

# Moser-Trudinger inequalities for singular Liouville systems

Luca Battaglia\*

## Abstract

In this paper we prove a Moser-Trudinger inequality for the Euler-Lagrange functional of general singular Liouville systems on a compact surface. We characterize the values of the parameters which yield coercivity for the functional, hence the existence of energy-minimizing solutions for the system, and we give necessary conditions for boundedness from below.

We also provide a sharp inequality under assuming the coefficients of the system to be non-positive outside the diagonal.

The proofs use a concentration-compactness alternative, Pohožaev-type identities and blow-up analysis.

## 1 Introduction

An essential tool in the study of the embeddings of Sobolev spaces is the Moser-Trudinger inequality, which gives compact embedding in any  $L^p$  space for finite  $p \geq 1$  and also exponential integrability. If we consider a 2-dimensional compact Riemannian manifold  $(\Sigma, g)$ , due to well-known works from Moser [18] and Fontana [13] we get

$$\log \int_{\Sigma} e^u dV_g - \int_{\Sigma} u dV_g \leq \frac{1}{16\pi} \int_{\Sigma} |\nabla u|^2 dV_g + C \quad \forall u \in H^1(\Sigma), \quad (1)$$

where  $\nabla = \nabla_g$  is the gradient given by the metric  $g$  and  $C = C_{\Sigma, g}$  is a constant depending only on  $\Sigma$  and  $g$ .

This inequality has fundamental importance in the study of the Liouville equations of the kind

$$-\Delta u = \rho \left( \frac{he^u}{\int_{\Sigma} he^u dV_g} - 1 \right), \quad (2)$$

where  $\Delta = \Delta_g$  is the Laplace-Beltrami operator,  $\rho$  a positive real parameter,  $h$  a positive smooth function and  $\Sigma$  is supposed, without loss of generality, to have area equal to  $|\Sigma| = 1$ . In fact, the solutions of (2) are critical points of the functional

$$I_{\rho}(u) = \frac{1}{2} \int_{\Sigma} |\nabla u|^2 dV_g - \rho \left( \log \int_{\Sigma} he^u dV_g - \int_{\Sigma} u dV_g \right);$$

using the inequality (1) we can control the last term by the Dirichlet energy, thus showing that  $I_{\rho}$  is bounded from below on  $H^1(\Sigma)$  if and only if  $\rho$  is smaller or equal to  $8\pi$ .

Equations like (2) have great importance in different contexts like the Gaussian curvature prescription problem (see for instance [6, 7]) and abelian Chern-Simons models in theoretical physics

\*Université Catholique de Louvain, Institut de Recherche en Mathématique et Physique, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve (Belgium) - luca.battaglia@uclouvain.be

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([21, 24]).

An extension of the inequality (1), which takes into consideration power-type weights, was given by Chen [8] and Trojanov [22]. For a given  $p \in \Sigma$  and  $\alpha \in (-1, 0]$ , they showed that

$$(1 + \alpha) \left( \log \int_{\Sigma} d(\cdot, p)^{2\alpha} e^u dV_g - \int_{\Sigma} u dV_g \right) \leq \frac{1}{16\pi} \int_{\Sigma} |\nabla u|^2 dV_g + C \quad \forall u \in H^1(\Sigma). \quad (3)$$

This inequality allows to treat singularities in the equation (2), that is to consider equations like

$$-\Delta u = \rho \left( \frac{he^u}{\int_{\Sigma} he^u dV_g} - 1 \right) - 4\pi \sum_{m=1}^M \alpha_m (\delta_{p_m} - 1), \quad (4)$$

where we take arbitrary  $p_1, \dots, p_M \in \Sigma$  and  $\alpha_m > -1$  for any  $m \in \{1, \dots, M\}$ .

This is a natural extension of (2), which allows to consider the same problems in a more general context. For instance, it arises in the Gaussian curvature prescription problem on surfaces with conical singularities and in Chern-Simons vortices theory.

Defining  $G_p$  as the Green function of  $-\Delta$  on  $\Sigma$  centered at a point  $p$ , through the change of variables

$$u \mapsto u + 4\pi \sum_{m=1}^M \alpha_m G_{p_m} \quad (5)$$

equation (4) becomes

$$-\Delta u = \rho \left( \frac{\tilde{h}e^u}{\int_{\Sigma} \tilde{h}e^u dV_g} - 1 \right)$$

with  $\tilde{h} = he^{-4\pi \sum_{m=1}^M \alpha_m G_{p_m}}$ .

Since  $G_p$  has the same behavior as  $\frac{1}{2\pi} \log \frac{1}{d(\cdot, p)}$  around  $p$ , then  $\tilde{h}$  behaves like  $d(\cdot, p_m)^{2\alpha_m}$  around each singular point  $p_m$ , hence the energy functional

$$I_{\rho}(u) = \frac{1}{2} \int_{\Sigma} |\nabla u|^2 dV_g - \rho \left( \log \int_{\Sigma} \tilde{h}e^u dV_g - \int_{\Sigma} u dV_g \right)$$

can be estimated, as in the regular case, using (3).

The purpose of this paper is to extend inequality (3) to singular Liouville systems of the type

$$-\Delta u_i = \sum_{j=1}^N a_{ij} \rho_j \left( \frac{h_j e^{u_j}}{\int_{\Sigma} h_j e^{u_j} dV_g} - 1 \right) - 4\pi \sum_{m=1}^M \alpha_{im} (\delta_{p_m} - 1), \quad i = 1, \dots, N,$$

where  $A = (a_{ij})$  is a  $N \times N$  symmetric positive definite matrix and  $\rho_i, h_i, \alpha_{im}$  are as before.

Applying, similarly to (5), the change of variables

$$u_i \mapsto u_i + 4\pi \sum_{m=1}^M \alpha_{im} G_{p_m},$$

the system becomes

$$-\Delta u_i = \sum_{j=1}^N a_{ij} \rho_j \left( \frac{\tilde{h}_j e^{u_j}}{\int_{\Sigma} \tilde{h}_j e^{u_j} dV_g} - 1 \right), \quad i = 1, \dots, N, \quad (6)$$

with  $\tilde{h}_j$  having the same behavior around the singular points.

The system has a variational formulation with the energy functional

$$J_{\rho}(u) := \frac{1}{2} \sum_{i,j=1}^N a^{ij} \int_{\Sigma} \nabla u_i \cdot \nabla u_j dV_g - \sum_{i=1}^N \rho_i \left( \log \int_{\Sigma} \tilde{h}_i e^{u_i} dV_g - \int_{\Sigma} u_i dV_g \right), \quad (7)$$

with  $a^{ij}$  indicating the entries of the inverse matrix  $A^{-1}$  of  $A$ .

A recent paper by the author and Malchiodi ([2]) gives an answer for the particular case of the  $SU(3)$  Toda system, that is  $N = 2$  and  $A$  is the Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

This is a particularly interesting case, due to its application in the description of holomorphic curves in  $\mathbb{C}\mathbb{P}^N$  in geometry ([3, 5, 9]) and in the non-abelian Chern-Simons theory in physics ([12, 21, 24]). The authors prove a sharp inequality, that is they show that the functional  $J_\rho$  is bounded from below if and only if both the parameters  $\rho_i$  are less or equal than  $4\pi \min \left\{ 1, 1 + \min_m \alpha_{im} \right\}$ , thus extending the result in the regular case from [15].

Concerning general regular Liouville systems, Wang [23] gave necessary and sufficient conditions for the boundedness from below of  $J_\rho$ , following previous results in [10, 11] for the problem on Euclidean domains with Dirichlet boundary conditions. Analogous results were given in [20] for the standard unit sphere  $(\mathbb{S}^2, g_0)$  and in [19] for a similar problem.

In these papers, the authors introduce, for any  $\mathcal{I} \subset \{1, \dots, N\}$ , the following function of the parameter  $\rho$ :

$$\Lambda_{\mathcal{I}}(\rho) = 8\pi \sum_{i \in \mathcal{I}} \rho_i - \sum_{i, j \in \mathcal{I}} a_{ij} \rho_i \rho_j.$$

What they prove is boundedness from below for  $J_\rho$  for any  $\rho \in \mathbb{R}_+^N$  which satisfies  $\Lambda_{\mathcal{I}}(\rho) > 0$  for all the subsets  $\mathcal{I}$  of  $\{1, \dots, N\}$ , whereas  $\inf_{H^1(\Sigma)^N} J_\rho = -\infty$  whenever  $\Lambda_{\mathcal{I}}(\rho) < 0$  for some  $\mathcal{I} \subset \{1, \dots, N\}$ .

The first main result of this paper is an extension of the results from [10, 11, 23] to the case of singularities.

Similarly to Liouville equation, we will have to multiply some quantities by  $1 + \alpha_{im}$ . Precisely, we have:

**Theorem 1.1.**

Let  $J_\rho$  be the functional defined by (7) and set, for  $\rho \in \mathbb{R}_{>0}^N$ ,  $x \in \Sigma$  and  $i \in \mathcal{I} \subset \{1, \dots, N\}$ :

$$\alpha_i(x) = \begin{cases} \alpha_{im} & \text{if } x = p_m \\ 0 & \text{otherwise} \end{cases} \quad \Lambda_{\mathcal{I},x}(\rho) := 8\pi \sum_{i \in \mathcal{I}} (1 + \alpha_i(x)) \rho_i - \sum_{i, j \in \mathcal{I}} a_{ij} \rho_i \rho_j \quad (8)$$

$$\Lambda(\rho) := \min_{\mathcal{I} \subset \{1, \dots, N\}, x \in \Sigma} \Lambda_{\mathcal{I},x}(\rho).$$

Then,  $J_\rho$  is bounded from below if  $\Lambda(\rho) > 0$ , whereas  $J_\rho$  is unbounded from below if  $\Lambda(\rho) < 0$ .

Notice that, in the definition of  $\Lambda$ , the minimum makes sense because it is taken in a finite set, since  $\alpha_i(x) = 0$  for all points of  $\Sigma$  but a finite number, and for all the former points  $\Lambda_{\mathcal{I},x}$  is defined in the same way.

As a consequence of this result, we obtain information about the existence of solutions for the system (6).

**Corollary 1.2.**

The functional  $J_\rho$  is coercive in  $\overline{H}^1(\Sigma)$  if and only if  $\Lambda(\rho) > 0$ .

Therefore, if this occurs, then  $J_\rho$  admits a minimizer  $u$  which solves (6).

Theorem 1.1 leaves an open question about what happens when  $\Lambda(\rho) = 0$ . In this case, as we will see in the following Sections, one encounters blow-up phenomena which are not yet fully known for general systems.

Anyway, we can say something more if we assume in addition  $a_{ij} \leq 0$  for any  $i, j \in \{1, \dots, N\}$  with  $i \neq j$ . First of all, we notice that in this case

$$\Lambda(\rho) = \min_{i \in \{1, \dots, N\}} (8\pi(1 + \tilde{\alpha}_i)\rho_i - a_{ii}\rho_i^2), \quad \text{where}$$

$$\tilde{\alpha}_i := \min_{m \in \{1, \dots, M\}, x \in \Sigma} \alpha_i(x) = \min \left\{ 0, \min_{m \in \{1, \dots, M\}} \alpha_{im} \right\}; \quad (9)$$

hence the sufficient condition in Theorem 1.1 is equivalent to assuming  $\rho_i < \frac{8\pi(1 + \tilde{\alpha}_i)}{a_{ii}}$  for any  $i$ .

With this assumption, studying what happens when  $\Lambda_{\mathcal{T}}(\rho) = 0$  is reduced to a single-component local blow-up, which can be treated by using an inequality from [1]. Therefore, we get the following sharp result:

**Theorem 1.3.**

Let  $J_\rho$  be defined by (7),  $\tilde{\alpha}_i$  as in (9) and  $\Lambda(\rho)$  as in Theorem 1.1, and suppose  $a_{ij} \leq 0$  for any  $i, j \in \{1, \dots, N\}$  with  $i \neq j$ .

Then,  $J_\rho$  is bounded from below on  $H^1(\Sigma)^N$  if and only if  $\Lambda(\rho) \geq 0$ , namely if and only if  $\rho_i \leq \frac{8\pi(1 + \tilde{\alpha}_i)}{a_{ii}}$  for any  $i \in \{1, \dots, N\}$ .

We remark that the assuming  $A$  to be positive definite is necessary. If it is not, then  $J_\rho$  is unbounded from below for any  $\rho$ .

In fact, suppose there exists  $v \in \mathbb{R}^N$  such that  $\sum_{i,j=1}^N a^{ij}v_i v_j \leq -\theta|v|^2$  for some  $\theta > 0$ . Then, we consider the family of functions  $u^\lambda(x) := \lambda v \cdot x$ ; by Jensen's inequality we get

$$\begin{aligned} J_\rho(u^\lambda) &\leq \frac{1}{2} \sum_{i,j=1}^N a^{ij} \int_\Sigma \nabla u_i^\lambda \cdot \nabla u_j^\lambda dV_g - \sum_{i=1}^N \rho_i \int_\Sigma \log \tilde{h}_i dV_g \\ &\leq -\frac{\theta}{2} \lambda^2 |v|^2 + C \\ &\xrightarrow{n \rightarrow +\infty} -\infty. \end{aligned}$$

We also notice that, with respect to the scalar case, in Theorem 1.1 and Corollary 1.2 the positive coefficients  $\alpha_{im}$ 's may affect the definition of  $\Lambda(\rho)$ , hence the conditions for coercivity and boundedness from below of  $J_\rho$ .

On the other hand, under the assumptions of Theorem 1.3, coercivity and boundedness from below only depend on the negative  $\alpha_{im}$ 's, just like for the scalar equation.

The plan of this paper is the following: in Section 2 we will introduce some notations and some preliminary results which will be used throughout the rest of the paper. In Section 3 we will show a sort of Concentration-compactness theorem, showing the possible non-compactness phenomena for solutions of the system (6). Finally, in Sections 4 and 5 we will give the proof of the two main theorems.

## 2 Notations and preliminaries

In this section, we will give some useful notation and some known preliminary results which will be needed to prove the two main theorems.

Given two points  $x, y \in \Sigma$ , we will indicate the metric distance on  $\Sigma$  between them as  $d(x, y)$ . We will indicate the open metric ball centered in  $p$  having radius  $r$  as

$$B_r(x) := \{y \in \Sigma : d(x, y) < r\}.$$

For any subset of a topological space  $A \subset X$  we indicate its closure as  $\overline{A}$  and its interior part as  $\overset{\circ}{A}$ .

Given a function  $u \in L^1(\Sigma)$ , the symbol  $\overline{u}$  will indicate the average of  $u$  on  $\Sigma$ . Since we assume  $|\Sigma| = 1$ , we can write:

$$\overline{u} = \int_{\Sigma} u dV_g = \oint_{\Sigma} u dV_g.$$

We will indicate the subset of  $H^1(\Sigma)$  which contains the functions with zero average as

$$\overline{H}^1(\Sigma) := \{u \in H^1(\Sigma) : \overline{u} = 0\}.$$

Since the functional  $J_\rho$  defined by (7) is invariant by addition of constants, it will not be restrictive to study it on  $\overline{H}^1(\Sigma)^N$  rather than on  $H^1(\Sigma)^N$ .

We will indicate with the letter  $C$  large constants which can vary among different lines and formulas. To underline the dependence of  $C$  on some parameter  $\alpha$ , we indicate with  $C_\alpha$  and so on.

We will denote as  $o_\alpha(1)$  quantities which tend to 0 as  $\alpha$  tends to 0 or to  $+\infty$  and we will similarly indicate bounded quantities as  $O_\alpha(1)$ , omitting in both cases the subscript(s) when it is evident from the context.

First of all, we need a result from Brezis and Merle [4]. It is a classical estimate about exponential integrability of solutions of some elliptic PDEs.

**Lemma 2.1.** ([4], Theorem 1)

Take  $r > 0$ ,  $\Omega := B_r(0) \subset \mathbb{R}^2$ ,  $f \in L^1(\Omega)$  with  $\|f\|_{L^1(\Omega)} < 4\pi$  and  $u$  solving

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}.$$

Then, for any  $q \in \left[1, \frac{4\pi}{\|f\|_{L^1(\Omega)}}\right)$  there exists a constant  $C = C_{q,r}$  such that  $\int_{\Omega} e^{q|u(x)|} dx \leq C$ .

A crucial role in the proof of both Theorem 1.1 and 1.3 will be played by the concentration values of the sequences of solutions of (6).

For a sequence  $u^n = \{u_1^n, \dots, u_N^n\}_{n \in \mathbb{N}}$  of solutions of (6) with  $\rho = \rho^n = \{\rho_1^n, \dots, \rho_N^n\}$ , we define (up to subsequences), for  $i \in \{1, \dots, N\}$ , the concentration value of its  $i^{\text{th}}$  component around a point  $x \in \Sigma$  as

$$\sigma_i(x) := \lim_{r \rightarrow 0} \lim_{n \rightarrow +\infty} \rho_i^n \frac{\int_{B_r(x)} \tilde{h}_i e^{u_i^n} dV_g}{\int_{\Sigma} \tilde{h}_i e^{u_i^n} dV_g}. \quad (10)$$

In a recent paper ([16], see also [14] for the regular case) it was proved, by a Pohožaev identity, that the concentration values satisfy the following algebraic relation, which involves the same quantities as in Theorem 1.1:

**Proposition 2.2.** ([14], Lemma 2.2; [16], Proposition 3.1)

Let  $\{u^n\}_{n \in \mathbb{N}}$  be a sequence of solutions of (6),  $\alpha_i(x)$  and  $\Lambda_{\mathcal{I},x}$  as in (8) and  $\sigma(x) = (\sigma_1(x), \dots, \sigma_N(x))$  as in (10). Then,

$$\Lambda_{\{1, \dots, N\}, x}(\sigma(x)) = 8\pi \sum_{i=1}^N (1 + \alpha_i(x)) \sigma_i(x) - \sum_{i,j=1}^N a_{ij} \sigma_i(x) \sigma_j(x) = 0.$$

To study the concentration phenomena of solutions of (6) we will use the following simple but useful calculus Lemma:

**Lemma 2.3.** ([15], Lemma 4.4)

Let  $\{a^n\}_{n \in \mathbb{N}}$  and  $\{b^n\}_{n \in \mathbb{N}}$  two sequences of real numbers satisfying

$$a^n \xrightarrow{n \rightarrow +\infty} +\infty \quad \lim_{n \rightarrow +\infty} \frac{b^n}{a^n} \leq 0.$$

Then, there exists a smooth function  $F : [0, +\infty) \rightarrow \mathbb{R}$  which satisfies, up to subsequences,

$$0 < F'(t) < 1 \quad \forall t > 0 \quad F'(t) \xrightarrow{t \rightarrow +\infty} 0 \quad F(a^n) - b^n \xrightarrow{n \rightarrow +\infty} +\infty.$$

Finally, as anticipated in the introduction, we will need a singular Moser-Trudinger inequality for Euclidean domains by Adimurthi and Sandeep [1], and its straightforward corollary.

**Theorem 2.4.** ([1], Theorem 2.1)

For any  $r > 0$ ,  $\alpha \in (-1, 0]$  there exists a constant  $C = C_{\alpha, r}$  such that if  $\Omega := B_r(0) \subset \mathbb{R}^2$  and  $u \in H_0^1(\Omega)$ , then

$$\int_{\Omega} |\nabla u(x)|^2 dx \leq 1 \quad \Rightarrow \quad \int_{\Omega} |x|^{2\alpha} e^{4\pi(1+\alpha)u(x)^2} dx \leq C$$

**Corollary 2.5.**

For any  $r > 0$ ,  $\alpha \in (-1, 0]$  there exists a constant  $C = C_{\alpha, r}$  such that if  $\Omega := B_r(0) \subset \mathbb{R}^2$  and  $u \in H_0^1(\Omega)$ , then

$$(1 + \alpha) \log \int_{\Omega} |x|^{2\alpha} e^{u(x)} dx \leq \frac{1}{16\pi} \int_{\Omega} |\nabla u(x)|^2 dx + C$$

*Proof.*

By the elementary inequality  $u \leq \theta u^2 + \frac{1}{4\theta}$  with  $\theta = \frac{4\pi(1+\alpha)}{\int_{\Omega} |\nabla u(y)|^2 dy}$  we get

$$\begin{aligned} (1 + \alpha) \log \int_{\Omega} |x|^{2\alpha} e^{u(x)} dx &\leq (1 + \alpha) \log \int_{\Omega} |x|^{2\alpha} e^{\theta u(x)^2 + \frac{1}{4\theta}} dx \\ &= \frac{1}{16\pi} \int_{\Omega} |\nabla u(y)|^2 dy + (1 + \alpha) \log \int_{\Omega} |x|^{2\alpha} e^{4\pi(1+\alpha) \left( \frac{u(x)}{\sqrt{\int_{\Omega} |\nabla u(y)|^2 dy}} \right)^2} dx \\ &\leq \frac{1}{16\pi} \int_{\Omega} |\nabla u(y)|^2 dy + C. \end{aligned}$$

□

### 3 A Concentration-compactness theorem

The aim of this section is to prove a result which describes the concentration phenomena for the solutions of (6), extending what was done for the two-dimensional Toda system in [2, 17].

We actually have to normalize such solutions to bypass the issue of the invariance by translation by constants and to have the parameter  $\rho$  multiplying only the constant term.

In fact, for any solution  $u$  of (6) the functions

$$v_i := u_i - \log \int_{\Sigma} \tilde{h}_i e^{u_i} dV_g + \log \rho_i \tag{11}$$

solve

$$\begin{cases} -\Delta v_i = \sum_{j=1}^N a_{ij} (\tilde{h}_j e^{v_j} - \rho_j) \\ \int_{\Sigma} \tilde{h}_i e^{v_i} dV_g = \rho_i \end{cases} \quad i = 1, \dots, N. \quad (12)$$

Moreover, we can rewrite in a shorter way (10) as

$$\sigma_i(x) = \lim_{r \rightarrow 0} \lim_{n \rightarrow +\infty} \int_{B_r(x)} \tilde{h}_i^n e^{v_i^n} dV_g.$$

For such functions, we get the following concentration-compactness alternative:

**Theorem 3.1.**

Let  $\{u^n\}_{n \in \mathbb{N}}$  be a sequence of solutions of (6) with  $\rho^n \xrightarrow{n \rightarrow +\infty} \rho \in \mathbb{R}_+^N$  and  $\tilde{h}_i^n = V_i^n \tilde{h}_i$  with  $V_i^n \xrightarrow{n \rightarrow +\infty} 1$  in  $C^1(\Sigma)^N$ ,  $\{v^n\}_{n \in \mathbb{N}}$  be defined as in (11) and  $\mathcal{S}_i$  be defined, for  $i \in \{1, \dots, N\}$ , by

$$\mathcal{S}_i := \left\{ x \in \Sigma : \exists x^n \xrightarrow{n \rightarrow +\infty} x \text{ such that } v_i^n(x^n) \xrightarrow{n \rightarrow +\infty} +\infty \right\}. \quad (13)$$

Then, up to subsequences, one of the following occurs:

- If  $\mathcal{S}_i = \emptyset$  for any  $i \in \{1, \dots, N\}$ , then  $v^n \xrightarrow{n \rightarrow +\infty} v$  in  $W^{2,q}(\Sigma)^N$  for some  $q > 1$  and some  $v$  which solves (12).
- If  $\mathcal{S}_i \neq \emptyset$  for some  $i$ , then it is a finite set for all such  $i$ 's. If this occurs, then there is a subset  $\mathcal{I} \subset \{1, \dots, N\}$  such that  $v_j^n \xrightarrow{n \rightarrow +\infty} -\infty$  in  $L_{\text{loc}}^\infty \left( \Sigma \setminus \bigcup_{j'=1}^N \mathcal{S}_{j'} \right)$  for any  $j \in \mathcal{I}$  and  $v_j^n \xrightarrow{n \rightarrow +\infty} v_j$  in  $W_{\text{loc}}^{2,q} \left( \Sigma \setminus \bigcup_{j'=1}^N \mathcal{S}_{j'} \right)$  for some  $q > 1$  and some suitable  $v_j$ , for any  $j \in \{1, \dots, N\} \setminus \mathcal{I}$ .

Since  $\tilde{h}_j$  is smooth outside the points  $p_m$ 's, the estimates in  $W^{2,q}(\Sigma)$  are actually in  $C_{\text{loc}}^{2,\alpha} \left( \Sigma \setminus \bigcup_{m=1}^M p_m \right)$  and the estimates in  $W_{\text{loc}}^{2,q} \left( \Sigma \setminus \bigcup_{j'=1}^N \mathcal{S}_{j'} \right)$  are actually in  $C_{\text{loc}}^{2,\alpha} \left( \Sigma \setminus \left( \bigcup_{j'=1}^N \mathcal{S}_{j'} \cup \bigcup_{m=1}^M p_m \right) \right)$ . Anyway, estimates in  $W^{2,q}$  will suffice in most of the paper.

To prove Theorem 3.1 we need two preliminary lemmas.

The first is a Harnack-type alternative for sequences of solutions of PDEs. It is inspired by [4, 17].

**Lemma 3.2.**

Let  $\Omega \subset \Sigma$  be a connected open subset,  $\{f^n\}_{n \in \mathbb{N}}$  a bounded sequence in  $L_{\text{loc}}^q(\Omega) \cap L^1(\Omega)$  for some  $q > 1$  and  $\{w^n\}_{n \in \mathbb{N}}$  bounded from above and solving  $-\Delta w^n = f^n$  in  $\Omega$ .

Then, up to subsequences, one of the following alternatives holds:

- $w^n$  is uniformly bounded in  $L_{\text{loc}}^\infty(\Omega)$ .
- $w^n \xrightarrow{n \rightarrow +\infty} -\infty$  in  $L_{\text{loc}}^\infty(\Omega)$ .

*Proof.*

Take a compact set  $\mathcal{K} \Subset \Omega$  and cover it with balls of radius  $\frac{r}{2}$ , with  $r$  smaller than the injectivity

radius of  $\Sigma$ . By compactness, we can write  $\mathcal{K} \subset \bigcup_{h=1}^H B_{\frac{r}{2}}(x_h)$ . If the second alternative does not occur, then up to relabeling we get  $\sup_{B_r(x_1)} w^n \geq -C$ .

Then, we consider the solution  $z^n$  of

$$\begin{cases} -\Delta z^n = f^n & \text{in } B_r(x_1) \\ z^n = 0 & \text{on } \partial B_r(x_1) \end{cases},$$

which is bounded in  $L^\infty(B_r(x_1))$  by elliptic estimates. This means that, for a large constant  $C$ , the function  $C - w^n + z^n$  is positive, harmonic and bounded from below on  $B_r(x_1)$ , and moreover its infimum is bounded from above; therefore, applying the Harnack inequality (which is allowed since  $r$  is small enough) we get that  $C - w^n + z^n$  is uniformly bounded in  $L^\infty(B_{\frac{r}{2}}(x_1))$ , hence  $w^n$  is.

At this point, by connectedness, we can relabel the index  $h$  in such a way that  $B_{\frac{r}{2}}(x_h) \cap B_{\frac{r}{2}}(x_{h+1}) \neq \emptyset$  for any  $h \in \{1, \dots, H-1\}$  and we repeat the argument for  $B_{\frac{r}{2}}(x_2)$ : since it has nonempty intersection with  $B_{\frac{r}{2}}(x_1)$ , we have  $\sup_{B_r(x_2)} w^n \geq -C$ , hence we get boundedness in  $L^\infty(B_{\frac{r}{2}}(x_2))$ . In

the same way, we obtain the same result in all the balls  $B_{\frac{r}{2}}(x_h)$ , whose union contains  $\mathcal{K}$ , therefore  $w^n$  must be uniformly bounded on  $\mathcal{K}$  and we get the conclusion.  $\square$

The second Lemma basically says that if all the concentration values in a point are under a certain threshold, and in particular if all of them equal zero, then compactness occurs around that point. On the other hand, if a point belongs to some set  $\mathcal{S}_i$ , then at least a fixed amount of mass has to accumulate around it; hence, being the total mass uniformly bounded from above, this can occur only for a finite number of points, so we deduce the finiteness of the  $\mathcal{S}_i$ 's.

Precisely, we have the following, inspired again by [17], Lemma 4.4:

**Lemma 3.3.**

Let  $\{v^n\}_{n \in \mathbb{N}}$  and  $\mathcal{S}_i$  be as in (13) and  $\sigma_i$  as in (10), and suppose  $\sigma_i(x) < \sigma_i^0$  for any  $i \in \{1, \dots, N\}$ , where

$$\sigma_i^0 := \frac{4\pi \min\{1, 1 + \min_{j \in \{1, \dots, N\}, m \in \{1, \dots, M\}} \alpha_{jm}\}}{\sum_{j=1}^N a_{ij}^+}.$$

Then,  $x \notin \mathcal{S}_i$  for any  $i \in \{1, \dots, N\}$ .

*Proof.*

First of all we notice that  $\sigma_i^0$  is well-defined for any  $i$  because  $a_{ii} > 0$ , hence  $\sum_{j=1}^N a_{ij}^+ > 0$ .

Under the hypotheses of the Lemma, for large  $n$  and small  $r$  we have

$$\int_{B_r(x)} \tilde{h}_i^n e^{v_i^n} dV_g < \sigma_i^0. \quad (14)$$

Let us consider  $w_i^n$  and  $z_i^n$  defined by

$$\begin{cases} -\Delta w_i^n = -\sum_{j=1}^N a_{ij} \rho_j^n & \text{in } B_r(x) \\ w_i^n = 0 & \text{on } \partial B_r(x) \end{cases}, \quad \begin{cases} -\Delta z_i^n = \sum_{j=1}^N a_{ij}^+ \tilde{h}_j^n e^{v_j^n} & \text{in } B_r(x) \\ z_i^n = 0 & \text{on } \partial B_r(x) \end{cases}. \quad (15)$$

Is it evident that the  $w_i^n$ 's are uniformly bounded in  $L^\infty(B_r(x))$ .

As for the  $z_i^n$ 's, we can suppose to be working on a Euclidean disc, up to applying a perturbation to  $\tilde{h}_i^n$  which is smaller as  $r$  is smaller, hence for  $r$  small enough we still have the strict estimate (14).

Therefore, we get

$$\|-\Delta z_i^n\|_{L^1(B_r(x))} = \sum_{j=1}^N a_{ij}^+ \int_{B_r(x)} \tilde{h}_j^n e^{v_j^n} dV_g < \sum_{j=1}^N a_{ij}^+ \sigma_j^0 \leq 4\pi \min\{1, 1 + \alpha_i(x)\}$$



and we can apply Lemma 2.1 to obtain  $\int_{B_r(x)} e^{q|z_i^n|} dV_g \leq C$  for some  $q > \frac{1}{\min\{1, 1 + \alpha_i(x)\}}$ .

If  $\alpha_i(x) \geq 0$ , then taking  $q \in \left(1, \frac{4\pi}{\|-\Delta z_i^n\|_{L^1(B_r(x))}}\right)$  we have

$$\int_{B_r(x)} \left(\tilde{h}_i^n e^{z_i^n}\right)^q dV_g \leq C_r \int_{B_r(x)} e^{q|z_i^n|} dV_g \leq C.$$

On the other hand, if  $\alpha_i(x) < 0$ , we choose

$$q \in \left(1, \frac{4\pi}{\|-\Delta z_i^n\|_{L^1(B_r(x))} - 4\pi\alpha_i(x)}\right) \quad q' \in \left(\frac{4\pi}{4\pi - q\|-\Delta z_i^n\|_{L^1(B_r(x))}}, \frac{1}{-\alpha_i(x)q}\right)$$

and, applying Hölder's inequality,

$$\begin{aligned} \int_{B_r(x)} \left(\tilde{h}_i^n e^{z_i^n}\right)^q dV_g &\leq C_r \int_{B_r(x)} d(\cdot, x)^{2q\alpha_i(x)} e^{qz_i^n} dV_g \\ &\leq C \left(\int_{B_r(x)} d(\cdot, x)^{2qq'\alpha_i(x)} dV_g\right)^{\frac{1}{q'}} \left(\int_{B_r(x)} e^{q\frac{q'}{q'-1}|z_i^n|} dV_g\right)^{1-\frac{1}{q'}} \\ &\leq C, \end{aligned}$$

because  $qq'\alpha_i(x) > -1$  and  $q\frac{q'}{q'-1}\alpha_i(x) < \frac{4\pi}{\|-\Delta z_i^n\|_{L^1(B_r(x))}}$ . Hence  $\tilde{h}_i^n e^{z_i^n}$  is uniformly bounded in  $L^q(B_r(x))$  for some  $q > 1$ .

Now, let us consider  $v_i^n - z_i^n - w_i^n$ : it is a subharmonic sequence by construction, so for any  $y \in B_{\frac{r}{2}}(x)$  we get

$$\begin{aligned} v_i^n(y) - z_i^n(y) - w_i^n(y) &\leq \int_{B_{\frac{r}{2}}(y)} (v_i^n - z_i^n - w_i^n) dV_g \\ &\leq C \int_{B_{\frac{r}{2}}(y)} (v_i^n - z_i^n - w_i^n)^+ dV_g \\ &\leq C \int_{B_r(x)} ((v_i^n - z_i^n)^+ + (w_i^n)^-) dV_g \\ &\leq C \left(1 + \int_{B_r(x)} (v_i^n - z_i^n)^+ dV_g\right). \end{aligned}$$

Moreover, since the maximum principle yields  $z_i^n \geq 0$ , taking  $\theta = \begin{cases} 1 & \text{if } \alpha_i(x) \leq 0 \\ \in \left(0, \frac{1}{1 + \alpha_i(x)}\right) & \text{if } \alpha_i(x) > 0 \end{cases}$ ,

we get

$$\begin{aligned} \int_{B_r(x)} (v_i^n - z_i^n)^+ dV_g &\leq \int_{B_r(x)} (v_i^n)^+ dV_g \\ &\leq \frac{1}{e\theta} \int_{B_r(x)} e^{\theta v_i^n} dV_g \\ &\leq C \left\| \left(\tilde{h}_i^n\right)^{-\theta} \right\|_{L^{\frac{1}{1-\theta}}(B_r(x))} \left(\int_{B_r(x)} \tilde{h}_i^n e^{v_i^n} dV_g\right)^\theta \\ &\leq C. \end{aligned}$$

Therefore, we showed that  $v_i^n - z_i^n - w_i^n$  is bounded from above in  $B_{\frac{r}{2}}(x)$ , that is  $e^{v_i^n - z_i^n - w_i^n}$  is uniformly bounded in  $L^\infty(B_{\frac{r}{2}}(x))$ . Since the same holds for  $e^{w_i^n}$  and  $\tilde{h}_i^n e^{z_i^n}$  is uniformly bounded in  $L^q(B_{\frac{r}{2}}(x))$  for some  $q > 1$ , we deduce that also

$$\tilde{h}_i^n e^{v_i^n} = \tilde{h}_i^n e^{z_i^n} e^{v_i^n - z_i^n - w_i^n} e^{w_i^n}$$

is bounded in the same  $L^q(B_{\frac{r}{2}}(x))$ .

Thus, we have an estimate on  $\|-\Delta z_i^n\|_{L^q(B_{\frac{r}{2}}(x))}$  for any  $i \in \{1, \dots, N\}$ , hence by standard elliptic estimates we deduce that  $z_i^n$  is uniformly bounded in  $L^\infty(B_{\frac{r}{2}}(x))$ . Therefore, we also deduce that

$$v_i^n = (v_i^n - z_i^n - w_i^n) + z_i^n + w_i^n$$

is bounded from above on  $B_{\frac{r}{2}}(x)$ , which is equivalent to saying  $x \notin \bigcup_{i=1}^N \mathcal{S}_i$ .  $\square$

From this proof, we notice that, under the assumptions of Theorem 1.3, the same result holds for any single index  $i \in \{1, \dots, N\}$ . In other words, the upper bound on one  $\sigma_i$  implies that  $x \notin \mathcal{S}_i$ .

**Corollary 3.4.**

Suppose  $a_{ij} \leq 0$  for any  $i \neq j$ .

Then, for any given  $i \in \{1, \dots, N\}$  the following conditions are equivalent:

- $x \in \mathcal{S}_i$ .
- $\sigma_i(x) \neq 0$ .
- $\sigma_i(x) \geq \sigma'_i = \frac{4\pi \min\{1, 1 + \min_m \alpha_{im}\}}{a_{ii}}$ .

*Proof.*

The third statement trivially implies the second and the second implies the first, since if  $v_i^n$  is bounded from above in  $B_r(x)$  then  $\tilde{h}_i^n e^{v_i^n}$  is bounded in  $L^q(B_r(x))$ . Finally, if  $\sigma_i(x) < \sigma'_i$  then the sequence  $\tilde{h}_i^n e^{z_i^n}$  defined by (15) is bounded in  $L^q$  for  $q > 1$ , so one can argue as in Lemma 3.3 to get boundedness from above of  $v_i^n$  around  $x$ , that is  $x \notin \mathcal{S}_i$ .  $\square$

We can now prove the main theorem of this Section.

*Proof of Theorem 3.1.*

If  $\mathcal{S}_i = \emptyset$  for any  $i$ , then  $e^{v_i^n}$  is bounded in  $L^\infty(\Sigma)$ , so  $-\Delta v_i^n$  is bounded in  $L^q(\Sigma)$  for any

$$q \in \left[1, \frac{1}{-\min_{j \in \{1, \dots, N\}, m \in \{1, \dots, M\}} \alpha_{jm}}\right).$$

Therefore, we can apply Lemma 3.2 to  $v_i^n$  on  $\Sigma$ , where we must have the first alternative for every  $i$ , since otherwise the dominated convergence would give  $\int_{\Sigma} \tilde{h}_i^n e^{v_i^n} dV_g \xrightarrow{n \rightarrow +\infty} 0$  which is absurd; standard elliptic estimates allow to conclude compactness in  $W^{2,q}(\Sigma)$ .

Suppose now  $\mathcal{S}_i \neq \emptyset$  for some  $i$ ; from Lemma 3.3 we deduce

$$|\mathcal{S}_i| \sigma_i^0 \leq \sum_{x \in \mathcal{S}_i} \max_j \sigma_j(x) \leq \sum_{j=1}^N \sum_{x \in \mathcal{S}_i} \sigma_j(x) \leq \sum_{j=1}^N \rho_j,$$

hence  $\mathcal{S}_i$  is finite.

For any  $j \in \{1, \dots, N\}$ , we can apply Lemma 3.2 on  $\Sigma \setminus \bigcup_{j'=1}^N \mathcal{S}_{j'}$  with  $f^n = \sum_{j'=1}^N a_{jj'} (\tilde{h}_{j'}^n e^{v_{j'}^n} - \rho_{j'}^n)$ ,

since the last function is bounded in  $L^q_{\text{loc}}\left(\Sigma \setminus \bigcup_{j'=1}^N \mathcal{S}_{j'}\right)$ .

Therefore, either  $v_j^n$  goes to  $-\infty$  or it is bounded in  $L^\infty_{\text{loc}}$ , and in the last case we get compactness in  $W^{2,q}_{\text{loc}}$  by applying again standard elliptic regularity.  $\square$

## 4 Proof of Theorem 1.1.

Here we will prove the theorem which gives sufficient and necessary conditions for the functional  $J_\rho$  to be bounded from below.

In other words, setting

$$E := \left\{ \rho \in \mathbb{R}_+^N : J_\rho \text{ is bounded from below on } H^1(\Sigma)^N \right\}, \quad (16)$$

we will prove that  $\{\Lambda > 0\} \subset E \subset \{\Lambda \geq 0\}$ .

As a first thing, we notice that the set  $E$  is not empty and it verifies a simple monotonicity condition.

### Lemma 4.1.

The set  $E$  defined by (16) is nonempty.

Moreover, for any  $\rho \in E$  then  $\rho' \in E$  provided  $\rho'_i \leq \rho_i$  for any  $i \in \{1, \dots, N\}$ .

*Proof.*

Let  $\theta > 0$  be the biggest eigenvalue of the matrix  $(a_{ij})$ . Then,

$$J_\rho(u) \geq \sum_{i=1}^N \left( \frac{1}{2\theta} \int_\Sigma |\nabla u_i|^2 dV_g - \rho_i \left( \log \int_\Sigma \tilde{h}_i e^{u_i} dV_g - \bar{u}_i \right) \right).$$

Therefore, from scalar Moser-Trudinger inequality (3), we deduce that  $J_\rho$  is bounded from below if  $\rho_i \leq \frac{8\pi(1 + \tilde{\alpha}_i)}{\theta}$ , hence  $E \neq \emptyset$ .

Suppose now  $\rho \in E$  and  $\rho'_i \leq \rho_i$  for any  $i$ . Then, through Jensen's inequality, we get

$$\begin{aligned} J_{\rho'}(u) &= J_\rho(u) + \sum_{i=1}^N (\rho_i - \rho'_i) \log \int_\Sigma e^{u_i - \bar{u}_i + \log \tilde{h}_i} dV_g \\ &\geq -C + \sum_{i=1}^N (\rho_i - \rho'_i) \int_\Sigma \log \tilde{h}_i dV_g \\ &\geq -C \end{aligned}$$

for any  $u \in H^1(\Sigma)^N$ , hence the claim.  $\square$

It is interesting to observe that a similar monotonicity condition is also satisfied by the set  $\{\Lambda > 0\}$  (although one can easily see that it is not true if we replace  $\Lambda$  with  $\Lambda_{\mathcal{I},x}$ ).

### Lemma 4.2.

Let  $\rho, \rho' \in \mathbb{R}_+^N$  be such that  $\Lambda(\rho) > 0$  and  $\rho'_i \leq \rho_i$  for any  $i \in \{1, \dots, N\}$ .

Then,  $\Lambda(\rho') > 0$ .

*Proof.*

Suppose by contradiction  $\Lambda(\rho') \leq 0$ , that is  $\Lambda_{\mathcal{I},x}(\rho') \leq 0$  for some  $\mathcal{I}, x$ .

This cannot occur for  $\mathcal{I} = \{i\}$  because it would mean  $\rho'_i \geq \frac{8\pi(1 + \alpha_i(x))}{a_{ii}}$ , so the same inequality would for  $\rho_i$ , hence  $\Lambda(\rho) \leq \Lambda_{\mathcal{I},x}(\rho) \leq 0$ .

Therefore, there must be some  $\mathcal{I}, x$  such that  $\Lambda_{\mathcal{I},x}(\rho') \leq 0$  and  $\Lambda_{\mathcal{I}\setminus\{i\},x}(\rho') > 0$  for any  $i \in \mathcal{I}$ ; this implies

$$\begin{aligned}
0 &< \Lambda_{\mathcal{I}\setminus\{i\},x}(\rho') - \Lambda_{\mathcal{I},x}(\rho') \\
&= 2 \sum_{j \in \mathcal{I}} a_{ij} \rho'_i \rho'_j - a_{ii} \rho_i'^2 - 8\pi(1 + \alpha_i(x)) \rho'_i \\
&< \rho'_i \left( 2 \sum_{j \in \mathcal{I}} a_{ij} \rho'_j - 8\pi(1 + \alpha_i(x)) \right).
\end{aligned} \tag{17}$$

It will be not restrictive to suppose, from now on,  $\rho'_1 \leq \rho_1$  and  $\rho'_i = \rho_i$  for any  $i \geq 2$ , since the general case can be treated by exchanging the indices and iterating.

Assuming this, we must have  $1 \in \mathcal{I}$ , therefore we obtain:

$$\begin{aligned}
0 &< \Lambda_{\mathcal{I},x}(\rho) - \Lambda_{\mathcal{I},x}(\rho') \\
&= 8\pi(1 + \alpha_1(x))(\rho_1 - \rho'_1) - a_{11}(\rho_1'^2 - \rho_1^2) - 2 \sum_{j \in \mathcal{I}\setminus\{1\}} a_{1j}(\rho'_1 - \rho_1)\rho_j \\
&= (\rho_1 - \rho'_1) \left( 8\pi(1 + \alpha_1(x)) - a_{11}(\rho'_1 + \rho_1) - 2 \sum_{j \in \mathcal{I}\setminus\{1\}} a_{1j} \rho_j \right) \\
&< (\rho_1 - \rho'_1) \left( 8\pi(1 + \alpha_1(x)) - 2 \sum_{j \in \mathcal{I}} a_{1j} \rho'_j \right),
\end{aligned}$$

which is negative by (17). We found a contradiction.  $\square$

We will now show that if the parameter  $\rho$  lies in the interior of  $E$  then not only the functional is bounded from below but it is coercive in the space of zero-average functions. In particular, this fact allows to deduce the “if” part in Corollary 1.2 from Theorem 1.1.

On the other hand, if  $\rho$  belongs to the boundary of  $E$ , then the scenario is quite different.

**Lemma 4.3.**

Suppose  $\rho \in \overset{\circ}{E}$ . Then, there exists a constant  $C = C_\rho$  such that

$$J_\rho(u) \geq \frac{1}{C} \sum_{i=1}^N \int_{\Sigma} |\nabla u_i|^2 dV_g - C.$$

Moreover,  $J_\rho$  admits a minimizer which solves (6).

*Proof.*

Choosing  $\delta \in \left(0, \frac{d(\rho, \partial E)}{\sqrt{N}|\rho|}\right)$  one has  $(1 + \delta)\rho \in E$ , so

$$\begin{aligned}
J_\rho(u) &= \frac{\delta}{2(1 + \delta)} \sum_{i,j=1}^N a^{ij} \int_{\Sigma} \nabla u_i \cdot \nabla u_j dV_g + \frac{1}{1 + \delta} J_{(1+\delta)\rho}(u) \\
&\geq \frac{\delta}{2\theta(1 + \delta)} \sum_{i=1}^N \int_{\Sigma} |\nabla u_i|^2 dV_g - C,
\end{aligned}$$

hence we get the former claim.

To get the latter, we notice that, due to invariance by translation, any minimizer can be supposed to be in  $\overline{H}^1(\Sigma)^N$ ; therefore, we can restrict  $J_\rho$  to this subspace. Here, the above inequality implies coercivity, and it is immediate to see that  $J_\rho$  is also lower semi-continuous, hence the existence of minimizers follows from direct methods of calculus of variations.  $\square$

**Lemma 4.4.**

Suppose  $\rho \in \partial E$ . Then, there exists a sequence  $\{u^n\}_{n \in \mathbb{N}} \subset H^1(\Sigma)^N$  such that

$$\sum_{i=1}^N \int_{\Sigma} |\nabla u_i^n|^2 dV_g \xrightarrow{n \rightarrow +\infty} +\infty \quad \lim_{n \rightarrow +\infty} \frac{J_{\rho}(u^n)}{\sum_{i=1}^N \int_{\Sigma} |\nabla u_i^n|^2 dV_g} \leq 0$$

*Proof.*

We first notice that  $(1 - \delta)\rho \in E$  for any  $\delta \in (0, 1)$ . In fact, otherwise, from Lemma 4.1 we would get  $\rho' \notin E$  as soon as  $\rho'_i \geq (1 - \delta)\rho_i$  for some  $i$ , hence  $\rho \notin \partial E$ .

Now, suppose by contradiction that for any sequence  $u^n$  one gets

$$\sum_{i=1}^N \int_{\Sigma} |\nabla u_i^n|^2 dV_g \xrightarrow{n \rightarrow +\infty} +\infty \quad \Rightarrow \quad \frac{J_{\rho}(u^n)}{\sum_{i=1}^N \int_{\Sigma} |\nabla u_i^n|^2 dV_g} \geq \varepsilon > 0.$$

Therefore, we would have

$$J_{\rho}(u) \geq \frac{\varepsilon}{2} \sum_{i=1}^N \int_{\Sigma} |\nabla u_i|^2 dV_g - C;$$

hence, indicating as  $\theta'$  the smallest eigenvalue of the matrix  $A$ , for small  $\delta$  we would get

$$\begin{aligned} J_{\rho}(u) &= (1 + \delta)J_{(1+\delta)\rho}(u) - \frac{\delta}{2} \sum_{i,j=1}^N a^{ij} \int_{\Sigma} \nabla u_i \cdot \nabla u_j dV_g \\ &\geq \left( (1 + \delta)\frac{\varepsilon}{2} - \frac{\delta}{2\theta'} \right) \sum_{i=1}^N \int_{\Sigma} |\nabla u_i|^2 - C \\ &\geq -C. \end{aligned}$$

So we obtain  $(1 + \delta)\rho \in E$ ; being also  $(1 - \delta)\rho \in E$  (by Lemma 4.1), we get a contradiction with  $\rho \in \partial E$ .  $\square$

To see what happens when  $\rho \in \partial E$ , we build an auxiliary functional using Lemma 2.3.

**Lemma 4.5.**

Fix  $\rho' \in \partial E$  and define:

$$a_{\rho'}^n := \frac{1}{2} \sum_{i,j=1}^N a^{ij} \int_{\Sigma} \nabla u_i^n \cdot \nabla u_j^n dV_g \quad b_{\rho'}^n := J_{\rho'}(u^n)$$

$$J'_{\rho',\rho}(u) = J_{\rho}(u) - F_{\rho'} \left( \frac{1}{2} \sum_{i,j=1}^N a^{ij} \int_{\Sigma} \nabla u_i \cdot \nabla u_j dV_g \right),$$

where  $u^n$  is given by Lemma 4.4 and  $F_{\rho'}$  by Lemma 2.3.

If  $\rho \in \overset{\circ}{E}$ , then  $J'_{\rho',\rho}$  is bounded from below on  $H^1(\Sigma)^N$  and its infimum is achieved by a solution of

$$-\Delta \left( u_i - \sum_{j=1}^N a^{ij} f u_j \right) = \sum_{j=1}^N a_{ij} \rho_j \left( \frac{\tilde{h}_j e^{u_j}}{\int_{\Sigma} \tilde{h}_j e^{u_j} dV_g} - 1 \right), \quad i = 1, \dots, N,$$

$$\text{with } f = (F_{\rho'})' \left( \frac{1}{2} \sum_{i,j=1}^N a^{ij} \int_{\Sigma} \nabla u_i \cdot \nabla u_j dV_g \right).$$

On the other hand,  $J'_{\rho',\rho}$  is unbounded from below.

*Proof.*

For  $\rho \in \overset{\circ}{E}$ , we can argue as in Lemma 4.3, since the continuity follows from the regularity of  $F$  and the coercivity from the behavior of  $F'$  at the infinity.

For  $\rho = \rho'$ , if we take  $u^n$  as in Lemma 4.4 we get

$$J'_{\rho',\rho'}(u^n) = b_{\rho'}^n - F_{\rho'}(a_{\rho'}^n) \xrightarrow{n \rightarrow +\infty} -\infty.$$

□

Now we can prove the first half of Theorem 1.1, that is  $J_\rho$  is bounded from below if  $\Lambda(\rho) > 0$ .

*Proof of  $\{\Lambda > 0\} \subset E$ .*

Suppose by contradiction there is some  $\rho' \in \partial E$  with  $\Lambda(\rho) > 0$  and take a sequence  $\rho^n \in E$  with  $\rho^n \xrightarrow{n \rightarrow +\infty} \rho'$ .

Then, by Lemma 4.5, the auxiliary functional  $J_{\rho',\rho^n}$  admits a minimizer  $u^n$ , so the functions  $v_i^n$  defined as in (11) solve

$$\begin{cases} -\Delta v_i^n = \sum_{j,j'=1}^N a_{ij} b^{jj',n} (\tilde{h}_j e^{v_j^n} - \rho_j^n) & i = 1, \dots, N \\ \int_{\Sigma} \tilde{h}_i^n e^{v_i^n} dV_g = \rho_i^n \end{cases}$$

where  $b^{ij,n}$  is the inverse matrix of  $b_{ij}^n := \delta_{ij} - a^{ij} f^n$ , hence  $b^{ij,n} \xrightarrow{n \rightarrow +\infty} \delta_{ij}$ .

We can then apply Theorem 3.1. The first alternative is excluded, since otherwise we would get, for any  $u \in H^1(\Sigma)^N$ ,

$$J'_{\rho',\rho'}(u) = \lim_{n \rightarrow +\infty} J'_{\rho',\rho^n}(u) \geq \lim_{n \rightarrow +\infty} J'_{\rho',\rho^n}(v^n) = J'_{\rho',\rho'}(v) > -\infty,$$

thus contradicting Lemma 4.5.

Therefore, blow up must occur; this means, by Lemma 3.3, that  $\sigma_i(p) \neq 0$  for some  $i \in \{1, \dots, N\}$  and some  $p \in \Sigma$ .

By Proposition 2.2 follows  $\Lambda(\sigma) \leq 0$ . On the other hand, since  $\sigma_i \leq \rho'_i$  for any  $i$ , Lemma 4.2 yields  $\Lambda(\rho') \leq 0$ , which contradicts our assumptions. □

To prove the unboundedness from below of  $J_\rho$  in the case  $\Lambda(\rho) < 0$  we will use suitable test functions, whose properties are described by the following:

**Lemma 4.6.**

Define, for  $x \in \Sigma$  and  $\lambda > 0$ ,  $\varphi = \varphi^{\lambda,x}$  as

$$\varphi_i := -2(1 + \alpha_i(x)) \log \max\{1, \lambda d(\cdot, x)\}.$$

Then, as  $\lambda \rightarrow +\infty$ , one has

$$\int_{\Sigma} \nabla \varphi_i \cdot \nabla \varphi_j dV_g = 8\pi(1 + \alpha_i(x))(1 + \alpha_j(x)) \log \lambda + O(1)$$

$$\overline{\varphi_i} = -2(1 + \alpha_i(x)) \log \lambda + O(1)$$

$$\int_{\Sigma} \tilde{h}_i e^{\sum_{j=1}^N \theta_j \varphi_j} dV_g \geq C \lambda^{-2(1+\alpha_i(x))} \quad \text{if} \quad \sum_{i=1}^N \theta_j (1 + \alpha_j(x)) > 1 + \alpha_i(x).$$

*Proof.*

It holds

$$\nabla\varphi_i = \begin{cases} 0 & \text{if } d(\cdot, x) < \frac{1}{\lambda} \\ -2(1 + \alpha_i(x)) \frac{\nabla d(\cdot, x)}{d(\cdot, x)} & \text{if } d(\cdot, x) > \frac{1}{\lambda} \end{cases}.$$

Therefore, being  $|\nabla d(\cdot, x)| = 1$  almost everywhere on  $\Sigma$ :

$$\begin{aligned} & \int_{\Sigma} \nabla\varphi_i \cdot \nabla\varphi_j dV_g \\ &= 4(1 + \alpha_i(x))(1 + \alpha_j(x)) \int_{\Sigma \setminus B_{\frac{1}{\lambda}}(x)} \frac{dV_g}{d(\cdot, x)^2} \\ &= 8\pi(1 + \alpha_i(x))(1 + \alpha_j(x)) \log \lambda + O(1). \end{aligned}$$

For the average of  $\varphi_i$ , we get

$$\int_{\Sigma} \varphi_i dV_g = -2(1 + \alpha_i(x)) \int_{\Sigma \setminus B_{\frac{1}{\lambda}}(x)} (\log \lambda + \log d(\cdot, x)) dV_g + O(1) = -2(1 + \alpha_i(x)) \log \lambda + O(1).$$

For the last estimate, choose  $r > 0$  such that  $\overline{B_{\delta}(x)}$  does not contain any of the points  $p_m$  for  $m = 1, \dots, M$ , except possibly  $x$ .

Then, outside such a ball,  $e^{\sum_{j=1}^N \theta_j \varphi_j} \leq C\lambda^{-2\sum_{j=1}^N \theta_j(1 + \alpha_j(x))}$ .

Therefore, under the assumptions of the Lemma,

$$\int_{\Sigma \setminus B_{\delta}(x)} \tilde{h}_i e^{\sum_{i=1}^N \theta_j \varphi_j} dV_g = o\left(\lambda^{-2(1 + \alpha_i(x))}\right),$$

hence

$$\begin{aligned} & \int_{\Sigma} \tilde{h}_i e^{\sum_{i=1}^N \theta_j \varphi_j} dV_g \\ & \geq \int_{B_{\delta}(x)} \tilde{h}_i e^{\sum_{i=1}^N \theta_j \varphi_j} dV_g \\ & \geq C \left( \int_{B_{\frac{1}{\lambda}}(x)} d(\cdot, x)^{2\alpha_i(x)} dV_g + \frac{1}{\lambda^{2\sum_{j=1}^N \theta_j(1 + \alpha_j(x))}} \int_{A_{\frac{1}{\lambda}, \delta}(x)} d(\cdot, x)^{2\alpha_i(x) - 2\sum_{i=1}^N \theta_j(1 + \alpha_j(x))} dV_g \right) \\ & \geq C\lambda^{-2(1 + \alpha_i(x))}, \end{aligned}$$

which concludes the proof.  $\square$

*Proof of  $E \subset \{\Lambda \geq 0\}$ .*

Take  $\rho, \mathcal{I}, x$  such that  $\Lambda_{\mathcal{I}, x}(\rho) < 0$  and  $\Lambda_{\mathcal{I} \setminus \{i\}, x}(\rho) \geq 0$  for any  $i \in \mathcal{I}$ , and consider the family of functions  $\{u^\lambda\}_{\lambda > 0}$  defined by

$$u_i^\lambda := \sum_{j \in \mathcal{I}} \frac{a_{ij} \rho_j}{4\pi(1 + \alpha_i(x))} \varphi_j^{\lambda, x}.$$

By Jensen's inequality we get

$$\begin{aligned} J_\rho(u^\lambda) & \leq \frac{1}{2} \sum_{i, j=1}^N a^{ij} \int_{\Sigma} \nabla u_i^\lambda \cdot \nabla u_j^\lambda dV_g + \sum_{i \in \mathcal{I}} \rho_i \left( \overline{u_i^\lambda} - \log \int_{\Sigma} \tilde{h}_i e^{u_i^\lambda} dV_g \right) + C \\ & = \frac{1}{2} \sum_{i, j \in \mathcal{I}} \frac{a_{ij} \rho_i \rho_j}{16\pi^2(1 + \alpha_i(x))(1 + \alpha_j(x))} \int_{\Sigma} \nabla\varphi_i \cdot \nabla\varphi_j dV_g \\ & \quad + \sum_{i, j \in \mathcal{I}} \frac{a_{ij} \rho_i \rho_j}{4\pi(1 + \alpha_j(x))} \overline{\varphi_j} - \sum_{i \in \mathcal{I}} \rho_i \log \int_{\Sigma} \tilde{h}_i e^{\sum_{j \in \mathcal{I}} \frac{a_{ij} \rho_j}{4\pi(1 + \alpha_j(x))} \varphi_j} dV_g + C. \end{aligned}$$

At this point, we would like to apply Lemma 4.6 to estimate  $J_\rho(u^\lambda)$ . To be able to do this, we have to verify that

$$\frac{1}{4\pi} \sum_{j \in \mathcal{I}} a_{ij} \rho_j > 1 + \alpha_i(x) \quad \forall i \in \mathcal{I}.$$

If  $\mathcal{I} = \{i\}$ , then  $\rho_i > \frac{8\pi(1 + \alpha_i(x))}{a_{ii}}$ , so it follows immediately. For the other cases, it follows from (17).

So we can apply Lemma 4.6 and we get from the previous estimates:

$$\begin{aligned} J_\rho(u^\lambda) &\leq \left( \frac{1}{4\pi} \sum_{i,j \in \mathcal{I}} a_{ij} \rho_i \rho_j - \frac{1}{2\pi} \sum_{i,j \in \mathcal{I}} a_{ij} \rho_i \rho_j + 2 \sum_{i \in \mathcal{I}} \rho_i (1 + \alpha_i(x)) \right) \log \lambda + C \\ &= -\frac{\Lambda_{\mathcal{I},x}(\rho)}{4\pi} \log \lambda + C \\ &\xrightarrow{n \rightarrow +\infty} -\infty. \end{aligned}$$

□

*Proof of Corollary 1.2.*

The coercivity in the case  $\Lambda < 0$ , hence the existence of minimizing solutions for (6) follows from Theorem 1.1 and Lemma 4.3.

If instead  $\Lambda(\rho) \geq 0$ , then one can find out the lack of coercivity by arguing as before with the sequence  $u^\lambda$ , which verifies

$$\sum_{i=1}^N \int_{\Sigma} |\nabla u_i^\lambda|^2 dV_g \xrightarrow{\lambda \rightarrow +\infty} +\infty \quad J_\rho(u^\lambda) \leq -\frac{\Lambda_{\mathcal{I},x}(\rho)}{4\pi} \log \lambda + C \leq C.$$

□

## 5 Proof of Theorem 1.3.

Here we will finally prove a sharp inequality in the case when the matrix  $a_{ij}$  has non-positive entries outside its main diagonal.

As already pointed out in the introduction, the function  $\Lambda(\rho)$  can be written in a much shorter form under these assumptions, so the condition  $\Lambda(\rho) \geq 0$  is equivalent to  $\rho_i \leq \frac{8\pi(1 + \tilde{\alpha}_i)}{a_{ii}}$  for any  $i \in \{1, \dots, N\}$ .

Moreover, thanks to Lemma 4.1, in order to prove Theorem 1.3 for all such  $\rho$ 's it will suffice to consider

$$\rho^0 := \left( \frac{8\pi(1 + \tilde{\alpha}_1)}{a_{11}}, \dots, \frac{8\pi(1 + \tilde{\alpha}_N)}{a_{NN}} \right). \quad (18)$$

By what we proved in the previous Section, for any sequence  $\rho^n \nearrow_{n \rightarrow +\infty} \rho^0$  one has

$$\inf_{H^1(\Sigma)^N} J_{\rho^n} = J_{\rho^n}(u^n) \geq -C_{\rho^n},$$

so Theorem 1.3 will follow by showing that, for a given sequence  $\{\rho^n\}_{n \in \mathbb{N}}$ , the constant  $C_n = C_{\rho^n}$  can be chosen independently of  $n$ .

As a first thing, we provide a Lemma which shows the possible blow-up scenarios for such a sequence  $u^n$ .

Here, the assumption on  $a_{ij}$  is crucial since it reduces largely the possible cases.



**Lemma 5.1.**

Let  $\rho^0$  be as in (18),  $\{\rho^n\}_{n \in \mathbb{N}}$  such that  $\rho^n \nearrow \rho^0$ ,  $u^n$  a minimizer of  $J_{\rho^n}$  and  $v^n$  as in (11). Then, up to subsequences, there exists a set  $\mathcal{I} \subset \{1, \dots, N\}$  such that:

- If  $i \in \mathcal{I}$ , then  $\mathcal{S}_i = \{x_i\}$  for some  $x_i \in \Sigma$  which satisfy  $\tilde{\alpha}_i = \alpha_i(x_i)$  and  $\sigma_i(x_i) = \rho_i^0$ , and  $v_i^n \xrightarrow{n \rightarrow +\infty} -\infty$  in  $L_{\text{loc}}^\infty \left( \Sigma \setminus \bigcup_{j \in \mathcal{I}} \{x_j\} \right)$ .
- If  $i \notin \mathcal{I}$ , then  $\mathcal{S}_i = \emptyset$  and  $v_i^n \xrightarrow{n \rightarrow +\infty} v_i$  in  $W_{\text{loc}}^{2,q} \left( \Sigma \setminus \bigcup_{j \in \mathcal{I}} \{x_j\} \right)$  for some  $q > 1$  and some suitable  $v_i$ .

Moreover, if  $a_{ij} < 0$  then  $x_i \neq x_j$ .

*Proof.*

From Theorem 3.1 we get a  $\mathcal{I} \subset \{1, \dots, N\}$  such that  $\mathcal{S}_i \neq \emptyset$  for  $i \in \mathcal{I}$ . If  $\mathcal{S}_i \neq \emptyset$ , then by Corollary 3.4 one gets

$$0 < \sigma_i(x) \leq \rho_i^0 \leq \frac{8\pi(1 + \alpha_i(x))}{a_{ii}}$$

for all  $x \in \mathcal{S}_i$ , hence

$$\begin{aligned} 0 &= \Lambda_{\{1, \dots, N\}, x}(\sigma(x)) \\ &\geq \sum_{j=1}^N (8\pi(1 + \alpha_j(x))\sigma_j(x) - a_{jj}\sigma_j(x)^2) \\ &\geq 8\pi(1 + \alpha_i(x))\sigma_i(x) - a_{ii}\sigma_i(x)^2 \\ &\geq 0. \end{aligned} \tag{19}$$

Therefore, all these inequalities must actually be equalities.

From the last, we have  $\sigma_i(x) = \rho_i^0 = \frac{8\pi(1 + \alpha_i(x))}{a_{ii}}$ , hence  $\alpha_i(x) = \tilde{\alpha}_i$ . On the other hand, since  $\sum_{x \in \mathcal{S}_i} \sigma_i(x) \leq \rho_i^0$ , it must be  $\sigma_i(x) = 0$  for all but one  $x_i \in \mathcal{S}_i$ , so Corollary 3.4 yields  $\mathcal{S}_i = \{x_i\}$ .

Let us now show that  $v_i^n \xrightarrow{n \rightarrow +\infty} -\infty$  in  $L_{\text{loc}}^\infty$ .

Otherwise, Theorem 3.1 would imply  $v_i^n \xrightarrow{n \rightarrow +\infty} v_i$  almost everywhere, therefore by Fatou's Lemma we would get the following contradiction:

$$\sigma_i(x_i) < \int_{\Sigma} \tilde{h}_i e^{v_i} dV_g + \sigma_i(x_i) \leq \int_{\Sigma} \tilde{h}_i^n e^{v_i^n} dV_g = \rho_i^n \leq \rho_i = \sigma_i(x_i).$$

Since also inequality (19) has to be an equality, we get  $a_{ij}\sigma_i(x_i)\sigma_j(x_i)$  for any  $i, j \in \mathcal{I}$ , so whenever  $a_{ij} < 0$  there must be  $\sigma_j(x_i) = 0$ , so  $x_i \neq x_j$ .

Finally, if  $\mathcal{S}_i = \emptyset$ , the convergence in  $W_{\text{loc}}^{2,q}$  follows from what we just proved and Theorem 3.1.  $\square$

We basically showed that if a component of the sequence  $v^n$  blows up, then all its mass concentrates at a single point which has the lowest singularity coefficient.

The next Lemma gives some more important information about the convergence or the blow-up of the components of  $v^n$ .

**Lemma 5.2.**

Let  $v_i^n$ ,  $v_i$ ,  $\rho^0$ ,  $\mathcal{I}$  and  $x_i$  as in Lemma 5.1. Then,

- If  $i \in \mathcal{I}$ , then the sequence  $v_i^n - \bar{v}_i^n$  converges to some  $G_i$  in  $W_{\text{loc}}^{2,q} \left( \Sigma \setminus \bigcup_{j \in \mathcal{I}} \{x_j\} \right)$  for some  $q > 1$  and weakly in  $W^{1,q'}(\Sigma)$  for any  $q' \in (1, 2)$ , and  $G_i$  solves:

$$\begin{cases} -\Delta G_i = \sum_{j \in \mathcal{I}} a_{ij} \rho_j^0 (\delta_{x_j} - 1) + \sum_{j \notin \mathcal{I}} a_{ij} (\tilde{h}_j e^{v_j} - \rho_j^0) \\ \bar{G}_i = 0 \end{cases}.$$

- If  $i \notin \mathcal{I}$ , then  $v_i^n \xrightarrow{n \rightarrow +\infty} v_i$  in the same space, and  $v_i$  solves:

$$\begin{cases} -\Delta v_i = \sum_{j \in \mathcal{I}} a_{ij} \rho_j^0 (\delta_{x_j} - 1) + \sum_{j \notin \mathcal{I}} a_{ij} (\tilde{h}_j e^{v_j} - \rho_j^0) \\ \int_{\Sigma} \tilde{h}_i e^{v_i} dV_g = \rho_i^0 \end{cases}. \quad (20)$$

*Proof.*

From Lemma 5.1 follows that, for  $i \in \mathcal{I}$ ,  $\tilde{h}_i^n e^{v_i^n} \xrightarrow{n \rightarrow \infty} \rho_i^0 \delta_{x_i}$  in the sense of measures; in fact, for any  $\phi \in C(\Sigma)$

$$\begin{aligned} \left| \int_{\Sigma} \tilde{h}_i^n e^{v_i^n} \phi dV_g - \rho_i^0 \phi(x_i) \right| &\leq \int_{\Sigma} \tilde{h}_i^n e^{v_i^n} |\phi - \phi(x_i)| dV_g + |\rho_i^n - \rho_i^0| |\phi(x_i)| \\ &\leq \varepsilon \int_{B_\delta(x_i)} \tilde{h}_i^n e^{v_i^n} dV_g + 2\|\phi\|_{L^\infty(\Sigma)} \int_{\Sigma \setminus B_\delta(x_i)} \tilde{h}_i^n e^{v_i^n} dV_g \\ &\quad + |\rho_i^n - \rho_i^0| \|\phi\|_{L^\infty(\Sigma)} \\ &\leq \varepsilon \rho_i^n + 2\|\phi\|_{L^\infty(\Sigma)} o(1) + o(1) \|\phi\|_{L^\infty(\Sigma)}, \end{aligned}$$

which is, choosing properly  $\varepsilon$ , arbitrarily small. Therefore,  $v_i$  solves (20).

On the other hand, if  $q' \in (1, 2)$ , then  $\frac{q'}{q' - 1} > 2$ , so any function  $\phi \in W^{1, \frac{q'}{q' - 1}}(\Sigma)$  is actually continuous, hence

$$\begin{aligned} \left| \int_{\Sigma} \nabla (v_i^n - \bar{v}_i^n - G_i) \cdot \nabla \phi dV_g \right| &= \left| \int_{\Sigma} (-\Delta v_i^n + \Delta G_i) \phi dV_h \right| \\ &\leq \sum_{j \in \mathcal{I}} a_{ij} \left| \int_{\Sigma} \tilde{h}_j e^{v_j^n} \phi dV_g - \rho_j^0 \phi(p) \right| \\ &\quad + \sum_{j \notin \mathcal{I}} a_{ij} \left| \int_{\Sigma} \tilde{h}_j (e^{v_j^n} - e^{v_j}) \phi dV_g \right| \\ &\xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

Therefore, we get weak convergence in  $W^{1,q'}(\Sigma)$  for any  $q' \in (1, 2)$ ; standard elliptic estimates yield convergence in  $W_{\text{loc}}^{2,q} \left( \Sigma \setminus \bigcup_{j \in \mathcal{I}} \{x_j\} \right)$ .

In the same way we prove the same convergence of  $v_i^n$  to  $v_i$ .  $\square$

From these information about the blow-up profile of  $v^n$  we deduce an important fact which will be used to prove the main Theorem:

**Corollary 5.3.**

Let  $v^n$  and  $x_i$  be as in Lemmas 5.1 and 5.2 and  $w^n$  be defined by  $w_i^n = \sum_{j=1}^N a^{ij} v_j^n$  for  $i \in \{1, \dots, N\}$ .

Then,  $w_i^n - \overline{w_i^n}$  is uniformly bounded in  $W_{\text{loc}}^{2,q}(\Sigma \setminus \{x_i\})$  for some  $q > 1$  if  $i \in \mathcal{I}$ , whereas if  $i \notin \mathcal{I}$  it is bounded in  $W^{2,q}(\Sigma)$ .

*Proof.*

Since  $-\Delta w_i^n = \tilde{h}_i^n e^{v_i^n} - \rho_i^n$ , the claim follows from the boundedness of  $e^{v_i^n}$  in  $L_{\text{loc}}^\infty(\Sigma \setminus \{x_i\})$  and from standard elliptic estimates.  $\square$

The last Lemma we need is a localized scalar Moser-Trudinger inequality for the blowing-up sequence.

**Lemma 5.4.**

Let  $w_i^n$  be as in Corollary 5.3 and  $x_i$  as in the previous Lemmas. Then, for any  $i \in \mathcal{I}$  and any small  $r > 0$  one has

$$\frac{a_{ii}}{2} \int_{B_r(x_i)} |\nabla w_i^n|^2 dV_g - \rho_i^n \left( \log \int_{B_r(x_i)} \tilde{h}_i e^{a_{ii} w_i^n} dV_g - a_{ii} \overline{w_i^n} \right) \geq -C_r.$$

*Proof.*

Since  $\Sigma$  is locally conformally flat, we can choose  $r$  small enough so that we can apply Corollary 2.5 up to modifying  $\tilde{h}_i^n$ . We also take  $r$  so small that  $\overline{B_r(x_i)}$  contains neither any  $x_j$  for  $x_j \neq x_i$  nor any  $p_m$  for  $m = 1, \dots, M$  (except possibly  $x_i$ ).

Let  $z_i^n$  be the solution of

$$\begin{cases} -\Delta z_i^n = \tilde{h}_i^n e^{v_i^n} - \rho_i^n & \text{in } B_r(x_i) \\ z_i^n = 0 & \text{on } \partial B_r(x_i) \end{cases}.$$

Then,  $w_i^n - \overline{w_i^n} - z_i^n$  is harmonic and it has the same value as  $w_i^n - \overline{w_i^n}$  on  $\partial B_r(x_i)$ , so from standard estimates

$$\|w_i^n - \overline{w_i^n} - z_i^n\|_{C^1(B_r(x_i))} \leq C \|w_i^n - \overline{w_i^n}\|_{C^1(\partial B_r(x_i))} \leq C.$$

From Lemma 5.2 we get

$$\begin{aligned} \left| \int_{B_r(x_i)} |\nabla w_i^n|^2 dV_g - \int_{B_r(x_i)} |\nabla z_i^n|^2 dV_g \right| &= \left| \int_{B_r(x_i)} |\nabla (w_i^n - z_i^n)|^2 dV_g \right. \\ &\quad \left. + 2 \int_{B_r(x_i)} \nabla w_i^n \cdot \nabla (w_i^n - z_i^n) dV_g \right| \\ &\leq \int_{B_r(x_i)} |\nabla (w_i^n - z_i^n)|^2 dV_g \\ &\quad + 2 \|\nabla w_i^n\|_{L^1(\Sigma)} \|\nabla (w_i^n - z_i^n)\|_{L^\infty(B_r(x_i))} \\ &\leq C_r. \end{aligned}$$

Moreover,

$$\begin{aligned} \int_{B_r(x_i)} \tilde{h}_i e^{a_{ii}(w_i^n - \overline{w_i^n})} dV_g &\leq e^{a_{ii} \|w_i^n - \overline{w_i^n} - z_i^n\|_{L^\infty(B_r(x_i))}} \int_{B_r(x_i)} \tilde{h}_i e^{a_{ii} z_i^n} dV_g \\ &\leq C_r \int_{B_r(x_i)} d(\cdot, x_i)^{2\tilde{\alpha}_i} e^{a_{ii} z_i^n} dV_g. \end{aligned}$$

Therefore, since  $\tilde{\alpha}_i \leq 0$  and  $a_{ii} \rho_i^n \leq 8\pi(1 + \tilde{\alpha}_i)$ , we can apply Corollary 2.5 to get the claim:

$$\begin{aligned} \frac{a_{ii}}{2} \int_{B_r(x_i)} |\nabla w_i^n|^2 dV_g - \rho_i^n \log \int_{B_r(x_i)} \tilde{h}_i e^{a_{ii}(w_i^n - \overline{w_i^n})} dV_g &\geq \frac{1}{2a_{ii}} \int_{B_r(x_i)} |\nabla (a_{ii} z_i^n)|^2 dV_g \\ &\quad - \rho_i^n \log \int_{B_r(x_i)} d(\cdot, x_i)^{2\tilde{\alpha}_i} e^{a_{ii} z_i^n} dV_g - C_r \\ &\geq -C_r \end{aligned}$$

$\square$

*Proof of Theorem 1.3.*

As noticed before, it suffices to prove the boundedness from below of  $J_{\rho^n}(u^n)$  for a sequence  $\rho^n \nearrow_{n \rightarrow +\infty} \rho^0$  and a sequence of minimizers  $u^n$  for  $J_{\rho^n}$ . Moreover, due to invariance by addition of constants, one can consider  $v^n$  in place of  $u^n$ .

Let us start by estimating the term involving the gradients.

From Corollary 5.3 we deduce that the integral of  $|\nabla w_i^n|^2$  outside a neighborhood of  $x_i$  is uniformly bounded for any  $i \in \mathcal{I}$ , and the integral on the whole  $\Sigma$  is bounded if  $i \notin \mathcal{I}$ .

For the same reason, the integral of  $a_{ij} \nabla w_i^n \cdot \nabla w_j^n$  on the whole surface is uniformly bounded. In fact, if  $a_{ij} \neq 0$ , then  $x_i \neq x_j$ , then

$$\begin{aligned} \left| \int_{\Sigma} \nabla w_i^n \cdot \nabla w_j^n dV_g \right| &\leq \int_{\Sigma \setminus B_r(x_j)} |\nabla w_i^n \cdot \nabla w_j^n| dV_g + \int_{\Sigma \setminus B_r(x_i)} |\nabla w_i^n \cdot \nabla w_j^n| dV_g \\ &\leq \|\nabla w_i^n\|_{L^{q'}(\Sigma)} \|\nabla w_j^n\|_{L^{q''}(\Sigma \setminus B_r\{x_j\})} + \|\nabla w_i^n\|_{L^{q''}(\Sigma \setminus B_r\{x_i\})} \|\nabla w_j^n\|_{L^{q'}(\Sigma)} \\ &\leq C_r, \end{aligned}$$

$$\text{with } q \text{ as in Corollary 5.3, } q' = \begin{cases} \frac{2q}{3q-2} < 2 & \text{if } q < 2 \\ 1 & \text{if } q \geq 2 \end{cases} \text{ and } q'' = \begin{cases} \frac{2q}{2-q} & \text{if } q < 2 \\ \infty & \text{if } q \geq 2 \end{cases}.$$

Therefore, we can write

$$\begin{aligned} \sum_{i,j=1}^N a^{ij} \int_{\Sigma} \nabla v_i^n \cdot \nabla v_j^n dV_g &= \sum_{i,j=1}^N a_{ij} \int_{\Sigma} \nabla w_i^n \cdot \nabla w_j^n dV_g \\ &\geq \sum_{i \in \mathcal{I}} a_{ii} \int_{B_r(x_i)} |\nabla w_i^n|^2 dV_g - C_r. \end{aligned}$$

To deal with the other term in the functional, we use the boundedness of  $w_i^n$  away from  $x_i$ : choosing  $r$  as in Lemma 5.4, we get

$$\begin{aligned} \int_{\Sigma} \tilde{h}_i^n e^{v_i^n - \bar{v}_i^n} dV_g &\leq 2 \int_{B_r(x_i)} \tilde{h}_i^n e^{v_i^n - \bar{v}_i^n} dV_g \\ &= 2 \int_{B_r(x_i)} \tilde{h}_i^n e^{\sum_{j=1}^N a_{ij} (w_j^n - \bar{w}_j^n)} dV_g \\ &\leq C_r \int_{B_r(x_i)} \tilde{h}_i^n e^{a_{ii} (w_i^n - \bar{w}_i^n)} dV_g. \end{aligned}$$

Therefore, using Lemma 5.4 we obtain

$$\begin{aligned} J_{\rho^n}(v^n) &= \frac{1}{2} \sum_{i,j=1}^N a^{ij} \int_{\Sigma} \nabla v_i^n \cdot \nabla v_j^n dV_g - \sum_{i=1}^N \rho_i^n \left( \log \int_{\Sigma} \tilde{h}_i^n e^{v_i^n} dV_g - \bar{v}_i^n \right) \\ &\geq \sum_{i \in \mathcal{I}} \left( \frac{a_{ii}}{2} \int_{B_r(x_i)} |\nabla w_i^n|^2 dV_g - \rho_i^n \left( \log \int_{B_r(x_i)} \tilde{h}_i^n e^{a_{ii} w_i^n} dV_g - a_{ii} \bar{w}_i^n \right) \right) - C_r \\ &\geq -C_r \end{aligned}$$

Since the choice of  $r$  does not depend on  $n$ , the proof is complete.  $\square$

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## References

- [1] Adimurthi and K. Sandeep. A singular Moser-Trudinger embedding and its applications. *NoDEA Nonlinear Differential Equations Appl.*, 13(5-6):585–603, 2007.
- [2] L. Battaglia and A. Malchiodi. A Moser-Trudinger inequality for the singular Toda system. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 9(1):1–23, 2014.
- [3] J. Bolton and L. M. Woodward. Some geometrical aspects of the 2-dimensional Toda equations. In *Geometry, topology and physics (Campinas, 1996)*, pages 69–81. de Gruyter, Berlin, 1997.
- [4] H. Brezis and F. Merle. Uniform estimates and blow-up behavior for solutions of  $-\Delta u = V(x)e^u$  in two dimensions. *Comm. Partial Differential Equations*, 16(8-9):1223–1253, 1991.
- [5] E. Calabi. Isometric imbedding of complex manifolds. *Ann. of Math. (2)*, 58:1–23, 1953.
- [6] S.-Y. A. Chang and P. C. Yang. Prescribing Gaussian curvature on  $S^2$ . *Acta Math.*, 159(3-4):215–259, 1987.
- [7] S.-Y. A. Chang and P. C. Yang. Conformal deformation of metrics on  $S^2$ . *J. Differential Geom.*, 27(2):259–296, 1988.
- [8] W. X. Chen. A Trüdinger inequality on surfaces with conical singularities. *Proc. Amer. Math. Soc.*, 108(3):821–832, 1990.
- [9] S. S. Chern and J. G. Wolfson. Harmonic maps of the two-sphere into a complex Grassmann manifold. II. *Ann. of Math. (2)*, 125(2):301–335, 1987.
- [10] M. Chipot, I. Shafrir, and G. Wolansky. On the solutions of Liouville systems. *J. Differential Equations*, 140(1):59–105, 1997.
- [11] M. Chipot, I. Shafrir, and G. Wolansky. Erratum: “On the solutions of Liouville systems” [J. Differential Equations **140** (1997), no. 1, 59–105; MR1473855 (98j:35053)]. *J. Differential Equations*, 178(2):630, 2002.
- [12] G. Dunne. *Self-dual Chern-Simons Theories*. Lecture notes in physics. New series m: Monographs. Springer, 1995.
- [13] L. Fontana. Sharp borderline Sobolev inequalities on compact Riemannian manifolds. *Comment. Math. Helv.*, 68(3):415–454, 1993.
- [14] J. Jost, C. Lin, and G. Wang. Analytic aspects of the Toda system. II. Bubbling behavior and existence of solutions. *Comm. Pure Appl. Math.*, 59(4):526–558, 2006.
- [15] J. Jost and G. Wang. Analytic aspects of the Toda system. I. A Moser-Trudinger inequality. *Comm. Pure Appl. Math.*, 54(11):1289–1319, 2001.
- [16] C.-S. Lin, J.-c. Wei, and L. Zhang. Classification of blowup limits for  $SU(3)$  singular Toda systems. *Anal. PDE*, 8(4):807–837, 2015.
- [17] M. Lucia and M. Nolasco.  $SU(3)$  Chern-Simons vortex theory and Toda systems. *J. Differential Equations*, 184(2):443–474, 2002.
- [18] J. Moser. A sharp form of an inequality by N. Trudinger. *Indiana Univ. Math. J.*, 20:1077–1092, 1970/71.
- [19] I. Shafrir and G. Wolansky. The logarithmic HLS inequality for systems on compact manifolds. *J. Funct. Anal.*, 227(1):200–226, 2005.
- [20] I. Shafrir and G. Wolansky. Moser-Trudinger and logarithmic HLS inequalities for systems. *J. Eur. Math. Soc. (JEMS)*, 7(4):413–448, 2005.
- [21] G. Tarantello. *Selfdual gauge field vortices*. Progress in Nonlinear Differential Equations and their Applications, 72. Birkhäuser Boston Inc., Boston, MA, 2008. An analytical approach.

- [22] M. Troyanov. Prescribing curvature on compact surfaces with conical singularities. *Trans. Amer. Math. Soc.*, 324(2):793–821, 1991.
- [23] G. Wang. Moser-Trudinger inequalities and Liouville systems. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(10):895–900, 1999.
- [24] Y. Yang. *Solitons in field theory and nonlinear analysis*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2001.