Monotonicity of Avoidance Coupling on K_N

Ohad N. Feldheim*

September 28, 2018

Abstract

Answering a question by Angel, Holroyd, Martin, Wilson and Winkler in [1], we show that the maximal number of non-colliding coupled simple random walks on the complete graph K_N, which take turns, moving one at a time, is monotone in N. We use this fact to couple $\lceil \frac{N}{4} \rceil$ such walks on K_N, improving the previous $\Omega(N/\log N)$ lower bound of Angel et al. We also introduce a new generalization of simple avoidance coupling which we call partially ordered simple avoidance coupling and provide a monotonicity result for this extension as well.

1 Introduction

Let $G = ([N], E)$ be a graph whose vertices are the set of integers $[N] = \{1, \ldots, N\}$. A *simple random walk* on this graph is a Markov chain $(X_t)_{t\in\mathbb{Z}}$ of elements in $[N]$ such that for all $t \in \mathbb{Z}$ the distribution of X_t is uniform on the neighbors of X_{t-1}.

A *Simple Avoidance Coupling (SAC)* of k walks on G is a sequence of random maps $(U_t)_{t\in\mathbb{Z}}$ from $[k]$ to $[N]$ which satisfy two conditions:

$$\forall i \in [k] : (U_t(i))_{t\in\mathbb{Z}} \text{ is a simple random walk on } G \tag{1}$$

$$\forall t \in \mathbb{Z},\ 1 \le i < j \le k \ :\ \mathbb{P}\Big(U_t(i) = U_t(j)\Big) = \mathbb{P}\Big(U_t(i) = U_{t-1}(j)\Big) = 0 \tag{2}$$

Angel, Holroyd, Martin, Wilson and Winkler introduce this notion in [1] in order to investigate couplings of k simple random walks which move in turns in discrete time and avoid collision.

One possible application of SACs on the complete graph K_N is semi-synchronous orthogonal frequency hopping. A communication network consists of several transmitters. As there are overlaps between the transmission ranges they wish to use distinct frequencies at every given time. Malicious adversaries, each located in the vicinity of one of these transmitters, are trying to interfere

*Institute of Mathematics and its Applications, University of Minnesota, College of Science and Engineering, Minneapolis, MN U.S.A. 55455 Email:ohad_f@netvision.net.il. This research was supported in part by the Institute for Mathematics and its Applications with funds provided by the National Science Foundation.

with the communication by noising several frequencies at every given time. Once an adversary hits his target transmitter's frequency he can tell that his interruption succeeded. In order to avoid persistent interference the transmitters wish to change frequencies often. Being unable to perfectly synchronize their clocks, the transmitter must take turns at hopping. In this scenario it is desirable for each transmitter to perform a simple random walk as this would make each of its frequency changes (hops) independent from the past with maximal entropy. Independence is desirable since the adversary has some access to the frequency history of its target transmitter. An ideal hopping scheme in this setting is a SAC.

An important result of [1] is that there exists a SAC of $\Omega(N/\log N)$ walks on K_N. The authors also show in [1, Theorem 6.1] that when $N = 2^\ell + 1$ for some $\ell \in \mathbb{N}$, there exists an avoidance coupling of $2^{\ell-1}$ walks on K_N. Angel, Holroyd, Martin, Wilson and Winkler ask: does the existence of an avoidance coupling of k walks on K_{N-1}, imply the existence of an avoidance coupling of k walks on K_N. Our main result is a positive answer to this question:

Theorem 1. *If there exists a simple avoidance coupling of k walks on K_{N-1}, then there exists a simple avoidance coupling of k walks on K_N.*

Combining this with [1, Theorem 6.1] we draw the following improved bound.

Theorem 2. *There exists a simple avoidance coupling of $\lceil N/4 \rceil$ walks on K_N.*

We find it interesting that a byproduct of the proof of Theorem 1 is that in the extended coupling on K_{N+1} one find another $k+1$-th special walk, which is a simple random walk as well, but does not obey the order in which the walkers move in every round. In Section 4 we investigate this observation and discuss a possible extensions of avoidance coupling to models where the order by which the walks move changes from one round to the next, subject to some restrictions.

1.1 Markovian Couplings

In [1] the authors give special attention to Markovian Simple Avoidance Couplings. These have the property that whenever a walker's turn to move arrives, he needs only to look at the current configuration walkers to determine the distribution of its next location. In particular the simple avoidance coupling of $\Omega(N \log N)$ walkers on K_N constructed in [1] has this property, as does the coupling of $2^{\ell-1}$ walkers on $K_{2^\ell+1}$. While our extension theorem does not preserve this property, we preserve the following weaker version. Consider a SAC in which each site of the underlying graph K_N is assigned a label. At the end of every round a random permutation is applied to these labels. Such a SAC is called *Label Markovian* if whenever a walker's turn to move arrives he needs only to look at the current configuration of the walkers and, in addition, at the current labels of the vertices. Observe that every Markovian SAC is also a Label Markovian SAC. It is straightforward to check that our construction preserves Label Markov property.

2 Background

Probabilistic coupling of several stochastic processes sharing the same distribution, has been introduced to probability theory mainly as a tool to study and prove various properties of that common distribution. Such methods have been successfully used in showing properties such as monotonicity, stochastic dominance and convergence.

Nevertheless, probabilistic coupling can also be a subject of study. In this context, the natural question is "in what sense a collection of coupled identically distributed stochastic processes, is different from a collection of independent processes with the same distribution?". A classical example is that of two random walks on some finite graph G. If two independent random walks move on G, then they are bound to collide with high probability after a polynomial number of steps. Collisions occur even if a scheduler is allowed to control the times in which each walk makes his move (see [4],[7]), and can be avoided only if the scheduler has some knowledge of the future of each walk, and only on special graphs (see [5]). On the other hand, there exist many graphs on which coupled random walks can easily avoid each other. On the cycle graph C_n for example, two walks which start on non-adjecent vertices can avoid each other by moving in the same direction at every step - either clockwise or counter-clockwise. Coupling of walks on K_N, the complete graph on N vertices, appears to be more difficult. In [1], the authors use various techniques inspired by discrete harmonic analysis to create an avoidance coupling of $\Omega(N/\log N)$ walks on K_N and of $N/2-1$ walks for an infinity collection of special N-s. They also investigate avoidance coupling on K_N^*, the complete graph with loops on N vertices, and obtain a lower bound of $N/4$ walks on this graph. The authors further show that no coupling exists for $N-1$ walks on K_N^*, if $N \geq 4$.

The research of avoidance couplings is closely related to that of Brownian motions which keep at least constant distance from each other. This subject and its relation to pursuit-evasion problems is investigated in [2], [3] and [6].

3 Extending an avoidance coupling

This section consists of the proof of Theorem 1. Let $\mathcal{U}_k^{N-1} = \left(U_t(j)\right)_{t\in\mathbb{Z}, j\in[k]}$ be a SAC of k walks on K_{N-1}. Our goal is to define $\mathcal{W}_k^N$, a SAC of k walks on K_N.

3.1 The extended coupling

We begin by introducing an auxiliary sequence of random permutations. Let $P_0 \in S_N$ be a uniformly chosen random permutation in S_N. Let $(a_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence where a_0 is a uniformly chosen element of $[N-1]$. For $t \in \mathbb{N}$ define inductively $P_t, P_{-t} \in S_N$ as follows.

$$P_t := P_{t-1} \circ (N\ a_t),$$
$$P_{-t} := P_{1-t} \circ (N\ a_{1-t}),$$

where $(a\ b)$ is the transposition of the two elements a and b.

Write $\mathcal{P}^N = (P_t)_{t\in\mathbb{Z}}$. It is straightforward to check that $\mathcal{P}^N$ is a stationary Markov chain on S_N which is independent from $\mathcal{U}_k^{N-1}$.

We define $\mathcal{W}_k^N = \left(W_t(j)\right)_{t\in\mathbb{Z}, j\in[k]}$ where $W_t : [k] \to [N]$, as follows:

$$W_t(j) = P_t U_t(j), \quad j \in [k],\ t \in \mathbb{Z}.$$

An example of $\mathcal{U}_2^5$, $\mathcal{P}^6$ and $\mathcal{W}_2^6$ is given in Figure 1. Below we prove that $\mathcal{W}_k^N$ is an avoidance coupling of k walks on K_N.

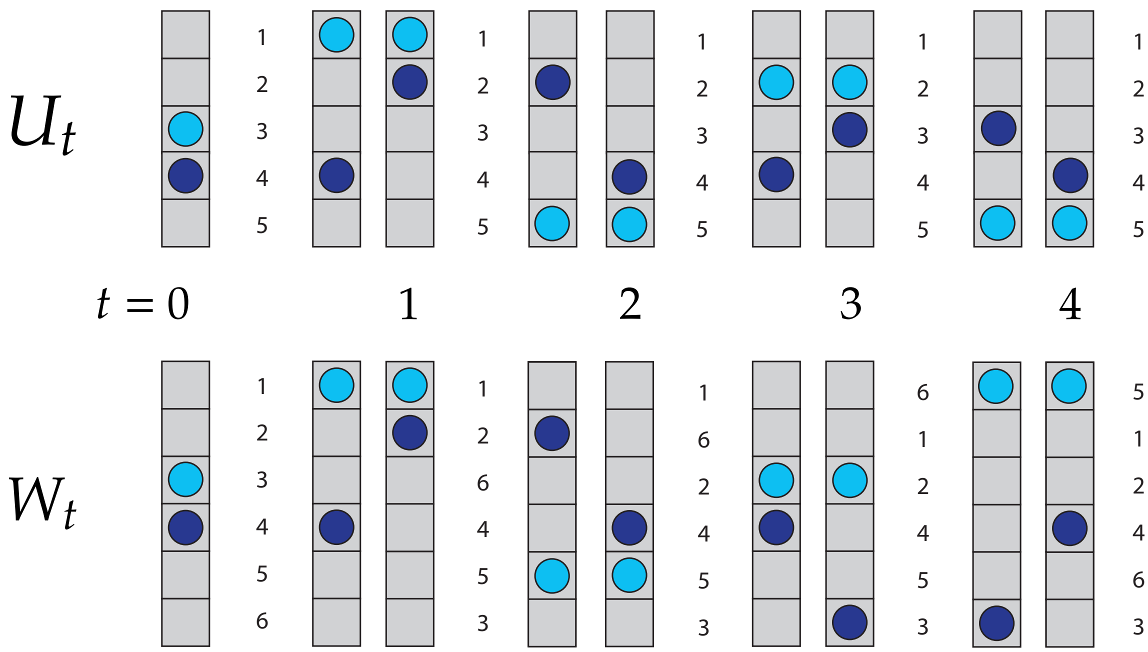

Figure 1: Above: U_t, a SAC of 2 walks on K_5. Below: W_t, the extended SAC on K_6. The label permutation P_t is given at the end of every time unit. Observe that the light blue walk always moves before the dark one. Also observe how W_t is determined by P_t and U_t.

3.2 The extension is a SAC

To show that $\mathcal{W}_k^N$ is a SAC we must show that is satisfies (1) and (2). We begin by showing (2).

Let $t \in \mathbb{Z}$, $1 \le i < j \le k$. We have

$$\mathbb{P}\Big(W_t(i) = W_t(j)\Big) = \mathbb{P}\Big(P_t U_t(i) = P_t U_t(j)\Big) = \mathbb{P}\Big(U_t(i) = U_t(j)\Big) = 0$$

Where the central equality uses the fact that P_t is a permutation and the right-most equality follow from the fact that $\mathcal{U}_k^{N-1}$ satisfies (2).

Recall the definition of the sequence $(a_t)_{t\in\mathbb{Z}}$ and write P'_t for the transposition $(N\ a_t)$. We have

$$\begin{aligned}\mathbb{P}\Big(W_t(i) = W_{t-1}(j)\Big) &= \mathbb{P}\Big(P_t U_t(i) = P_{t-1} U_{t-1}(j)\Big) = \mathbb{P}\Big(P_{t-1} \circ P'_t) U_t(i) = P_{t-1} U_{t-1}(j)\Big) \\ &= \mathbb{P}\Big(P'_t U_t(i) = U_{t-1}(j)\Big) = 0,\end{aligned} \tag{3}$$

where the the last equality follows from the fact that $\mathcal{U}_k^{N-1}$ satisfies (2), and from the fact that $U_t(i), U_{t-1}(j) \in [N-1]$.

We are left with showing that $\mathcal{W}_k^N$ satisfies (1). Fix $j \in [k]$, we must show that $W_t(j)$ is a simple random walk on K_N. Equivalently – for every $\ell \in \mathbb{N}$, every history $w_{t-\ell}, ..., w_{t-1} \in [N]$ such that

$$\mathbb{P}\Big(W_{t-1}(j) = w_{t-1}, \ldots, W_{t-\ell}(j) = w_{t-\ell}\Big) > 0,$$

and for every $v \ne w_{t-1}$, we have

$$\mathbb{P}\Big(W_t(j) = v \,\Big|\, W_{t-1}(j) = w_{t-1}, \ldots, W_{t-\ell}(j) = w_{t-\ell}\Big) = \frac{1}{N-1}. \tag{4}$$

To obtain this we show a stronger claim. Fix $\ell \in\ N$ and let $p = (p_{t-\ell}, ..., p_{t-1}) \in (S_N)^\ell$, $u = (u_{t-\ell}, ..., u_{t-1}) \in [N+1]^\ell$. Consider the event

$$A_t^{p,u} = \Big\{U_{t-1}(j) = u_{t-1}, \ldots, U_{t-\ell}(j) = u_{t-\ell} \text{ and } P_{t-1}(j) = p_{t-1}, \ldots, P_{t-\ell}(j) = p_{t-\ell}\Big\}.$$

We show that for all p, u such that $\mathbb{P}(A_t^{p,u}) \ne 0$ and for all $v \ne p_{t-1}(u_{t-1})$ we have

$$\mathbb{P}\Big(W_t(j) = v \,\Big|\, A_t^{p,u}\Big) = 1/N. \tag{5}$$

Indeed, (5) is stronger than (4), as the values of $P_{t-1}, \ldots, P_{t-\ell}$ and $U_{t-1}(j), \ldots, U_{t-\ell}(j)$ determine the values of $W_{t-1}(j), \ldots, W_{t-\ell}(j)$.

Since $w_{t-1} = p_{t-1}(u_{t-1}) \ne p_{t-1}(N)$ and using the fact that by (2) we have

$$\sum_{n\in[N]\setminus w_{t-1}} \mathbb{P}\Big(W_t(j) = n \,\Big|\, A_t^{p,u}\Big) = 1,$$

it would suffice to show (5) in the case $v \ne p_{t-1}(N)$. Thus, let $v \in [N] \setminus \{p_{t-1}(N), w_{t-1}\}$ and use the total probability formula to write

$$\begin{aligned}
\mathbb{P}\Big(W_t(j) = v \,\Big|\, A_t^{p,u}\Big) &= \mathbb{P}\Big(W_t(j) = v \,\Big|\, A_t^{p,u}, P_t(N) = v\Big)\mathbb{P}\Big(P_t(N) = v\Big) \\
&\quad + \mathbb{P}\Big(W_t(j) = v \,\Big|\, A_t^{p,u}, P_t(N) \neq v\Big)\mathbb{P}\Big(P_t(N) \neq v\Big) \\
&= \mathbb{P}\Big(U_t(j) = N \,\Big|\, A_t^{p,u}, P_t(N) = v\Big) \cdot \frac{1}{N-1} \\
&\quad + \mathbb{P}\Big(U_t(j) = P_t^{-1}(v) \,\Big|\, A_t^{p,u}, P_t(N) \neq v\Big) \cdot \frac{N-2}{N-1} \\
&= 0 \cdot \frac{1}{N-1} + \mathbb{P}\Big(U_t(j) = P_t^{-1}(v) \,\Big|\, A_t^{p,u}, P_t(N) \neq v\Big) \cdot \frac{N-2}{N-1}. \qquad (6)
\end{aligned}$$

We now observe that

$$\begin{aligned}
&\mathbb{P}\Big(U_t(j) = P_t^{-1}(v) \,\Big|\, A_t^{p,u}, v \neq P_t(N)\}\Big) = \\
&\mathbb{P}\Big(U_t(j) = p_{t-1}^{-1}(v) \,\Big|\, A_t^{p,u}, v \neq P_t(N)\}\Big) = \frac{1}{N-2}. \qquad (7)
\end{aligned}$$

where the first equality follows from the fact that for all $v \notin \{P_t(N), P_{t-1}(N)\}$, we have $P_t^{-1}(v) = P_{t-1}^{-1}(v)$, and the last equality uses our assumption that $v \neq w_{t-1} = P_{t-1}U_{t-1}(j)$.

Plugging (7) into (6) we deduce (5), concluding the proof. □

4 Partially ordered avoidance coupling

Consider the following generalization of an avoidance coupling. Let R be a partial order on $[k]$. An *R Partially Ordered Avoidance Coupling* (POSAC) of k walks on G is a sequence of random maps

$$U_t : [m] \to [N], t \in \mathbb{Z},$$

such that there exists a sequence of permutations $\sigma_t \in S_m$ which respect R (i.e., $i <_R j \to \sigma(i) < \sigma(j)$) such that $(U_t)_{t\in\mathbb{Z}}$ and σ_t satisfy two conditions:

1. $\forall i \in [m] : (U_t[i])_{t\in\mathbb{Z}}$ is a simple random walk on G, (8)
2. $\forall t \in \mathbb{Z},\ 1 \leq \sigma_t(i) < \sigma_t(j) \leq m \ :\ \mathbb{P}\Big(U_t(i) = U_{t-1}(j)\Big) = \mathbb{P}\Big(U_t(i) = U_t(j)\Big) = 0.$ (9)

A POSAC is a generalization of a SAC to a situation where the order in which the walks take turns can change from one round to the next, restricted by some partial order constraint (in the application to orthogonal hoping consider a situation where two transmitters can alter the order of their hops only if they receive each other's signal).

The proof of Theorem 1 extends in this case to the following.

Theorem 3. *If there exists an R POSAC of k walks on K_{N-1}, then there exists an R POSAC of $k+1$ walks on K_N.*

Observe that in this case, although the extension does not allow adding additional relations it does allow increasing the number of walks.

4.1 Extending a POSAC

This section is dedicated to the proof of Theorem 3. Let R be a partial order on $[k]$, let $\mathcal{U}_{k,R}^{N-1}$ be an R POSAC of k walks on K_{N-1}, and let $(s_t)_{t\in\mathbb{Z}}$ be a sequence of permutations which respect R and satisfy (9).

Let $\mathcal{P}^N = (P_t)_{t\in\mathbb{Z}}$ and $\mathcal{W}_k^N = \big(W_t(j)\big)_{t\in\mathbb{Z},j\in[k]}$ be as in section 3.1, and define $\mathcal{W}_{k+1,R}^N = \big(W_t(j)\big)_{t\in\mathbb{Z},j\in[k+1]}$ with $W_t(k+1) := P_t(N)$.

Observe that given $t \in \mathbb{Z}$, for any distinct $i, j \in [k+1]$ we have

$$\mathbb{P}\Big(W_t(i) = W_t(j)\Big) = \mathbb{P}\Big(P_t U_t(i) = P_t U_t(j)\Big) = \mathbb{P}\Big(U_t(i) = U_t(j)\Big) = 0, \tag{10}$$

as before. Using this we define $(\sigma_t)_{t\in\mathbb{Z}}$ in the following way. If there exists $b \in [k]$ such that $W_{t-1}(b) = W_t(k+1)$ we set

$$\sigma_t(j) = \begin{cases} s_t(j) & s_t(j) \le s_t(b) \\ s_t(b)+1 & j = m+1 \\ s_t(j)+1 & s_t(j) > s_t(b) \end{cases} \tag{11}$$

while otherwise we set

$$\sigma_t(j) = \begin{cases} s_t(j)+1 & j \le m \\ 1 & j = m+1 \quad . \end{cases} \tag{12}$$

Our purpose is to show that $\mathcal{W}_k^N$ and $(\sigma_t)_{t\in\mathbb{Z}}$ satisfy (8) and (9). An example of $\mathcal{U}_{2,R}^5$, $\mathcal{P}^6$, $\mathcal{W}_{3,R}^6$ and $(\sigma_t)_{t\in\mathbb{Z}}$ is given in Figure 2.

We begin by showing (8). Since the first k walks of $\mathcal{W}_{k+1,R}^N$ are defined in exactly the same way as these of $\mathcal{W}_k^N$, the proof that each of these walks performs a simple random walk is identical to the proof of this fact for $\mathcal{W}_{k+1}^N$ and we omit it. The fact that $\{W_t(k+1)\}_{t\in\mathbb{Z}}$ is a simple random walk is straightforward from fact that $W_t(k+1) = P_t(N)$ and from the definition of P_t.

Next let us show that $\mathcal{W}_{k+1,R}^N$ satisfies (9). Observe that we have obtained the first part of (9) in 10. For the second part, consider the event

$$B_t^{i,j} = \{\sigma_t(i) < \sigma_t(j)\},$$

and write again P'_t for the transposition $(N\ a_t)$. For $i, j \in [k+1]$ we have

$$\begin{aligned} \mathbb{P}\Big(W_t(i) = W_{t-1}(j), B_t^{i,j}\Big) &= \mathbb{P}\Big(P_t U_t(i) = P_{t-1} U_{t-1}(j), B_t^{i,j}\Big) = \mathbb{P}\Big(P_{t-1} \circ P'_t U_t(i) = P_{t-1} U_{t-1}(j), B_t^{i,j}\Big) \\ &= \mathbb{P}\Big(P'_t U_t(i) = U_{t-1}(j), B_t^{i,j}\Big) = 0, \end{aligned}$$

following similar arguments to those used in (4.1).

We thus are left with the case $k+1 \in \{i, j\}$. However, if $i = k+1$ and $W_t(k+1) = W_{t-1}(j)$, then by the definition of σ_t we would have $\sigma_t(j) = \sigma_t(k+1) = s_t(i) + 1$ and $\sigma_t(i) = s_t(i)$. Thus

$$\mathbb{P}\Big(W_t(k+1) = W_{t-1}(j), B_t^{i,j}\Big) = 0.$$

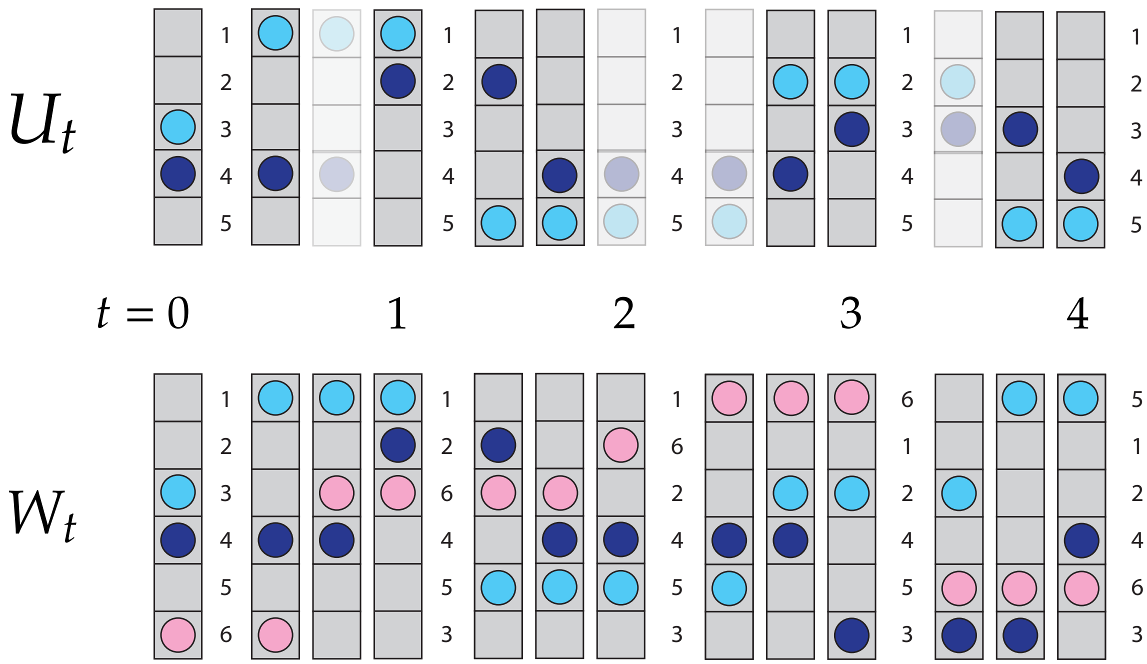

Figure 2: Above: U_t, the same SAC of 2 walks on K_5 as in Figure 1. Faded are duplicates of previous steps used to synchronize with the extension below. Below: W_t, a POSAC of 3 walks, under the partial order of the light blue walker walking before the dark blue one. The permutation is given at the end of every time unit while the order can be inferred from the diagram. Observe how the order of the blue walk change with respect to the extended pink walk between different time units. The rules is that the pink walk waits until his new place is clear and then movs. Also notice that the pink walk always ends his motion in place number 6.

Finally consider the case $j = k+1$. If $B_t^{i,k+1}$ holds then, by the definition of σ_t there must exist some $b \in [k]$ which satisfies $B_t^{i,b}$ such that $W_{t-1}(b) = W_t(j) = P_t(N)$. This b satisfies $W_{t-1}(b) = P_{t-1}U_{t-1}(b) = P_t(N)$ and hence, by the definition of P_t, we have $P_tU_{t-1}(b) = P_{t-1}(N)$.

We get that

$$\begin{aligned}\mathbb{P}\Big(W_t(i) = W_{t-1}(k+1), B_t^{i,j}\Big) = \mathbb{P}\Big(W_t(i) = P_{t-1}(N), B_t^{i,j}\Big) = \mathbb{P}\Big(\exists b \in [k] \ : \ P_tU_t(i) = P_tU_{t-1}(b), B_t^{i,b}\Big) \\ = \mathbb{P}\Big(\exists b \in [k] \ : \ U_t(i) = U_{t-1}(b), B_t^{i,b}\Big) = 0.\end{aligned}$$

Where the last equality follows from the fact that $\mathcal{U}_{k,R}^{N-1}$ satisfies (9). Theorem 3 follows.

5 Acknowledgments

The author would like to thank David Wilson for introducing him to the problem and for useful discussions and to Omer Angel for his corrections and comments.

References

[1] O. Angel, A. Holroyd, J. Martin, D. Wilson and P. Winkler, *Avoidance Coupling*, Electron. Comm. in Prob. 18 (2013), no. 58, 1–13.

[2] I. Benjamini, K. Burdzy, and Z. Chen. *Shy couplings,* Probab. Theory & Related Fields. 137 (2007), no. 3–4, 345-377.

[3] M. Bramson, K. Burdzy, and W. Kendall. *Shy couplings, CAT(0) spaces, and the lion and man.* Ann. Probab. 41 (2010), no. 2, 744–784.

[4] D. Coppersmith, P. Tetali and P. Winkler, *Collisions among random walks on a graph*, SIAM J. Discrete Math. 6 (1993), no. 3, 363–374.

[5] P. Gács, *Clairvoyant scheduling of random walks*, Random Structures & Algorithms 39 (2011), no. 4, 413–485.

[6] W. Kendall. *Brownian couplings, convexity, and shy-ness*, Electron. Commun. Probab. 14 (2009), 6680.

[7] P. Tetali and P. Winkler, *Simultaneous reversible Markov chains*, Combinatorics, Paul Erdős is eighty, Vol. 1, 433–451, János Bolyai Math. Soc., Budapest, 1993.