Representation growth of maximal class groups: various exceptional cases

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Abstract

This paper is a sequel to *Representation growth of maximal class groups: non-exceptional primes* [6]. We use a constructive method to calculate some exceptional cases of p-local representation zeta functions of a family of finitely generated nilpotent groups M_n with maximal nilpotency class. Using the machinery of the constructive method from the prequel paper we construct all irreducible representations of degree p^N for all $N \in \mathbb{N}$ for the group M_{p+1} for a fixed prime p. We also construct all irreducible representations of degree 2^N for the group M_4. Together with the main result from the prequel, this gives us a complete understanding of the irreducible representations of the groups M_3 and M_4, along with their global representation zeta functions.

1 Introduction

Representation growth is a burgeoning area within group theory where one studies (usually infinite) groups by considering the sequence of the number of (sometimes equivalence classes of) irreducible complex representations of degree n for all $n \in \mathbb{N}$. More formally, for a group G and for all $n \in \mathbb{N}$ let $r_n(G)$ be the number of irreducible representations of degree n. If all $r_n(G)$ are finite, we can study this sequence by embedding them as coefficients in a zeta function.

Throughout this paper, all representations are complex representations. Two representations of a group G, say ρ_1 and ρ_2, are said to be *twist-equivalent* if $\rho_1 = \chi \otimes \rho_2$ where χ is a representation of G of degree 1. It is easy to see that there is only one twist isoclass of degree 1. For a finitely generated torsion-free nilpotent group G all $r_n(G)$ are infinite. However, Lubotzky and Magid [12] show that if we redefine $r_n(G)$ to be the equivalence classes of representations of G of degree n up to twist equivalence and isomorphism then all $r_n(G)$ are finite. These are the objects we count in this paper. Additionally, they show

1

that each irreducible representation of G is twist equivalent to one that factors through a finite quotient of G. See, for example, [6, Introduction] for more information about representation growth of $\mathcal{T}$-groups.

Let G be a $\mathcal{T}$-group and let the complex series

$$\zeta_G^{irr}(s) = \sum_{n=1}^{\infty} r_n(G)n^{-s}. \tag{1}$$

We call $\zeta_G^{irr}(s)$ the *(global) representation zeta function* of G. Additionally, let

$$\zeta_{G,p}^{irr}(s) = \sum_{n=0}^{\infty} r_{p^n}(G)p^{-ns}. \tag{2}$$

We call this the *p-local representation zeta function* of G. Since all irreducible representations factor through a finite quotient and finite nilpotent groups are direct products of their Sylow p-subgoups, it follows that

$$\zeta_G^{irr}(s) = \prod_p \zeta_{G,p}^{irr}(s). \tag{3}$$

In [16, Theorem D], Voll shows that for almost all primes, p-local representation zeta functions of a $\mathcal{T}$-group G satisfy a functional equation in the variable p, depending only on the Hirsch length of the derived subgroup of G. This functional equation has been refined in [15, Theorem A] to include number theoretic information for groups that arise as, for a given number field, integer points of unipotent group schemes.

Let $M_n := \langle a_1, \ldots, a_n, b \mid [a_i, b] = a_{i+1} \rangle$, where all commutators that do not appear in (or follow from) the group relations are trivial. This paper is a continuation of [6] in which we construct both the irreducible representations and the p-local zeta functions of the family of groups M_n for almost all primes p. While in the previous paper we calculated the p-local zeta functions when p was non-exceptional (see [6, Section 8] for definition), in this paper we study some cases when p is indeed exceptional. For the rest of the paper, and with a slight abuse of notation, we denote the subgroup $\langle a_{n-k+1}, a_{n-k+2}, \ldots, a_n, b \rangle$ as M_k.

The techniques of representation growth have been used to study various types of groups, including $\mathcal{T}$-groups [9, 16, 15], compact Lie groups [17], arithmetic and p-adic analytic groups [13, 10, 2, 3, 1], wreath products of finite groups [4], and profinite and pro-p groups [5]. For more details, see [7, Introduction] and [11].

1.1 Main Result

Let q be an arbitrary prime. The main result of this paper is the calculation of the q-local representation zeta function for M_{q+1}, and the 2-local zeta function for M_4. Combining these results with the non-exceptional p- local representation zeta functions in [6], this gives us the global representation zeta functions for M_3 and M_4. Note the uniformity in all of the p-local zeta functions for both groups. It is striking that the exceptional prime-local zeta functions are of the same form as the non-exceptional prime-local zeta functions. We have the following theorem:

Theorem 1.1. *Let p be a prime. Then the p-local representation zeta function of the group M_{p+1} is*

$$\zeta^{irr}_{M_{p+1},p}(s) = \frac{(1-p^{-s})^2}{(1-p^{((p+1)-2)-s})(1-p^{1-s})}.$$

Theorem 1.2. *The 2-local representation zeta function of the group M_4 is*

$$\zeta^{irr}_{M_4,2}(s) = \frac{(1-2^{-s})^2}{(1-2^{1-s})(1-2^{2-s})}.$$

Combining the previous two theorems with the main result from [6] and [14, Theorem 5] we have the following corollary.

Corollary 1.3. *Let $n \in \{2,3,4\}$. Then the global representation zeta function of M_n is*

$$\zeta^{irr}_{M_n}(s) = \frac{\zeta(s-1)\zeta(s-(n-2))}{(\zeta(s))^2}.$$

1.2   Kirillov Orbit Method vs. Constructive Method

There are two methods currently in use to study $\mathcal{T}$-groups using the tools of representation growth. These are the constructive method, which appears in [7], [6], and this paper (and trivially in [14]), and the Kirillov orbit method, first used in [16] and later employed in [15]. We use the constructive method particularly to study the behaviour of some exceptional p-local zeta functions of M_n. In this section we compare and contrast these two different methods of calculating representation zeta functions of $\mathcal{T}$-groups.

The constructive method is quite general. Other than the choice of group, nothing else is assumed. The general technique could theoretically be used to calculate the representation zeta function of any $\mathcal{T}$-group. However, as the complexity of the eigenspace structure of the irreducible representations increases, the complexity of the calculation may increase as well. We do note that this method relies on less mathematical machinery than the Kirillov orbit method and thus can be appreciated with minimal technical background. This method explicitly constructs all twist isoclasses of dimension p^N for a chosen prime p and $N \in \mathbb{N}$, allowing one to easily read off the coefficients $r_{p^N}(G)$ of the representation zeta function.

Unlike the Kirillov orbit method, the main benefit of the constructive method is that primes are not excluded by the method itself. While there may be special cases that occur in the calculation for certain primes, the p-local representation zeta function of these primes are still able to be calculated. We call primes of this nature *constructive-exceptional primes*. Provided one can do the calculation, one can understand the entire representation theory of irreducibles of a $\mathcal{T}$-group by the constructive method. We are able to calculate all irreducible representations of maximal class groups M_3 and M_4 and thus their representation zeta functions. This is not possible using the Kirillov orbit method.

The main idea of the Kirillov orbit method applied to $\mathcal{T}$-groups is to count irreducible representations by exploiting the 1-to-1 correspondence, up to twisting and isomorphism, between the irreducible characters of a $\mathcal{T}$-group G of degree p^N for $N \in \mathbb{N}$ and the co-adjoint orbits of its associated Lie algebra. This

method, including the use of p-adic integration to aid in counting, is outlined in [16]. However, this correspondence holds for all but finitely many primes and by disregarding these primes we are unable to construct the p-local representation zeta function for finitely many primes p and, by extension, the global representation zeta function. We say we *lose* a prime p_* if the hypotheses of the Kirillov orbit method do not apply for p_*. We call such primes Kirillov-exceptional.

If we pass to an appropriate finite index subgroup of our $\mathcal{T}$-group G, say H, then, by [16, Section 3.4], we lose all p_* such that $p_* \mid |G : H|$. Secondly, by Howe's parametrization [8, Theorem 1.a] we lose all primes p_* such that $p_* \mid 2|G'_s : G'|$ where H_s is the isolator of $H \leq G$ and G' is the commutator subgroup of G. Next, by [16, Corollary 3.1], for all but finitely many primes we have index conditions on subgroups of G and subalgebras of L, the Lie algebra associated to G. These conditions are $|G : G_\psi| = |L : \mathrm{Rad}_\psi|$ and $|G : G_{\psi,2}| = |L : L_{\psi,2}|$, where $G_\psi, \mathrm{Rad}_\psi, G_{\psi,2}$, and $L_{\psi,2}$ are defined in [16, Section 3.4]. Thus, we lose all primes p_* where these equalities do not hold. Also, by assuming that L has the appropriate basis structure to apply the hypotheses of the Kirillov orbit method we lose finitely many primes p_*. Finally, by [16, Section 2.2], we lose primes p_* such that the antisymmetric matrix $\mathcal{R}$ encoding the commutator structure of L is a zero matrix mod p_*.

While it is true that there are only finitely many of these Kirillov-exceptional primes, comparing the p-local representation zeta functions of non-exceptional primes may not be sufficient to distinguish two $\mathcal{T}$-groups from each other. The constructive method allows for the calculation of all p-local representation zeta functions, and thus one has a finer invariant of $\mathcal{T}$-groups.

Compared to the constructive method, the comparatively mathematically deeper Kirillov orbit method allows for easier computations in many cases since one counts representations without constructing them explicitly. The machinery that appears in [16] allows one to calculate p-local representation zeta functions by, essentially, linear algebra. The Howe correspondence [8, Theorem 1.a] allows one, for almost all primes, to linearize the computation of calculating the number of p^N-dimensional irreducible representations. However, Voll's method does not explicitly (without using a linear recurrence relation) give the coefficients $r_{p^N}(G)$, for some non-exceptional p and some N. This is because it parameterizes representations in a different way than to dimension of twist isoclass.

A strength of the Kirillov orbit method is its use in studying p-local representation zeta functions in more generality than the constructive method is currently able to. Indeed, the functional equation given in [16], and its generalization in [15], is proved via the Kirillov orbit method. As it presently stands, the constructive method seems unable to prove such a result. In fact, using the Kirillov orbit method, one can understand much about p-local representation zeta functions by understanding antisymmetric matrices over the ring $\mathbb{Z}/p^N\mathbb{Z}$ for each N. This translates the problem of counting representations to linear algebra over the ring of p-adic integers.

Also, as shown in [15], the Kirillov orbit method is able to use number-theoretic information about a $\mathcal{T}$-group to help construct the p-local representation zeta functions. Indeed, the representation zeta function of the group $H_{\sqrt{d}}$ studied in [7] can be fully calculated by the Kirillov orbit method that appears in [15]. The constructive method, in its current form, "forgets" any number-theoretic structure and thus treats all $\mathcal{T}$-groups the same way.

2 Important Results from Prequel

In order to keep this paper relatively self-contained, we give a list of results and definitions that appear in [6] that are used in this paper. For the proofs of the following results, see the aforementioned paper.

Definition 2.1. Let S_p^∞ be the all complex p^ℓth roots of unity for all $\ell \in \mathbb{N}$ and S_p^k be the p^kth roots of unity (and note that $S_p^k \backslash S_p^{k-1}$ are the primitive p^kth roots of unity). Define $s : S_p^\infty \to \mathbb{N}$ such that $s(\lambda) = k$ if and only if $\lambda \in S_p^k \backslash S_p^{k-1}$. If $s(\lambda) = k$ we say that λ has *depth k*.

Let $T_0(0) = 1$, $T_0(j) = 1$, and $T_j(0) = 0$ for $j \in \mathbb{N}$ and recursively define $T_k(j) = \sum_{l=1}^j T_{k-1}(l) = T_k(j-1) + T_{k-1}(j)$ for $k \in \mathbb{N}$. The next lemma lists some properties of these numbers that are needed for this paper. We state these without proof.

Lemma 2.2. *Let $i, j, k, b \in \mathbb{N}$ and $T_k(j)$ be defined as above.*

i. $T_k(i) = \binom{i+k-1}{k} = \frac{i(i+1)\dots(i+k-1)}{k!}$.

ii. Let $p > k$. Then for any $b \in \mathbb{N}$ and α such that $1 \le \alpha \le p - 1$ we have $T_k(\alpha p^b + j) = T_k(j) \mod p^b$.

iii. If $p > k$ then $T_k(p^N - 1) = 0 \mod p^N$.

As a corollary of (ii) we have the following.

Corollary 2.3. *Let p be a prime, let $k < p$, let $N \ge 1$, let $1 \le m \le N$, let $\alpha \in \mathbb{N}$ such that $p \nmid \alpha$, and, for $j \ge 0$, let*

$$\Gamma(k, j) = \alpha p^m T_k(j-1). \tag{4}$$

Then we have that $\Gamma(k, \beta p^{N-m} + j + 1) = \Gamma(k, j+1) \mod p^N$ for all β such that $1 \le \beta < p^m$ and all j such that $0 \le j \le p^{N-m} - 1$.

Lemma 2.4. *For $1 \le i \le n-1$ we have that $\lambda_{i,j} = \prod_{k=i}^n \lambda_k^{T_{k-i}(j-1)}$ and thus the matrix x_i has the structure*

$$x_i = \begin{pmatrix} \lambda_i & & & \\ & \prod_{k=i}^n \lambda_k^{T_{k-i}(1)} & & \\ & & \ddots & \\ & & & \prod_{k=i}^n \lambda_k^{T_{k-i}(p^N-1)} \end{pmatrix}. \tag{5}$$

Moreover we have that

$$\lambda_i^{p^N} \prod_{k=i+1}^n \lambda_k^{T_{k-i}(p^N-1)} = 1. \tag{6}$$

Definition 2.5. The matrices $x_1, \dots, x_n, y$ are in *standard form* if the x_i are in the form of Lemma 2.4 and y is in the form

$$y = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}. \tag{7}$$

We say ρ is in *standard form* if, under a chosen basis, the matrices $x_1, \ldots, x_n, y$ are in standard form.

Let $\lambda_{i,j}$ be the jth diagonal entry of x_i and let $\lambda_i = \lambda_{i,1}$. Let V_{p^k} be the subspace spanned by $\langle y \rangle \cdot (e_1 + e_{p^k+1} + \ldots + e_{(p^{N-k}-1)p^k+1})$. Also, we define the n-tuple $\Lambda_n(k) := (\lambda_{1,k}, \ldots, \lambda_{n,k})$ where k is considered mod p^N.

Lemma 2.6. *For any k_1, k_2 if $\Lambda_n(k_1) = \Lambda_n(k_2)$ then $\Lambda_n(k_1 + 1) = \Lambda_n(k_2 + 1)$.*

Corollary 2.7. *Let β such that $0 \leq \beta \leq p^{N-j} - 1$, and let j be the minimal power such that $\Lambda_n(k) = \Lambda_n(\beta p^j + k)$. Then V_{p^j} is a stable subspace of ρ and $V_{p^{j-1}}$ is not stable.*

We define notation to this effect. Let $H \leq M_n$ and let $\mathcal{V}(\rho|_H)$ be the the minimal stable subspace V_{p^j}, as in Corollary 2.7, of $\rho|_H$. We say that $\mathcal{V}(\rho) = \mathcal{V}\big(\rho(M_n)\big)$.

Corollary 2.8. *The following are equivalent:*

1. *The number j is minimal such that $\Lambda_n(1) = \Lambda_n(p^j + 1)$*

2. *$\mathcal{V}(\rho) = V_{p^j}$.*

Corollary 2.9. *Let $\rho : M_n \to GL_{p^N}(\mathbb{C})$ be a representation. Then, for $k < n$ if $\mathcal{V}(\rho|_{M_k}) = V_{p^j}$ then $\mathcal{V}(\rho) = V_{p^\ell}$ for some ℓ such that $\ell \geq k$.*

We know that if V_{p^k} is ρ-stable then so is V_{p^j} for $j \geq k$. Thus, we obtain the following corollary:

Corollary 2.10. *Let ρ be a representation of M_n. The representation ρ is irreducible if and only if $V_{p^{N-1}}$ is not ρ-stable.*

Throughout this paper we use Corollary 2.10 to check if a representation ρ is irreducible. We use Corollary 2.8 to determine the number of isomorphic representations in standard form in one twist isoclass.

Lemma 2.11. *Let S_ρ be the twist isoclass represented by ρ and let $\mathcal{V}(\rho|_{M_{n-1}}) = V_{p^m}$. Then there are p^m representations in standard form in S_ρ that are twist-and-shout equivalent to ρ.*

Definition 2.12. *Let ρ be irreducible and let x_i, y, for i such that $1 \leq i \leq n$, be in standard form as defined earlier in the section. A *shout* is a matrix P such that, up to twisting, PyP^{-1} and Px_iP^{-1} for $i = 1 \ldots n$ are in standard form. The representations ρ and $P\rho P^{-1}$ (note that Px_1P^{-1} may not be in standard form) are said to be equivalent under *shouting*.*

We use the following proposition and lemma to help count twist-and-shout equivalent representations in each isoclass.

Proposition 2.13. *Let $p \geq n$ and ρ be a p^N-dimensional representation of M_n with corresponding matrices in standard form. Then ρ is irreducible if and only if there exists a λ_i such that $s(\lambda_i) = N$, where $2 \leq i \leq n$.*

We slightly change the form of this lemma from its version in [6].

Lemma 2.14. *For $p \geq n - 1$ let ρ be an irreducible p^N-dimensional representation of M_n and let $k = \max\{s(\lambda_3), \ldots s(\lambda_n)\}$. Then there are p^k representations in standard form equivalent to ρ under twisting and shouting.*

3 The p-local Representation Zeta Function for M_{p+1}

For a prime p, we study the p-local representations of M_n when $n = p + 1$. We calculate the exceptional prime representation growth zeta function $\zeta_{M_{p+1},p}^{irr}(s)$. Note that, unlike the non-exceptional calculation in [6], p is fixed by our choice of group for this calculation.

Let ρ be a p^N-dimensional representation. We will determine the choices of λ_i for which ρ is irreducible. We can choose a basis of the form in Lemma 2.4.

We divide this calculation into two cases: when $s(\lambda_{p+1}) = N$ and when $s(\lambda_{p+1}) \leq N - 1$. Furthermore, we break the second case into two sub-cases: when there is a λ_i with $3 \leq i \leq p$ such that $s(\lambda_i) = N$ and when there is no such λ_i. Call these cases Case 1, and Case 2, respectively. We call Case 2's respective subcases Case 2.1 and Case 2.2. Note that, since p is not exceptional when considering $\rho|_{M_p}$ (and we remind the reader that $M_p = \langle y, x_2, \ldots, x_n \rangle$), we can apply Lemma 2.14 when determining the number of representations twist-and-shout equivalent to some irreducible ρ.

Case 1 Assume that $s(\lambda_{p+1}) = N$.
By [14], we have that $\rho|_{M_2}$ is an irreducible representation and thus ρ is irreducible. We now must determine for which choices of λ_i the representation is well defined.

By Lemma 2.2(iii), it is clear that we can use Equation 6 in Lemma 2.4 to show that $s(\lambda_i) \leq N$ for $i \neq 2$ and

$$\lambda_2^{p^N} \prod_{k=3}^{p+1} \lambda_k^{T_{k-1}(p^N - 1)} = 1. \tag{8}$$

Since $s(\lambda_i) \leq N$ for $3 \leq i \leq p$ by Lemma 2.2(iii) the preceding equation simplifies to

$$\lambda_2^{p^N} \lambda_{p+1}^{T_p(p^N - 1)} = 1. \tag{9}$$

Cancelling out the p from the denominator of $T_p(p^N - 1)$, we have that

$$T_p(p^N - 1) = \alpha p^{N-1} \tag{10}$$

for some α coprime to p. Thus $s(\lambda_{p+1}^{\alpha p^{N-1}}) = 1$ and by Equation 9 we have that $s(\lambda_2) = N + 1$. There are p^N choices for λ_2 so that Equation 9 holds. Thus, there are $(1 - p^{-1})p^N$ choices for λ_{p+1} and p^N choices for each λ_i where $2 \leq i \leq p$. By Lemma 2.14 we must divide by p^N to take shouting into account. Therefore in this case there are

$$(1 - p^{-1})p^N p^{(p-1)N} p^{-N} = (1 - p^{-1})p^{(p-1)N} \tag{11}$$

twist isoclasses. Note that the right hand side of Equation 11 is also the contribution to r_{p^N} in the non-exceptional case in [6] for when $s(\lambda_{p+1}) = N$.

Case 2 Now assume $s(\lambda_{p+1}) \leq N - 1$.
It is clear, since $T_i(p^N - 1) = 0 \mod p^N$ for $i < p$ by Lemma 2.2(iii) and since $\lambda_{p+1}^{T_p(p^N - 1)} = 1$ by Equation 10, that we can say that $s(\lambda_i) \leq N$ for $2 \leq i \leq p+1$. We now break this case into subcases.

Case 2.1 For i such that $3 \leq i \leq p$, assume one of $s(\lambda_i) = N$, say λ_k. Then, since $p \geq k$, by Proposition 2.13 we have that $\rho|_{M_{p+1-k+2}}$ is an irreducible representation and thus ρ is irreducible. In this case there are $(1 - p^{-(p-2)})p^{(p-2)N}$ choices for λ_i, p^N choices for λ_2, and p^{N-1} choices for λ_{p+1}. By Lemma 2.14 we must divide by p^N to take shouting into account. Thus there are

$$(1 - p^{-(p-2)})p^{(p-2)N}p^{N-1}p^{N}p^{-N} = (1 - p^{-(p-2)})p^{(p-1)N-1} \tag{12}$$

twist isoclasses in this case. We note that the contribution to r_{p^N} in this case is the same contribution to r_{p^N} for non-exceptional primes [6, Section 9].

Case 2.2 Assume $s(\lambda_i) \leq N - 1$ where $3 \leq i \leq p$.
Note that in this case $\rho|_{M_p}$ has $V_{p^{N-1}}$ as a proper stable subspace so by Lemma 2.10 it is not irreducible. If $s(\lambda_{p+1}) = 0$ then M_{p+1} is isomorphic to M_p and by Proposition 2.13 the representation ρ is irreducible if and only if $s(\lambda_2) = N$.

Now let $s(\lambda_{p+1}) \geq 1$. We choose λ_* such that $s(\lambda_*) = N$ and write each λ_i in terms of it; that is, let $\lambda_i = \lambda_*^{\alpha_i p^{m_i}}$, $p \nmid \alpha_i$, $m_2 \geq 0$, and $m_i \geq 1$ for $3 \leq i \leq p+1$.

We appeal to Lemma 2.8 and determine when $\langle y, x_1 \rangle$ does not have $V_{p^{N-1}}$ as a proper stable subspace. This is the case exactly when $\lambda_{1,1} \neq \lambda_{1,p^{N-1}+1}$.

Consider $\lambda_{1,p^{N-1}+1}$. If $N = 1$, then in order for ρ not to be trivial we have that $s(\lambda_2) = 1$ and it is easily verified that x_1 is not scalar and thus ρ is irreducible. Now, for $N \geq 2$, we have that

$$\Lambda := \log_{\lambda_*}(\lambda_{1,p^{N-1}+1}) \tag{13}$$
$$= \alpha_2 p^{m_2}(p^{N-1}) + \alpha_3 p^{m_3}\frac{(p^{N-1})(\beta p^{N-1} + 1)}{2} + \dots$$
$$+ \alpha_{p+1}p^{m_{p+1}}\frac{(p^{N-1})\dots(\beta p^{N-1} + p - 1)}{p!} \mod p^N.$$

By Corollary 2.3 and keeping in mind that $m_i \geq 1$ for $3 \leq i \leq p+1$ this simplifies to the following:

$$\Lambda = \alpha_2 p^{m_2}p^{N-1} + \alpha_{p+1}p^{m_{p+1}-1}\frac{p^{N-1}\dots(p^{N-1} + p - 1)}{(p-1)!} \mod p^N \tag{14}$$
$$= p^{N-1}\left(\alpha_2 p^{m_2} + \alpha_{p+1}p^{m_{p+1}-1}\frac{(p^{N-1} + 1)\dots(p^{N-1} + p - 1)}{(p-1)!}\right) \mod p^N$$

Note that the last term has a denominator of $(p-1)!$ since the factor of p was subtracted from m_{p+1}.

We want ρ to be irreducible. Thus, by Wilson's Theorem it must be that

$$\alpha_2 p^{m_2} + \alpha_{p+1}p^{m_{p+1}-1} \neq 0 \mod p. \tag{15}$$

We now enumerate the cases when we do not have a factor of p, and thus an irreducible representation. By Equation 15 this is precisely when $m_{p+1} = 1$ or $m_2 = 0$ except when $m_{p+1} = 1$, $m_2 = 0$, and $\alpha_2 \neq -\alpha_{p+1} \mod p$.

We still need to take shouting into account. Therefore, by Lemma 2.14, we must divide our count, if we enumerated the representations in this case at this stage, by p^{m_*} where $m_* = \max\{s(\lambda_3), \dots, s(\lambda_{p+1})\}$. Note that, since p is non-exceptional when considering $\rho|_{M_p}$, the shouting behaviour is the same as

8

in the non-exceptional case. We will count these later.

This ends the case distinctions.

We note that the only difference between the r_{p^N} for this exceptional prime and the r_{p^N} for non-exceptional primes is the situation when we can choose λ_2 and λ_{p+1} such that (still thinking of all λ_i written as powers of λ_*) $m_{p+1} = 1$ and $m_2 \geq 1$, which gives us additional irreducible representations, and when $m_{p+1} = 1, m_2 = 0$, and $\alpha_2 \neq -\alpha_{p+1} \mod p$, which gives us representations that are no longer irreducible. Therefore, starting with r_{p^N} calculated for non-exceptional primes, we can add the cases where our choices of λ_i give us additional representations and subtract the cases where we lose representations.

Let C be r_{p^N} for non-exceptional primes, that is the sum in [6, Find Equation!!!!] The situation where $m_{p+1} = 1$ and $m_2 \geq 1$ does not correspond to irreducible representations for non-exceptional primes, but does for exceptional primes. There are $(1 - p^{-1})p^{N-1}$ choices for λ_{p+1} and p^{N-1} choices for λ_2 in this case. Remembering that we assumed that $s(\lambda_i) \leq N - 1$ for $3 \leq i \leq p$ then there are $p^{(p-2)(N-1)}$ choices for these λ_i. By Lemma 2.14 we must divide by p^{N-1} to take shouting into account. Therefore we must add

$$(1 - p^{-1})p^{N-1}p^{N-1}p^{(p-2)(N-1)}p^{-(N-1)} = (1 - p^{-1})p^{(p-1)(N-1)} \qquad (16)$$

to C.

The situation where $m_{p+1} = 1$, $m_2 = 0$, and $\alpha_2 = -\alpha_{p+1} \mod p$ does correspond to irreducible representations for non-exceptional primes, but does not for exceptional primes. There are $(1 - p^{-1})p^N$ choices for λ_2 and, given our choice for λ_2, there are p^{N-2} choices for λ_{p+1} in this case. Remembering that we assumed that $s(\lambda_i) \leq N - 1$ for $3 \leq i \leq p$ there are $p^{(p-2)(N-1)}$ choices for these λ_i. By Lemma 2.14 we must divide by p^{N-1} to take shouting into account. Therefore we must subtract

$$(1 - p^{-1})p^N p^{N-2} p^{(p-2)(N-1)} p^{-(N-1)} = (1 - p^{-1})p^{(p-1)(N-1)} \qquad (17)$$

from C. Notice that $(16) = (17)$. Therefore

$$r_{p^N} = C \qquad (18)$$

and

$$\zeta_{M_{p+1},p}^{irr}(s) = \frac{(1 - p^{-s})^2}{(1 - p^{((p+1)-2)-s})(1 - p^{1-s})} \qquad (19)$$

by [6, Equation 31].

This result, and the result from [6], gives us the entire irreducible representation theory, as well as the representation zeta function, of M_3. In fact, we can say that

$$\zeta_{M_3}^{irr}(s) = \left(\frac{\zeta(s-1)}{\zeta(s)} \right)^2 . \qquad (20)$$

4 The 2-local Representation Zeta Function for M_4

We now have a complete understanding of the irreducible representations of M_3. The aim of this section is to do the same for M_4. Our previous work leaves us with only one p-local zeta function to calculate; the previous section calculates the 3-local zeta function and 2 and 3 are the only exceptional primes. Therefore once we calculate the 2-local representation zeta function we have $\zeta_{M_4}^{irr}(s)$ in its entirety.

The complexity of this calculation lies partly in the inability to use Lemma 2.14 in many cases. Thus, some work is to be done to calculate the correct factor by which we are overcounting.

For ease of computation, we calculate $r_2(M_4)$ separately later in this section. Until noted otherwise we assume the condition that $N \geq 2$. In keeping with the style of the general cases earlier, and for elucidation if one wishes to generalize this calculation, we do not simplify the expressions $(1 - 2^{-1})$ to 2^{-1} as far in the calculation as possible.

Let $\rho : M_4 \to GL_{2^N}(\mathbb{C})$ be a representation. By Equation 6 in Lemma 2.4 we have that

$$\lambda_4^{2^N} = 1, \tag{21}$$

$$\lambda_3^{2^N} \lambda_4^{T_2(2^N-1)} = 1, \tag{22}$$

and

$$\lambda_2^{2^N} \lambda_3^{T_2(2^N-1)} \lambda_4^{T_3(2^N-1)} = 1. \tag{23}$$

Therefore, by Equation 21, we have that $s(\lambda_4) \leq N$.

Before we begin counting twist isoclasses we must determine the possible depths of λ_2 and λ_3. We remind the reader of Equation 10. Assume $s(\lambda_4) \leq N-1$. Then $\lambda_4^{T_2(2^N-1)} = \lambda_4^{T_3(2^N-1)} = 1$ and by Equation 22 we have that $\lambda_3^{2^N} = 1$ and thus $s(\lambda_3) \leq N$. If $s(\lambda_3) \leq N-1$ then $\lambda_3^{T_2(2^N-1)} = 1$ and by Equation 23 we have that $\lambda_2^{2^N} = 1$ so $s(\lambda_2) \leq N$. If $s(\lambda_3) = N$ then $\lambda_3^{T_2(2^N-1)} = \lambda_3^{-2^{N-1}}$ and thus $s(\lambda_3^{-2^{N-1}}) = 1$. By Equation 23 we have that $\lambda_2^{2^N} = \lambda_3^{2^{N-1}}$ and thus $\lambda_2^{2^N}$ must satisfy this equation. So $\lambda_2^{2^N} = -1$ and $s(\lambda_2) = N+1$.

Now assume $s(\lambda_4) = N$. Then $\lambda_4^{T_2(2^N-1)} = \lambda_4^{T_3(2^N-1)} = \lambda_4^{-2^{N-1}}$ and thus $s(\lambda_4^{-2^{N-1}}) = 1$. By Equation 22 we have that

$$\lambda_3^{2^N} = \lambda_4^{2^{N-1}} \tag{24}$$

and thus $\lambda_3^{2^N}$ must satisfy this equation. So $\lambda_3^{2^N} = -1$ and $s(\lambda_3) = N+1$. We have that $\lambda_3^{T_2(2^N-1)} = \lambda_3^{-2^{N-1}}$ and by Equations 23 and 24 we have that $\lambda_2^{2^N} = \lambda_3^{2^{N-1}} \lambda_4^{2^{N-1}} = \lambda_3^{2^N + 2^{N-1}} = \lambda_3^{(1+2)2^{N-1}}$. Note that we leave $(1+2)$ in this form since we wish to keep the form $(1+p)$. Thus $s(\lambda_3^{(1+2)2^{N-1}}) = 2$ and $\lambda_2^{2^N}$ must satisfy Equation 23. So $\lambda_2^{2^N} = \pm\sqrt{-1}$ and $s(\lambda_2) = N+2$.

We break our computation into eight cases, with Cases 6 and 7 being further broken down into subcases. Tables 1,2, and 3 show, repsectively, the number of twist isoclasses in each case, for the subcases of Case 6, and for the subcases of Case 7. We leave the computation of Cases 1,3, and 4 to the reader; these follow almost immediately from previous computations.

Table I: Table of Cases for M_4

Case	$s(\lambda_4)$	$s(\lambda_3)$	$s(\lambda_2)$	Other Conditions	No. of twist isoclasses, $N \geq 2$
1	$= N$	$= N+1$	$= N+2$		$(1-2^{-1})^4 2^{2N+3}$
2	$= N-1$	$= N$	$= N+1$	$\alpha_3 = 3 \mod 4$	$(1-2^{-1})^3 2^{2N}$
3	$= N-1$	$\leq N-1$	$\leq N$		$(1-2^{-1})2^{2N-2}$
4	$\leq N-2$	$= N$	$= N+1$		$(1-2^{-1})^2 2^{2N-1}$
5	$\leq N-2$	$= N-1$	$= N$		0
6	$\leq N-2$	$\leq N-2$	$= N$		See Table II on page 11
7	$\leq N-2$	$= N-1$	$\leq N-1$		See Table III on page 11
8	$\leq N-2$	$\leq N-2$	$\leq N-1$		0

Table II: Case 6 of Table I

N	Case	Relationship of $s(\lambda_3)$ and $s(\lambda_4)$	No. of twist isoclasses
$= 2$	6.4	$s(\lambda_3) = s(\lambda_4) = 0$	2
$=3$	6.2	$s(\lambda_3) \leq 1, s(\lambda_4) = 1$	$(1-2^{-1})^2 2^3$
	6.4	$s(\lambda_4) = 0$	$(1-2^{-1})2^3(1 + (1-2^{-1}))$
≥ 4	6.1	$s(\lambda_3) > s(\lambda_4) + 1, s(\lambda_3) \geq 2, s(\lambda_4) \neq 0$	$\left[(1 - 2^{-1})2^N\left((2^{N-4} - 1) - (1-2^{-1})(N-4)\right)\right]$
	6.2	$s(\lambda_3) < s(\lambda_4) + 1, s(\lambda_4) \neq 0$	$(1-2^{-1})2^N(2^{N-3} - 2^{-1})$
	6.3	$s(\lambda_3) = s(\lambda_4) + 1, s(\lambda_4) \neq 0$	$(1-2^{-1})^2 2^N(2^{N-2} - 2)$
	6.4	$s(\lambda_4) = 0$	$(1-2^{-1})^2 2^N(1+(1-2^{-1})(N-2))$

Table III: Case 7 of Table I

N	Case	No. of twist isoclasses
$= 2$		1
≥ 3	7.1	$(1-2^{-1})2^{2N-4}$
	7.2	$(1-2^{-1})^2 2^{2N-2}$

11

Case 2: By Case 2.2 of Section 4 we have that $\rho|_{M_3}$ is reducible. Appealing to Lemma 2.10 we must check whether $V_{2^{N-1}}$ is a stable subspace of $\langle y, x_1 \rangle$. We write each root of unity in terms of a primitive 2^{N+1}th one. Let $\lambda_* = \lambda_2$, $\lambda_i = \lambda_*^{\alpha_i 2^{m_i}}$ for some α_i such that $2 \nmid \alpha_i, m_4 = 2, m_3 = 1$, and $i \in \{3, 4\}$.

Using Corollary Corollary 2.8 and noting that $2 \cdot (2^{N-1})^2 = 0 \mod 2^{N+1}$ for $N = 2$, consider $\lambda_{1,2^{N-1}+1}$:

$$
\begin{aligned}
\log_{\lambda_*} & (\lambda_{1,2^{N-1}+1}) \hspace{4cm} (25)\\
&= (2^{N-1}) + 2^{1-1}\alpha_3(2^{N-1})(2^{N-1}+1) \\
&\quad + \alpha_4 2^{2-1} \frac{(2^{N-1})(2^{N-1}+1)(2^{N-1}+2)}{3} \quad \mod 2^{N+1} \\
&= 2^{N-1}\left(1 + \alpha_3 + \alpha_4 2^1 \frac{2}{3}\right) \quad \mod 2^{N+1} \\
&= \log_{\lambda_*}(\lambda_1) + 2^{N-1}\left[1 + \alpha_3\right] \quad \mod 2^{N+1}
\end{aligned}
$$

So the expression in the square brackets above is a multiple of 4 if and only if $V_{2^{N-1}}$ is a $\langle y, x_1 \rangle$-stable subspace. Let Q be the aforementioned expression. It is clear that $Q = 0 \mod 4$ precisely when $\alpha_3 = 3 \mod 4$. This means that we are only free to choose half of the elements of S_2^N / S_2^{N-1} for λ_3. Thus, there are $(1 - 2^{-1})2^{N-1}$ choices for λ_3, $(1 - 2^{-1})2^{N+1}$ choices for λ_2, and $(1 - 2^{-1})2^{N-1}$ choices for λ_4. Since $\rho|_{M_3}$ is not irreducible it has at least $V_{2^{N-1}}$ as a stable subspace. But since $s(\lambda_4) = N - 1$, by Corollary 2.9 we have that $\mathcal{V}(\rho|_{M_3}) = V_{2^{N-1}}$. Thus, by Lemma 2.11 we must divide by 2^{N-1} to take shouting into account. So in this case we have

$$
\begin{aligned}
(1 - 2^{-1})2^{N-1}(1 - 2^{-1})&2^{N+1}(1 - 2^{-1})2^{N-1}2^{-(N-1)} \hspace{1cm} (26)\\
&= (1 - 2^{-1})^3 2^{2N}
\end{aligned}
$$

twist isoclasses.

Cases 5 and 6: We note $s(\lambda_2) = N$ and $s(\lambda_4) \leq N - 2$ for both cases. We have, by Case 2.2 of Section 4, that $\rho|_{M_3}$ has $V_{2^{N-1}}$ as a proper stable subspace. Appealing to Lemma 2.10, we check whether $V_{2^{N-1}}$ is a stable subspace of $\langle y, x_1 \rangle$. We let $\lambda_* = \lambda_2$ and write each λ_i as a power of λ_*; that is, let $\lambda_4 = \lambda_*^{\alpha_4 2^{m_4}}$ and $\lambda_3 = \lambda_*^{\alpha_3 2^{m_3}}$ such that $2 \nmid \alpha_i$ $m_3 \geq 1$, $m_4 \geq 2$, and $i \in \{3, 4\}$. If $m_4 = N$ then by Case 2.2 of Section 4 we have that ρ is irreducible if and only if $m_3 \neq 1$. If $m_3 = N$ it is easy to show that $\log_{\lambda_*}(\lambda_{1,2^{N-1}+1}) \neq 1$. We leave this to the reader. Assume that $m_3, m_4 \neq N$.

Appealing to Corollary 2.8, consider $\lambda_{1,2^{N-1}+1}$, noting that $2^{2N-2} = 0 \mod 2^N$:

$$
\begin{aligned}
\Lambda := \log_{\lambda_*} & (\lambda_{1,2^{N-1}+1}) \hspace{4cm} (27)\\
&= (2^{N-1}) + \alpha_3 2^{m_3-1}(2^{N-1})(2^{N-1}+1) \\
&\quad + \alpha_4 2^{m_4-1}\frac{2^{N-1}(2^{N-1}+1)(2^{N-1}+2)}{3} \quad \mod 2^N \\
&= \log_{\lambda_*}(\lambda_1) + 2^{N-1}\left[1 + \alpha_3 2^{m_3-1}\right] \quad \mod 2^N.
\end{aligned}
$$

So when the term in the square brackets above, say Q, is not $0 \mod 2$ then $\lambda_1 \neq \lambda_{1,2^{N-1}+1}$. It follows that $V_{2^{N-1}}$ is not a stable subspace of ρ and therefore

ρ is irreducible. Thus Q is $0 \mod 2$ when $m_3 = 1$; that is when $s(\lambda_3) = N - 1$. So in Case 5 there are no irreducible representations.

If $m_3 \geq 2$ it is clear that $Q \neq 0 \mod 2$. Thus, in Case 6 there are 2^{N-2} choices for λ_4, 2^{N-2} choices for λ_3, and $(1 - 2^{-1})2^N$ choices for λ_2.

We now need to analyze the shouting behaviour for this case. It is clear, since $\mathcal{V}(\rho|_{M_2}) = V_{2^{s(\lambda_4)}}$ by Lemma 2.14 and Corollary 2.9, that there are at least $2^{s(\lambda_4)} = 2^{N-m_4}$ representations twist and shout-equivalent to ρ. We now determine $\mathcal{V}(\rho|_{M_3})$ for each possible choice of m_3 and m_4. Let $m_4 \neq N$. We deal with the case $m_4 = N$ in the next lemma. Also, note that we use the power of Corollary 2.8 for this computation.

Consider, for some k such that $1 \leq k \leq m_4$,

$$
\begin{aligned}
&\log_{\lambda_*}(\lambda_{2,2^{N-k}+1}) - \log_{\lambda_*}(\lambda_{2,1}) \qquad\qquad\qquad (28)\\
&= \alpha_3 2^{m_3} 2^{N-k} + \alpha_4 2^{m_4-1} 2^{N-k}(2^{N-k} + 1) \mod 2^N\\
&= 2^{N-k}\left[\alpha_3 2^{m_3} + \alpha_4 2^{m_4-1}(2^{N-k} + 1)\right] \mod 2^N.
\end{aligned}
$$

For the following lemma let Q be the sum in the square brackets above. By Lemma 2.8, if $Q = 0 \mod 2^k$ then $\lambda_{2,1} = \lambda_{2,2^{N-k}+1}$ and V_{p^k} is a proper stable subspace of $\rho|_{M_3}$.

Lemma 4.1. *Let $m_4 \neq N$ and let $m_* = \min\{m_3, m_4 - 1\}$. If $m_3 \neq m_4 - 1$ then $\mathcal{V}(\rho|_{M_3}) = V_{2^{N-m_*}}$. If $m_3 = m_4 - 1$ then $\mathcal{V}(\rho|_{M_3}) = V_{2^{N-m_4}}$.*
If $m_4 = N$ then $\mathcal{V}(\rho|_{M_3}) = V_{2^{N-m_3}}$.

Proof. Assume $m_4 \neq N$. If $m_3 \neq m_4 - 1$ the maximum value of k such that $Q = 0 \mod 2^k$ is $\min\{m_3, m_4 - 1\}$. If $m_3 = m_4 - 1$ then, since both terms in Q are of the same 2-adic valuation, the maximal value of k is at least m_4. However, since $\mathcal{V}(\rho|_{M_2}) = V_{2^{s(\lambda_4)}}$, by Corollary 2.9 it follows that $\mathcal{V}(\rho|_{M_3}) = V_{2^{s(\lambda_4)}}$.

Now let $m_4 = N$. Then,

$$
\begin{aligned}
&\log_{\lambda_*}(\lambda_{2,2^{N-k}+1}) - \log_{\lambda_*}(\lambda_{2,1}) \qquad\qquad\qquad (29)\\
&= \alpha_3 2^{m_3} 2^{N-k} = 0 \mod 2^N
\end{aligned}
$$

when $k \leq m_3$. Thus k is maximal when $k = m_3$ and $\mathcal{V}(\rho|_{M_3}) = V_{2^{s(\lambda_3)}}$ when $m_4 = N$. $\qquad\square$

We now count the number of twist isoclasses. To do this we break the computation into four subcases. Note that, in all subcases, there are $(1-2^{-1})2^N$ choices for λ_2. For the first three subcases we assume that $s(\lambda_4) \neq 0$.

Case 6.1 For some M such that $2 \leq M \leq N-2$, let $s(\lambda_3) = M > s(\lambda_4)+1$. We have that there are $(1 - 2^{-1})2^M$ choices for λ_3 and $2^{M-2} - 1$ choices for λ_4. Since $s(\lambda_3) = M > s(\lambda_4) + 1$ by Lemmas 2.11 and 4.1 we must divide by 2^M to take shouting into account. Thus, in this subcase there are

$$
\begin{aligned}
&(1 - 2^{-1})2^N \sum_{M=2}^{N-2} (1 - 2^{-1})2^M (2^{M-2} - 1)2^{-M} \qquad\qquad (30)\\
&= (1 - 2^{-1})2^N\left((2^{N-4} - 1) - (1 - 2^{-1})(N - 4)\right)
\end{aligned}
$$

twist isoclasses. Note that when $M = 2$ we have that $(2^{M-2} - 1) = 0$.

13

Case 6.2 For some M such that $1 \leq M \leq N - 2$, let $s(\lambda_4) = M$ and $s(\lambda_3) < s(\lambda_4) + 1 = M + 1$. There are $(1 - 2^{-1})2^M$ choices for λ_4 and 2^M choices for λ_3. By Lemmas 2.11 and 4.1 we must divide by 2^{M+1} to take shouting into account. Thus in this subcase there are

$$(1 - 2^{-1})2^N \sum_{M=1}^{N-2} (1 - 2^{-1})2^M 2^M 2^{-(M+1)} \tag{31}$$
$$= (1 - 2^{-1})2^N(2^{N-3} - 2^{-1})$$

twist isoclasses.

Case 6.3 For some M such that $2 \leq M \leq N - 2$, let $s(\lambda_3) = M = s(\lambda_4) + 1$. We have that there are $(1 - 2^{-1})2^M$ choices for λ_3 and $(1 - 2^{-1})2^{M-1}$ choices for λ_4. Since $s(\lambda_3) = M = s(\lambda_4) + 1$ by Lemmas 2.11 and 4.1 we must divide by 2^{M-1} to take shouting into account. Thus, in this subcase there are

$$(1 - 2^{-1})2^N \sum_{M=2}^{N-2} (1 - 2^{-1})^2 2^M 2^{M-1} 2^{-(M-1)} \tag{32}$$
$$= (1 - 2^{-1})^2 2^N(2^{N-2} - 2)$$

twist isoclasses.

Case 6.4 Assume $s(\lambda_4) = 0$. Let $s(\lambda_3) = M$ for $0 \leq M \leq N - 2$. If $M > 0$ there are $(1 - 2^{-1})2^M$ choices for λ_3 and there is 1 choice for λ_3 if $M = 0$. There is only 1 choice for λ_4. By Lemmas 2.11 and 4.1 we must divide by 2^M to take shouting into account. Thus in this subcase there are

$$(1 - 2^{-1})2^N \left(1 + \sum_{M=1}^{N-2} (1 - 2^{-1})2^M 2^{-M}\right) \tag{33}$$
$$= (1 - 2^{-1})2^N \left(1 + (1 - 2^{-1})(N - 2)\right)$$

twist isoclasses.

This ends the subcase distinctions.

Thus, summing together all subcases we have that there are

$$(1 - 2^{-1})2^N \left((2^{N-4} - 1) - (1 - 2^{-1})(N - 4)\right) + (1 - 2^{-1})2^N(2^{N-3} - 2^{-1}) \tag{34}$$
$$+ (1 - 2^{-1})^2 2^N(2^{N-2} - 2) + (1 - 2^{-1})2^N \left(1 + (1 - 2^{-1})(N - 2)\right)$$
$$= (1 - 2^{-1})2^N(2^{N-2} + 2^{N-4} - 2^{-1})$$

twist isoclasses in Case 6 when $N \geq 4$. When $N = 3$ we sum together Cases 6.2 and 6.4. Thus there are

$$(1 - 2^{-1})2^3 \left((1 - 2^{-1}) + 1 + (1 - 2^{-1})\right) \tag{35}$$
$$= 8$$

twist isoclasses in Case 6. When $N = 2$ we only include Case 6.4 and thus there are

$$(1 - 2^{-1})2^2(1) = 2 \tag{36}$$

twist isoclasses in Case 6.

Cases 7 and 8: We note for both cases $s(\lambda_2) \leq N-1$ and $s(\lambda_4) \leq N-2$. By Case 2.2 of Section 4, we have that $\rho|_{M_3}$ has $V_{2^{N-1}}$ as a proper stable subspace, as with the previous two cases. We check whether $V_{2^{N-1}}$ is a stable subspace of $\langle y, x_1 \rangle$. As usual, we choose a $\lambda_* \in S_2^N / S_2^{N-1}$ and write each λ_i as a power of λ_*; that is $\lambda_4 = \lambda_*^{\alpha_i 2^{m_i}}$ such that $2 \nmid \alpha_i$, $m_2 \geq 1$, $m_3 \geq 1$, $m_4 \geq 2$, and $i \in \{2, 3, 4\}$.

Appealing to Corollary 2.8, consider $\lambda_{1,2^{N-1}+1}$, noting that $2^{2N-2} = 0$ mod 2^N:

$$
\begin{aligned}
\Lambda := {} & \log_{\lambda_*}(\lambda_{1,2^{N-1}+1}) && (37) \\
= {} & \alpha_2 2^{m_2} 2^{N-1} + \alpha_3 2^{m_3-1} 2^{N-1}(2^{N-1}+1) \\
& + \alpha_4 2^{m_4-1}\frac{2^{N-1}(2^{N-1}+1)(2^{N-1}+2)}{3} \quad \text{mod } 2^N \\
= {} & \log_{\lambda_*}(\lambda_1) + 2^{N-1}[\alpha_2 2^{m_2} + \alpha_3 2^{m_3-1}] \quad \text{mod } 2^N.
\end{aligned}
$$

Clearly if $m_3 \geq 2$ then the expression in the square brackets above, say Q, is $0 \mod 2$ and $V_{2^{N-1}}$ is indeed a stable subspace of ρ. If $m_3 = 1$ then Q is not $0 \mod 2$ and $V_{2^{N-1}}$ is not a stable subspace of ρ. Therefore ρ is irreducible. So we have that in Case 8 there are no twist isoclasses. In Case 7 there are 2^{N-2} choices for λ_4, $(1-2^{-1})2^{N-1}$ choices for λ_3, and 2^{N-1} choices for λ_2.

We now determine the behaviour of shouting in this case. It is easy to see that $\mathcal{V}(\rho|_{M_2}) = V_{2^{s(\lambda_4)}}$ and thus $\mathcal{V}(\rho|_{M_3})$ is no smaller than $V_{2^{s(\lambda_4)}}$.

Let $N \geq 3$; we calculate the case when $N = 2$ separately later in the section. We write λ_3, λ_4 in terms of some $\lambda_* \in S_2^N \backslash S_2^{N-1}$ in the usual way, with $m_3 = 1$ and m_4 such that $2 \leq m_4 \leq N$. If $m_4 = N$ then it is easy to show that $\mathcal{V}(\rho|_{M_3}) = V_{2^{N-1}}$. Now assume $m_4 \neq N$. As in Case 6, we use the power of Corollary 2.8. Consider $\Lambda := \log_{\lambda_*}(\lambda_{2,2^{N-k}+1}) - \log_{\lambda_*}(\lambda_{2,1})$ for k such that $1 \leq k \leq m_4$. Then

$$
\begin{aligned}
\Lambda &= \alpha_3 2 \cdot 2^{N-k} + \alpha_4 2^{m_4-1} 2^{N-k}(2^{N-k}+1) \quad \text{mod } 2^N && (38) \\
&= 2^{N-k}[\alpha_3 2 + \alpha_4 2^{m_4-1}(2^{N-k}+1)]
\end{aligned}
$$

Let Q be the terms in the square brackets above. We have that $Q = 0 \mod 2^k$ if and only if $\lambda_{2,1} = \lambda_{2,2^{N-k}+1}$ and thus $V_{2^{N-k}}$ is a proper stable subspace of $\langle y, x_2 \rangle$. We break this computation into two subcases.

Case 7.1 Assume $m_4 > 2$.
It is clear that if $m_4 > 2$ then, since $m_3 = 1$, by Equation 38 the maximal k such that $\Lambda = 0 \mod 2^N$ is when $k = 1$. Thus $\mathcal{V}(\rho|_{M_3}) = V_{2^{N-1}}$. Note that $V_{2^{N-1}}$ is also minimal when $m_4 = N$. Let $s(\lambda_4) = M$ where $M \leq N-3$. In this subcase there are 2^{N-1} choices for λ_2, $(1-2^{-1})2^{N-1}$ choices for λ_3 and 2^{N-3} choices for λ_4. By Lemma 2.11 we must divide by 2^{N-1} to take shouting into account. Thus, in this subcase, there are

$$
2^{N-1}(1-2^{-1})2^{N-1}2^{N-3}2^{-(N-1)} = (1-2^{-1})2^{2N-4} \qquad (39)
$$

twist isoclasses.

Case 7.2 Assume $m_4 = 2$
If $m_4 = 2$ then we have that $Q = 0 \mod 2^2$ and, since $\mathcal{V}(\rho|_{M_2}) = V_{2^{N-2}}$, then

15

by Corollary 2.9 $\mathcal{V}(\rho|_{M_3}) = V_{2^{N-2}}$. There are 2^{N-1} choices for λ_2, $(1-2^{-1})2^{N-1}$ choices for λ_3, and $(1-2^{-1})2^{N-2}$ choices for λ_4. By Lemma 2.11 we must divide by 2^{N-2} to take shouting into account. Thus, in this subcase there are

$$(1-2^{-1})^2 2^{N-1} 2^{N-1} 2^{N-2} 2^{-(N-2)} = (1-2^{-1})^2 2^{2N-2} \qquad (40)$$

twist isoclasses.

This ends the subcase distinctions.

Summing together these two subcases there are, for $N \geq 3$,

$$(1-2^{-1})2^{2N-4} + (1-2^{-1})^2 2^{2N-2} = (1-2^{-1})2^{2N-4}(1 + 2^2(1-2^{-1})) \quad (41)$$

twist isoclasses.

Now assume $N = 2$. Then we have that $s(\lambda_4) = 0$. A short calculation shows that $\lambda_{2,1} = \lambda_{2,3}$ and by Corollary 2.8 we have that V_2 is a minimal stable subspace. By Lemma 2.11 we must divide by 2 to take shouting into account. There are 2^1 choices for λ_2, $(1-2^{-1})2^1 = 1$ choice for λ_3, and 1 choice for λ_4. Thus, in this subcase there is

$$2 \cdot 1 \cdot 1 \cdot 2^{-1} = 1 \qquad (42)$$

twist isoclass.

This ends the case distinctions.

We now consider the case when $N = 1$. Note that, for clarity, we will call ι the square root of -1. By Equation 21 we have that $s(\lambda_4) \leq 1$; that is, $\lambda_4 \in \{1, -1\}$. If $\lambda_4 = -1$, then by Equation 22 we have that $\lambda_3 \in \{\iota, -\iota\}$ and by Equation 23 we have that $\lambda_2 \in \{\pm\sqrt{\iota}, \pm\sqrt{\iota}\}$ such that $\lambda_2^2 = -\lambda_3$.

If $\lambda_4 = 1$ then by Equation 22 we have that $\lambda_3 \in \{1, -1\}$. If $\lambda_3 = 1$ then by Equation 23 and, since ρ is not the identity representation, we have that $\lambda_2 = -1$. If $\lambda_3 = -1$ then by Equation 23 we have that $\lambda_2 \in \{\iota, -\iota\}$.

A set of choices of the λ_i gives us an irreducible representation if and only if $\lambda_{i,1} \neq \lambda_{i,2}$ holds for at least one $1 \leq i \leq 3$; that is, one of the following is true:

$$\lambda_4 \neq 1 \qquad (43)$$
$$\lambda_3\lambda_4 \neq 1 \qquad (44)$$
$$\lambda_2\lambda_3\lambda_4 \neq 1. \qquad (45)$$

It is easy to see that all of our choices of sets of λ_i give us irreducible representations. For triples $(\lambda_4, \lambda_3, \lambda_2)$ it is easy to check that the pairs $\left[(-1, \iota, \sqrt{\iota}), (-1, -\iota, -\sqrt{-\iota})\right], \left[(-1, \iota, -\sqrt{\iota}), (-1, -\iota, \sqrt{-\iota})\right], \left[(1, -1, \iota), (1, -1, -\iota)\right]$ are twist-and-shout equivalent. Therefore we can say that

$$r_2(M_4) = 4. \qquad (46)$$

We count the number of twist isoclasses for $N = 2, 3$ separately as well. Summing Cases 1 through 8 for $N = 2, 3$ we have $r_4(M_4) = 17$ and $r_8(M_4) = 70$.

We can now compute the 2-local representation growth zeta function of M_4 by summing together the number of twist isoclasses from Cases 1 through 8:

$$\zeta_{M_4,2}^{irr}(s) = 1 + 4 \cdot 2^{-s} + 17 \cdot 2^{-2s} + 70 \cdot 2^{-3s} \tag{47}$$

$$+ \sum_{N=4}^{\infty} \left[(1 - 2^{-1})^4 2^{2N+3} + (1 - 2^{-1})^3 2^{2N} \right.$$

$$+ (1 - 2^{-1}) 2^{2N-2} + (1 - 2^{-1})^2 2^{2N-1}$$

$$+ (1 - 2^{-1}) 2^N (2^{N-2} + 2^{N-4} - 2^{-1})$$

$$\left. + (1 - 2^{-1}) 2^{2N-4} (1 + 2^2 (1 - 2^{-1})) \right] 2^{-Ns}.$$

Simplifying the expression above, we obtain

$$\zeta_{M_4,2}^{irr}(s) = \frac{(1 - 2^{-s})^2}{(1 - 2^{1-s})(1 - 2^{2-s})}. \tag{48}$$

Note that this result is the same as the zeta function for non-exceptional primes in [6]. It is then easy to check that this does satisfy the functional equation in [].

Now that we have the p-local representation zeta functions of M_4 we can now state the global representation zeta function:

$$\zeta_{M_4}^{irr}(s) = \frac{\zeta(s-1)\zeta(s-2)}{(\zeta(s))^2}. \tag{49}$$

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