Cross-bifix-free sets via Motzkin paths generation

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Abstract

Cross-bifix-free sets are sets of words such that no prefix of any word is a suffix of any other word. In this paper, we introduce a general constructive method for the sets of cross-bifix-free q-ary words of fixed length. It enables us to determine a cross-bifix-free words subset which has the property to be non-expandable.

1 Introduction

A *cross-bifix-free set* of words (also called *cross-bifix-free code*) is a set where, given any two words over an alphabet, possibly the same, any prefix of the first one is not a suffix of the second one and vice-versa. Cross-bifix-free sets are involved in the study of frame synchronization which is an essential requirement in a digital communication systems to establish and maintain a connection between a transmitter and a receiver.

Analytical approaches to the synchronization acquisition process and methods for the construction of sequences with the best aperiodic autocorrelation properties [1, 2, 3, 4] have been the subject of numerous analyses in the digital transmission.

The historical engineering approach started with the introduction of bifix, a name proposed by J. L. Massey as acknowledged in [5]. It denotes a subsequence that is both a prefix and suffix of a longer observed sequence.

In [4] the notion of a *distributed sequences* is introduced, where the synchronization word is not a contiguous sequence of symbols but is instead interleaved into the data stream. In [6] is showed that the distributed sequence entails a simultaneous search for a set of synchronization words. Each word in the set of sequences is required to be bifix-free. In addition, they arises a new requirement that no prefix of any length of any word in the set is a suffix of any other word in the set. This property of the set of synchronization words was termed as *cross-bifix-free.*

The problem of determining such sets is also related to several other scientific applications, for instance in pattern matching [7] and automata theory [8].

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Several methods for constructing cross-bifix-free sets have been recently proposed as in [9, 10, 11]. In particular, once the cardinality q of the alphabet and the length n of the words are fixed, a matter is the construction of a cross-bifix-free set with the cardinality as large as possible. An interesting method has been proposed in [9] for words on a binary alphabet. This specific construction reveals interesting connections to the Fibonacci sequence of numbers. In a recent paper [11] the authors revisit the construction in [9] and generalize it obtaining cross-bifix-free sets having greater cardinality over an alphabet of any size q. They also show that their cross-bifix-free sets have a cardinality close to the maximum possible. To our knowledge this is the best result in the literature about the greatest size of cross-bifix-free sets.

For the sake of completeness we note that an intermediate step between the original method [9] and its generalization [11] has been proposed in [10] and it is constituted by a different construction of binary cross-bifix-free sets based on lattice paths which allows to obtain greater values of cardinality if compared to the ones in [9].

In this study, we revisit the construction in [10]. We give a new construction of cross-bifix-free sets that generalizes the construction of [10] in order to extend the construction to q-ary alphabets for any q, $q > 2$. This approach enables us to obtain cross-bifix-free sets having greater cardinality than the ones presented in [11], for the initial values of n. This new result extends the theory of cross-bifix-free sets and it could be used to improve some technical applications.

This paper is organized as follows. In Section 2 we give some preliminaries and describe the adopted notation. In Section 3 we present a new construction of cross-bifix-free sets in the q-ary alphabet and in Section 4 we analyze the sizes of the sets of our construction in comparison to the ones in the literature.

2 Basic definitions and notations

Let $\mathbb{Z}_q = \{0, 1, \cdots, q-1\}$ be an alphabet of q elements. A (finite) sequence of elements in $\mathbb{Z}_q$ is called (finite) *word*. The set of all words over $\mathbb{Z}_q$ having length n is denoted by $\mathbb{Z}_q^n$. A consecutive sequence of m element $a \in \mathbb{Z}_q$ is denoted by the short form a^m. Let $w \in \mathbb{Z}_q^n$, then $|w|_a$ denotes the number of occurrences of a in w, being $a \in \mathbb{Z}_q$. Let $w = uzv$ then u is called a *prefix* of w and v is called a *suffix* of w. A *bifix* of w is a subsequence of w that is both its prefix and suffix.

A word $w \in \mathbb{Z}_q^n$ is said to be *bifix-free* or *unbordered* [12] if and only if no prefix of w is also a suffix of w. Therefore, w is bifix-free if and only if $w \neq uzu$, being u any necessarily non-empty word and z any word. Obviously, a necessary condition for w to be bifix-free is that the first and the last letters of w must be different.

Example 2.1 *In* $\mathbb{Z}_2 = \{0, 1\}$*, the word* 111010100 *of length* $n = 9$ *is bifix-free, while the word* 101001010 *contains two bifixes,* 10 *and* 1010*.*

Let $BF_q(n)$ denote the set of all bifix-free words of length n over an alphabet of fixed size q (for more details about this topic see [12]).

Given $q > 1$ and $n > 1$, two distinct words $w, w' \in BF_q(n)$ are said to be *cross-bifix-free* [6] if and only if no strict prefix of w is also a suffix of w' and vice-versa.

Example 2.2 *The binary words* 111010100 *and* 110101010 *in* $BF_2(9)$ *are cross-bifix-free, while the binary words* 111001100 *and* 110011010 *in* $BF_2(9)$ *have the cross-bifix* 1100.

A subset of $BF_q(n)$ is said to be a *cross-bifix-free set* if and only if for each w, w', with $w \neq w'$, in this set, w and w' are cross-bifix-free. This set is said to be *non-expandable* on $BF_q(n)$ if and only if the set obtained by adding any other word in $BF_q(n)$ is not a cross-bifix-free set. A non-expandable cross-bifix-free set on $BF_q(n)$ having maximal cardinality is called a *maximal cross-bifix-free set* on $BF_q(n)$.

In a recent paper [11] the authors provide a general construction of cross-bifix-free sets over a q-ary alphabet. Below, we recall such generation for the family of cross-bifix-free sets in $\mathbb{Z}_q^n$.

For any $2 \leq k \leq n-2$, the cross-bifix-free set $\mathcal{S}_{k,q}(n)$ in [11] is the set of all words $s = s_1 s_2 \cdots s_n$ in $\mathbb{Z}_q^n$ that satisfy the following two properties:

1) $s_1 = \cdots = s_k = 0$, $s_{k+1} \neq 0$ and $s_n \neq 0$,

2) the subsequence $s_{k+2} \ldots s_{n-1}$ does not contain k consecutive 0's.

Let

$$F_{k,q}(n) = \begin{cases} q^n & \text{if } 0 \leq n < k, \\ (q-1)\sum_{l=1}^{k} F_{k,q}(n-l) & \text{if } n \geq k, \end{cases}$$

be the sequence enumerating the words in $\mathbb{Z}_q^n$ avoiding k consecutive zero's [13]. Then, from the above definition of $\mathcal{S}_{k,q}(n)$, we have

$$|S_{n,q}^{(k)}| = (q-1)^2 F_{k,q}(n-k-2) \ .$$

For any fixed n and q, the largest size of $|S_{n,q}^{(k)}|$ is denoted by $S(n,q)$ and it is given by the following expression as in [11]

$$S(n,q) = \max\{(q-1)^2 F_{k,q}(n-k-2) : 2 \leq k \leq n-2\}.$$

This result allows to obtain non-expandable cross-bifix-free sets in the q-ary alphabet having cardinality close to the maximum.

In the present paper we introduce an alternative constructive method for the generation of cross-bifix-free set in $\mathbb{Z}_q$. Our approach is based on the study of lattice path in the discrete plane and it moves from the construction in [10].

Each word $w \in \mathbb{Z}_q^n$ can be represented as a lattice path of $\mathbb{N}^2$ running from $(0,0)$ to $(n,0)$ having the following properties:

- the element 0 corresponds to a *fall step* which is defined by $(1, -1)$,
- the element 1 corresponds to a *rise step* which is defined by $(1, 1)$,
- the elements $2, \ldots, q-1$ correspond respectively to a *colored level step* which is defined by $(1, 0)$ and it is labeled by one of the $q-2$ fixed colors.

For example, in Table 1 and Table 2 is showed an equivalence between elements and steps of lattice paths in the alphabets $\mathbb{Z}_3$ and $\mathbb{Z}_4$, respectively.

Table 1: Equivalence between symbols and steps for $\mathbb{Z}_3 = \{0, 1, 2\}$.

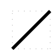
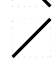
Symbol	Step	Color	Representation
0	$(1, -1)$	-	
1	$(1, 1)$	-	
2	$(1, 0)$	Black	

Table 2: Equivalence between symbols and steps for $\mathbb{Z}_4 = \{0, 1, 2, 3\}$.

Symbol	Step	Color	Representation
0	$(1, -1)$	-	
1	$(1, 1)$	-	
2	$(1, 0)$	Red	
3	$(1, 0)$	Green	

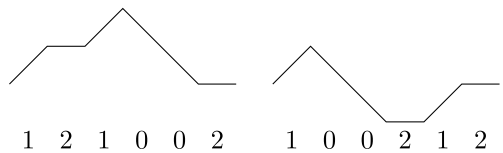
From now on, we will refer interchangeably to words or their graphical representations on the discrete plane, that is paths. The definition of bifix-free and cross-bifix-free can be easily extended to paths.

A *k-colored Motzkin path* of length n is a lattice path of $\mathbb{N}^2$ running from $(0, 0)$ to $(n, 0)$ that never goes below the x-axis and whose admitted steps are rise steps, fall steps and k-colored level steps (for more details about this copy see [14]).

For example, the left side of Fig. 1 shows a Motzkin path in $\mathbb{Z}_3$ having length 6, while the path in its right side is not a Motzkin path since it crosses the x-axis.

We denote by $\mathcal{M}_k(n)$ the set of all k-colored Motzkin paths of length n, and let $M_k(n)$ be the size of $\mathcal{M}_k(n)$.

Figure 1: Words 121002, 100212 and the equivalent paths. The first one is a Motzkin word.



Proposition 2.1 *For any* $n \geq 0$ *and* $k \geq 3$, $M_k(n)$ *is given by the following expression*

$$M_k(n+1) = kM_k(n) + \sum_{i=0}^{n-1} M_k(i)M_k(n-1-i)$$

with $M_k(0) = 1$ *and* $M_k(1) = k$.

Proof. If $n = 0$, $\mathcal{M}_k(n)$ contains the empty path only, then $M_k(0) = 1$. If $n = 1$, $\mathcal{M}_k(n)$ only contains those paths obtained by a level step, thus $M_k(1) = k$.
Let $n \geq 1$ and $w \in \mathcal{M}_k(n+1)$. There are two cases: w begins with a level step or w begins with a rise step. In the first case we have that $w = h\alpha$ where h is a level step and $\alpha \in \mathcal{M}_k(n)$, then the number of this first kind of paths is equal to $kM_k(n)$.

Otherwise, we have that $w = u\alpha d\beta$ where u is a rise step, d is a fall step, $\alpha \in \mathcal{M}_k(i)$ and $\beta \in \mathcal{M}(n-1-i)$ with $0 \leq i \leq n-1$. Then the number of this latter kind of paths is equal to $\sum_{i=0}^{n-1} M_k(i)M_k(n-1-i)$.

Thus,

$$M_k(n+1) = kM_k(n) + \sum_{i=0}^{n-1} M_k(i)M_k(n-1-i).$$

■

A word $w \in \mathbb{Z}_q^n$ is called $(q-2)$-*colored Motzkin word* if the equivalent lattice path is a $(q-2)$-colored Motzkin path.

For our purposes, it is useful to denote by $\hat{\mathcal{M}}_{q-2}(n)$ the set of all *elevated* $(q-2)$-*colored Motzkin words* of length n, defined as

$$\hat{\mathcal{M}}_{q-2}(n) = \{1\alpha 0 : \alpha \in \mathcal{M}_{q-2}(n-2)\}.$$

For example, in Fig. 2 two words in $\hat{\mathcal{M}}_1(6)$ are depicted.

In the next section of the present paper we are interested in determining one among all the possible non-expandable cross-bifix-free sets of words of fixed length $n > 1$ on $\mathbb{Z}_q^n$. We denote this set by $\mathcal{CBFS}_q(n)$.

Figure 2: An example of elevated Motzkin words

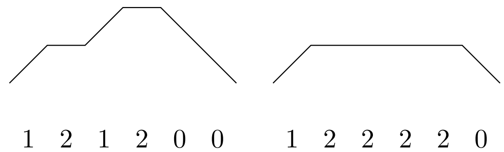
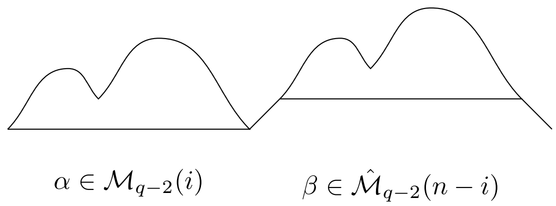


Figure 3: Graphical representation of the set $\mathcal{A}_q(n)$, $n \geq 3$



3 On the non-expandability of $\mathcal{CBFS}_q(n)$

In this section we define the set $\mathcal{CBFS}_q(n)$ which is formed by the union of three sets of $(q-2)$-colored Motzkin paths denoted by $\mathcal{A}_q(n), \mathcal{B}_q(n)$ and $\mathcal{C}_q(n)$, with $q \geq 3$ and $n \geq 3$, respectively.

Let

$$\mathcal{A}_q(n) = \left\{\alpha\beta : \alpha \in \mathcal{M}_{q-2}(i), \beta \in \hat{\mathcal{M}}_{q-2}(n-i)\right\} \setminus \left\{\alpha\beta : \alpha, \beta \in \hat{\mathcal{M}}_{q-2}\left(\frac{n}{2}\right)\right\}$$

with $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, be the set of words composed by a $(q-2)$-colored Motzkin word α of length i, and a elevated $(q-2)$-colored Motzkin word β of length $n-i$ (see Fig. 3). If n is even, we need to remove the words composed by two elevated subwords of the same length. On the other side, if n is odd, we assume the set $\left\{\alpha\beta : \alpha, \beta \in \hat{\mathcal{M}}_{q-2}\left(\frac{n}{2}\right)\right\}$ empty, since it does not exists any path of non-integer length.

Then, the enumeration of the set $\mathcal{A}_q(n)$ is given by the following expression

$$|\mathcal{A}_q(n)| = \sum_{i=0}^{\lfloor n/2 \rfloor} M_{q-2}(i) M_{q-2}(n-i-2) - \left[M_{q-2}\left(\frac{n}{2}-2\right)\right]^2.$$

Let

$$\mathcal{B}_q(n) = \left\{1\alpha\beta : \alpha \in \mathcal{M}_{q-2}(i), \beta \in \hat{\mathcal{M}}_{q-2}(n-i-1)\right\}$$

with $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$, be the set of words composed by a rise step, a $(q-2)$-colored Motzkin word α of length i, and a elevated $(q-2)$-colored Motzkin word β of length $n-i-1$ (see Fig. 4).

Figure 4: Graphical representation of the set $\mathcal{B}_q(n)$, $n \geq 3$

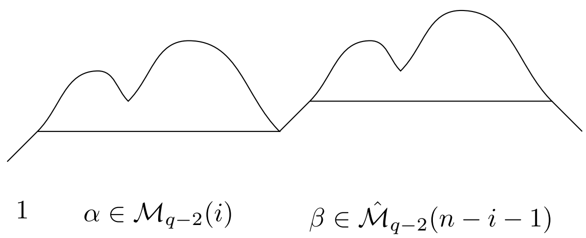
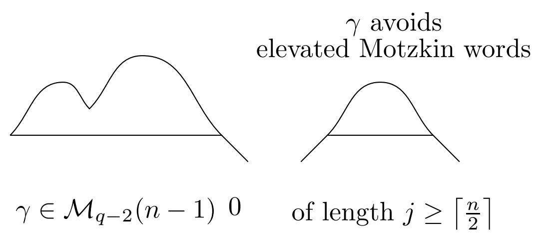


Figure 5: Graphical representation of the set $\mathcal{C}_q(n)$, $n \geq 3$



Then, the enumeration of the set $\mathcal{B}_q(n)$ is given by the following expression

$$|\mathcal{B}_q(n)| = \sum_{i=0}^{\lfloor n/2 \rfloor - 1} M_{q-2}(i) M_{q-2}(n-i-3).$$

Let

$$\mathcal{C}_q(n) = \left\{ \gamma 0 : \gamma \in \mathcal{M}_{q-2}(n-1), \gamma \neq u\beta v, \beta \in \hat{\mathcal{M}}_{q-2}(j) \right\}$$

with $j \geq \lceil \frac{n}{2} \rceil$, be the set of words composed by a $(q-2)$-colored Motzkin word γ of length $n-1$ that avoids elevated $(q-2)$-colored Motzkin words of length j, and a fall step (see Fig. 5).

Then, the enumeration of the set $\mathcal{C}_q(n)$ is given by the following expression

$$|\mathcal{C}_q(n)| = M_{q-2}(n-1) - \sum_{k=\lceil n/2 \rceil}^{n-1} \sum_{i=0}^{n-1-k} M_{q-2}(i) M_{q-2}(k-2) M_{q-2}(n-1-i-k).$$

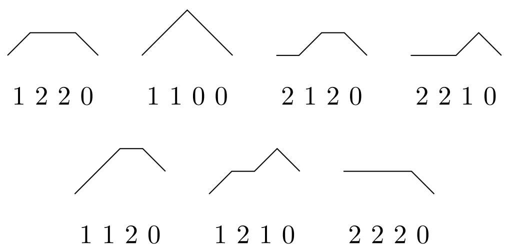
Note that, in order to obtain the size $|\mathcal{C}_q(n)|$ we need to subtract from all words γ of length $n-1$ those containing a elevated Motzkin subword β of length greater than or equal to $\lceil n/2 \rceil$, and γ can contain one of those subwords at most. Then, for $k = \lceil n/2 \rceil, \ldots, n-1$ we need to remove the words $u\beta v$, with $u \in \mathcal{M}_{q-2}(i)$, $\beta \in \hat{\mathcal{M}}_{q-2}(k)$, $v \in \mathcal{M}_{q-2}(n-1-i-k)$ and $0 \leq i \leq n-1-k$.

At this point, we define the set $\mathcal{CBFS}_q(n)$ as follows

$$\mathcal{CBFS}_q(n) = \mathcal{A}_q(n) \cup \mathcal{B}_q(n) \cup \mathcal{C}_q(n)$$

that is the union of the above described sets. For instance, in Fig. 6 the set $\mathcal{CBFS}_3(4)$ is depicted.

Figure 6: Graphical representation of the set $\mathcal{CBFS}_3(4)$



Proposition 3.1 *The set $\mathcal{CBFS}_q(n)$ is a cross-bifix-free set on $BF_q(n)$, for any $q \geq 3$ and $n \geq 3$.*

Proof. Let $w, w' \in \mathcal{CBFS}_q(n)$. Let u be a prefix of w, and v be a suffix of w' such that $|u| = |v|$. We need to check that in each case the prefix u does not match with the suffix v.

1. Let $w \in \mathcal{A}_q(n)$ and $w' \in \mathcal{A}_q(n) \cup \mathcal{B}_q(n)$.
   For each prefix u of w we have $|u|_0 \leq |u|_1$ and if $|u| > \lfloor \frac{n}{2} \rfloor$, then $|u|_0 < |u|_1$. For each suffix v of w' we have $|v|_0 \geq |v|_1$ and if $|v| < \lfloor \frac{n+1}{2} \rfloor$, then $|v|_0 > |v|_1$.
   Let $|u| = |v| = l$, if either $l < \lfloor \frac{n+1}{2} \rfloor$ or $l > \lfloor \frac{n}{2} \rfloor$, then u does not match with v. So we have to check the case $\lfloor \frac{n+1}{2} \rfloor \leq l \leq \lfloor \frac{n}{2} \rfloor$.
   If n is odd, it does not exist an integer l satisfying $\lfloor \frac{n+1}{2} \rfloor \leq l \leq \lfloor \frac{n}{2} \rfloor$, otherwise if n is even, the case $\lfloor \frac{n+1}{2} \rfloor \leq l \leq \lfloor \frac{n}{2} \rfloor$ is verified only for $l = \frac{n}{2}$. Therefore let n be even and $l = \frac{n}{2}$. In this case $|u|_0 \leq |u|_1$ and $|v|_0 \geq |v|_1$. At this point u can match with v only if $|v|_0 = |v|_1$, and this can happen only if v is a elevated Motzkin word. Suppose now that $u = v$, so u should be a elevated Motzkin word too, and they have both length $\frac{n}{2}$. In this case, w should be a word composed of two elevated Motzkin subwords of the same length, but such a word does not exists in $\mathcal{CBFS}_q(n)$ since the set $\left\{\alpha\beta : \alpha, \beta \in \hat{\mathcal{M}}_{q-2}\left(\frac{n}{2}\right)\right\}$ is not included in it, thus u does not match with v.

2. Let $w \in \mathcal{B}_q(n)$ and $w' \in \mathcal{A}_q(n) \cup \mathcal{B}_q(n)$.
   For each prefix u of w we have $|u|_0 < |u|_1$, and for each suffix v of w' we have $|v|_0 \geq |v|_1$, thus u does not match with v.

3. Let $w \in \mathcal{C}_q(n)$ and $w' \in \mathcal{A}_q(n) \cup \mathcal{B}_q(n)$.
   For each prefix u of w we have $|u|_0 \leq |u|_1$. For each suffix v of w' we have

$|v|_0 \geq |v|_1$ and if $|v| < \lfloor \frac{n+1}{2} \rfloor$, then $|v|_0 > |v|_1$.
Let $|u| = |v| = l$. If $l < \lfloor \frac{n+1}{2} \rfloor$, then u does not match with v. So we have to check the case $l \geq \lfloor \frac{n+1}{2} \rfloor$. In this case v contains a elevated Motzkin subword of length $\lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n}{2} \rceil$ at least, and u does not match with v, since u avoids such subwords.

4. Let $w \in \mathcal{CBFS}_q(n)$ and $w' \in \mathcal{C}_q(n)$.
For each prefix u of w we have $|u|_0 \leq |u|_1$, and for each suffix v of w' we have $|v|_0 > |v|_1$, thus u cannot match with v.

We proved that $\mathcal{CBFS}_q(n)$ is a cross-bifix-free set on $BF_q(n)$, for any $q \geq 3$ and $n \geq 3$. ■

Proposition 3.2 *The set $\mathcal{CBFS}_q(n)$ is a non-expandable cross-bifix-free set on $BF_q(n)$, for any $q \geq 3$ and $n \geq 3$.*

Proof. Let $w \in BF_q(n) \setminus \mathcal{CBFS}_q(n)$ and $W = \mathcal{CBFS}_q(n) \cup \{w\}$. If w begins with 0 then W is not cross-bifix-free since any word in $\mathcal{CBFS}_q(n)$ ends with 0. If w ends with 1 then W is not cross-bifix-free since any word in $\mathcal{A}_q(n)$ begins with 1. If w ends with a letter $k \neq 0, 1$ then W is not cross-bifix-free since the suffix k of w matches, for instance, with the prefix k of the word $k^{n-1}0 \in \mathcal{C}_q(n)$. Consequently we have to consider w as a word beginning with a non-zero letter and ending with 0.

Let $h = |w|_1 - |w|_0$ be the ordinate of the last point of the path corresponding to w. We now need to distinguish three different cases: $h > 0$, $h < 0$ and $h = 0$.

If $h > 0$, w can be written as (see Fig. 7)

$$w = \phi\ 1\ \mu_1\ 1\ \mu_2\ \cdots\ 1\ \mu_h,$$

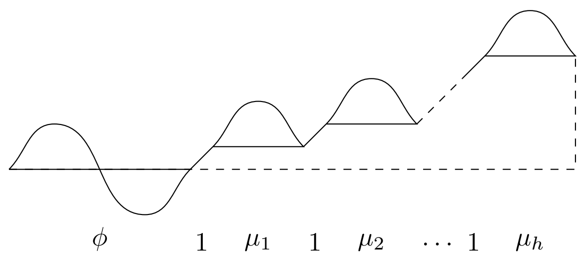
where ϕ is a word satisfying $|\phi|_1 = |\phi|_0$ and not beginning with 0, and $\mu_1, \ldots, \mu_h$ are $(q-2)$-colored Motzkin words with μ_h non-empty as w ends with 0.

In this case, if $|\mu_h| = l \leq n-2$, considering for instance the word $u = 1\mu_h 2^{n-l-2}0 \in \mathcal{A}_q(n)$ we can clearly see that $1\mu_h$ is a cross-bifix between w and u, and then W is not cross-bifix-free. On the other hand, if $|\mu_h| = n-1$, then necessarily $h = 1$ and $w = 1\mu_1$. So, w can be written as $w = 1\alpha\beta$, where $\alpha \in \mathcal{M}_{q-2}(i)$, $\beta \in \hat{\mathcal{M}}_{q-2}(n-i-1)$ with $i > \lfloor \frac{n}{2} \rfloor$ (otherwise $w \in \mathcal{B}_q(n)$). In this case, for instance, the word $\beta 1 2^{i-1} 0 \in \mathcal{A}_q(n)$ has a cross-bifix with w, thus W is not a cross-bifix-free-set.

If $h < 0$, w can be written as (see Fig. 8)

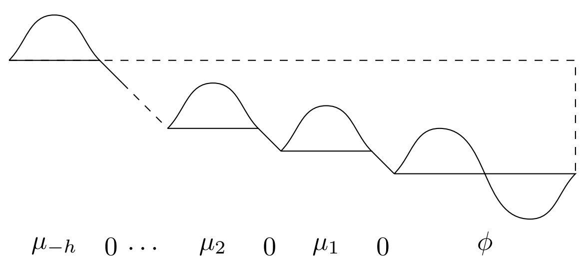
$$w = \mu_{-h}\ 0\ \cdots\ \mu_2\ 0\ \mu_1\ 0\ \phi$$

Figure 7: Graphical representation of w, in the case $h > 0$



where ϕ is a word satisfying $|\phi|_1 = |\phi|_0$ and ending with 0, and $\mu_1, \ldots, \mu_{-h}$ are $(q-2)$-colored Motzkin words with μ_{-h} non-empty as w begins with a non-zero letter.

Figure 8: Graphical representation of w, in the case $h < 0$



In this case, if $|\mu_{-h}| = l \leq n-2$, considering for instance the word $u = 12^{n-l-2}\mu_{-h}0 \in \mathcal{A}_q(n)$ we can clearly see that $\mu_{-h}0$ is a cross-bifix between w and u, and then W is not cross-bifix-free. On the other hand, if $|\mu_{-h}| = n-1$, then necessarily $h = -1$ and $w = \mu_1 0$. So, w can be written as $w = \alpha\beta\delta 0$, where $\beta \in \hat{\mathcal{M}}_{q-2}(j)$ with $j \geq \lceil \frac{n}{2} \rceil$ (otherwise $w \in \mathcal{C}_q(n)$), and α, δ any two $(q-2)$-colored Motzkin words of the appropriate length. In this case, for instance, the word $2^{n-j-|\alpha|}\alpha\beta \in \mathcal{A}_q(n)$ has a cross-bifix with w, thus W is not a cross-bifix-free-set.

Finally, if $h = 0$, the path associated to w can either remain above x-axis or fall below it.

In the first case let i, with $\lfloor \frac{n}{2} \rfloor \leq i < n$, be the last x-coordinate of the path intercepting the x-axis. Notice that i can not be less than $\lfloor \frac{n}{2} \rfloor$, otherwise $w \in \mathcal{A}_q(n)$. We can write $w = \alpha\beta$, where α is a non-empty word in $\mathcal{M}_{q-2}(i)$ and $\beta \in \hat{\mathcal{M}}_{q-2}(n-i)$. We now need to take into consideration two different cases: $i = \lfloor \frac{n}{2} \rfloor$ and $i > \lfloor \frac{n}{2} \rfloor$. In the first case $\alpha \in \hat{\mathcal{M}}_{q-2}(\frac{n}{2})$, otherwise $w \in \mathcal{A}_q(n)$, then, for instance, the word $2^{n/2}\alpha \in \mathcal{A}_q(n)$ has a cross-bifix with w. In the latter case, for instance, the word $\beta 2^{i-1}0 \in \mathcal{C}_q(n)$ has a cross-bifix with w, so that W is not a cross-bifix-free-set.

In the other case the path associated to w crosses the x-axis. Let i, with

$0 < i < n$, be the first x-coordinate of the path crossing x-axis. We can write $w = \alpha 0 \phi$, where α is a non-empty word in $\mathcal{M}_{q-2}(i)$. In this case, for instance, the word $12^{n-i-2}\alpha 0 \in \mathcal{A}_q(n)$ has a cross-bifix with w, then W is not a cross-bifix-free-set.

We proved that $\mathcal{CBFS}_q(n)$ is a non-expandable cross-bifix-free set on $BF_q(n)$, for any $q \geq 3$ and $n \geq 3$.

■

4 Sizes of Cross-Bifix-Free sets for Small Lengths

In this section we present some interesting results concerning the size of $\mathcal{CBFS}_q(n)$ compared to the ones in [11].

For fixed n and q, we recall that the size of q-ary cross-bifix-free sets given in [11] is obtained by

$$S(n,q) = \max\{(q-1)^2 F_{k,q}(n-k-2) : 2 \leq k \leq n-2\}$$

which is proved to be nearly optimal.

In Table III is shown the values of $S(n,q)$ and $|\mathcal{CBFS}_q(n)|$ for $3 \leq q \leq 6$ and $n \leq 16$. For the initial values of n, we can observe that the sizes obtained by our construction are greater than the size $S(n,q)$. In particular, the number of the initial values of n for which $|\mathcal{CBFS}_q(n)|$ is greater grows with q and this trend can be easily verified by experimental results.

In order to improve the values of the size $S(n,q)$ for the initial size of n, we can consider the following expression

$$S^*(n,q) = \max\{(q-1)^2 F_{k,q}(n-k-2) : 1 \leq k \leq n-2\},$$

where k can assume also the value 1. When $k = 1$, in the case of small n and large q, we obtain cross-bifix-free sets having cardinality greater than the one proposed in [11].

In Table IV is shown the values of $S^*(n,q)$ and $|\mathcal{CBFS}_q(n)|$ for $3 \leq q \leq 6$ and $n \leq 16$. Also in this situation, we can observe that the sizes obtained by our construction are greater than the size $S(n,q)$ in a range of values of n. In particular, the range of values of n for which $|\mathcal{CBFS}_q(n)|$ is greater grows with q and this trend can be easily verified by experimental results.

Table 3: Comparing the values from [11] with $\mathcal{CBFS}_q(n)$, for $3 \leq q \leq 6$

n	$\lvert\mathcal{CBFS}_3(n)\rvert$	$S(n,3)$	$\lvert\mathcal{CBFS}_4(n)\rvert$	$S(n,4)$
3	4	4	9	9
4	**7**	4	**25**	9
5	**16**	12	**72**	36
6	**36**	32	**223**	135
7	87	88	**712**	513
8	210	240	**2 334**	1 944
9	535	656	**7 868**	7 371
10	1 350	1 792	26 731	27 945
11	3 545	4 896	93 175	105 948
12	9 205	13 376	324 520	401 679
13	24 698	36 544	1 157 031	1 522 881
14	65 467	99 840	4 104 449	5 773 680
15	178 375	272 768	14 874 100	21 889 683
16	480 197	745 216	53 514 974	82 990 089

n	$\lvert\mathcal{CBFS}_5(n)\rvert$	$S(n,5)$	$\lvert\mathcal{CBFS}_6(n)\rvert$	$S(n,6)$
3	16	16	25	25
4	**61**	16	**121**	25
5	**224**	80	**550**	150
6	**900**	384	**2 739**	875
7	**3 595**	1 856	**13 260**	5 125
8	**15 014**	8 960	**67 740**	30 000
9	**63 135**	43 264	**342 676**	175 625
10	**271 136**	208 896	**1 787 415**	1 028 125
11	**1 178 677**	1 008 640	**9 324 647**	6 018 750
12	**5 167 953**	4 870 144	**49 456 240**	35 234 375
13	22 986 100	23 515 136	**263 776 127**	206 265 625
14	102 403 229	113 541 120	**1 417 981 855**	1 207 500 000
15	463 098 075	548 225 024	**7 688 015 908**	7 068 828 125
16	2 089 302 415	2 647 064 576	**41 785 951 916**	41 381 640 625

Table 4: Comparing the values from $S^*(n,q)$ with $\mathcal{CBFS}_q(n)$, for $3 \leq q \leq 6$

n	$\lvert\mathcal{CBFS}_3(n)\rvert$	$S^*(n,3)$	$\lvert\mathcal{CBFS}_4(n)\rvert$	$S^*(n,4)$
3	4	4	9	9
4	7	8	25	27
5	16	16	72	81
6	**36**	32	223	243
7	87	88	712	729
8	210	240	**2 334**	2 187
9	535	656	**7 868**	7 371
10	1 350	1 792	26 731	27 945
11	3 545	4 896	93 175	105 948
12	9 205	13 376	324 520	401 679
13	24 698	36 544	1 157 031	1 522 881
14	65 467	99 840	4 104 449	5 773 680
15	178 375	272 768	14 874 100	21 889 683
16	480 197	745 216	53 514 974	82 990 089

n	$\lvert\mathcal{CBFS}_5(n)\rvert$	$S^*(n,5)$	$\lvert\mathcal{CBFS}_6(n)\rvert$	$S^*(n,6)$
3	16	16	25	25
4	61	64	121	125
5	224	256	550	625
6	900	1 024	2 739	3 125
7	3 595	4 096	13 260	15 625
8	15 014	16 384	67 740	78 125
9	63 135	65 536	342 676	390 625
10	**271 136**	262 144	1 787 415	1 953 125
11	**1 178 677**	1 048 576	9 324 647	9 765 625
12	**5 167 953**	4 870 144	**49 456 240**	48 828 125
13	22 986 100	23 515 136	**263 776 127**	244 140 625
14	102 403 229	113 541 120	**1 417 981 855**	1 220 703 125
15	463 098 075	548 225 024	**7 688 015 908**	7 068 828 125
16	2 089 302 415	2 647 064 576	**41 785 951 916**	41 381 640 625

5 Conclusions and further developments

In this paper, we introduce a general constructive method for cross-bifix-free sets in the q-ary alphabet based upon the study of lattice paths on the discrete plane. This approach enables us to obtain the cross-bifix-free set $\mathcal{CBFS}_q(n)$ having greater cardinality than the ones proposed in [11], for the initial values of n.

Moreover, we prove that $\mathcal{CBFS}_q(n)$ is a non-expandable cross-bifix-free set on $BF_q(n)$, i.e. $\mathcal{CBFS}_q(n) \cup \{w\}$ is not a cross-bifix-free set on $BF_q(n)$, for any $w \in BF_q(n) \backslash \mathcal{CBFS}_q(n)$.

The non-expandable property is obviously a necessary condition to obtain a maximal cross-bifix-free set on $BF_q(n)$, anyway the problem of determine maximal cross-bifix-free sets is still open and no general solution has been found yet.

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