding as in [35] we can see that $\mathcal{L}^{i\gamma}_{\alpha,\beta;a}$ is a Calderón-Zygmund operator in the sense of a space of homogeneous type $((0,\pi), d\theta, |\cdot|)$, where $|\cdot|$ stands for the Euclidean metric. Then, $\mathcal{L}^{i\gamma}_{\alpha,\beta;a}$ defines a bounded operator from $L^p_w(0,\pi)$ into itself, for every $1 < p < \infty$ and $w \in A_p(0,\pi)$. Moreover, classical arguments (see for instance, [19, Chapter 7, Section 4]) allow us to obtain that, for every $1 < p < \infty$ and $w \in A_p(0,\pi)$,

$$\|\mathcal{L}^{i\gamma}_{\alpha,\beta;a}\|_{L^p_w(0,\pi)\to L^p_w(0,\pi)} \le C_{p,w} e^{\pi|\gamma|/2}, \tag{16}$$

where $C_{p,w} > 0$ does not depend on γ. Estimation (16) shows an exponential increase with respect to $|\gamma|$ of the operator norm $\|\mathcal{L}^{i\gamma}_{\alpha,\beta;a}\|_{L^p_w(0,\pi)\to L^p_w(0,\pi)}$ which is not sufficient to obtain our multiplier result. Actually, the exponential behavior in (16) can be replaced by a polynomial growth. Indeed, according to [11, Theorem 1.3 and Remarks 1.4 and 1.5] we have that, for every $1 < p < \infty$ and $w \in A_p(0,\pi)$,

$$\|\mathcal{L}^{i\gamma}_{\alpha,\beta;a}\|_{L^p_w(0,\pi)\to L^p_w(0,\pi)} \le C_{p,w}(1 + |\gamma|),$$

where $C_{p,w} > 0$ does not depend on γ.

We now establish our result concerning the $L^p_w(0,\pi)$-boundedness of spectral multipliers for the operator $\mathcal{L}_{\alpha,\beta;a}$.

Proposition 3.1. *Let $1 < p < \infty$, $\alpha, \beta \ge -1/2$ and $-\infty < a < \left(\frac{\alpha+\beta+1}{2}\right)^2$. Assume that:*

(i) m is a bounded holomorphic function on $\{z \in \mathbb{C} \,:\, \mathrm{Re}\ z > 0\}$; or
(ii) $m \in C^\infty(0,\pi)$ and for every $\ell \in \mathbb{N}$

$$\sup_{x\in(0,\infty)} \left| x^\ell \frac{d^\ell}{dx^\ell} m(x) \right| < \infty. \tag{17}$$

Then, the spectral multiplier $m(\mathcal{L}_{\alpha,\beta;a})$ related to the operator $\mathcal{L}_{\alpha,\beta;a}$ given by

$$m(\mathcal{L}_{\alpha,\beta;a})f = \sum_{k=0}^{\infty} m(\lambda_k^{\alpha,\beta;a}) c_k^{\alpha,\beta}(f)\phi_k^{\alpha,\beta}, \tag{18}$$

is bounded from $L^p_w(0,\pi)$ into itself, for every $w \in A_p(0,\pi)$.

This result can be proved as in [29, Theorem 3 or Corollary 1]. By using now [15, Theorem 1.3] we deduce from Proposition 3.1 the following $L^{p(\cdot)}$-boundedness result for spectral multipliers associated with $\mathcal{L}_{\alpha,\beta;a}$.

Proposition 3.2. *Let $\alpha, \beta \ge -1/2$ and $-\infty < a < \left(\frac{\alpha+\beta+1}{2}\right)^2$. Assume that $p \in \mathcal{B}(0,\pi)$. If m satisfies condition (i) or (ii) of Proposition 3.1, then the spectral multiplier $m(\mathcal{L}_{\alpha,\beta;a})$ given by (18) defines a bounded operator from $L^{p(\cdot)}(0,\pi)$ into itself.*

The negative powers of $\mathcal{L}_{\alpha,\beta}$ defined in (3) are spectral multipliers for the Jacobi operator that will be useful in the sequel. Suppose that $\gamma > 0$ and $\alpha + \beta \ne -1$. Since $\lambda_k^{\alpha,\beta} \ge \left(\frac{\alpha+\beta+1}{2}\right)^2$, $k \in \mathbb{N}$, the operator $\mathcal{L}^{-\gamma}_{\alpha,\beta}$ is bounded from $L^2(0,\pi)$ into itself.

We take $a = \frac{1}{2}\left(\frac{\alpha+\beta+1}{2}\right)^2$. We can write

$$\mathcal{L}^{-\gamma}_{\alpha,\beta} f = \sum_{k=0}^{\infty} (\lambda_k^{\alpha,\beta;a} + a)^{-\gamma} c_k^{\alpha,\beta}(f)\phi_k^{\alpha,\beta} = T^{\alpha,\beta;a}_{m_\gamma}(f), \quad f \in L^2(0,\pi),$$

where $m_\gamma(z) = (z + a)^{-\gamma}$, $z \in \mathbb{C}$, $\mathrm{Re}\ z > 0$. Since m_γ is a bounded holomorphic function on $\{z \in \mathbb{C} \,:\, \mathrm{Re}\ z > 0\}$ from Propositions 3.1 and 3.2 we deduce the following.

Proposition 3.3. *Let $\gamma > 0$ and $\alpha, \beta \ge -1/2$ such that $\alpha + \beta \ne -1$.*

(a) *If* $1 < p < \infty$ *and* $w \in A_p(0,\pi)$*, then* $\mathcal{L}_{\alpha,\beta}^{-\gamma}$ *can be extended from* $L^2(0,\pi) \cap L^p_w(0,\pi)$ *to* $L^p_w(0,\pi)$ *as a bounded operator from* $L^p_w(0,\pi)$ *into itself.*

(b) *If* $p \in \mathcal{B}(0,\pi)$*, then* $\mathcal{L}_{\alpha,\beta}^{-\gamma}$ *defines a bounded operator from* $L^{p(\cdot)}(0,\pi)$ *into itself.*

We also have the injectivity of $\mathcal{L}_{\alpha,\beta}^{-\gamma}$ on $L^p_w(0,\pi)$ and $L^{p(\cdot)}(0,\pi)$.

Proposition 3.4. *Let* $\gamma > 0$ *and* $\alpha, \beta \geq -1/2$ *such that* $\alpha + \beta \neq -1$.

(a) *If* $1 < p < \infty$ *and* $w \in A_p(0,\pi)$*, then* $\mathcal{L}_{\alpha,\beta}^{-\gamma}$ *is one to one on* $L^p_w(0,\pi)$.

(b) *Assume that* $p \in \mathcal{B}(0,\pi)$*. Then,* $\mathcal{L}_{\alpha,\beta}^{-\gamma}$ *is one to one on* $L^{p(\cdot)}(0,\pi)$.

Proof. We prove (b). Property (a) can be shown in a similar way. It is clear that if $f \in S_{\alpha,\beta}$ we have that

$$c_k^{\alpha,\beta}(\mathcal{L}_{\alpha,\beta}^{-\gamma} f) = (\lambda_k^{\alpha,\beta})^{-\gamma} c_k^{\alpha,\beta}(f), \quad k \in \mathbb{N}. \tag{19}$$

Since $\mathcal{L}_{\alpha,\beta}^{-\gamma}$ is bounded from $L^{p(\cdot)}(0,\pi)$ into itself (see Proposition 3.3); for every $k \in \mathbb{N}$, $\phi_k^{\alpha,\beta} \in L^{p'(\cdot)}(0,\pi) = \big(L^{p(\cdot)}(0,\pi)\big)^*$ ([17, Theorem 3.4.6]) and $S_{\alpha,\beta}$ is dense in $L^{p(\cdot)}(0,\pi)$ (Proposition 2.1), we conclude that (19) holds for every $f \in L^{p(\cdot)}(0,\pi)$. Then, from Corollary 2.1 we deduce that $f = 0$ provided that $\mathcal{L}_{\alpha,\beta}^{-\gamma} f = 0$. □

By using Proposition 2.2 we obtain the following characterization of the potential space $H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi)$.

Proposition 3.5. *Let* $\gamma > 0$ *and* $\alpha, \beta \geq -1/2$ *such that* $\alpha + \beta \neq -1$*. Assume that* $p \in \mathcal{B}(0,\pi)$*. A function* $f \in L^{p(\cdot)}(0,\pi)$ *is in* $H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi)$ *if, and only if, the series* $\sum_{n=0}^\infty (\lambda_n^{\alpha,\beta})^\gamma c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta}$ *converges in* $L^{p(\cdot)}(0,\pi)$*. Moreover, for every* $f \in H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi)$,

$$\|f\|_{H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi)} = \Big\| \sum_{n=0}^\infty (\lambda_n^{\alpha,\beta})^\gamma c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} \Big\|_{L^{p(\cdot)}(0,\pi)}.$$

Proof. Let $f \in L^{p(\cdot)}(0,\pi)$. Suppose that $f \in H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi)$. Then, there exists $g \in L^{p(\cdot)}(0,\pi)$ such that $f = \mathcal{L}_{\alpha,\beta}^{-\gamma} g$. Thus, by (19) we have that $c_n^{\alpha,\beta}(f) = (\lambda_n^{\alpha,\beta})^{-\gamma} c_n^{\alpha,\beta}(g)$, $n \in \mathbb{N}$. Hence, according to Proposition 2.2, the series

$$\sum_{n=0}^\infty (\lambda_n^{\alpha,\beta})^\gamma c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} = \sum_{n=0}^\infty c_n^{\alpha,\beta}(g)\phi_n^{\alpha,\beta}$$

converges in $L^{p(\cdot)}(0,\pi)$.

Assume now that the series $F = \sum_{n=0}^\infty (\lambda_n^{\alpha,\beta})^\gamma c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta}$ converges in $L^{p(\cdot)}(0,\pi)$. Then, by Proposition 3.4, $\mathcal{L}_{\alpha,\beta}^{-\gamma} F = f$ and $f \in H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi)$. □

As an immediate consequence of Proposition 3.5 we establish the density of $S_{\alpha,\beta}$ in $H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi)$.

Corollary 3.1. *Let* $\gamma > 0$ *and* $\alpha, \beta \geq -1/2$ *such that* $\alpha + \beta \neq -1$*. Assume that* $p \in \mathcal{B}(0,\pi)$*. Then, for every* $f \in H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi)$,

$$f = \lim_{n \to \infty} \sum_{k=0}^n c_k^{\alpha,\beta}(f)\phi_k^{\alpha,\beta},$$

in the sense of convergence in $H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi)$.

Proof. Let $f \in H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi)$. We have that $f = \mathcal{L}_{\alpha,\beta}^{-\gamma} g$, where

$$g = \sum_{k=0}^\infty (\lambda_k^{\alpha,\beta})^\gamma c_k^{\alpha,\beta}(f)\phi_k^{\alpha,\beta},$$

in the sense of convergence in $L^{p(\cdot)}(0,\pi)$. Then,

$$\Big\| f - \sum_{k=0}^n c_k^{\alpha,\beta}(f)\phi_k^{\alpha,\beta} \Big\|_{H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi)} = \Big\| g - \sum_{k=0}^n (\lambda_k^{\alpha,\beta})^\gamma c_k^{\alpha,\beta}(f)\phi_k^{\alpha,\beta} \Big\|_{L^{p(\cdot)}(0,\pi)} \longrightarrow 0, \quad \text{as } n \to \infty.$$

□

4. Boundedness of the higher order Riesz transforms

This section has to do with the proof of assertions (b) and (c) of Section 1.

Firstly, we establish that $R^k_{\alpha,\beta}$ and $R^{k,*}_{\alpha,\beta}$ are composition of Jacobi Riesz transforms of order one.

Lemma 4.1. *Let $k \in \mathbb{N}$ and $\alpha, \beta \geq -1/2$ such that $\alpha + \beta \neq -1$. Then,*

$$R^k_{\alpha,\beta} f = R^1_{\alpha+k-1,\beta+k-1} \circ R^1_{\alpha+k-2,\beta+k-2} \circ \ldots.. \circ R^1_{\alpha,\beta} f, \quad f \in S_{\alpha,\beta}, \tag{20}$$

and

$$R^{k,*}_{\alpha,\beta} f = R^{1,*}_{\alpha,\beta} \circ R^{1,*}_{\alpha+1,\beta+1} \circ \ldots.. \circ R^{1,*}_{\alpha+k-1,\beta+k-1} f, \quad f \in S_{\alpha+k,\beta+k}. \tag{21}$$

Proof. We are going to prove (20), (21) can be shown in a similar way. It is sufficient to see that (20) is true when $f = \phi^{\alpha,\beta}_l$, for every $l \in \mathbb{N}$.

Let $l \in \mathbb{N}$. According to [24, Lemma 3.1] we have that

$$\mathbb{D}^k_{\alpha,\beta} \phi^{\alpha,\beta}_l = (-1)^k \sqrt{(l-k+1)_k (l+\alpha+\beta+1)_k}\ \phi^{\alpha+k,\beta+k}_{l-k}. \tag{22}$$

Recall the definition of the Pochhammer symbol in (15) and by convention $\phi^{\alpha,\beta}_n = 0$, $n \in \mathbb{Z}$, $n < 0$. Hence,

$$R^k_{\alpha,\beta} \phi^{\alpha,\beta}_l = (-1)^k \sqrt{\frac{(l-k+1)_k (l+\alpha+\beta+1)_k}{(\lambda^{\alpha,\beta}_l)^k}}\ \phi^{\alpha+k,\beta+k}_{l-k}.$$

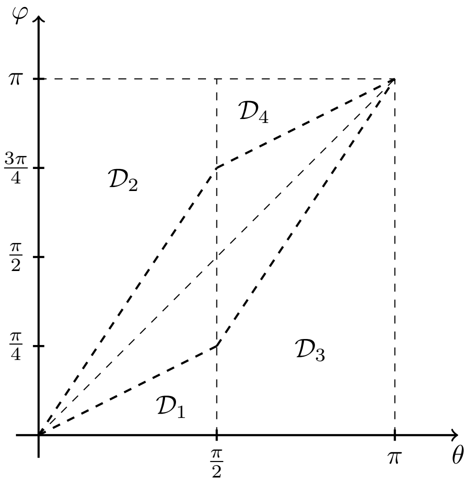
Since $\lambda^{\alpha,\beta}_l = \lambda^{\alpha+n,\beta+n}_{l-n}$, $0 \leq n \leq l$, we can write

$$\begin{aligned} R^k_{\alpha,\beta} \phi^{\alpha,\beta}_l &= (-1)^k \prod_{n=0}^{k-1} \sqrt{\frac{(l-n)(l+\alpha+\beta+1+n)}{\lambda^{\alpha+n,\beta+n}_{l-n}}}\ \phi^{\alpha+k,\beta+k}_{l-k} \\ &= (-1)^{k-1} \prod_{n=0}^{k-2} \sqrt{\frac{(l-n)(l+\alpha+\beta+1+n)}{\lambda^{\alpha+n,\beta+n}_{l-n}}}\ R^1_{\alpha+k-1,\beta+k-1} \phi^{\alpha+k-1,\beta+k-1}_{l-k+1} \\ &= R^1_{\alpha+k-1,\beta+k-1} \circ R^1_{\alpha+k-2,\beta+k-2} \circ \ldots.. \circ R^1_{\alpha,\beta} \phi^{\alpha,\beta}_l, \end{aligned}$$

and (20) is established. □

We are going to prove that $R^k_{\alpha,\beta}$ and $R^{k,*}_{\alpha,\beta}$ define bounded operators from $L^p_w(0,\pi)$ into itself for every $1 < p < \infty$ and $w \in A_p(0,\pi)$. As consecuence of the next lemma, we only need to study the corresponding local operators (see [10] and [13]).

We consider the domain $\mathcal{D} = \cup_{j=1}^4 \mathcal{D}_j$ represented in the figure bellow



$$\mathcal{D}_1 = \Big\{(\theta,\varphi)\ :\ 0 < \varphi < \frac{\theta}{2},\ 0 < \theta < \frac{\pi}{2}\Big\},$$
$$\mathcal{D}_2 = \Big\{(\theta,\varphi)\ :\ 0 < \frac{3\theta}{2} < \varphi < \pi,\ 0 < \theta < \frac{\pi}{2}\Big\},$$
$$\mathcal{D}_3 = \Big\{(\theta,\varphi)\ :\ 0 < \varphi < \frac{3\theta-\pi}{2},\ \frac{\pi}{2} < \theta < \pi\Big\},$$
$$\mathcal{D}_4 = \Big\{(\theta,\varphi)\ :\ \frac{\theta+\pi}{2} < \varphi < \pi,\ \frac{\pi}{2} < \theta < \pi\Big\}.$$

Figure 1. Global regions

Lemma 4.2. *Suppose that $K : (0,\pi) \times (0,\pi) \setminus \{(\theta,\theta)\ :\ \theta \in (0,\pi)\} \longrightarrow \mathbb{R}$ is a measurable function such that*

$$|K(\theta,\varphi)| \leq \frac{C}{|\theta-\varphi|}, \quad \theta,\varphi \in (0,\pi),\ \theta \neq \varphi.$$

Then, for every $1 < p < \infty$ and $w \in A_p(0,\pi)$ the operator H defined by

$$Hf(\theta) = \int_0^\pi K(\theta,\varphi)\chi_{\mathcal{D}}(\theta,\varphi)f(\varphi)d\varphi, \quad \theta \in (0,\pi),$$

is bounded from $L^p_w(0,\pi)$ into $L^p_w(0,\pi)$.

Proof. We define

$$H_j f(\theta) = \int_0^\pi K(\theta,\varphi)\chi_{\mathcal{D}_j}(\theta,\varphi)f(\varphi)d\varphi, \quad \theta \in (0,\pi),\ j = 1,2,3,4.$$

Thus, $H = \sum_{j=1}^{4} H_j$.

By $\mathcal{M}$ we denote the Hardy-Littlewood maximal function on $(0,\pi)$. We have that

$$|H_1 f(\theta)| \leq \int_0^{\theta/2} \frac{|f(\varphi)|}{|\theta-\varphi|}d\varphi \leq \frac{C}{\theta}\int_0^{\theta/2}|f(\varphi)|d\varphi \leq C\mathcal{M}(f)(\theta), \quad \theta \in (0,\pi),$$

and

$$|H_4 f(\theta)| \leq \int_{(\theta+\pi)/2}^{\pi} \frac{|f(\varphi)|}{|\theta-\varphi|}d\varphi \leq \frac{C}{\pi-\theta}\int_{\pi-3(\pi-\theta)/2}^{\pi}|f(\varphi)|d\varphi \leq C\mathcal{M}(f)(\theta), \quad \theta \in (0,\pi).$$

By using the classical maximal theorem we deduce that H_1 and H_4 are bounded from $L^p_w(0,\pi)$ into itself, for every $1 < p < \infty$ and $w \in A_p(0,\pi)$.

The adjoint operator H_2^* of H_2 is defined by

$$H_2^* g(\varphi) = \chi_{(0,\frac{3\pi}{4})}(\varphi)\int_0^{2\varphi/3} K(\theta,\varphi)g(\theta)d\theta + \chi_{(\frac{3\pi}{4},\pi)}(\varphi)\int_0^{\pi/2} K(\theta,\varphi)g(\theta)d\theta, \quad \varphi \in (0,\pi).$$

If $1 < p < \infty$ and $w \in A_p(0,\pi)$, we deduce that

$$\begin{aligned}\|H_2^* g\|_{L^p_w(0,\pi)} &\leq C\left\{\left(\int_0^{3\pi/4} w(\varphi)\left(\int_0^{2\varphi/3}\frac{|g(\theta)|}{|\theta-\varphi|}d\theta\right)^p d\varphi\right)^{1/p} + \left(\int_{3\pi/4}^{\pi} w(\varphi)\left(\int_0^{\pi/2}\frac{|g(\theta)|}{|\theta-\varphi|}d\theta\right)^p d\varphi\right)^{1/p}\right\}\\ &\leq C\left\{\left(\int_0^\pi w(\varphi)|\mathcal{M}(|g|)(\varphi)|^p d\varphi\right)^{1/p} + \left(\int_{3\pi/4}^{\pi} w(\varphi)d\varphi\right)^{1/p}\int_0^\pi |g(\theta)|d\theta\right\}\\ &\leq C\|g\|_{L^p_w(0,\pi)}, \quad g \in L^p_w(0,\pi).\end{aligned}$$

Hence, H_2 is bounded from $L^p_w(0,\pi)$ into itself for every $1 < p < \infty$ and $w \in A_p(0,\pi)$. On the other hand, the adjoint operator H_3^* of H_3 is given by

$$H_3^* g(\varphi) = \chi_{(0,\frac{\pi}{4})}(\varphi)\int_{\pi/2}^{\pi} K(\theta,\varphi)g(\theta)d\theta + \chi_{(\frac{\pi}{4},\pi)}(\varphi)\int_{(2\varphi+\pi)/3}^{\pi} K(\theta,\varphi)g(\theta)d\theta.$$

If $1 < p < \infty$ and $w \in A_p(0,\pi)$, we get

$$\begin{aligned}\|H_3^* g\|_{L^p_w(0,\pi)} &\leq C\left\{\left(\int_0^{\pi/4} w(\varphi)d\varphi\right)^{1/p}\int_0^\pi |g(\theta)|d\theta + \left(\int_{\pi/4}^{\pi} w(\varphi)\left(\int_{(2\varphi+\pi)/3}^{\pi}\frac{|g(\theta)|}{|\theta-\varphi|}d\theta\right)^p d\varphi\right)^{1/p}\right\}\\ &\leq C\left\{\|g\|_{L^p_w(0,\pi)} + \left(\int_{\pi/4}^{\pi} w(\varphi)\left(\frac{1}{\pi-\varphi}\int_{\pi-4(\pi-\varphi)/3}^{\pi}|g(\theta)|d\theta\right)^p d\varphi\right)^{1/p}\right\}\\ &\leq C\left(\|g\|_{L^p_w(0,\pi)} + \|\mathcal{M}(|g|)\|_{L^p_w(0,\pi)}\right) \leq C\|g\|_{L^p_w(0,\pi)}, \quad g \in L^p_w(0,\pi).\end{aligned}$$

We conclude that H_3 is bounded from $L^p_w(0,\pi)$ into itself, for every $1 < p < \infty$ and $w \in A_p(0,\pi)$.

Thus, the proof of this lemma is finished. □

By using Lemmas 4.1 an 4.2 we will deduce the $L^p_w(0,\pi)$-boundedness of $R^m_{\alpha,\beta}$ and $R^{m,*}_{\alpha,\beta}$ from the corresponding property of $R^1_{\alpha,\beta}$ and $R^{1,*}_{\alpha,\beta}$, respectively.

Proposition 4.1. *Let $1 < p < \infty$, $w \in A_p(0,\pi)$ and $\alpha,\beta \geq -1/2$ such that $\alpha+\beta \neq -1$. The Jacobi Riesz transforms $R^1_{\alpha,\beta}$ and $R^{1,*}_{\alpha,\beta}$ define bounded operators from $L^p_w(0,\pi)$ into itself.*

We are going to use local Calderón-Zygmund theory for singular integrals (see [13]). We are inspired in the arguments developed by Nowak and Sjögren in [35].

Proof of Proposition 4.1; the case of $R^1_{\alpha,\beta}$. By (22) we have that

$$R^1_{\alpha,\beta}f=-\sum_{k=0}^{\infty}\sqrt{\frac{k(k+\alpha+\beta+1)}{\lambda_k^{\alpha,\beta}}}\;c_k^{\alpha,\beta}(f)\;\phi_{k-1}^{\alpha+1,\beta+1},\quad f\in L^2(0,\pi).$$

According to Plancherel's theorem, $R^1_{\alpha,\beta}$ is bounded from $L^2(0,\pi)$ into itself. By using [12, Theorem 2.4] we can write

$$R^1_{\alpha,\beta}f(\theta)=\lim_{\varepsilon\to 0^+}\int_{0,\ |\theta-\varphi|>\varepsilon}^{\pi}R^1_{\alpha,\beta}(\theta,\varphi)f(\varphi)d\varphi,\quad a.e.\ \theta\in(0,\pi),$$

for every $f\in C_c^\infty(0,\pi)$. Here the kernel $R^1_{\alpha,\beta}(\theta,\varphi)$ is defined by

$$R^1_{\alpha,\beta}(\theta,\varphi)=\int_0^\infty D_{\alpha,\beta}P_t^{\alpha,\beta}(\theta,\varphi)dt,\quad \theta,\varphi\in(0,\pi),\ \theta\neq\varphi.$$

According to [13, Theorem 2.4] and Lemma 4.2, to prove that $R^1_{\alpha,\beta}$ is bounded from $L^p_w(0,\pi)$ into itself, it is enough to show that

$$|R^1_{\alpha,\beta}(\theta,\varphi)|\leq\frac{C}{|\theta-\varphi|},\quad \theta,\varphi\in(0,\pi),\ \theta\neq\varphi,\tag{23}$$

and

$$|\partial_\theta R^1_{\alpha,\beta}(\theta,\varphi)|+|\partial_\varphi R^1_{\alpha,\beta}(\theta,\varphi)|\leq\frac{C}{|\theta-\varphi|^2},\quad (\theta,\varphi)\in(0,\pi)^2\backslash\mathcal{D},\ \theta\neq\varphi,\tag{24}$$

where $\mathcal{D}$ is the domain in Figure 1.

According to [35, Proposition 4.1] and [36, (3)] we have that for every $\theta,\varphi\in(0,\pi)$ and $t>0$,

$$P_t^{\alpha,\beta}(\theta,\varphi)=C_{\alpha,\beta}\left(\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\right)^{\beta+1/2}\sinh\frac{t}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{d\Pi_\alpha(u)d\Pi_\beta(v)}{(\cosh\frac{t}{2}-1+q(\theta,\varphi,u,v))^{\alpha+\beta+2}},\tag{25}$$

where $C_{\alpha,\beta}=\frac{2^{-\alpha-\beta-1}}{\int_0^\pi(\sin\frac{\theta}{2})^{2\alpha+1}(\cos\frac{\theta}{2})^{2\beta+1}d\theta}$, $d\Pi_\alpha(u)=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}(1-u^2)^{\alpha-1/2}du$, and

$$q(\theta,\varphi,u,v)=1-u\sin\frac{\theta}{2}\sin\frac{\varphi}{2}-v\cos\frac{\theta}{2}\cos\frac{\varphi}{2}.$$

By proceeding as in [35, Proof of Theorem 2.4; the case of $R_1^{\alpha,\beta}$] and using [35, Lemma 4.4 and trigonometric identities in p. 738] we get that

$$\begin{aligned}
&|R^1_{\alpha,\beta}(\theta,\varphi)|\\
&\leq C\int_0^\infty\sinh\frac{t}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{\left(\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\right)^{\beta+1/2}|\partial_\theta q(\theta,\varphi,u,v)|}{(\cosh\frac{t}{2}-1+q(\theta,\varphi,u,v))^{\alpha+\beta+3}}d\Pi_\alpha(u)d\Pi_\beta(v)dt\\
&\leq C\int_{-1}^{1}\int_{-1}^{1}\frac{\left(\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\right)^{\beta+1/2}}{q^{\alpha+\beta+3/2}(\theta,\varphi,u,v)}d\Pi_\alpha(u)d\Pi_\beta(v)\\
&\leq C\int_{-1}^{1}\frac{\left(\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\right)^{\beta+1/2}d\Pi_\alpha(u)}{(1-u\sin\frac{\theta}{2}\sin\frac{\varphi}{2})^{\beta+1/2}(1-u\sin\frac{\theta}{2}\sin\frac{\varphi}{2}-\cos\frac{\theta}{2}\cos\frac{\varphi}{2})^{\alpha+1}}\\
&\leq C\frac{\left(\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\right)^{\beta+1/2}}{(1-\sin\frac{\theta}{2}\sin\frac{\varphi}{2})^{\beta+1/2}}\frac{1}{(1-\sin\frac{\theta}{2}\sin\frac{\varphi}{2}-\cos\frac{\theta}{2}\cos\frac{\varphi}{2})^{1/2}}\frac{\left(\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\right)^{\alpha+1/2}}{\left(1-\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\right)^{\alpha+1/2}}\\
&\leq C\left(\frac{\cos\frac{\theta}{2}\cos\frac{\varphi}{2}}{1-\sin\frac{\theta}{2}\sin\frac{\varphi}{2}-\cos\frac{\theta}{2}\cos\frac{\varphi}{2}+\cos\frac{\theta}{2}\cos\frac{\varphi}{2}}\right)^{\beta+1/2}\frac{1}{\left(1-\cos\frac{\theta-\varphi}{2}\right)^{1/2}}\\
&\leq C\left(\frac{\cos\frac{\theta}{2}\cos\frac{\varphi}{2}}{1-\cos\frac{\varphi-\theta}{2}+\cos\frac{\theta}{2}\cos\frac{\varphi}{2}}\right)^{\beta+1/2}\frac{1}{|\theta-\varphi|}\leq\frac{C}{|\theta-\varphi|},\quad\theta,\varphi\in(0,\pi).
\end{aligned}\tag{26}$$

Then (23) is proved.

Also, we have that

$$\partial_\theta R^1_{\alpha,\beta}(\theta,\varphi) = \left(\frac{2\alpha+1}{4}\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} - \frac{2\beta+1}{4}\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right) R^1_{\alpha,\beta}(\theta,\varphi)$$
$$+ \left(\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\right)^{\beta+1/2} T_{\alpha,\beta}(\theta,\varphi), \quad \theta,\varphi\in(0,\pi), \tag{27}$$

where

$$T_{\alpha,\beta}(\theta,\varphi) = C_{\alpha,\beta}\partial_\theta^2\int_0^\infty \sinh\frac{t}{2}\int_{-1}^1\int_{-1}^1 \frac{d\Pi_\alpha(u)d\Pi_\beta(v)}{(\cosh\frac{t}{2}-1+q(\theta,\varphi,u,v))^{\alpha+\beta+2}}dt, \quad \theta,\varphi\in(0,\pi).$$

We can write by [35, Lemma 4.7] and proceeding as in [35, Proof of Theorem 2.4; the case of $R_N^{\alpha,\beta}$],

$$\left|\left(\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\right)^{\beta+1/2} T_{\alpha,\beta}(\theta,\varphi)\right|$$
$$\leq C\int_{-1}^1\int_{-1}^1 \frac{\left(\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\right)^{\beta+1/2}}{q(\theta,\varphi,u,v)^{\alpha+\beta+2}}d\Pi_\alpha(u)d\Pi_\beta(v) \leq \frac{C}{|\theta-\varphi|^2}, \quad \theta,\varphi\in(0,\pi). \tag{28}$$

On the other hand

$$\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}\cos\frac{\varphi}{2}}{\sin\frac{\theta}{2}\cos\frac{\varphi}{2}} = \frac{\cos\frac{\theta}{2}\cos\frac{\varphi}{2}}{\sin\frac{\theta}{2}\cos\frac{\varphi}{2}-\sin\frac{\varphi}{2}\cos\frac{\theta}{2}+\sin\frac{\varphi}{2}\cos\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}\cos\frac{\varphi}{2}}{\sin\frac{\theta-\varphi}{2}+\sin\frac{\varphi}{2}\cos\frac{\theta}{2}}$$
$$\leq \frac{1}{\sin\frac{\theta-\varphi}{2}}, \quad 0<\varphi<\theta<\pi. \tag{29}$$

If $\varphi\in(0,\pi)$, $\theta\in(0,\pi/2)$ and $\theta<\varphi<3\theta/2$, then $\sin\varphi/3<\sin\theta/2$ and

$$\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \leq \frac{\cos\frac{\theta}{2}}{\sin\frac{\varphi}{3}} = \frac{\cos\frac{\theta}{2}\cos\frac{\theta}{3}}{\sin\frac{\varphi}{3}\cos\frac{\theta}{3}-\sin\frac{\theta}{3}\cos\frac{\varphi}{3}+\sin\frac{\theta}{3}\cos\frac{\varphi}{3}} \leq \frac{1}{\sin\frac{\theta-\varphi}{3}}. \tag{30}$$

Also, we get

$$\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \leq \frac{1}{\sin\frac{\pi}{4}} \leq \frac{C}{\sin|\frac{\theta-\varphi}{2}|}, \quad 0<\varphi<\pi, \quad \pi/2<\theta<\pi. \tag{31}$$

By combining (26), (29), (30) and (31) we obtain

$$\left|\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}R^1_{\alpha,\beta}(\theta,\varphi)\right| \leq \frac{C}{|\theta-\varphi|^2}, \quad (\theta,\varphi)\in(0,\pi)^2\backslash\mathcal{D}. \tag{32}$$

We can write

$$\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = -\frac{\cos\frac{\pi-\theta}{2}}{\sin\frac{\pi-\theta}{2}}, \quad \theta\in(0,\pi), \tag{33}$$

and by symmetries reasons and proceeding as above we get

$$\left|\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}R^1_{\alpha,\beta}(\theta,\varphi)\right| \leq \frac{C}{|\theta-\varphi|^2}, \quad (\theta,\varphi)\in(0,\pi)^2\backslash\mathcal{D}. \tag{34}$$

From (27), (28), (32) and (34) we conclude that

$$|\partial_\theta R^1_{\alpha,\beta}(\theta,\varphi)| \leq \frac{C}{|\theta-\varphi|^2}, \quad (\theta,\varphi)\in(0,\pi)^2\backslash\mathcal{D}.$$

In a similar way, we can see that

$$|\partial_\varphi R^1_{\alpha,\beta}(\theta,\varphi)| \leq \frac{C}{|\theta-\varphi|^2}, \quad (\theta,\varphi)\in(0,\pi)^2\backslash\mathcal{D}.$$

Thus, (24) is established. □

Proof of Proposition 4.1; the case of $R^{1,*}_{\alpha,\beta}$. We have that

$$R^{1,*}_{\alpha,\beta}f=-\sum_{k=0}^{\infty}\sqrt{\frac{(k+1)(k+\alpha+\beta+2)}{\lambda_k^{\alpha+1,\beta+1}}}\;c_k^{\alpha+1,\beta+1}(f)\;\phi_{k+1}^{\alpha,\beta},\quad f\in L^2(0,\pi).$$

From Plancherel's theorem we deduce that $R^{1,*}_{\alpha,\beta}$ is a bounded operator from $L^2(0,\pi)$ into itself.

If $f\in C_c^\infty(0,\pi)$, then for every $m\in\mathbb{N}$ there exists C_m such that

$$|c_k^{\alpha+1,\beta+1}(f)|\le C_m(k+1)^{-m},\quad k\in\mathbb{N}.$$

Suppose that $f,g\in C_c^\infty(0,\pi)$. Partial integration leads to

$$\int_0^\pi R^{1,*}_{\alpha,\beta}f(\theta)g(\theta)d\theta=\int_0^\pi f(\varphi)\mathcal{L}^{-1/2}_{\alpha+1,\beta+1}\left(D_{\alpha,\beta}g\right)(\varphi)d\varphi.$$

By taking into account the rapid decay of the sequence $\left(c_k^{\alpha,\beta}(g)\right)_{k\in\mathbb{N}}$ and [24, Lemma 3.1] we write

$$D_{\alpha,\beta}g(\theta)=-\sum_{k=0}^{\infty}\sqrt{k(k+\alpha+\beta+1)}c_k^{\alpha,\beta}(g)\phi_{k-1}^{\alpha+1,\beta+1}(\theta),\quad \theta\in(0,\pi),$$

and

$$\mathcal{L}^{-1/2}_{\alpha+1,\beta+1}\left(D_{\alpha,\beta}g\right)(\theta)=-\sum_{k=0}^{\infty}\sqrt{\frac{k(k+\alpha+\beta+1)}{\lambda_k^{\alpha,\beta}}}\;c_k^{\alpha,\beta}(g)\;\phi_{k-1}^{\alpha+1,\beta+1}(\theta)=R^1_{\alpha,\beta}g(\theta),\quad \theta\in(0,\pi).$$

Hence, $R^{1,*}_{\alpha,\beta}$ is the adjoint of $R^1_{\alpha,\beta}$ (fact justifying the notation). Thus, $R^{1,*}_{\alpha,\beta}$ defines a bounded operator from $L^p_w(0,\pi)$ into itself, for every $1<p<\infty$ and $w\in A_p(0,\pi)$. □

Combining [15, Theorem 1.3] with Lemma 4.1 and Proposition 4.1 we obtain the following.

Proposition 4.2. *Let $k\in\mathbb{N}$ and $\alpha,\beta\ge-1/2$ such that $\alpha+\beta\ne-1$. Suppose that $p\in\mathcal{B}(0,\pi)$. Then, $R^k_{\alpha,\beta}$ and $R^{k,*}_{\alpha,\beta}$ define bounded operators from $L^{p(\cdot)}(0,\pi)$ into itself.*

According to [24, Lemma 3.1] we get, for every $f\in S_{\alpha,\beta}$,

$$R^{k,*}_{\alpha,\beta}R^k_{\alpha,\beta}f=\sum_{n=k}^{\infty}\frac{(n-k+1)_k(n+\alpha+\beta+1)_k}{(\lambda_n^{\alpha,\beta})^k}\;c_k^{\alpha,\beta}(f)\;\phi_n^{\alpha,\beta}.$$

Notice that, for every $n\in\mathbb{N}$, $n\ge k$,

$$(n-k+1)_k=\left(\sqrt{\lambda_n^{\alpha,\beta}}-\sqrt{\lambda_{k-1}^{\alpha,\beta}}\right)\left(\sqrt{\lambda_n^{\alpha,\beta}}-\sqrt{\lambda_{k-2}^{\alpha,\beta}}\right)\cdot\ldots\cdot\left(\sqrt{\lambda_n^{\alpha,\beta}}-\sqrt{\lambda_0^{\alpha,\beta}}\right),$$

and

$$(n+\alpha+\beta+1)_k=\left(\sqrt{\lambda_n^{\alpha,\beta}}+\sqrt{\lambda_0^{\alpha,\beta}}\right)\left(\sqrt{\lambda_n^{\alpha,\beta}}+\sqrt{\lambda_1^{\alpha,\beta}}\right)\cdot\ldots\cdot\left(\sqrt{\lambda_n^{\alpha,\beta}}+\sqrt{\lambda_{k-1}^{\alpha,\beta}}\right).$$

We consider the function M given by

$$M(x)=\frac{x^k}{\prod_{j=0}^{k-1}\left(x-\lambda_j^{\alpha,\beta}\right)},\qquad x\ne\lambda_j^{\alpha,\beta},\ j=0,\ldots,k-1,$$

and we choose a smooth function ϕ on $(0,\infty)$ such that

$$\phi(x)=\begin{cases}0, & 0<x<\lambda_{k-1}^{\alpha,\beta}+\frac{\alpha+\beta+1}{8},\\ 1, & x\ge\lambda_k^{\alpha,\beta}-\frac{\alpha+\beta+1}{8}.\end{cases}$$

Take $m=\phi M$. Then,

$$m(\mathcal{L}_{\alpha,\beta})R^{k,*}_{\alpha,\beta}R^k_{\alpha,\beta}f=f,\quad f\in S_{\alpha,\beta}.\tag{35}$$

It is not hard to see that m satisfies condition (17) of proposition 3.1. Hence, by Proposition 3.2 (with $a=0$) we infer the following.

Proposition 4.3. *Let $\alpha,\beta\ge-1/2$ such that $\alpha+\beta\ne-1$. Suppose that $p\in\mathcal{B}(0,\pi)$. Then, the Jacobi spectral multiplier $m(\mathcal{L}_{\alpha,\beta})$, where $m=\phi M$ is as above, defines a bounded operator from $L^{p(\cdot)}(0,\pi)$ into itself.*

5. Proof of Theorem 1.2

First of all we establish the following lemma where we define some Jacobi spectral multipliers that will be useful in the sequel.

Lemma 5.1. *Let $\varepsilon, \gamma > 0$, $r \in \mathbb{N}$ with $r > \gamma$ and $\alpha, \beta \geq -1/2$ such that $\alpha + \beta \neq -1$. Assume that $p \in \mathcal{B}(0,\pi)$. We define, for each $t > 0$, the functions*

$$Y_\varepsilon(t) = (1-e^{-\varepsilon t})^r, \quad M_\varepsilon(t) = \frac{(1-e^{-\varepsilon t})^r}{(\varepsilon t)^{\gamma/2}} \quad \text{and} \quad H_\varepsilon(t) = \int_{\varepsilon t}^\infty \frac{(1-e^{-u})^r}{u^{1+\gamma}} du. \tag{36}$$

By m_ε we represent Y_ε, M_ε or H_ε. Then, m_ε defines a Jacobi spectral multiplier on $L^{p(\cdot)}(0,\pi)$. Moreover,

$$\sup_{\varepsilon>0} \|m_\varepsilon(\mathcal{L}_{\alpha,\beta})\|_{L^{p(\cdot)}(0,\pi)} < \infty.$$

Proof. Straightforward manipulations allow us to show that, for every $\ell \in \mathbb{N}$, there exists $C > 0$ such that

$$\sup_{\varepsilon>0} \left| t^\ell \frac{d^\ell}{dt^\ell} m_\varepsilon(t) \right| \leq C,$$

where C does not depend on ε. Then, by Proposition 3.2 (taken with $a = 0$) we concluded the desired results. □

Proposition 5.1. *Let $\varepsilon, \gamma > 0$, $r \in \mathbb{N}$ with $r > \gamma$ and $\alpha, \beta \geq -1/2$ such that $\alpha + \beta \neq -1$. Assume that $p \in \mathcal{B}(0,\pi)$. Then, the operator $I_\varepsilon^{\gamma,r}$ defined in (5) is bounded from $L^{p(\cdot)}(0,\pi)$ into itself.*

Proof. Let $f \in S_{\alpha,\beta}$. We can write

$$(I - W_u^{\alpha,\beta})^r f = \sum_{n=0}^\infty Y_u\left(\lambda_n^{\alpha,\beta}\right) c_n^{\alpha,\beta}(f) \phi_n^{\alpha,\beta} = Y_u(\mathcal{L}_{\alpha,\beta}) f, \quad u > 0,$$

where the series is actually a finite sum. According to Lemma 5.1, we deduce that,

$$\|I_\varepsilon^{\gamma,r} f\|_{L^{p(\cdot)}(0,\pi)} \leq C \sup_{u>0} \|Y_u(\mathcal{L}_{\alpha,\beta}) f\|_{L^{p(\cdot)}(0,\pi)} \int_\varepsilon^\infty \frac{du}{u^{\gamma+1}} \leq C \|f\|_{L^{p(\cdot)}(0,\pi)}.$$

Taking into account that $S_{\alpha,\beta}$ is a dense subspace of $L^{p(\cdot)}(0,\pi)$ (Proposition 2.1) the conclusion follows. □

Proof of Theorem 1.2. Suppose that $f \in D_{p(\cdot)}(\mathcal{L}_{\alpha,\beta}^\gamma)$ and call $g = \lim_{\varepsilon\to 0^+} I_\varepsilon^{\gamma,r} f$. Since $\mathcal{L}_{\alpha,\beta}^{-\gamma}$ is a bounded operator from $L^{p(\cdot)}(0,\pi)$ into itself (Proposition 3.3), we have that

$$\begin{aligned}
\mathcal{L}_{\alpha,\beta}^{-\gamma} g &= C_{\gamma,r} \lim_{\varepsilon\to 0^+} \mathcal{L}_{\alpha,\beta}^{-\gamma} \int_\varepsilon^\infty \frac{(I - W_u^{\alpha,\beta})^r f}{u^{1+\gamma}} du \\
&= C_{\gamma,r} \lim_{\varepsilon\to 0^+} \int_\varepsilon^\infty \frac{(I - W_u^{\alpha,\beta})^r}{u^{1+\gamma}} \mathcal{L}_{\alpha,\beta}^{-\gamma} f \, du, \quad \text{in } L^{p(\cdot)}(0,\pi).
\end{aligned}$$

We can write

$$\frac{(I - W_u^{\alpha,\beta})^r f}{u^{\gamma/2}} \mathcal{L}_{\alpha,\beta}^{-\gamma} f = \sum_{n=0}^\infty \frac{\left(1 - e^{-u\lambda_n^{\alpha,\beta}}\right)^r}{\left(u\lambda_n^{\alpha,\beta}\right)^{\gamma/2} \left(\lambda_n^{\alpha,\beta}\right)^{\gamma/2}} c_n^{\alpha,\beta}(f) \, \phi_n^{\alpha,\beta} = M_u(\mathcal{L}_{\alpha,\beta}) \mathcal{L}_{\alpha,\beta}^{-\gamma/2} f, \quad u > 0,$$

where M_u was defined in (36). According to Lemma 5.1 and Propositions 2.2 and 3.3, there exists $C > 0$ such that

$$\begin{aligned}
\left\| M_u(\mathcal{L}_{\alpha,\beta}) \mathcal{L}_{\alpha,\beta}^{-\gamma/2} \Big(\sum_{n=0}^\ell c_n^{\alpha,\beta}(f) \phi_n^{\alpha,\beta} \Big) \right\|_{L^{p(\cdot)}(0,\pi)} &\leq C \left\| \sum_{n=0}^\ell c_n^{\alpha,\beta}(f) \phi_n^{\alpha,\beta} \right\|_{L^{p(\cdot)}(0,\pi)} \\
&\leq C \|f\|_{L^{p(\cdot)}(0,\pi)}, \quad \ell \in \mathbb{N} \text{ and } u > 0.
\end{aligned}$$

Also, since $u^{-1-\gamma} \in L^1(\varepsilon, \infty)$, $\varepsilon > 0$, we obtain

$$\begin{aligned}
\mathcal{L}_{\alpha,\beta}^{-\gamma} \int_\varepsilon^\infty \frac{(I - W_u^{\alpha,\beta})^r f}{u^{1+\gamma}} du &= \sum_{n=0}^\infty \int_\varepsilon^\infty \frac{\left(1 - e^{-u\lambda_n^{\alpha,\beta}}\right)^r}{\left(u\lambda_n^{\alpha,\beta}\right)^\gamma} \frac{du}{u} \, c_n^{\alpha,\beta}(f) \, \phi_n^{\alpha,\beta} \\
&= \sum_{n=0}^\infty \int_{\varepsilon\lambda_n^{\alpha,\beta}}^\infty \frac{(1 - e^{-u})^r}{u^{1+\gamma}} du \, c_n^{\alpha,\beta}(f) \, \phi_n^{\alpha,\beta} = H_\varepsilon(\mathcal{L}_{\alpha,\beta}) f, \quad \varepsilon > 0,
\end{aligned}$$

where H_ε was defined in (36).

Suppose that $F \in S_{\alpha,\beta}$. We can write, for every $l \in \mathbb{N}$,

$$\lim_{\varepsilon\to 0^+} H_\varepsilon(\mathcal{L}_{\alpha,\beta})F = \lim_{\varepsilon\to 0^+} \sum_{n=0}^{\ell} \int_{\varepsilon\lambda_n^{\alpha,\beta}}^\infty \frac{(1-e^{-u})^r}{u^{1+\gamma}} du\, c_n^{\alpha,\beta}(F)\phi_n^{\alpha,\beta} = \frac{1}{C_{\gamma,r}} \sum_{n=0}^{\ell} c_n^{\alpha,\beta}(F)\phi_n^{\alpha,\beta} = \frac{F}{C_{\gamma,r}},$$

in the sense of convergence in $L^{p(\cdot)}(0,\pi)$. Since $S_{\alpha,\beta}$ is dense in $L^{p(\cdot)}(0,\pi)$ (Proposition 2.1), Lemma 5.1 leads to

$$\lim_{\varepsilon\to 0^+} H_\varepsilon(\mathcal{L}_{\alpha,\beta})(f) = \frac{f}{C_{\gamma,r}}.$$

Thus, we conclude that $\mathcal{L}_{\alpha,\beta}^{-\gamma} g = f$.

On the other hand, take $f \in H_{\alpha,\beta}^{\gamma,p(\cdot)}(0,\pi)$ such that $f = \mathcal{L}_{\alpha,\beta}^{-\gamma} g$, with $g \in L^{p(\cdot)}(0,\pi)$. Then, as it has just been proved,

$$\lim_{\varepsilon\to 0^+} I_\varepsilon^{\gamma,r} f = C_{\gamma,r} \lim_{\varepsilon\to 0^+} \int_\varepsilon^\infty \frac{(I - W_u^{\alpha,\beta})^r}{u^{1+\gamma}} \mathcal{L}_{\alpha,\beta}^{-\gamma} g\, du = g,$$

in the sense of convergence in $L^{p(\cdot)}(0,\pi)$. □

Remark 5.1. *A careful reading of the above proof reveals that we can consider any $r \in \mathbb{N}$, $r > \gamma$ (not necessarily $r < \gamma \le r+1$). This fact implies that the operator $\mathcal{L}_{\alpha,\beta}^{\gamma}$ can be defined by* (6)*, for any $r \in \mathbb{N}$, $r > \gamma$.*

6. Proof of Theorem 1.3

Assume that $\gamma > 0$. It is not hard to see that $\partial_t^\gamma e^{-at} = e^{i\pi\gamma} a^\gamma e^{-at}$, $t, a > 0$. Thus, we have that, for every $f \in S_{\alpha,\beta} \cup C_c^\infty(0,\pi)$,

$$\partial_t^\gamma P_t^{\alpha,\beta} f(\theta) = \sum_{n=0}^\infty e^{i\pi\gamma} (\lambda_n^{\alpha,\beta})^{\gamma/2} e^{-t\sqrt{\lambda_n^{\alpha,\beta}}} c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta}(\theta), \quad \theta \in (0,\pi).$$

Hence, for every $f \in S_{\alpha,\beta} \cup C_c^\infty(0,\pi)$,

$$g_{\alpha,\beta}^\gamma(f)(\theta) < \infty, \quad \theta \in (0,\pi).$$

Our first objective is to establish L_ω^p-boundedness properties of $g_{\alpha,\beta}^\gamma$-functions.

Proposition 6.1. *Let $\gamma > 0$ and $\alpha, \beta \ge -1/2$. Then, $g_{\alpha,\beta}^\gamma$ defines a bounded (quasi-linear) operator from $L_w^p(0,\pi)$ into itself, for every $1 < p < \infty$ and $w \in A_p(0,\pi)$.*

Proof. For every $N \in \mathbb{N}$, we define

$$\mathcal{G}_{\alpha,\beta}^{\gamma,N}(f)(\theta) = \left(\int_{1/N}^N \left| t^\gamma \partial_t^\gamma P_t^{\alpha,\beta}(f)(\theta) \right|^2 \frac{dt}{t} \right)^{1/2}, \quad \theta \in (0,\pi).$$

We will show that, for every $1 < p < \infty$ and $w \in A_p(0,\pi)$, there exists $C > 0$ independent of $N \in \mathbb{N}$, such that

$$\|\mathcal{G}_{\alpha,\beta}^{\gamma,N}(f)\|_{L_w^p(0,\pi)} \le C\|f\|_{L_w^p(0,\pi)}, \quad f \in L_w^p(0,\pi). \tag{37}$$

From (37), by using monotone convergence theorem, we deduce that for every $1 < p < \infty$ and $w \in A_p(0,\pi)$, there exists $C > 0$ satisfying that

$$\|g_{\alpha,\beta}^\gamma(f)\|_{L_w^p(0,\pi)} \le C\|f\|_{L_w^p(0,\pi)}, \quad f \in L_w^p(0,\pi).$$

In order to show (37) we apply the local Calderón-Zygmund theory [13] in a Banach valued setting [38].

By proceeding as in [6, Proposition 2.1] we obtain

$$\frac{2^{2\gamma}}{\Gamma(2\gamma)} \int_0^\pi \int_0^\infty t^\gamma \partial_t^\gamma P_t^{\alpha,\beta} f(\theta) t^\gamma \partial_t^\gamma P_t^{\alpha,\beta}(\bar{g})(\theta) d\theta \frac{dt}{t} = \int_0^\pi f(\theta)\bar{g}(\theta) d\theta, \quad f, g \in S_{\alpha,\beta}. \tag{38}$$

Thus, for every $N \in \mathbb{N}$, we get

$$\|\mathcal{G}_{\alpha,\beta}^{\gamma,N}(f)\|_{L^2(0,\pi)}^2 = \frac{\Gamma(2\gamma)}{2^{2\gamma}} \|f\|_{L^2(0,\pi)}^2, \quad f \in S_{\alpha,\beta}. \tag{39}$$

Hence, $g_{\alpha,\beta}^\gamma$ and $\mathcal{G}_{\alpha,\beta}^{\gamma,N}$, $N \in \mathbb{N}$, can be extended from $S_{\alpha,\beta}$ to $L^2(0,\pi)$ as a bounded operators from $L^2(0,\pi)$ into itself.

Let $m \in \mathbb{N}$. According to [3, Lemma 4] we have that

$$\left|\partial_t^m [te^{-t^2/4u}]\right| \leq Ce^{-t^2/4u}u^{(1-m)/2}, \quad t,u \in (0,\infty). \tag{40}$$

By (2) and by taking into account that (13) and (40) the differentiation under the integral sign is justified, so we can write

$$\partial_t^m P_t^{\alpha,\beta}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}} \int_0^\infty \partial_t^m \Big[\frac{te^{-t^2/4u}}{u^{3/2}}\Big] W_u^{\alpha,\beta}(\theta,\varphi)du, \quad t>0 \text{ and } \theta,\varphi \in (0,\pi).$$

From (13) and (40) it follows that

$$\begin{aligned}\left|\partial_t^m P_t^{\alpha,\beta}(\theta,\varphi)\right| \leq & C\int_0^\infty \frac{e^{-c(t^2+(\theta-\varphi)^2)/u}}{u^{(m+3)/2}}du \\ \leq & \frac{C}{(t^2+(\theta-\varphi)^2)^{(m+1)/2}}, \quad t>0 \text{ and } \theta,\varphi \in (0,\pi).\end{aligned} \tag{41}$$

Let $f \in L^2(0,\pi)$. By (41) we obtain

$$\partial_t^m P_t^{\alpha,\beta} f(\theta) = \int_0^\pi \partial_t^m P_t^{\alpha,\beta}(\theta,\varphi) f(\varphi) d\varphi, \quad t>0 \text{ and } \theta \in (0,\pi).$$

Thus, if $m-1 \leq \gamma < m$, (41) leads to

$$\begin{aligned}\left|\partial_t^\gamma P_t^{\alpha,\beta} f(\theta)\right| &\leq C\int_0^\infty \int_0^\pi \left|\partial_t^m P_{t+s}^{\alpha,\beta}(\theta,\varphi)\right| \, |f(\varphi)| d\varphi s^{m-\gamma-1} ds \\ &\leq C\int_0^\infty \int_0^\pi \frac{|f(\varphi)|}{[(t+s)^2+(\theta-\varphi)^2]^{(m+1)/2}} d\varphi s^{m-\gamma-1} ds \\ &\leq C\int_0^\infty \frac{s^{m-\gamma-1}}{(t+s)^{m+1}} ds \, \|f\|_{L^2(0,\pi)} \leq \frac{C}{t^{\gamma+1}}\|f\|_{L^2(0,\pi)}, \quad t>0 \text{ and } \theta \in (0,\pi).\end{aligned}$$

Hence, we obtain, for every $N \in \mathbb{N}$,

$$\mathcal{G}_{\alpha,\beta}^{\gamma,N}(f)(\theta) \leq C\Big(\int_{1/N}^N \frac{dt}{t^3}\Big)^{1/2} \|f\|_{L^2(0,\pi)}, \quad \theta \in (0,\pi). \tag{42}$$

This estimate shows that, for every $N \in \mathbb{N}$, $\mathcal{G}_{\alpha,\beta}^{\gamma,N}$ is a bounded operator from $L^2(0,\pi)$ into itself. By (39) we conclude that, for every $N \in \mathbb{N}$,

$$\|\mathcal{G}_{\alpha,\beta}^{\gamma,N}(f)\|_{L^2(0,\pi)}^2 = \frac{\Gamma(2\gamma)}{2^{2\gamma}}\|f\|_{L^2(0,\pi)}^2, \quad f \in L^2(0,\pi). \tag{43}$$

Note that (43), in contrast with (42), shows that the family $\{\mathcal{G}_{\alpha,\beta}^{\gamma,N}\}_{N\in\mathbb{N}}$ is bounded in $\mathcal{L}(L^2(0,\pi))$, the space of bounded operators from $L^2(0,\pi)$ into itself.

Let $N \in \mathbb{N}$. We consider the operator

$$T_{\alpha,\beta}^{\gamma,N}(f)(\theta) = \int_0^\pi K_{\alpha,\beta}^{\gamma,N}(\theta,\varphi) f(\varphi) d\varphi,$$

where, for every $\theta,\varphi \in (0,\pi)$, $\theta \neq \varphi$,

$$[K_{\alpha,\beta}^{\gamma,N}(\theta,\varphi)](t) = t^\gamma \partial_t^\gamma P_t^{\alpha,\beta}(\theta,\varphi), \quad t \in (1/N,N),$$

and the integral is understood in the $L^2((1/N,N),dt/t)$-Böchner sense.

From (41) we deduce that

$$\begin{aligned}\left\|K_{\alpha,\beta}^{\gamma,N}(\theta,\varphi)\right\|_{L^2((1/N,N),dt/t)} &\leq C\Big(\int_{1/N}^N \Big|t^\gamma \int_0^\infty \frac{s^{m-\gamma-1}}{((t+s)^2+(\theta-\varphi)^2)^{(m+1)/2}} ds\Big|^2 \frac{dt}{t}\Big)^{1/2} \\ &\leq C\Big(\int_{1/N}^N \frac{t^{2\gamma-1}}{(t+|\theta-\varphi|)^{2\gamma+2}} dt\Big)^{1/2} \leq \frac{C}{|\theta-\varphi|}, \quad \theta,\varphi \in (0,\pi), \ \theta \neq \varphi.\end{aligned} \tag{44}$$

Here $C>0$ does not depend on $N \in \mathbb{N}$.

Let $f \in L^2(0,\pi)$ and $\theta \notin \operatorname{supp}(f)$. If $h \in L^2((1/N,N),dt/t)$, (44) allows us to write

$$\begin{aligned}\int_{1/N}^N h(t)[T_{\alpha,\beta}^{\gamma,N}(f)(\theta)](t)\frac{dt}{t} &= \int_0^\pi f(\varphi)\int_{1/N}^N h(t)[K_{\alpha,\beta}^{\gamma,N}(\theta,\varphi)](t)\frac{dt}{t}d\varphi \\ &= \int_0^\pi f(\varphi)\int_{1/N}^N h(t) t^\gamma \partial_t^\gamma P_t^{\alpha,\beta}(\theta,\varphi)\frac{dt}{t}d\varphi = \int_{1/N}^N h(t)\int_0^\pi t^\gamma \partial_t^\gamma P_t^{\alpha,\beta}(\theta,\varphi)f(\varphi)d\varphi\frac{dt}{t}.\end{aligned}$$

Thus, we obtain

$$[T^{\gamma,N}_{\alpha,\beta}(f)(\theta)](t) = t^\gamma \partial_t^\gamma P_t^{\alpha,\beta}(f)(\theta), \quad \text{a.e. } t \in (1/N, N).$$

We are going to show, for every $N \in \mathbb{N}$ and $(\theta,\varphi) \in (0,\pi)^2 \setminus \mathcal{D}$, $\theta \neq \varphi$,

$$\Big\| \partial_\theta \Big(t^\gamma \partial_t^\gamma P_t^{\alpha,\beta}(\theta,\varphi)\Big)\Big\|_{L^2((1/N,N),dt/t)} + \Big\| \partial_\varphi \Big(t^\gamma \partial_t^\gamma P_t^{\alpha,\beta}(\theta,\varphi)\Big)\Big\|_{L^2((1/N,N),dt/t)} \leq \frac{C}{|\theta-\varphi|^2}, \tag{45}$$

for a certain $C > 0$ which does not depend on N and the domain $\mathcal{D}$ is as in Figure 1.

To simplify we call

$$\Phi_{\alpha,\beta}(t,z) = \frac{\sinh \frac{t}{2}}{(\cosh \frac{t}{2} - 1 + z)^{\alpha+\beta+2}}, \quad t, z > 0,$$

to one of the terms appearing in (25). According to [35, Lemma 4.8] we have that, for every $m \in \mathbb{N}$,

$$\Big|\partial_t^m \Phi_{\alpha,\beta}(t,z)\Big| \leq C \begin{cases} (\cosh \frac{t}{2} - 1 + z)^{-\alpha-\beta-(m+3)/2}, & t \leq 1,\ z > 0 \\ (\cosh \frac{t}{2} - 1 + z)^{-\alpha-\beta-1}, & t > 1,\ z > 0, \end{cases} \tag{46}$$

and

$$\begin{aligned} &\Big|\partial_\theta \partial_t^m \Phi_{\alpha,\beta}(t, q(\theta,\varphi,u,v))\Big| + \Big|\partial_\varphi \partial_t^m \Phi_{\alpha,\beta}(t, q(\theta,\varphi,u,v))\Big| \\ &\quad \leq C \begin{cases} (\cosh \frac{t}{2} - 1 + q(\theta,\varphi,u,v))^{-\alpha-\beta-(m+4)/2}, & t \leq 1,\ \theta,\varphi \in (0,\pi),\ -1 < u,v < 1 \\ (\cosh \frac{t}{2} - 1 + q(\theta,\varphi,u,v))^{-\alpha-\beta-3/2}, & t > 1,\ \theta,\varphi \in (0,\pi),\ -1 < u,v < 1. \end{cases} \end{aligned} \tag{47}$$

Let $m \in \mathbb{N}$. By using (46) and [35, Lemma 4.4] we get

$$\begin{aligned} &\int_{-1}^1 \int_{-1}^1 \Big|\partial_t^m \Phi_{\alpha,\beta}(t, q(\theta,\varphi,u,v))\Big| d\Pi_\alpha(u) d\Pi_\beta(v) \\ &\qquad \leq C \begin{cases} \displaystyle\int_{-1}^1 \int_{-1}^1 \frac{d\Pi_\alpha(u) d\Pi_\beta(v)}{(\cosh \frac{t}{2} - 1 + q(\theta,\varphi,u,v))^{\alpha+\beta+(m+3)/2}}, & t \leq 1 \\ \displaystyle\int_{-1}^1 \int_{-1}^1 \frac{d\Pi_\alpha(u) d\Pi_\beta(v)}{(\cosh \frac{t}{2} - 1 + q(\theta,\varphi,u,v))^{\alpha+\beta+1}}, & t > 1 \end{cases} \\ &\qquad \leq \frac{C}{(\cosh \frac{t}{2} - 1)^{\alpha+\beta+1}}, \quad t > 0 \text{ and } \theta,\varphi \in (0,\pi). \end{aligned}$$

Thus, from (25) we can write for each $\theta, \varphi \in (0,\pi)$ and $t > 0$,

$$\begin{aligned} &\partial_t^m P_t^{\alpha,\beta}(\theta,\varphi) \\ &\quad = C_{\alpha,\beta} \Big(\sin\frac{\theta}{2} \sin\frac{\varphi}{2}\Big)^{\alpha+1/2} \Big(\cos\frac{\theta}{2} \cos\frac{\varphi}{2}\Big)^{\beta+1/2} \int_{-1}^1 \int_{-1}^1 \partial_t^m \Phi_{\alpha,\beta}(t, q(\theta,\varphi,u,v)) d\Pi_\alpha(u) d\Pi_\beta(v). \end{aligned}$$

Assume that $m \in \mathbb{N}$ is such that $m - 1 \leq \gamma < m$. From (47) and [35, trigonometric identities in p. 738] we deduce, for every $\theta, \varphi \in (0,\pi)$ and $t > 0$,

$$\begin{aligned} &\int_0^\infty s^{m-\gamma-1} \int_{-1}^1 \int_{-1}^1 \Big|\partial_\theta \partial_t^m \Phi_{\alpha,\beta}(t+s, q(\theta,\varphi,u,v))\Big| d\Pi_\alpha(u) d\Pi_\beta(v) ds \\ &\qquad \leq C \Big\{ \int_0^{\max\{0,1-t\}} \frac{s^{m-\gamma-1}}{(\cosh \frac{t+s}{2} - 1 + 2\sin^2 \frac{\theta-\varphi}{4})^{\alpha+\beta+(m+4)/2}} ds \\ &\qquad\quad + \int_{\max\{0,1-t\}}^1 \frac{s^{m-\gamma-1}}{(\cosh \frac{t+s}{2} - 1 + 2\sin^2 \frac{\theta-\varphi}{4})^{\alpha+\beta+3/2}} ds \\ &\qquad\quad + \int_1^\infty s^{m-\gamma-1} e^{-c(\alpha+\beta+3/2)(t+s)} ds \Big\} < \infty. \end{aligned}$$

Hence, we can write for $\theta, \varphi \in (0,\pi)$ and $t > 0$,

$$\begin{aligned} t^\gamma \partial_\theta \partial_t^\gamma P_t^{\alpha,\beta}(\theta,\varphi) &= C_{\alpha,\beta} \Big(\sin\frac{\theta}{2} \sin\frac{\varphi}{2}\Big)^{\alpha+1/2} \Big(\cos\frac{\theta}{2} \cos\frac{\varphi}{2}\Big)^{\beta+1/2} \frac{e^{-i(m-\gamma)\pi}}{\Gamma(m-\gamma)} t^\gamma \\ &\times \Big[\int_0^\infty s^{m-\gamma-1} \int_{-1}^1 \int_{-1}^1 \partial_\theta \partial_t^m \Phi_{\alpha,\beta}(t+s, q(\theta,\varphi,u,v)) d\Pi_\alpha(u) d\Pi_\beta(v) ds \\ &+ \Big(\frac{2\alpha+1}{4} \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} - \frac{2\beta+1}{4} \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \Big) \int_0^\infty s^{m-\gamma-1} \int_{-1}^1 \int_{-1}^1 \partial_t^m \Phi_{\alpha,\beta}(t+s, q(\theta,\varphi,u,v)) d\Pi_\alpha(u) d\Pi_\beta(v) ds \Big]. \end{aligned}$$

By proceeding as in [35, pp. 747-748] (see also the proof of Proposition 4.1), (47) and Minkowski's inequality leads to

$$\begin{aligned}
&\Big\|\Big(\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\Big)^{\alpha+1/2}\Big(\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\Big)^{\beta+1/2}t^{\gamma}\\
&\qquad\times\int_0^\infty s^{m-\gamma-1}\int_{-1}^1\int_{-1}^1\partial_\theta\partial_t^m\Phi_{\alpha,\beta}(t+s,q(\theta,\varphi,u,v))d\Pi_\alpha(u)d\Pi_\beta(v)ds\Big\|_{L^2((0,\infty),dt/t)}\\
&\quad\le\Big(\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\Big)^{\alpha+1/2}\Big(\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\Big)^{\beta+1/2}\int_0^\infty s^{m-\gamma-1}\int_{-1}^1\int_{-1}^1\\
&\qquad\times\Big\|t^\gamma\partial_\theta\partial_t^m\Phi_{\alpha,\beta}(t+s,q(\theta,\varphi,u,v))\Big\|_{L^2((0,\infty),dt/t)}d\Pi_\alpha(u)d\Pi_\beta(v)ds\\
&\quad\le C\Big(\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\Big)^{\alpha+1/2}\Big(\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\Big)^{\beta+1/2}\int_{-1}^1\int_{-1}^1\frac{d\Pi_\alpha(u)d\Pi_\beta(v)ds}{q(\theta,\varphi,u,v)^{\alpha+\beta+2}}\\
&\quad\le\frac{C}{|\theta-\varphi|^2},\quad\theta,\varphi\in(0,\pi),\ \theta\neq\varphi. \qquad (48)
\end{aligned}$$

In a similar way, by using (46) we obtain

$$\begin{aligned}
&\Big\|\Big(\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\Big)^{\alpha+1/2}\Big(\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\Big)^{\beta+1/2}t^{\gamma}\\
&\qquad\times\int_0^\infty s^{m-\gamma-1}\int_{-1}^1\int_{-1}^1\partial_t^m\Phi_{\alpha,\beta}(t+s,q(\theta,\varphi,u,v))d\Pi_\alpha(u)d\Pi_\beta(v)ds\Big\|_{L^2((0,\infty),dt/t)}\\
&\quad\le C\Big(\sin\frac{\theta}{2}\sin\frac{\varphi}{2}\Big)^{\alpha+1/2}\Big(\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\Big)^{\beta+1/2}\int_{-1}^1\int_{-1}^1\frac{d\Pi_\alpha(u)d\Pi_\beta(v)ds}{q(\theta,\varphi,u,v)^{\alpha+\beta+3/2}}\\
&\quad\le\frac{C}{|\theta-\varphi|}\le\frac{C}{|\theta-\varphi|^2},\quad\theta,\varphi\in(0,\pi),\ \theta\neq\varphi. \qquad (49)
\end{aligned}$$

Combining (48) and (49) with (29), (30), (31) and (33), we deduce that

$$\Big\|\partial_\theta K^\gamma_{\alpha,\beta}(\theta,\varphi)\Big\|_{L^2((0,\infty),dt/t)}\le\frac{C}{|\theta-\varphi|^2},\quad(\theta,\varphi)\in(0,\pi)^2\setminus\mathcal{D}.$$

The same procedure allows us to prove that

$$\Big\|\partial_\varphi K^\gamma_{\alpha,\beta}(\theta,\varphi)\Big\|_{L^2((0,\infty),dt/t)}\le\frac{C}{|\theta-\varphi|^2},\quad(\theta,\varphi)\in(0,\pi)^2\setminus\mathcal{D}.$$

Thus, (45) is established.

By using now the local Calderón-Zygmund theory for singular integrals (see [13]) in the $L^2((1/N,N),dt/t)$-setting and by taking into account Lemma 4.2, we conclude that, for every $1<p<\infty$ and $w\in A_p(0,\pi)$, the operator $T^{\gamma,N}_{\alpha,\beta}$ can be extended from $L^2(0,\pi)\cap L^p_w(0,\pi)$ to $L^p_w(0,\pi)$ as a bounded operator $\widetilde{T}^{\gamma,N}_{\alpha,\beta}$ from $L^p_w(0,\pi)$ into $L^p_w\big((0,\pi);L^2((1/N,N),dt/t)\big)$, and there exists $C>0$, which does not depend on N, such that

$$\Big\|\widetilde{T}^{\gamma,N}_{\alpha,\beta}(f)\Big\|_{L^p_w\left((0,\pi);L^2((1/N,N),dt/t)\right)}\le C\|f\|_{L^p_w(0,\pi)},\quad f\in L^p_w(0,\pi). \qquad (50)$$

Let $f\in L^p_w(0,\pi)$ where $1<p<\infty$ and $w\in A_p(0,\pi)$. We take a sequence $(f_n)_{n\in\mathbb{N}}\subseteq L^p_w(0,\pi)\cap L^2(0,\pi)$ such that

$$f_n\longrightarrow f,\quad\text{as } n\to\infty,\ \text{in } L^p_w(0,\pi).$$

As in (42) we obtain that

$$\mathcal{G}^{\gamma,N}_{\alpha,\beta}(f-f_n)(\theta)\le C\|f-f_n\|_{L^p_w(0,\pi)},\quad n\in\mathbb{N}\text{ and }\theta\in(0,\pi).$$

Hence,

$$\mathcal{G}^{\gamma,N}_{\alpha,\beta}(f_n)(\theta)\longrightarrow\mathcal{G}^{\gamma,N}_{\alpha,\beta}(f)(\theta),\quad\text{as } n\to\infty\text{ for every }\theta\in(0,\pi).$$

On the other hand,

$$\widetilde{T}^{\gamma,N}_{\alpha,\beta}(f)=\lim_{n\to\infty}T^{\gamma,N}_{\alpha,\beta}(f_n),\quad\text{in } L^p_w\big((0,\pi);L^2((1/N,N),dt/t)\big).$$

Then, there exists a monotone function $\phi:\mathbb{N}\longrightarrow\mathbb{N}$ such that

$$T^{\gamma,N}_{\alpha,\beta}(f_{\phi(n)})(\theta)\longrightarrow\widetilde{T}^{\gamma,N}_{\alpha,\beta}(f)(\theta),\quad\text{as } n\to\infty,\ \text{in } L^2((1/N,N),dt/t),$$

for almost every $\theta \in (0,\pi)$. This implies that

$$\mathcal{G}^{\gamma,N}_{\alpha,\beta}(f_{\phi(n)})(\theta) \longrightarrow \left\|\widetilde{T}^{\gamma,N}_{\alpha,\beta}(f)(\theta)\right\|_{L^2((1/N,N),dt/t)}, \quad \text{as } n\to\infty,$$

for almost every $\theta \in (0,\pi)$. We conclude that

$$\mathcal{G}^{\gamma,N}_{\alpha,\beta}(f)(\theta) = \left\|\widetilde{T}^{\gamma,N}_{\alpha,\beta}(f)(\theta)\right\|_{L^2((1/N,N),dt/t)}, \quad \text{a.e. } \theta \in (0,\pi),$$

and from (50) we deduce (37).

Thus the proof of this proposition is completed. □

By using [15, Theorem 1.3] from Proposition 6.1 we infer the following.

Corollary 6.1. *Let $\alpha,\beta \geq -1/2$ and $\gamma > 0$. Suppose that $p \in \mathcal{B}(0,\pi)$. Then, the fractional square function $g^\gamma_{\alpha,\beta}$ defines a bounded (quasi-linear) operator from $L^{p(\cdot)}(0,\pi)$ into itself.*

Also Proposition 6.1 and the polarization formula (38) allow us to obtain the converse inequality for $g^\gamma_{\alpha,\beta}$.

Corollary 6.2. *Let $\alpha,\beta \geq -1/2$ and $\gamma > 0$.*

(*a*) *If $1 < p < \infty$ and $w \in A_p(0,\pi)$ then, for a certain $C > 0$,*

$$\|f\|_{L^p_w(0,\pi)} \leq C\|g^\gamma_{\alpha,\beta}(f)\|_{L^p_w(0,\pi)}, \quad f \in L^p_w(0,\pi).$$

(*b*) *If $p \in \mathcal{B}(0,\pi)$, then there exits $C > 0$ such that*

$$\|f\|_{L^{p(\cdot)}(0,\pi)} \leq C\|g^\gamma_{\alpha,\beta}(f)\|_{L^{p(\cdot)}(0,\pi)}, \quad f \in L^{p(\cdot)}(0,\pi).$$

Proof. We are going to prove (*b*), (*a*) can be deduced in a similar way.

For every $f \in L^{p(\cdot)}(0,\pi)$ and $g \in L^{p'(\cdot)}(0,\pi)$, we consider the bilinear operators

$$T(f,g) = \int_0^\pi f(\theta)\overline{g}(\theta)d\theta,$$

and

$$L(f,g) = \frac{2^{2\gamma}}{\Gamma(2\gamma)}\int_0^\pi\int_0^\infty t^\gamma \partial_t^\gamma P_t^{\alpha,\beta} f(\theta) t^\gamma \partial_t^\gamma P_t^{\alpha,\beta}(\overline{g})(\theta)\frac{dt}{t}d\theta.$$

By using Hölder's inequality in the variable exponent setting (see [17, Lemma 3.2.20]) we can see that T and L are bounded from $L^{p(\cdot)}(0,\pi) \times L^{p'(\cdot)}(0,\pi)$ into $\mathbb{C}$. Since $S_{\alpha,\beta}$ is a dense subspace of $L^{p(\cdot)}(0,\pi)$ and $L^{p'(\cdot)}(0,\pi)$ (Proposition 2.1), equality (38) holds for every $f \in L^{p(\cdot)}(0,\pi)$ and $g \in L^{p'(\cdot)}(0,\pi)$.

Let $f \in L^{p(\cdot)}(0,\pi)$. According to the norm conjugate formula ([17, Corollary 3.2.14]), by Proposition 6.1 we can write

$$\begin{aligned}
\|f\|_{L^{p(\cdot)}(0,\pi)} &\leq 2 \sup_{\substack{g\in L^{p'(\cdot)}(0,\pi)\\ \|g\|_{L^{p'(\cdot)}(0,\pi)}\leq 1}} \Big|\int_0^\pi f(\theta)\overline{g}(\theta)d\theta\Big| \\
&\leq C \sup_{\substack{g\in L^{p'(\cdot)}(0,\pi)\\ \|g\|_{L^{p'(\cdot)}(0,\pi)}\leq 1}} \Big|\int_0^\pi\int_0^\infty t^\gamma \partial_t^\gamma P_t^{\alpha,\beta} f(\theta) t^\gamma \partial_t^\gamma P_t^{\alpha,\beta}(\overline{g})(\theta)\frac{dt}{t}d\theta\Big| \\
&\leq C \sup_{\substack{g\in L^{p'(\cdot)}(0,\pi)\\ \|g\|_{L^{p'(\cdot)}(0,\pi)}\leq 1}} \int_0^\pi g^\gamma_{\alpha,\beta}(f)(\theta) g^\gamma_{\alpha,\beta}(\overline{g})(\theta)d\theta \\
&\leq C \sup_{\substack{g\in L^{p'(\cdot)}(0,\pi)\\ \|g\|_{L^{p'(\cdot)}(0,\pi)}\leq 1}} \|g^\gamma_{\alpha,\beta}(f)\|_{L^{p(\cdot)}(0,\pi)}\, \|g^\gamma_{\alpha,\beta}(\overline{g})\|_{L^{p'(\cdot)}(0,\pi)} \leq C\|g^\gamma_{\alpha,\beta}(f)\|_{L^{p(\cdot)}(0,\pi)}.
\end{aligned}$$

□

Remark 6.1. *Note that Proposition 6.1 together with Corollaries 6.1 and 6.2 tell us that the new norms $|||\cdot|||_{L^p_w(0,\pi)}$ and $|||\cdot|||_{L^{p(\cdot)}(0,\pi)}$ defined by*

$$|||f|||_{L^p_w(0,\pi)} = \|g^\gamma_{\alpha,\beta}(f)\|_{L^p_w(0,\pi)}, \quad f \in L^p_w(0,\pi),$$
$$|||f|||_{L^{p(\cdot)}(0,\pi)} = \|g^\gamma_{\alpha,\beta}(f)\|_{L^{p(\cdot)}(0,\pi)}, \quad f \in L^{p(\cdot)}(0,\pi),$$

are equivalent to $\|\cdot\|_{L^p_w(0,\pi)}$ *on* $L^p_w(0,\pi)$ *and to* $\|\cdot\|_{L^{p(\cdot)}(0,\pi)}$ *on* $L^{p(\cdot)}(0,\pi)$*, respectively, provided that the specified conditions are satisfied.*

Proof of Theorem 1.3. We first establish that $H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi) \subseteq T^{\gamma,k,p(\cdot)}_{\alpha,\beta}(0,\pi)$. Assume that $f, g \in S_{\alpha,\beta}$ are such that $f = \mathcal{L}^{-\gamma}_{\alpha,\beta} g$. We can write

$$\partial_t^k P_t^{\alpha,\beta}(\mathcal{L}^{-\gamma/2}_{\alpha,\beta} g) = (-1)^k \sum_{n=0}^{\infty} \frac{e^{-t\sqrt{\lambda_n^{\alpha,\beta}}}}{(\lambda_n^{\alpha,\beta})^{(\gamma-k)/2}} c_n^{\alpha,\beta}(g)\phi_n^{\alpha,\beta} = e^{i\pi\gamma}\partial_t^{k-\gamma} P_t^{\alpha,\beta} g, \quad t>0,$$

because $\partial_t^\delta e^{-at} = e^{i\pi\delta} a^\delta e^{-at}$, $\delta, a, t > 0$. Hence, we get

$$g^{\gamma,k}_{\alpha,\beta}(\mathcal{L}^{-\gamma/2}_{\alpha,\beta} g) = g^{k-\gamma}_{\alpha,\beta}(g). \tag{51}$$

From (51) and Corollaries 6.1 and 6.2 we deduce that, for every $f \in S_{\alpha,\beta}$,

$$\frac{1}{C}\|f\|_{H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)} \le \|g^{\gamma,k}_{\alpha,\beta}(f)\|_{L^{p(\cdot)}(0,\pi)} \le C\|f\|_{H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)}, \tag{52}$$

for a certain $C > 0$. Since $S_{\alpha,\beta}$ is a dense subspace of $H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)$, $g^{\gamma,k}_{\alpha,\beta}$ can be extended to $H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)$ as a bounded operator $\widetilde{g}^{\gamma,k}_{\alpha,\beta}$ from $H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)$ into $L^{p(\cdot)}(0,\pi)$. Moreover, (52) holds for every $f \in H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)$ when $g^{\gamma,k}_{\alpha,\beta}$ is replaced by $\widetilde{g}^{\gamma,k}_{\alpha,\beta}$.

We are going to see that $\widetilde{g}^{\gamma,k}_{\alpha,\beta} = g^{\gamma,k}_{\alpha,\beta}$. For every $N \in \mathbb{N}$, we define

$$\mathcal{G}^{\gamma,k,N}_{\alpha,\beta}(f)(\theta) = \Big(\int_{1/N}^{N} \big|t^{k-\gamma}\partial_t^k P_t^{\alpha,\beta} f(\theta)\big|^2 \frac{dt}{t}\Big)^{1/2}, \quad \theta \in (0,\pi).$$

Let $N \in \mathbb{N}$. From (52) it follows that $\mathcal{G}^{\gamma,k,N}_{\alpha,\beta}$ can be extended to $H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)$ as a bounded operator $\widetilde{\mathcal{G}}^{\gamma,k,N}_{\alpha,\beta}$ from $H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)$ into $L^{p(\cdot)}(0,\pi)$ and

$$\|\widetilde{\mathcal{G}}^{\gamma,k,N}_{\alpha,\beta}(f)\|_{L^{p(\cdot)}(0,\pi)} \le C\|f\|_{H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)}, \quad f \in H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi).$$

Note that C does not depend on N. Let $f \in H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)$. We choose a sequence $(f_n)_{n\in\mathbb{N}} \subseteq S_{\alpha,\beta}$ such that

$$f_n \longrightarrow f, \quad \text{as } n \to \infty, \text{ in } H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi).$$

Then,

$$\mathcal{G}^{\gamma,k,N}_{\alpha,\beta}(f_n) \longrightarrow \widetilde{\mathcal{G}}^{\gamma,k,N}_{\alpha,\beta}(f), \quad \text{as } n \to \infty, \text{ in } L^{p(\cdot)}(0,\pi).$$

Since, $L^{p(\cdot)}(0,\pi) \subseteq L^{p_-}(0,\pi)$, there exists a monotone function $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$\mathcal{G}^{\gamma,k,N}_{\alpha,\beta}(f_{\phi(n)})(\theta) \longrightarrow \widetilde{\mathcal{G}}^{\gamma,k,N}_{\alpha,\beta}(f)(\theta), \quad \text{as } n \to \infty, \quad \text{a.e. } \theta \in (0,\pi).$$

By proceeding as in (42) we deduce that

$$\mathcal{G}^{\gamma,k,N}_{\alpha,\beta}(f_{\phi(n)})(\theta) \longrightarrow \mathcal{G}^{\gamma,k,N}_{\alpha,\beta}(f)(\theta), \quad \text{as } n \to \infty, \quad \theta \in (0,\pi).$$

Then, $\widetilde{\mathcal{G}}^{\gamma,k,N}_{\alpha,\beta} = \mathcal{G}^{\gamma,k,N}_{\alpha,\beta}$ and

$$\|\mathcal{G}^{\gamma,k,N}_{\alpha,\beta}(f)\|_{L^{p(\cdot)}(0,\pi)} \le C\|f\|_{H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)}.$$

Since

$$\lim_{N\to\infty} \mathcal{G}^{\gamma,k,N}_{\alpha,\beta}(f)(\theta) = g^{\gamma,k}_{\alpha,\beta}(f)(\theta), \quad \theta \in (0,\pi),$$

Fatou's Lemma in variable exponent $L^{p(\cdot)}$-spaces (see [17, p. 77]) leads to

$$\|g^{\gamma,k}_{\alpha,\beta}(f)\|_{L^{p(\cdot)}(0,\pi)} \le C\|f\|_{H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)}. \tag{53}$$

From (52) we also deduce now that

$$\|f\|_{H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)} \le C\|g^{\gamma,k}_{\alpha,\beta}(f)\|_{L^{p(\cdot)}(0,\pi)}, \quad f \in H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi). \tag{54}$$

By (53) it follows that $H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)$ is contained in $T^{\gamma,k,p(\cdot)}_{\alpha,\beta}(0,\pi)$ and by Proposition 3.3

$$\|f\|_{T^{\gamma,k,p(\cdot)}_{\alpha,\beta}(0,\pi)} \le C\|f\|_{H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)}, \quad f \in H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi).$$

Suppose now that $f \in T^{\gamma,k,p(\cdot)}_{\alpha,\beta}(0,\pi)$. In order to show that $f \in H^{\gamma/2,p(\cdot)}_{\alpha,\beta}(0,\pi)$ we can follow the procedure developed in the proof of [6, Proposition 4.1]. Indeed, that method works because the following properties hold:

(i) There exists $C>0$ such that, for every $n\in\mathbb{N}$,

$$\|\phi_n^{\alpha,\beta}\|_{L^{p(\cdot)}(0,\pi)} \leq C(n+1)^{\alpha+\beta+5/2}.$$

Indeed, according to [17, Theorem 3.3.11], $L^{p_+}(0,\pi)$ is continuously contained in $L^{p(\cdot)}(0,\pi)$. Then, from [35, (3)] it follows that

$$\|\phi_n^{\alpha,\beta}\|_{L^{p(\cdot)}(0,\pi)} \leq C\|\phi_n^{\alpha,\beta}\|_{L^{p_+}(0,\pi)} \leq C(n+1)^{\alpha+\beta+5/2}, \quad n\in\mathbb{N}.$$

Assume that $h\in L^{p(\cdot)}(0,\pi)$. Hölder's inequality ([17, Lemma 3.2.20]) implies that

$$|c_n^{\alpha,\beta}(h)| \leq C(n+1)^{\alpha+\beta+5/2}\|h\|_{L^{p(\cdot)}(0,\pi)}, \quad n\in\mathbb{N}.$$

(ii) For every $\delta>0$, we define $f_\delta = P_\delta^{\alpha,\beta}(f)$ and

$$F_\delta = \sum_{n=0}^{\infty} (\lambda_n^{\alpha,\beta})^{\gamma/2} e^{-\delta\sqrt{\lambda_n^{\alpha,\beta}}} c_n^{\alpha,\beta}(f_\delta)\phi_n^{\alpha,\beta}.$$

Property (i) implies that $F_\delta\in L^{p(\cdot)}(0,\pi)$ and $f_\delta = \mathcal{L}_{\alpha,\beta}^{-\gamma/2}F_\delta \in H_{\alpha,\beta}^{\gamma/2,p(\cdot)}(0,\pi)$, $\delta>0$. We choose $\ell\in\mathbb{N}$ such that $2(\ell-\gamma)>1$ and $\ell>k$. (54) allows us to write

$$\|F_\delta\|_{L^{p(\cdot)}(0,\pi)} = \|f_\delta\|_{H_{\alpha,\beta}^{\gamma/2,p(\cdot)}(0,\pi)} \leq C\|g_{\alpha,\beta}^{\gamma,\ell}(f_\delta)\|_{L^{p(\cdot)}(0,\pi)}, \quad \delta>0.$$

(iii) As in [6, Proposition 2.6] we can prove that

$$\|g_{\alpha,\beta}^{\gamma,\ell}(f)\|_{L^{p(\cdot)}(0,\pi)} \leq C\|g_{\alpha,\beta}^{\gamma,k}(f)\|_{L^{p(\cdot)}(0,\pi)}.$$

Moreover, straightforward manipulations lead to

$$g_{\alpha,\beta}^{\gamma,\ell}(f_\delta)(\theta) \leq g_{\alpha,\beta}^{\gamma,\ell}(f)(\theta), \quad \theta\in(0,\pi),\ \delta>0,$$

because $2(\ell-\gamma)>1$. Then, we obtain

$$\|F_\delta\|_{L^{p(\cdot)}(0,\pi)} \leq C\|g_{\alpha,\beta}^{\gamma,k}(f)\|_{L^{p(\cdot)}(0,\pi)}, \quad \delta>0.$$

(iv) By using Banach-Alaoglu's Theorem, Proposition 3.3 and [17, Theorem 3.2.13] we conclude that $f=\mathcal{L}_{\alpha,\beta}^{-\gamma/2}F$, for a certain $F\in L^{p(\cdot)}(0,\pi)$ such that

$$\|F\|_{L^{p(\cdot)}(0,\pi)} \leq C\|g_{\alpha,\beta}^{\gamma,k}(f)\|_{L^{p(\cdot)}(0,\pi)}.$$

Thus, we prove that $f\in H_{\alpha,\beta}^{\gamma/2,p(\cdot)}(0,\pi)$ and

$$\|f\|_{H_{\alpha,\beta}^{\gamma/2,p(\cdot)}(0,\pi)} \leq C\|f\|_{T_{\alpha,\beta}^{\gamma,k,p(\cdot)}(0,\pi)}.$$

□

7. Proof of Theorem 1.4

In order to establish this theorem we use the ideas developed in the proof of [33, Proposition 4.3]. First of all, we introduce some spectral multipliers of Hörmander type, associated with the Jacobi operator.

Lemma 7.1. *Let $\gamma>0$, $1<p<\infty$, $w\in A_p(0,\pi)$ and $\alpha,\beta\geq -1/2$ such that $\alpha+\beta\neq -1$. We consider, for each $t>0$, the functions*

- $m_\varepsilon^\ell(t) = \displaystyle\sum_{j=0}^{\ell} \frac{\varepsilon_j 2^{j\gamma}}{(t+1)^\gamma}\mathfrak{a}\left(\frac{t}{2^{j-1}}\right)$, $\quad \ell\in\mathbb{N}$ *and* $\varepsilon=(\varepsilon_j)_{j=0}^{\ell}\in\{-1,1\}^{\ell+1}$.
- $M(t) = \left(\dfrac{t+1}{t}\right)^\gamma \phi(t)$, *where $\phi\in C^\infty(0,\infty)$ is such that $\phi(t)=0$, $0<t<\lambda_0^{\alpha,\beta}/2$; and $\phi(t)=1$, $t\geq\lambda_0^{\alpha,\beta}$.*

Then, the spectral multipliers $m_\varepsilon^\ell(\mathcal{L}_{\alpha,\beta})$ and $M(\mathcal{L}_{\alpha,\beta})$ define bounded operators in $L_w^p(0,\pi)$. Moreover,

$$\sup_{\ell,\varepsilon}\|m_\varepsilon^\ell(\mathcal{L}_{\alpha,\beta})\|_{L_w^p(0,\pi)\to L_w^p(0,\pi)} < \infty.$$

Proof. By Proposition 3.1, it is enough to notice that, for every $k \in \mathbb{N}$, there exists $C > 0$ such that

$$\sup_{t>0} \left| t^k \frac{d^k}{dt^k} m_\varepsilon^\ell(t) \right| \leq C, \quad \ell \in \mathbb{N} \text{ and } \varepsilon \in \{-1,1\}^{\ell+1}.$$

and

$$\sup_{t>0} \left| t^k \frac{d^k}{dt^k} M(t) \right| \leq C.$$

□

Proof of Theorem 1.4; the case of $H_{\alpha,\beta}^{\gamma,p(\cdot)}(0,\pi) \subseteq F_{\alpha,\beta}^{\gamma,2,p(\cdot)}(0,\pi)$. Let $\varepsilon = (\varepsilon_j)_{j=0}^\ell \in \{-1,1\}^{\ell+1}$ with $\ell \in \mathbb{N}$. We can write,

$$\begin{aligned}
\sum_{n=0}^\infty m_\varepsilon^\ell(\lambda_n^{\alpha,\beta})(\lambda_n^{\alpha,\beta}+1)^\gamma c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} &= \sum_{n=0}^\infty (\lambda_n^{\alpha,\beta}+1)^\gamma c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} \sum_{j=0}^\ell \frac{\varepsilon_j 2^{j\gamma}}{(\lambda_n^{\alpha,\beta}+1)^\gamma} \mathfrak{a}\left(\frac{\lambda_n^{\alpha,\beta}}{2^{j-1}}\right) \\
&= \sum_{j=0}^\ell \varepsilon_j 2^{j\gamma} \sum_{n=0}^\infty \mathfrak{a}\left(\frac{\lambda_n^{\alpha,\beta}}{2^{j-1}}\right) c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} \\
&= \sum_{j=0}^\ell \varepsilon_j 2^{j\gamma} \Phi_j^{\alpha,\beta}(f), \quad f \in L_w^p(0,\pi). \qquad (55)
\end{aligned}$$

Note that the serie $\sum_{n=0}^\infty$ is actually a finite sum. From Lemma 7.1, it follows that

$$\begin{aligned}
\Big\| \sum_{j=0}^\ell \varepsilon_j 2^{j\gamma} \Phi_j^{\alpha,\beta}(f) \Big\|_{L_w^p(0,\pi)} &= \Big\| \sum_{n=0}^\infty m_\varepsilon^\ell(\lambda_n^{\alpha,\beta}) M(\lambda_n^{\alpha,\beta})(\lambda_n^{\alpha,\beta})^\gamma c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} \Big\|_{L_w^p(0,\pi)} \\
&\leq C \Big\| \sum_{n=0}^\infty (\lambda_n^{\alpha,\beta})^\gamma c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} \Big\|_{L_w^p(0,\pi)}, \quad f \in L_w^p(0,\pi),
\end{aligned}$$

provided that $\sum_{n=0}^\infty (\lambda_n^{\alpha,\beta})^\gamma c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} \in L_w^p(0,\pi)$. Also, we get

$$\Big\| \sum_{j=0}^\ell \varepsilon_j 2^{j\gamma} \Phi_j^{\alpha,\beta}(f) \Big\|_{L_w^p(0,\pi)} \leq C \Big\| \sum_{\substack{n\in\mathbb{N} \\ \lambda_n^{\alpha,\beta} \leq 2^\ell}} (\lambda_n^{\alpha,\beta})^\gamma c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} \Big\|_{L_w^p(0,\pi)}, \quad f \in L_w^p(0,\pi). \qquad (56)$$

Observe that, the constant $C > 0$ does not depend on ε or ℓ.

By using Khintchine's inequality ([45, Vol. I, p. 213]) from (56) we deduce that,

$$\Big\| \Big(\sum_{j=0}^\ell (2^{j\gamma} |\Phi_j^{\alpha,\beta}(f)|)^2 \Big)^{1/2} \Big\|_{L_w^p(0,\pi)} \leq C \Big\| \sum_{\substack{n\in\mathbb{N} \\ \lambda_n^{\alpha,\beta} \leq 2^\ell}} (\lambda_n^{\alpha,\beta})^\gamma c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} \Big\|_{L_w^p(0,\pi)}, \quad f \in L_w^p(0,\pi),$$

where $C > 0$ does not depend on ℓ. According to [15, Theorem 1.3], there exists $C > 0$ such that

$$\Big\| \Big(\sum_{j=0}^\ell (2^{j\gamma} |\Phi_j^{\alpha,\beta}(f)|)^2 \Big)^{1/2} \Big\|_{L^{p(\cdot)}(0,\pi)} \leq C \Big\| \sum_{\substack{n\in\mathbb{N} \\ \lambda_n^{\alpha,\beta} \leq 2^\ell}} (\lambda_n^{\alpha,\beta})^\gamma c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} \Big\|_{L^{p(\cdot)}(0,\pi)}, \quad f \in L^{p(\cdot)}(0,\pi).$$

We have taken into account that:

(*a*) For every $n \in \mathbb{N}$, the mapping $f \longmapsto c_n^{\alpha,\beta}(f)$ is bounded from $L^{p(\cdot)}(0,\pi)$ into $\mathbb{C}$.

(*b*) For every $j \in \mathbb{N}$, the mapping $f \longmapsto \Phi_j^{\alpha,\beta}(f)$ is bounded from $L^{p(\cdot)}(0,\pi)$ into itself (Proposition 3.1). Also, we used that $\sqrt{a^2+b^2} \leq a+b$, $a,b \geq 0$.

(*c*) $S_{\alpha,\beta}$ is dense in $L^{p(\cdot)}(0,\pi)$ (Proposition 2.1).

Taking $\ell \to \infty$, Proposition 3.5 allow us to deduce that

$$\Big\| \Big(\sum_{j=0}^\infty (2^{j\gamma} |\Phi_j^{\alpha,\beta}(f)|)^2 \Big)^{1/2} \Big\|_{L^{p(\cdot)}(0,\pi)} \leq C \|f\|_{H_{\alpha,\beta}^{\gamma,p(\cdot)}(0,\pi)}, \quad f \in H_{\alpha,\beta}^{\gamma,p(\cdot)}(0,\pi).$$

□

Next, we prove the converse inclusion of Theorem 1.4. As before, we need to study previously some Jacobi spectral multipliers. It is convenient to introduce the following notation. We define,

$$\mathbb{N}_s = \{4\ell + s \ : \ \ell \in \mathbb{N}\}\backslash\{0\}, \quad s = 0,1,2,3.$$

Also we consider the function

$$\mathfrak{b}(t) = \mathfrak{a}(t/2) + \mathfrak{a}(t) + \mathfrak{a}(2t), \quad t > 0.$$

Note that $\operatorname{supp}\mathfrak{b} \subseteq [1/4,4]$ and $\mathfrak{b}(t) = 1$, $t \in [1/2,2]$, because $\mathfrak{a}(t) + \mathfrak{a}(2t) = 1$, $t \in [1/2,1]$, and $\operatorname{supp}\mathfrak{a} \subseteq [1/2,2]$.

Lemma 7.2. *Let $1 < p < \infty$, $w \in A_p(0,\pi)$ and $\alpha,\beta \geq -1/2$ such that $\alpha+\beta \neq -1$. We consider, for each $t > 0$, the functions*

- $m^\ell_{\varepsilon,s}(t) = \displaystyle\sum_{j=0,\ j\in\mathbb{N}_s}^{\ell} \varepsilon_j \mathfrak{b}\left(\frac{t}{2^{j-1}}\right), \quad s = 0,1,2,3, \quad \ell \in \mathbb{N}$ *and* $\varepsilon = (\varepsilon_j)_{j=0}^\ell \in \{-1,1\}^{\ell+1}$;
- $M_\ell(t) = \displaystyle\sum_{j=0}^{\ell} \frac{2^{j\gamma}}{(t+1)^\gamma}\mathfrak{a}\left(\frac{t}{2^{j-1}}\right), \quad \ell \in \mathbb{N}$;
- $R_\ell(t) = \phi/M_\ell(t)$, *where ϕ is as in Lemma 7.1;*
- $R(t) = \left(\dfrac{t}{t+1}\right)^\gamma$.

Then, the spectral multipliers $m^\ell_{\varepsilon,s}(\mathcal{L}_{\alpha,\beta})$, $M_\ell(\mathcal{L}_{\alpha,\beta})$, $R_\ell(\mathcal{L}_{\alpha,\beta})$ and $R(\mathcal{L}_{\alpha,\beta})$ define bounded operators in $L^p_w(0,\pi)$. Moreover,

$$\sup_{s,\ell,\varepsilon} \|m^\ell_{\varepsilon,s}(\mathcal{L}_{\alpha,\beta})\|_{L^p_w(0,\pi)\to L^p_w(0,\pi)} < \infty,$$

and

$$\sup_\ell \Big(\|M_\ell(\mathcal{L}_{\alpha,\beta})\|_{L^p_w(0,\pi)\to L^p_w(0,\pi)} + \|R_\ell(\mathcal{L}_{\alpha,\beta})\|_{L^p_w(0,\pi)\to L^p_w(0,\pi)}\Big) < \infty.$$

Proof. Again, by Proposition 3.1, it suffices to take into account that, for every $k \in \mathbb{N}$ there exists $C > 0$ for which

$$\sup_{t\in(0,\infty)} \left|t^k \frac{d^k}{dt^k} m^\ell_{\varepsilon,s}(t)\right| \leq C,$$

where $C > 0$ does not depend on s, ℓ or ε. Also, $M_\ell = m^\ell_\varepsilon$ in Lemma 7.1, for $\varepsilon = (1)_{j=0}^\ell$. Finally, for every $k \in \mathbb{N}$, there exists $C > 0$ such that

$$\sup_{t\geq\lambda_0^{\alpha,\beta}/2} \left|t^k \frac{d^k}{dt^k}\frac{1}{M_\ell(t)}\right| \leq C,$$

where $C > 0$ does not depend on ℓ. □

Proof of Theorem 1.4; the case of $F^{\gamma,2,p(\cdot)}_{\alpha,\beta}(0,\pi) \subseteq H^{\gamma,p(\cdot)}_{\alpha,\beta}(0,\pi)$. Suppose that $s \in \{0,1,2,3\}$ and $n \in \mathbb{N}\backslash\{0\}$. We define

$$g^{\alpha,\beta}_{s,\ell}(f) = \sum_{j=0,\ j\in\mathbb{N}_s}^{\ell} 2^{j\gamma}\Phi^{\alpha,\beta}_j(f), \quad \ell \in \mathbb{N} \text{ and } f \in L^1(0,\pi).$$

There exists at most an unique $j_n \in \mathbb{N}_s$ such that $\lambda^{\alpha,\beta}_n \in [2^{j_n-2}, 2^{j_n})$. Hence,

$$\mathfrak{b}\left(\frac{\lambda^{\alpha,\beta}_n}{2^{j_n-1}}\right) = 1 \qquad \text{and} \qquad \mathfrak{b}\left(\frac{\lambda^{\alpha,\beta}_n}{2^{j-1}}\right) = \mathfrak{a}\left(\frac{\lambda^{\alpha,\beta}_n}{2^{j-1}}\right) = 0, \quad j \in \mathbb{N}_s,\ j \neq j_n.$$

Observe that $m^\ell_{\varepsilon,s}(\lambda^{\alpha,\beta}_n) = \varepsilon_{j_n}$, provided that $j_n \leq \ell$, and $m^\ell_{\varepsilon,s}(\lambda^{\alpha,\beta}_n) = 0$, otherwise. We can write

$$g^{\alpha,\beta}_{s,\ell}(f) = \sum_{j=0,\ j\in\mathbb{N}_s}^{\ell} 2^{j\gamma}\sum_{n=0}^\infty \mathfrak{a}\left(\frac{\lambda^{\alpha,\beta}_n}{2^{j-1}}\right) c^{\alpha,\beta}_n(f)\phi^{\alpha,\beta}_n = \sum_{n=0}^\infty a_n\ c^{\alpha,\beta}_n(f)\phi^{\alpha,\beta}_n, \quad f \in L^1(0,\pi).$$

where $a_n = 2^{j_n\gamma}\mathfrak{a}\left(\lambda_n^{\alpha,\beta}/2^{j_n-1}\right)$, if $j_n \le \ell$, and $a_n = 0$, otherwise. Note that the above serie is actually a finite sum. Also, we have that

$$\begin{aligned} m^\ell_{\varepsilon,s}(\mathcal{L}_{\alpha,\beta})g^{\alpha,\beta}_{s,\ell}(f) &= \sum_{n=0}^\infty m^\ell_{\varepsilon,s}\left(\lambda_n^{\alpha,\beta}\right) a_n\ c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} = \sum_{n=0}^\infty \varepsilon_{j_n} a_n\ c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} \\ &= \sum_{j=0,\ j\in\mathbb{N}_s}^{\ell} 2^{j\gamma}\varepsilon_j \sum_{n=0}^\infty \mathfrak{a}\left(\frac{\lambda_n^{\alpha,\beta}}{2^{j-1}}\right) c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} = \sum_{j=0,\ j\in\mathbb{N}_s}^{\ell} 2^{j\gamma}\varepsilon_j \Phi_j^{\alpha,\beta}(f). \end{aligned}$$

Then,

$$m^\ell_{\varepsilon,s}(\mathcal{L}_{\alpha,\beta})m^\ell_{\varepsilon,s}(\mathcal{L}_{\alpha,\beta})g^{\alpha,\beta}_{s,\ell}(f) = \sum_{n=0}^\infty a_n\ c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} = g^{\alpha,\beta}_{s,\ell}(f).$$

Assume that $1 < p < \infty$ and $w \in A_p(0,\infty)$. From Lemma 7.2 we get

$$\|g^{\alpha,\beta}_{s,\ell}(f)\|_{L^p_w(0,\pi)} \le C\|m^\ell_{\varepsilon,s}(\mathcal{L}_{\alpha,\beta})g^{\alpha,\beta}_{s,\ell}(f)\|_{L^p_w(0,\pi)} \le C\Big\|\sum_{j=0,\ j\in\mathbb{N}_s}^{\ell} 2^{j\gamma}\varepsilon_j\Phi_j^{\alpha,\beta}(f)\Big\|_{L^p_w(0,\pi)}, \quad f\in L^p_w(0,\pi),$$

where $C>0$ does not depend on ε or ℓ. By using Khintchine's inequality argument we obtain

$$\|g^{\alpha,\beta}_{s,\ell}(f)\|_{L^p_w(0,\pi)} \le C\Big\|\sum_{j=0,\ j\in\mathbb{N}_s}^{\ell} (2^{j\gamma}\varepsilon_j|\Phi_j^{\alpha,\beta}(f)|)^2)^{1/2}\Big\|_{L^p_w(0,\pi)}, \quad f\in L^p_w(0,\pi),$$

where $C>0$ does not depend on ℓ. According to [15, Theorem 1.3],

$$\|g^{\alpha,\beta}_{s,\ell}(f)\|_{L^{p(\cdot)}(0,\pi)} \le C\Big\|\sum_{j=0,\ j\in\mathbb{N}_s}^{\ell} (2^{j\gamma}\varepsilon_j|\Phi_j^{\alpha,\beta}(f)|)^2)^{1/2}\Big\|_{L^{p(\cdot)}(0,\pi)}, \quad f\in S_{\alpha,\beta},$$

where $C>0$ does not depend on ℓ. As in the proof of the first inclusion we obtain

$$\|g^{\alpha,\beta}_{s,\ell}(f)\|_{L^{p(\cdot)}(0,\pi)} \le C\Big\|\sum_{j=0,\ j\in\mathbb{N}_s}^{\ell} (2^{j\gamma}\varepsilon_j|\Phi_j^{\alpha,\beta}(f)|)^2)^{1/2}\Big\|_{L^{p(\cdot)}(0,\pi)}, \quad f\in L^{p(\cdot)}(0,\pi). \tag{57}$$

According to (55) we have that, for every $f\in L^1(0,\pi)$,

$$\sum_{n=0}^\infty M_\ell(\lambda_n^{\alpha,\beta})(\lambda_n^{\alpha,\beta}+1)^\gamma c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} = \sum_{n=0,\lambda_n^{\alpha,\beta}\le 2^\ell}^\infty \varepsilon_{j_n} c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} = \sum_{j=0}^{\ell} 2^{j\gamma}\Phi_j^{\alpha,\beta}(f).$$

By using (57), Lemma 7.2 and [15, Theorem 1.3] we can write

$$\begin{aligned} &\Big\|\sum_{n=0,\lambda_n^{\alpha,\beta}\le 2^\ell}^\infty (\lambda_n^{\alpha,\beta})^\gamma c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta}\Big\|_{L^{p(\cdot)}(0,\pi)} \\ &\quad= \Big\|\sum_{n=0}^\infty R(\lambda_n^{\alpha,\beta})(\lambda_n^{\alpha,\beta}+1)^\gamma R_\ell(\lambda_n^{\alpha,\beta})M_\ell(\lambda_n^{\alpha,\beta})c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta}\Big\|_{L^{p(\cdot)}(0,\pi)} \\ &\quad\le C\Big\|\sum_{j=0}^{\ell} 2^{j\gamma}\Phi_j^{\alpha,\beta}(f)\Big\|_{L^{p(\cdot)}(0,\pi)} \\ &\quad\le C\Big(\sum_{s=0}^{3}\Big\|\sum_{j=0,\ j\in\mathbb{N}_s}^{\ell} 2^{j\gamma}\Phi_j^{\alpha,\beta}(f)\Big\|_{L^{p(\cdot)}(0,\pi)} + \|\Phi_0^{\alpha,\beta}\|_{L^{p(\cdot)}(0,\pi)}\Big) \\ &\quad\le C\Big(\sum_{s=0}^{3}\Big\|\Big(\sum_{j=0,\ j\in\mathbb{N}_s}^{\ell} (2^{j\gamma}|\Phi_j^{\alpha,\beta}(f)|)^2\Big)^{1/2}\Big\|_{L^{p(\cdot)}(0,\pi)} + \|\Phi_0^{\alpha,\beta}\|_{L^{p(\cdot)}(0,\pi)}\Big) \\ &\quad\le C\Big\|\Big(\sum_{j=0}^{\ell}(2^{j\gamma}|\Phi_j^{\alpha,\beta}(f)|)^2\Big)^{1/2}\Big\|_{L^{p(\cdot)}(0,\pi)}, \quad f\in L^{p(\cdot)}(0,\pi). \qquad (58) \end{aligned}$$

Suppose now that $f = \sum_{n=m}^{\ell} c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta}$, where $m, \ell \in \mathbb{N}$, $m \leq \ell$. Since $\operatorname{supp} \mathfrak{a} \subseteq [1/2, 2]$, we have that

$$\Phi_j^{\alpha,\beta} = \sum_{n=0}^{\infty} \mathfrak{a}\left(\frac{\lambda_n^{\alpha,\beta}}{2^{j-1}}\right) c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} = \sum_{n=m}^{\ell} \mathfrak{a}\left(\frac{\lambda_n^{\alpha,\beta}}{2^{j-1}}\right) c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} = 0,$$

provided that $j > 2 + \log_2 \ell$ or $j < \log_2 m$. Then, from (58) we deduce that

$$\Big\| \sum_{n=m}^{\ell} (\lambda_n^{\alpha,\beta})^{\gamma} c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta} \Big\|_{L^{p(\cdot)}(0,\pi)} \leq C \Big\| \Big(\sum_{j=\log_2 m}^{2+\log_2 \ell} (2^{j\gamma} |\Phi_j^{\alpha,\beta}(f)|)^2 \Big)^{1/2} \Big\|_{L^{p(\cdot)}(0,\pi)}. \tag{59}$$

Let $f \in F_{\alpha,\beta}^{\gamma,2,p(\cdot)}(0,\pi)$. By (59), the series $\sum_{n=m}^{\ell} (\lambda_n^{\alpha,\beta})^{\gamma} c_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta}$ converges in $L^{p(\cdot)}(0,\pi)$. Hence, $f \in H_{\alpha,\beta}^{\gamma,p(\cdot)}(0,\pi)$ and by (58) and Proposition 3.5, we conclude that

$$\|f\|_{H_{\alpha,\beta}^{\gamma,p(\cdot)}(0,\pi)} \leq C \|f\|_{F_{\alpha,\beta}^{\gamma,2,p(\cdot)}(0,\pi)}.$$

□

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References

[1] C. Balderrama and W. Urbina, *Fractional integration and fractional differentiation for Jacobi expansions*, Divulg. Mat., 15 (2007), pp. 93–113.

[2] H. Berens, P. L. Butzer, and U. Westphal, *Representations of fractional powers of infinitesimal generators of semigroups*, Bull. Amer. Math. Soc., 74 (1968), pp. 191–196.

[3] J. J. Betancor, A. J. Castro, J. Curbelo, J. C. Fariña, and L. Rodríguez-Mesa, *Square functions in the Hermite setting for functions with values in UMD spaces*, Ann. Mat. Pura Appl. (4), 193 (2014), pp. 1397–1430.

[4] J. J. Betancor, J. C. Fariña, L. Rodríguez-Mesa, R. Testoni, and J. L. Torrea, *Fractional square functions and potential spaces II.* Preprint 2012.

[5] J. J. Betancor, J. C. Fariña, L. Rodrríguez-Mesa, R. Testoni, and J. L. Torrea, *A choice of Sobolev spaces associated with ultraspherical expansions*, Publ. Mat., 54 (2010), pp. 221–242.

[6] ———, *Fractional square functions and potential spaces*, J. Math. Anal. Appl., 386 (2012), pp. 487–504.

[7] A. Bonami and J.-L. Clerc, *Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques*, Trans. Amer. Math. Soc., 183 (1973), pp. 223–263.

[8] B. Bongioanni and J. L. Torrea, *Sobolev spaces associated to the harmonic oscillator*, Proc. Indian Acad. Sci. Math. Sci., 116 (2006), pp. 337–360.

[9] ———, *What is a Sobolev space for the Laguerre function systems?*, Studia Math., 192 (2009), pp. 147–172.

[10] D. Buraczewski, T. Martinez, J. L. Torrea, and R. Urban, *On the Riesz transform associated with the ultraspherical polynomials*, J. Anal. Math., 98 (2006), pp. 113–143.

[11] F. Cacciafesta and P. D'Ancona, *Weighted L^p estimates for powers of selfadjoint operators*, Adv. Math., 229 (2012), pp. 501–530.

[12] A. J. Castro, A. Nowak, and T. Szarek, *Riesz-Jacobi transforms as principal value integrals.* Preprint 2014 (arXiv:1405.7069v1).

[13] Ó. Ciaurri, A. Nowak, and K. Stempak, *Jacobi transplantation revisited*, Math. Z., 257 (2007), pp. 355–380.

[14] W. C. Connett and A. L. Schwartz, *A multiplier theorem for Jacobi expansions*, Studia Math., 52 (1974/75), pp. 243–261.

[15] D. Cruz-Uribe, A. Fiorenza, J. M. Martell, and C. Pérez, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math., 31 (2006), pp. 239–264.

[16] L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl., 7 (2004), pp. 245–253.

[17] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, vol. 2017 of Lecture Notes in Mathematics, Springer, Heidelberg, 2011.

[18] L. Diening and M. Růžička, *Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics*, J. Reine Angew. Math., 563 (2003), pp. 197–220.

[19] J. Duoandikoetxea, *Fourier analysis*, vol. 29 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2001. Translated and revised from the 1995 Spanish original by David Cruz-Uribe.

[20] G. Gasper and W. Trebels, *Multiplier criteria of Marcinkiewicz type for Jacobi expansions*, Trans. Amer. Math. Soc., 231 (1977), pp. 117–132.

[21] R. Hunt, B. Muckenhoupt, and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc., 176 (1973), pp. 227–251.

[22] R. A. Kerman, *Strong and weak weighted convergence of Jacobi series*, J. Approx. Theory, 88 (1997), pp. 1–27.

[23] G. Kyriazis, P. Petrushev, and Y. Xu, *Jacobi decomposition of weighted Triebel-Lizorkin and Besov spaces*, Studia Math., 186 (2008), pp. 161–202.

[24] B. Langowski, *Sobolev spaces associated with Jacobi expansions*, J. Math. Anal. Appl., 420 (2014), pp. 1533–1551.
[25] Z. Li, *Conjugate Jacobi series and conjugate functions*, J. Approx. Theory, 86 (1996), pp. 179–196.
[26] Z. Li, F. Song, and K. Li, *Characterizations of smoothness of functions in terms of their Jacobi expansions*, Results Math., 59 (2011), pp. 13–34.
[27] J.-L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math., (1964), pp. 5–68.
[28] T. Martinez, *Multipliers of Laplace transform type for ultraspherical expansions*, Math. Nachr., 281 (2008), pp. 978–988.
[29] S. Meda, *A general multiplier theorem*, Proc. Amer. Math. Soc., 110 (1990), pp. 639–647.
[30] B. Muckenhoupt, *Mean convergence of Jacobi series*, Proc. Amer. Math. Soc., 23 (1969), pp. 306–310.
[31] ———, *Transplantation theorems and multiplier theorems for Jacobi series*, Mem. Amer. Math. Soc., 64 (1986), pp. iv+86.
[32] B. Muckenhoupt and E. M. Stein, *Classical expansions and their relation to conjugate harmonic functions*, Trans. Amer. Math. Soc., 118 (1965), pp. 17–92.
[33] F. Narcowich, P. Petrushev, and J. Ward, *Decomposition of Besov and Triebel-Lizorkin spaces on the sphere*, J. Funct. Anal., 238 (2006), pp. 530–564.
[34] A. Nowak and L. Roncal, *Potential operators associated with Jacobi and Fourier–Bessel expansions*, J. Math. Anal. Appl., 422 (2015), pp. 148–184.
[35] A. Nowak and P. Sjögren, *Calderón-Zygmund operators related to Jacobi expansions*, J. Fourier Anal. Appl., 18 (2012), pp. 717–749.
[36] ———, *Sharp estimates of the Jacobi heat kernel*, Studia Math., 218 (2013), pp. 219–244.
[37] A. Nowak, P. Sjögren, and T. Szarek, *Analysis related to all admissible type parameters in Jacobi setting.* To appear in Constr. Approx. (arXiv:1211.3270v1).
[38] J. L. Rubio de Francia, F. J. Ruiz, and J. L. Torrea, *Calderón-Zygmund theory for operator-valued kernels*, Adv. in Math., 62 (1986), pp. 7–48.
[39] S. Samko, *On some classical operators of variable order in variable exponent spaces*, in Analysis, partial differential equations and applications, vol. 193 of Oper. Theory Adv. Appl., Birkhäuser Verlag, Basel, 2009, pp. 281–301.
[40] C. Segovia and R. L. Wheeden, *On certain fractional area integrals*, J. Math. Mech., 19 (1969/1970), pp. 247–262.
[41] E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Annals of Mathematics Studies, No. 63, Princeton University Press, Princeton, N.J., 1970.
[42] K. Stempak, *Jacobi conjugate expansions*, Studia Sci. Math. Hungar., 44 (2007), pp. 117–130.
[43] G. Szegő, *Orthogonal polynomials*, American Mathematical Society, Providence, R.I., fourth ed., 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.
[44] B. Wróbel, *Multivariate spectral multipliers for tensor product orthogonal expansions*, Monatsh. Math., 168 (2012), pp. 125–149.
[45] A. Zygmund, *Trigonometric series. 2nd ed. Vols. I, II*, Cambridge University Press, New York, 1959.

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