

# How we pass from fuzzy $po$ -semigroups to fuzzy $po$ - $\Gamma$ -semigroups

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**ABSTRACT.** The results on fuzzy ordered semigroups (or on fuzzy semigroups) can be transferred to fuzzy ordered gamma (or to fuzzy gamma) semigroups. We show the way we pass from fuzzy ordered semigroups to fuzzy ordered gamma semigroups.

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The results of fuzzy ordered semigroups can be transferred into ordered gamma semigroups in the way indicated in the present paper. Let us give some results to justify what we say.

As we know, a fuzzy subset  $f$  of an ordered groupoid  $S$  is called a *fuzzy quasi-ideal* of  $S$  if

- (1)  $(f \circ 1) \wedge (1 \circ f) \leq f$  and
- (2) if  $x \leq y$ , then  $f(x) \geq f(y)$ .

While the quasi-ideals are defined via the multiplication and the order of fuzzy sets, the right, the left ideals and the bi-ideals are defined via the multiplication of  $S$  as follows:

Let  $(S, \cdot, \leq)$  be a  $po$ -groupoid. A fuzzy subset  $f$  of  $S$  is called a *fuzzy right ideal* of  $S$  if

- (1)  $f(xy) \geq f(x)$  for every  $x, y \in S$  and
- (2) if  $x \leq y$ , then  $f(x) \geq f(y)$ .

A fuzzy subset  $f$  of  $S$  is called a *fuzzy left ideal* of  $S$  if

- (1)  $f(xy) \geq f(y)$  for every  $x, y \in S$  and
- (2) if  $x \leq y$ , then  $f(x) \geq f(y)$ .

A fuzzy subset  $f$  of an ordered semigroup  $S$  is called a *bi-ideal* of  $S$  if

- (1)  $f(xyz) \geq \min\{f(x), f(z)\}$  for every  $x, y, z \in S$  and

(2) if  $x \leq y$ , then  $f(x) \geq f(y)$ .

We have seen in [3] that a fuzzy subset of a groupoid  $S$  is a fuzzy right (resp. left) ideal of  $S$  if and only if the following assertions are satisfied:

(1)  $f \circ 1 \preceq f$  (resp.  $1 \circ f \preceq f$ ) and

(2) if  $x \leq y$ , then  $f(x) \geq f(y)$ .

For a  $po$ - $\Gamma$ -groupoid  $(M, \Gamma, \leq)$  we naturally have the following definitions: Any mapping  $f$  from  $M$  into the real closed interval  $[0,1]$  of real numbers is called a *fuzzy subset* of  $M$ . For a subset  $A$  of  $M$  the fuzzy subset  $f_A$  is the characteristic function defined by:

$$f_A : M \rightarrow [0, 1] \mid x \rightarrow f_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

For  $A = \{a\}$  we write  $f_a$  instead of  $f_{\{a\}}$ , and we have

$$f_a : M \rightarrow [0, 1] \mid x \rightarrow f_a(x) := \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a. \end{cases}$$

For an element  $a$  of  $M$  we denote by  $A_a$  the relation on  $M$  defined by

$$A_a := \{(y, z) \mid a \leq y\gamma z \text{ for some } \gamma \in \Gamma\}.$$

For two fuzzy subsets  $f$  and  $g$  of  $M$  the multiplication  $f \circ g$  is defined as follows:

$$f \circ g : M \rightarrow [0, 1] \mid a \rightarrow \begin{cases} \bigvee_{(y,z) \in A_a} \min\{f(y), g(z)\} & \text{if } A_a \neq \emptyset \\ 0 & \text{if } A_a = \emptyset \end{cases}$$

and in the set of all fuzzy subsets of  $M$  we define the order relation as follows:

$$f \preceq g \text{ if and only if } f(a) \leq g(a) \text{ for all } a \in M.$$

We also denote

$$(f \wedge g)(a) := \min\{f(a), g(a)\} \text{ for all } a \in M.$$

We denote by  $1$  the fuzzy subset of  $M$  defined by:  $1 : M \rightarrow [0, 1] \mid x \rightarrow 1(x) := 1$  and this is the greatest element of the set of fuzzy subsets of  $M$ . If  $M$  is a  $po$ - $\Gamma$ -groupoid (resp.  $po$ - $\Gamma$ -semigroup), then the set of all fuzzy subsets of  $M$  with the multiplication “ $\circ$ ” and the order “ $\preceq$ ” above is a  $po$ - $\Gamma$ -groupoid (resp.  $po$ - $\Gamma$ -semigroup).

For a subset  $H$  of  $M$  we denote by  $(H]$  the subset of  $M$  defined by:

$$(H] := \{t \in M \mid t \leq a \text{ for some } a \in H\}.$$

**Definition 1.** Let  $M$  be a  $po$ - $\Gamma$ -groupoid. A fuzzy subset  $f$  of  $M$  is called a *fuzzy right ideal* of  $M$  if

- (1)  $f(x\gamma y) \geq f(x)$  for every  $x, y \in M$  and every  $\gamma \in \Gamma$  and
- (2) if  $x \leq y$ , then  $f(x) \geq f(y)$ .

**Definition 2.** Let  $M$  be a  $po$ - $\Gamma$ -groupoid. A fuzzy subset  $f$  of  $M$  is called a *fuzzy left ideal* of  $M$  if

- (1)  $f(x\gamma y) \geq f(y)$  for every  $x, y \in M$  and every  $\gamma \in \Gamma$  and
- (2) if  $x \leq y$ , then  $f(x) \geq f(y)$ .

**Definition 3.** A fuzzy subset  $f$  of a  $po$ - $\Gamma$ -semigroup  $M$  is called a *fuzzy bi-ideal* of  $S$  if

- (1)  $f(xyz) \geq \min\{f(x), f(z)\}$  for every  $x, y, z \in M$  and
- (2) if  $x \leq y$ , then  $f(x) \geq f(y)$ .

**Proposition 4.** (cf. also [3]) *Let  $M$  be a  $po$ - $\Gamma$ -groupoid. A fuzzy subset  $f$  of  $M$  is a fuzzy right ideal of  $M$  if and only if*

- (1)  $f \circ 1 \preceq f$  and
- (2) if  $x \leq y$ , then  $f(x) \geq f(y)$ .

**Proof.**  $\implies$ . Let  $a \in S$ . Then  $(f \circ 1)(a) \leq f(a)$ . In fact: If  $A_a = \emptyset$ , then  $(f \circ 1)(a) := 0 \leq f(a)$ . Let  $A_a \neq \emptyset$ . Then

$$(f \circ 1)(a) := \bigvee_{(y,z) \in A_a} \min\{f(y), 1(z)\} = \bigvee_{(y,z) \in A_a} \min\{f(y)\}.$$

On the other hand,

$$f(y) \leq f(a) \text{ for every } (y, z) \in A_a.$$

Indeed: Let  $(y, z) \in A_a$ . Then  $a \leq y\gamma z$  for some  $\gamma \in \Gamma$ . Since  $f$  is a fuzzy right ideal of  $M$ , we have  $f(a) \geq f(y\gamma z) \geq f(y)$ . Thus we have

$$(f \circ 1)(a) = \bigvee_{(y,z) \in A_a} \min\{f(y)\} \leq f(a).$$

$\impliedby$ . Let  $x, y \in M$  and  $\gamma \in \Gamma$ . Then  $f(x\gamma y) \geq f(x)$ . Indeed: By hypothesis, we have  $f(x\gamma y) \geq (f \circ 1)(x\gamma y)$ . Since  $(x, y) \in A_{x\gamma y}$ , we have

$$(f \circ 1)(x\gamma y) := \bigvee_{(u,v) \in A_{x\gamma y}} \min\{f(u), 1(v)\} \geq \min\{f(x), 1(y)\} = f(x).$$

Thus  $f(x\gamma y) \geq f(x)$ . □

In a similar way one can prove the following propositions:

**Proposition 5.** *Let  $M$  be a  $po$ - $\Gamma$ -groupoid. A fuzzy subset  $f$  of  $M$  is a fuzzy left ideal of  $M$  if and only if*

- (1)  $1 \circ f \preceq f$  and
- (2) if  $x \leq y$ , then  $f(x) \geq f(y)$ .

**Proposition 6.** *Let  $M$  be a  $po$ - $\Gamma$ -groupoid. A fuzzy subset  $f$  of  $M$  is a fuzzy bi-ideal of  $M$  if and only if*

- (1)  $f \circ 1 \circ f \preceq f$  and
- (2) if  $x \leq y$ , then  $f(x) \geq f(y)$ .

We characterize now the regular and the intra-regular  $po$ - $\Gamma$ -semigroups in terms of fuzzy right and fuzzy left ideals.

**Definition 7.** A  $po$ - $\Gamma$ -semigroup  $M$  is called *regular* if

$$a \in (a\Gamma M\Gamma a] \text{ for every } a \in M.$$

**Lemma 8.** *Let  $M$  be a regular  $po$ - $\Gamma$ -semigroup. Then for every fuzzy right ideal  $f$  and every fuzzy subset  $g$  of  $M$ , we have  $f \wedge g \preceq f \circ g$ .*

**Proof.** Let  $f$  be a fuzzy right ideal,  $g$  a fuzzy subset of  $M$  and  $a \in M$ . Since  $M$  is regular, there exist  $x \in M$  and  $\gamma, \mu \in \Gamma$  such that  $a \leq (a\gamma x)\mu a$ . Since  $(a\gamma x, a) \in A_a$ , we have

$$(f \circ g)(a) = \bigvee_{(y,z) \in A_a} \min\{f(y), g(z)\} \geq \min\{f(a\gamma x), g(a)\}.$$

Since  $f$  is a right ideal of  $M$ , we have  $f(a\gamma x) \geq f(a)$ . Then we have

$$(f \circ g)(a) \geq \min\{f(a\gamma x), g(a)\} \geq \min\{f(a), g(a)\} = (f \wedge g)(a).$$

This holds for every  $a \in M$ , thus we have  $f \wedge g \preceq f \circ g$ . □

In a similar way we prove the following

**Lemma 9.** *Let  $M$  be a regular  $po$ - $\Gamma$ -semigroup. Then for every fuzzy subset  $f$  and every fuzzy left ideal  $g$  of  $M$ , we have  $f \wedge g \preceq f \circ g$ .*

**Lemma 10.** *Let  $M$  be a  $po$ - $\Gamma$ -groupoid,  $f$  a fuzzy right ideal and  $g$  a fuzzy left ideal of  $M$ . Then we have  $f \circ g \preceq f \wedge g$ .*

**Proof.** Let  $a \in M$ . Then  $(f \circ g)(a) \leq (f \wedge g)(a)$ . In fact: If  $A_a = \emptyset$ , then  $(f \circ g)(a) := 0$ . Since  $a \in M$  and  $f \wedge g$  is a fuzzy subset of  $M$ , we have  $(f \wedge g)(a) \geq 0$ , thus we have  $(f \circ g)(a) \leq (f \wedge g)(a)$ . Let  $A_a \neq \emptyset$ . Then

$$(f \circ g)(a) := \bigvee_{(y,z) \in A_a} \min\{f(y), g(z)\}.$$

On the other hand,

$$\min\{f(y), g(z)\} \leq (f \wedge g)(a) \quad \forall (y, z) \in A_a.$$

Indeed: Let  $(y, z) \in A_a$ . Then  $y, z \in M$  and  $a \leq y\gamma z$  for some  $\gamma \in \Gamma$ . Since  $f$  is a fuzzy right ideal of  $M$ , we have  $f(a) \geq f(y\gamma z) \geq f(y)$ . Since  $g$  is a fuzzy left ideal of  $M$ , we have  $g(a) \geq g(y\gamma z) \geq g(z)$ . Thus

$$\min\{f(y), g(z)\} \leq \min\{f(a), g(a)\} = (f \wedge g)(a).$$

Hence we obtain

$$(f \circ g)(a) = \bigvee_{(y,z) \in A_a} \min\{f(y), g(z)\} \leq (f \wedge g)(a),$$

and the proof is complete.  $\square$

**Lemma 11.** (cf. also [1]) *Let  $M$  be a po- $\Gamma$ -groupoid. Then  $A$  is a right (resp. left) ideal of  $M$  if and only if the characteristic function  $f_A$  is a fuzzy right (resp. fuzzy left) ideal of  $M$ .*

For an element  $a$  of  $M$ , we denote by  $R(a)$ ,  $L(a)$  the right and the left ideal of  $M$  generated by  $a$ . We have  $R(a) = (a \cup a\Gamma M]$  and  $L(a) = (a \cup M\Gamma a]$ .

**Lemma 12.** *A po- $\Gamma$ -semigroup  $M$  is regular if and only if*

$$R(a) \cap L(a) \subseteq \left( R(a)\Gamma L(a) \right] \text{ for every } a \in M.$$

**Theorem 13.** (see [2]) *A po- $\Gamma$ -semigroup  $M$  is regular if and only if for every fuzzy right ideal  $f$  and every fuzzy left ideal  $g$  of  $M$ , we have*

$$f \wedge g \preceq f \circ g, \text{ equivalently } f \wedge g = f \circ g.$$

**Proof.** Let  $f$  be a fuzzy right ideal and  $g$  a fuzzy left ideal of  $M$ . By Lemma 8, we have  $f \wedge g \preceq f \circ g$ . By Lemma 10, we have  $f \circ g \preceq f \wedge g$ . Then  $f \wedge g = f \circ g$ . Suppose  $f \wedge g \preceq f \circ g$  for every right ideal  $f$  and every left ideal  $g$  of  $M$ . Then  $M$  is regular. In fact: By Lemma 12, it is enough to prove that  $R(a) \cap L(a) \subseteq (R(a)\Gamma L(a)]$  for every  $a \in M$ . Let  $a \in M$  and  $b \in R(a) \cap L(a)$ . By Lemma 11,  $f_{R(a)}$  is a fuzzy right ideal and  $f_{L(a)}$  a fuzzy left ideal of  $M$ . By hypothesis, we have

$$1 = \min \left\{ f_{R(a)}(b), f_{L(a)}(b) \right\} = \left( f_{R(a)} \wedge f_{L(a)} \right)(b) \leq \left( f_{R(a)} \circ f_{L(a)} \right)(b),$$

then

$$1 \leq \left( f_{R(a)} \circ f_{L(a)} \right)(b).$$

If  $A_b = \emptyset$ , then  $(f_{R(a)} \circ f_{L(a)})(b) := 0$  which is impossible. If  $A_b \neq \emptyset$ , then we have

$$(f_{R(a)} \circ f_{L(a)})(b) := \bigvee_{(y,z) \in A_a} \min\{f_{R(a)}(y), f_{L(a)}(z)\}.$$

If  $y \notin R(a)$  or  $z \notin L(a)$  for every  $(y, z) \in A_b$ , then

$$\min\{f_{R(a)}(y), f_{L(a)}(z)\} = 0 \text{ for every } (y, z) \in A_b,$$

and then  $(f_{R(a)} \circ f_{L(a)})(b) = 0$  which is impossible. Thus there exists  $(y, z) \in A_b$  such that  $y \in R(a)$  or  $z \in L(a)$ . And so, there exists  $\gamma \in \Gamma$  such that  $b \leq y\gamma z \in R(a)\Gamma L(a)$ , that is  $b \in (R(a)\Gamma L(a))$ .  $\square$

**Definition 14.** A  $po$ - $\Gamma$ -semigroup  $M$  is called *intra-regular* if

$$a \in (M\Gamma a\Gamma a\Gamma M] \quad \forall a \in M.$$

**Lemma 15.** A  $po$ - $\Gamma$ -semigroup  $M$  is *intra-regular* if and only if

$$R(a) \cap L(a) \subseteq (L(a)\Gamma R(a)] \text{ for every } a \in M.$$

**Theorem 16.** A  $po$ - $\Gamma$ -semigroup  $M$  is *intra-regular* if and only if for every fuzzy right ideal  $f$  and every fuzzy left ideal  $g$  of  $M$ , we have

$$f \wedge g \preceq g \circ f.$$

**Proof.**  $\implies$ . Let  $f$  be a fuzzy right,  $g$  a fuzzy left ideal of  $M$  and  $a \in M$ . Since  $M$  is *intra-regular*, there exist  $x, y \in M$  and  $\gamma, \mu, \rho \in \Gamma$  such that  $a \leq x\gamma a\mu a\rho y$ . Since  $(x\gamma a, a\rho y) \in A_a$ , we have

$$(g \circ f)(a) := \bigvee_{(y,z) \in A_a} \min\{g(y), f(z)\} \geq \min\{g(x\gamma a), f(a\rho y)\}.$$

Since  $g$  is a fuzzy left ideal of  $M$ ,  $g(x\gamma a) \geq g(a)$ . Since  $f$  is a fuzzy right ideal of  $M$ ,  $f(a\rho y) \geq f(a)$ . Thus we have

$$(g \circ f)(a) \geq \min\{g(x\gamma a), f(a\rho y)\} \geq \min\{g(a), f(a)\} = (f \wedge g)(a),$$

then  $f \wedge g \preceq g \circ f$ .

$\impliedby$ . Let  $a \in M$  and  $b \in R(a) \cap L(a)$ . By hypothesis, we have

$$1 = \min\{f_{R(a)}(b), f_{L(a)}(b)\} = (f_{R(a)} \wedge f_{L(a)})(b) \leq (f_{R(a)} \circ f_{L(a)})(b).$$

Then  $A_b \neq \emptyset$ , and there exists  $(y, z) \in A_b$  such that  $y \in L(b)$  and  $z \in R(b)$ . Then there exists  $\gamma \in \Gamma$  such that  $b \leq y\gamma z \in L(b)\Gamma R(b)$ , so  $b \in \left(L(a)\Gamma R(a)\right]$ . By Lemma 15,  $M$  is intra-regular.  $\square$

Let us characterize now the intra-regular, the regular and the left (right) regular  $po$ - $\Gamma$ -semigroups in terms of fuzzy subsets.

**Lemma 17.** *Let  $M$  be a  $po$ - $\Gamma$ -groupoid,  $f, g$  fuzzy subsets of  $M$ , and  $a \in M$ . The following are equivalent:*

- (1)  $(f \circ g)(a) \neq 0$ .
- (2) *There exists  $(x, y) \in A_a$  such that  $f(x) \neq 0$  and  $g(y) \neq 0$ .*

**Proof.** (1)  $\implies$  (2). If  $A_a = \emptyset$ , then  $(f \circ g)(a) = 0$  which is impossible. So there exists  $(x, y) \in A_a$ , then

$$(f \circ g)(a) := \bigvee_{(u,v) \in A_a} \min\{f(u), g(v)\} \geq \min\{f(x), g(y)\}.$$

If  $f(x) = 0$  or  $g(y) = 0$ , then  $(f \circ g)(a) = 0$ . Since  $f \circ g$  is a fuzzy subset of  $M$ , we have  $(f \circ g)(a) \leq 0$ . Then  $(f \circ g)(a) = 0$  which is impossible. Hence we have  $f(x) \neq 0$  and  $g(y) \neq 0$ .  $\square$

**Corollary 18.** *Let  $M$  be a  $po$ - $\Gamma$ -groupoid,  $f$  a fuzzy subset of  $M$ , and  $a \in M$ . The following are equivalent:*

- (1)  $(f \circ 1)(a) \neq 0$ .
- (2) *There exists  $(x, y) \in A_a$  such that  $f(x) \neq 0$ .*

**Corollary 19.** *Let  $M$  be a  $po$ - $\Gamma$ -groupoid,  $f$  a fuzzy subset of  $M$ , and  $a \in M$ . The following are equivalent:*

- (1)  $(1 \circ g)(a) \neq 0$ .
- (2) *There exists  $(x, y) \in A_a$  such that  $g(y) \neq 0$ .*

**Lemma 20.** *Let  $M$  be a  $po$ - $\Gamma$ -groupoid and  $a, b \in M$ . Then we have*

$$b \leq a\gamma a \text{ for some } \gamma \in \Gamma \iff (f_a \circ f_a)(b) \neq 0.$$

**Proof.**  $\implies$ . Let  $b \leq a\gamma a$  for some  $\gamma \in \Gamma$ . Since  $(a, a) \in A_b$ , we have

$$(f_a \circ f_a)(b) := \bigvee_{(u,v) \in A_b} \min\{f_a(u), f_a(v)\} \geq \min\{f_a(a), f_a(a)\} = f_a(a) = 1$$

$\impliedby$ . Since  $(f_a \circ f_a)(b) \neq 0$ , by Lemma 17, there exists  $(x, y) \in A_b$  such that  $f_a(x) \neq 0$  and  $f_a(y) \neq 0$ . Then  $f_a(x) = f_a(y) = 1$ , and  $x = y = a$ . Since  $b \leq x\mu y$  for some  $\mu \in \Gamma$ , we have  $b \leq a\mu a$ .  $\square$

**Lemma 21.** *Let  $M$  be a  $po$ - $\Gamma$ -groupoid and  $f$  a fuzzy subset of  $M$ . Then*

$$f(a) \leq (f \circ f)(a\gamma a) \quad \forall a \in M \quad \forall \gamma \in \Gamma.$$

**Proof.** Let  $a \in M$  and  $\gamma \in \Gamma$ . Since  $(a, a) \in A_a$ , we have

$$(f \circ f)(a\gamma a) := \bigvee_{(u,v) \in A_{a\gamma a}} \min\{f(u), f(v)\} \geq \min\{f(a), f(a)\} = f(a).$$

Thus  $f(a) \leq (f \circ f)(a\gamma a)$ . □

We denote  $f^2 = f \circ f$ .

**Lemma 22.** *Let  $M$  be a  $po$ - $\Gamma$ -semigroup,  $f$  a fuzzy subset of  $M$  and  $a \in M$ . If  $a \leq x\mu a\gamma a\rho y$  for some  $x, y \in M$ ,  $\mu, \gamma, \rho \in \Gamma$ , then  $f(a) \leq (1 \circ f^2 \circ 1)(a)$ .*

**Proof.** Since  $(x\mu a\gamma a, y) \in A_a$ , we have

$$\begin{aligned} (1 \circ f^2 \circ 1)(a) &:= \bigvee_{(u,v) \in A_a} \min\{(1 \circ f^2)(u), 1(v)\} \\ &\geq \min\{(1 \circ f^2)(x\mu a\gamma a), 1(y)\} \\ &= (1 \circ f^2)(x\mu a\gamma a). \end{aligned}$$

Since  $(x, a\gamma a) \in A_{x\mu a\gamma a}$ , we have

$$\begin{aligned} (1 \circ f^2)(x\mu a\gamma a) &:= \bigvee_{(w,t) \in A_{x\mu a\gamma a}} \min\{1(w), f^2(t)\} \\ &\geq \min\{1(x), f^2(a\gamma a)\} \\ &= f^2(a\gamma a). \end{aligned}$$

Then, by Lemma 21, we have  $(1 \circ f^2 \circ 1)(a) \geq f^2(a\gamma a) \geq f(a)$ . □

**Theorem 23.** *A  $po$ - $\Gamma$ -semigroup  $M$  is regular if and only if for every fuzzy subset  $f$  of  $M$ , we have  $f \preceq f \circ 1 \circ f$ .*

**Proof.**  $\implies$ . Let  $f$  be a fuzzy subset of  $M$  and  $a \in M$ . Since  $M$  is regular, there exist  $x \in M$  and  $\gamma, \mu \in \Gamma$  such that  $a \leq a\gamma x\mu a$ . Since  $(a\gamma x, a) \in A_a$ , we have

$$\begin{aligned} (f \circ 1 \circ f)(a) &:= \bigvee_{(y,z) \in A_a} \min\{(f \circ 1)(y), f(z)\} \\ &\geq \min\{(f \circ 1)(a\gamma x), f(a)\}. \end{aligned}$$

Since  $(a, x) \in A_{a\gamma x}$ , we have

$$(f \circ 1)(a\gamma x) := \bigvee_{(u,v) \in A_{a\gamma x}} \min\{f(u), 1(v)\} \geq \min\{f(a), 1(x)\} = f(a).$$



Then

$$(f \circ 1 \circ f)(a) \geq \{\min\{f(a), f(a)\} = f(a),$$

so  $f \preceq f \circ 1 \circ f$ .

$\Leftarrow$ . let  $a \in M$ . Since  $f_a$  is a fuzzy subset of  $M$ , by hypothesis, we have  $a = f_a(a) \leq (f_a \circ 1 \circ f_a)(a) \leq 1$ , so  $((f_a \circ 1) \circ f_a)(a) \neq 0$ . By Lemma 17, there exists  $(x, y) \in A_a$  such that  $(f_a \circ 1)(x) \neq 0$  and  $f_a(y) \neq 0$ . By Corollary 18, there exists  $(u, v) \in A_x$  such that  $f_a(x) \neq 0$ . Since  $a \leq x\gamma y$  for some  $\gamma \in \Gamma$  and  $x = y = a$ , we have  $a \leq a\gamma a \leq (a\gamma a)\gamma a \in a\Gamma M\Gamma a$ . Then  $a \in (a\Gamma M\Gamma a]$ , and  $M$  is regular.  $\square$

**Theorem 24.** (see [4]) *A po- $\Gamma$ -semigroup  $M$  is intra-regular if and only if, for every fuzzy subset  $f$  of  $M$ , we have*

$$f \preceq 1 \circ f^2 \circ 1.$$

**Proof.**  $\Rightarrow$ . Let  $f$  be a fuzzy subset of  $M$  and  $a \in M$ . Since  $M$  is intra-regular, there exist  $x, y \in M$  and  $\mu, \gamma, \rho \in \Gamma$  such that  $a \leq x\mu a\gamma a\rho y$ . Then, by Lemma 22, we have  $f(a) \leq (1 \circ f^2 \circ 1)(a)$ , so  $f \preceq 1 \circ f^2 \circ 1$ .

$\Leftarrow$ . Let  $a \in M$ . Since  $f_a$  is a fuzzy subset of  $M$ , by hypothesis, we have

$$1 = f_a(a) \leq (1 \circ f_a^2 \circ 1)(a) \leq 1,$$

hence  $((1 \circ f_a^2) \circ 1)(a) \neq 0$ . By Corollary 18, there exists  $(x, y) \in A_a$  such that  $(1 \circ f_a^2)(x) \neq 0$ . By Corollary 19, there exists  $(u, v) \in A_x$  such that  $f_a^2(v) \neq 0$ . By Lemma 20, there exists  $\gamma \in \Gamma$  such that  $v \leq a\gamma a$ . In addition,  $a \leq x\mu y$  and  $x \leq u\rho v$  for some  $\mu, \rho \in \Gamma$ . Then we have

$$a \leq x\mu y \leq (u\rho v)\mu y \leq u\rho(a\gamma a)\mu y \in M\Gamma a\Gamma a\Gamma M,$$

and  $a \in (M\Gamma a\Gamma a\Gamma M]$ , thus  $M$  is intra-regular.  $\square$

**Definition 25.** A po- $\Gamma$ -semigroup  $M$  is called *right regular* if

$$a \in (a\Gamma a\Gamma M] \quad \forall a \in M.$$

**Theorem 26.** *A po- $\Gamma$ -semigroup  $M$  is right regular if and only if, for every fuzzy subset  $f$  of  $M$ , we have*

$$f \preceq f^2 \circ 1.$$

**Proof.**  $\Rightarrow$ . Let  $a \in M$ . Since  $M$  is right regular, there exist  $x \in M$  and  $\gamma, \mu \in \Gamma$  such that  $a \leq a\gamma a\mu x$ . Since  $(a\gamma a, x) \in A_a$ , we have

$$\begin{aligned} (f^2 \circ 1)(a) &:= \bigvee_{(u,v) \in A_a} \min\{f^2(u), 1(v)\} \\ &\geq \min\{f^2(a\gamma a), 1(x)\} \\ &= f^2(a\gamma a). \end{aligned}$$

Since  $(a, a) \in A_{a\gamma a}$ , we have

$$f^2(a\gamma a) = \bigvee_{(w,t) \in A_{a\gamma a}} \min\{f(w), f(t)\} \geq \min\{f(a), f(a)\} = f(a).$$

So  $(f^2 \circ 1)(a) \geq f(a)$ , then  $f \preceq f^2 \circ 1$ .

$\Leftarrow$ . Let  $a \in M$ . By hypothesis, we have  $1 = f_a(a) \leq (f_a^2 \circ 1)(a)$ , so  $(f_a^2 \circ 1)(a) = 1$ . By Corollary 18, there exists  $(x, y) \in A_a$  such that  $f_a^2(x) \neq 0$ . By Lemma 20, there exists  $\lambda \in \Gamma$  such that  $x \leq a\lambda a \leq (a\lambda a)\lambda a \in a\Gamma a\Gamma M$ , so  $a \in (a\Gamma a\Gamma M]$ .  $\square$

**Definition 27.** A  $po$ - $\Gamma$ -semigroup  $M$  is called *left regular* if

$$a \in (M\Gamma a\Gamma a] \quad \forall a \in M.$$

**Theorem 28.** A  $po$ - $\Gamma$ -semigroup  $M$  is left regular if and only if, for every fuzzy subset  $f$  of  $M$ , we have

$$f \preceq 1 \circ f^2.$$

**Proposition 29.** If  $M$  is a  $po$ - $\Gamma$ -groupoid and  $f$  a fuzzy right (resp. fuzzy left) ideal of  $M$ , then  $f \circ 1 \preceq f$  (resp.  $1 \circ f \preceq f$ ).

**Proof.** Let  $f$  be a fuzzy right ideal of  $M$  and  $a \in M$ . Then  $(f \circ 1)(a) \leq f(a)$ . Indeed: If  $A_a = \emptyset$ , then  $(f \circ 1)(a) := 0 \leq f(a)$ . Let  $A_a \neq \emptyset$ . Then

$$(f \circ 1)(a) := \bigvee_{(x,y) \in A_a} \min\{f(x), 1(y)\} = \bigvee_{(x,y) \in A_a} \min\{f(x)\}.$$

On the other hand,  $f(x) \leq f(a)$  for every  $(x, y) \in A_a$ . Indeed: Let  $(x, y) \in A_a$ . Then  $a \leq x\gamma y$  for some  $\gamma \in \Gamma$ . Since  $f$  be a fuzzy right ideal of  $M$ , we have

$$f(a) \geq f(x\gamma y) \geq f(x).$$

Then  $f \circ 1 \preceq f$ .  $\square$

**Corollary 30.** If  $M$  is a  $po$ - $\Gamma$ -groupoid, then the fuzzy right (and the fuzzy left) ideals of  $M$  are subidempotent.

**Proof.** Let  $f$  be a fuzzy right ideal of  $M$ . Then  $f^2 \preceq f \circ 1 \preceq f$ .  $\square$

**Corollary 31.** If  $M$  is a regular  $po$ - $\Gamma$ -semigroup, then the fuzzy right (and the fuzzy left) ideals of  $M$  are idempotent.

**Proof.**  $\Rightarrow$ . Let  $f$  be a fuzzy right ideal of  $M$ . Since  $M$  is regular, by Theorem 23, we have  $f \preceq (f \circ 1) \circ f \preceq f^2$ . By Corollary 30,  $f^2 \preceq f$ , thus we have  $f = f^2$ .  $\square$

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