Strong law of large number for branching Hunt processes [1]

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Abstract

In this paper we prove that, under certain conditions, a strong law of large number holds for a class of branching particle systems X corresponding to the parameters (Y, β, ψ), where Y is a Hunt process and ψ is the generating function for the offspring. The main tool of this paper is the spine decomposition and we only need a $L \log L$ condition.

AMS Subject Classifications (2000): Primary 60J80; Secondary 60F15

Key words and Phrases : branching Hunt process, strong law of large numbers

1 Introduction

1.1 Motivation

In recent years, many people (see [1, 2, 3, 4, 5, 6, 7] and the reference therein) have studied limit theorems for branching Markov processes or superprocesses using the principal eigenvalue and ground state of the linear part of the characteristic equations. For superprocesses, the second moment condition on the branching mechnisms can be weaken, see [1, 7]. However, for branching Markov processes, all the papers in the literature assumed that the branching mechanisms satisfy a second moment condition or (and), they assume that the underlying process is symmetric.

In [8], Asmussen and Hering established a Kesten-Stigum $LlogL$ type theorem for a class of branching diffusion processes under a condition which is later called a positive regular property in [9]. In [10, 11], Liu, Ren and Song established Kesten-Stigum $LlogL$ type theorem for super-diffusions and branching Hunt processes respectively. As a natural continuation of [10], Liu, Ren and Song give a strong law of large number for super-diffusions, see [7]. This paper concerns with the case of branching Markov processes. We establish a strong law of large numbers for a class of branching Hunt processes. The main tool is the spine decomposition. We only assume that the branching mechanisms satisfy a $L \log L$ condition and the underlying process need not to be symmetric.

We first introduce the setup in this paper. Let E be a locally compact separable metric space. Denote by $E_\Delta := E \cup \{\Delta\}$ the one point compactification of E. Let $\mathcal{B}(E)$ denote both the Borel σ-fields on E and the space of functions measurable with respect to itself. Write $\mathcal{B}_b(E)$

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(respectively, $\mathcal{B}^+(E)$) for the space of bounded (respectively, non-negative) $\mathcal{B}(E)$-measurable functions on E. Let $M_p(E)$ be the space of finite point measures on E, that is,

$$M_p(E) = \left\{ \sum_{i=1}^{n} \delta_{x_i} : n \in \mathbb{N} \text{ and } x_i \in E, i = 1, 2, \ldots, n \right\}.$$

As usual, $\langle f, \mu \rangle := \int_E f(x)\mu(dx)$ for any function f on E and any measure $\mu \in M_p(E)$.

As a continuation of [11], the model in this paper is the same as in that paper, we will state it in the next subsection for reader's convenience.

1.2 Model

Let $Y = \{Y_t, \Pi_x, \zeta\}$ be a Hunt process on E, where $\zeta = \inf\{t > 0 : Y_t = \Delta\}$ is the lifetime of Y. Let $\{P_t, t \geq 0\}$ be the transition semigroup of Y:

$$P_t f(x) = \Pi_x[f(Y_t)] \quad \text{for } f \in \mathcal{B}^+(E).$$

Let m be a positive Radon measure on E with full support. $\{P_t, t \geq 0\}$ can be extended to a strongly continuous semigroup on $L^2(E, m)$. Let $\{\widehat{P}_t, t \geq 0\}$ be the dual semigroup of $\{P_t, t \geq 0\}$ on $L^2(E, m)$ satisfy

$$\int_E f(x) P_t g(x) m(dx) = \int_E g(x) \widehat{P}_t f(x) m(dx), \quad f, g \in L^2(E, m).$$

Throught this paper we assume that

Assumption 1.1 (i) There exists a family of continuous strictly positive functions $\{p(t, \cdot, \cdot); t > 0\}$ on $E \times E$ such that for any $(t, x) \in (0, \infty) \times E$, we have

$$P_t f(x) = \int_E p(t, x, y) f(y) m(dy), \quad \widehat{P}_t f(x) = \int_E p(t, y, x) f(y) m(dy).$$

(ii)The semigroups $\{P_t, t \geq 0\}$ and $\{\widehat{P}_t, t \geq 0\}$ are ultracontractive, that is, for any $t > 0$, there exists a constant $c_t > 0$ such that

$$p(t, x, y) \leq c_t \;\text{ for any } (x, y) \in E \times E.$$

Suppose that $\psi \in \mathcal{B}(E \times [-1, 1])$ and ψ is the generating function for each $x \in E$, that is

$$\psi(x, z) = \sum_{n=0}^{\infty} p_n(x) z^n, \quad |z| \leq 1,$$

where $p_n(x) \geq 0$ and $\sum_{n=0}^{\infty} p_n(x) = 1$. The branching system we are going to study determined by the following properties:

1. The particles in E move independently according to the law of Y, and each particle has a random birth and a random death time.
2. Given the path Y of a particle and given that the particle is alive at time t, its probability of dying in the interval $[t, t + dt)$ is $\beta(Y_t)dt + o(dt)$.

3. When a particle dies at $x \in E$, it splits into n particles at x with probability $p_n(x)$. The point Δ is a cemetery. When a particle reaches Δ, it stays at Δ for ever and there is no branching at Δ.

We assume that the functions $\beta(x)$ and $A(x) := \psi'(x,1) = \sum_{n=0}^{\infty} np_n(x)$ are bounded $\mathcal{B}(E)$-measurable and that $p_0(x) + p_1(x) = 0$ on E. The last condition implies $A(x) \geq 2$ on E. The assumption $p_0(x) = 0$ on E is essential for the probabilistic proof of this paper since we need the spine to be defined for all $t \geq 0$. The assumption $p_1(x) = 0$ on E is just for convenience as the case $p_1(x) > 0$ can be reduced to the case $p_1(x) = 0$ by changing the parameters β and ψ of the branching Hunt process.

Let $X_t(B)$ be the number of particles located in $B \in \mathcal{B}(E)$ at time t. Then $X = \{X_t, t \geq 0\}$ is a Markov process in $M_p(E)$ which is called a (Y, β, ψ)-branching process. The process X has probabilities $\{\mathbb{P}_\mu : \mu \in M_p(E)\}$, and $\mathbb{E}_\mu$ is expectation with respect to $\mathbb{P}_\mu$. Then we have

$$\mathbb{E}_\mu[\langle f, X_t\rangle] = \Pi_\mu[e_{(1-A)\beta}(t) f(Y_t)], \quad f \in \mathcal{B}_b^+(E),$$

where $e_c(t) = \exp(-\int_0^t c(Y_s)ds)$ for any $c \in \mathcal{B}_b(E)$. We use $\{P_t^{(1-A)\beta}, t \geq 0\}$ to denote the following Feynman-Kac semigroup

$$P_t^{(1-A)\beta} f(x) := \Pi_x[e_{(1-A)\beta}(t) f(Y_t)], \quad f \in \mathcal{B}(E).$$

Under Assumption 1.1, we can show that $\{P_t^{(1-A)\beta}\}$ is strongly continuous on $L^2(E,m)$ and for any $t > 0$, $P_t^{(1-A)\beta}$ admits a density $p_t^{(1-A)\beta}(t,x,y)$ which is jointly continuous in (x,y).

Let $\{\widehat{P}_t^{(1-A)\beta}, t \geq 0\}$ be the dual semigroup of $\{P_t^{(1-A)\beta}, t \geq 0\}$ defined by

$$\widehat{P}_t^{(1-A)\beta} f(x) = \int_E p_t^{(1-A)\beta}(t,y,x) f(y) m(dy), \quad f \in \mathcal{B}^+(E).$$

write $\mathbf{A}$ and $\hat{\mathbf{A}}$ for the generators of $\{P_t\}$ and $\{\widehat{P}_t\}$. Then the generators of $\{P_t^{(1-A)\beta}\}$ and $\{\widehat{P}_t^{(1-A)\beta}\}$ can be formally written as $\mathbf{A} + (A-1)\beta$ and $\hat{\mathbf{A}} + (A-1)\beta$ respectively.

Let $\sigma(\mathbf{A} + (A-1)\beta)$ and $\sigma(\hat{\mathbf{A}} + (A-1)\beta)$ be the spectrum of $\{P_t^{(1-A)\beta}\}$ and $\{\widehat{P}_t^{(1-A)\beta}\}$, respectively. It follow from Jentzch's Theorem (Theorem V.6.6 on p.333 of [12]) and the strong continuity of $\{P_t^{(1-A)\beta}\}$ and $\{\widehat{P}_t^{(1-A)\beta}\}$ that the common value $\lambda_1 := \sup Re(\sigma(\mathbf{A} + (A-1)\beta)) = \sup Re(\sigma(\hat{\mathbf{A}} + (A-1)\beta))$ is an eigenvalue of multiplicity 1 for both $\mathbf{A} + (A-1)\beta$ and $\hat{\mathbf{A}} + (A-1)\beta$. Let ϕ be an eigenfunction of $\mathbf{A} + (A-1)\beta$ associated with λ_1 and $\widetilde{\phi}$ be an eigenfunction of $\hat{\mathbf{A}} + (A-1)\beta$ associated with λ_1. By (Proposition 2.3 in [13]) we know that ϕ and $\widetilde{\phi}$ are strictly positive and continuous on E. We choose ϕ and $\widetilde{\phi}$ so that $\int_E \phi\widetilde{\phi} m(dx) = 1$. Then

$$\phi(x) = e^{-\lambda_1 t} P_t^{(1-A)\beta}\phi(x), \quad \widetilde{\phi}(x) = e^{-\lambda_1 t}\widehat{P}_t^{(1-A)\beta}\widetilde{\phi}(x), \quad x \in E.$$

Throughout this paper we also assume that

Assumption 1.2 $\lambda_1 > 0$ and $\int_E \phi^2(y)\widetilde{\phi}(y) m(dy) < \infty$.

The assumption $\lambda_1 > 0$ is the condition for supercriticality of the branching Hunt process.

Assumption 1.3 The semigroups $\{P_t^{(1-A)\beta}\}$ and $\{\widehat{P}_t^{(1-A)\beta}\}$ are intrinsic ultracontrative, that is, for any $t > 0$ there exists a constant c_t such that

$$p^{(1-A)\beta}(t,x,y) \leq c_t \phi(x)\widetilde{\phi}(y), \quad x, y \in E.$$

We refer to [11] for examples satisfy the above assumptions.

1.3 Spine Decomposition

For the convenience of state our main result, we shortly recall the spine decomposition in [11]. First we extend the probability measure $\mathbb{P}_{\delta_x}$ to a probability measure $\widetilde{\mathbb{P}}_{\delta_x}$ under which:

1. a single particle, $\widetilde{Y} = \{\widetilde{Y}_t\}_{t\geq 0}$, referred to as the spine, initially starts at x moves according to the measure Π_x.
2. Given the trajectory $\widetilde{Y}_{\cdot}$, the fission time ζ_u of node u on the spine is distributed according to $L^{\beta(\widetilde{Y})}$, where $L^{\beta(\widetilde{Y})}$ is the law of the Poisson random measure with intensity $\beta(\widetilde{Y}_t)dt$.
3. At the fission time ζ_u of node u in the spine, the single spine particle is replaced by a random number r_u of offspring with r_u being distributed according to the law $P(\widetilde{Y}_{\zeta_u}) = (p_k(\widetilde{Y}_{\zeta_u}))_{k\geq 1}$.
4. The spine is chosen uniformly from the r_u particles at the fission time of u.
5. Each of the remaining $r_u - 1$ particles gives rise to independent copys of a P-branching Hunt process started at its space-time point of creation.

Let $\xi = \{\xi_0 = \phi, \xi_1, \xi_2, \ldots\}$ be the selected line of decent in the spine, let $N = (N_t : t \geq 0)$ to denote the counting process of fission times along the spine. Write $node_t(\xi)$ for the node in the spine that is alive at time t. It is clear that $node_t(\xi) = \xi_{N_t}$. Define the natural filtration of the motion and the birth process along the spine by

$$\mathcal{G}_t := \sigma((\widetilde{Y}_s, s \leq t), (node_s(\xi) : s \leq t), (\zeta_u, u < \xi_{N_t}), (r_u : u < \xi_{N_t})),$$

and define $\mathcal{G} = \bigcup_{t\geq 0} \mathcal{G}_t$. Let $\widetilde{\mathcal{F}}_t := \sigma((X_s, s \leq t), (node_s(\xi) : s \leq t))$ and $\widetilde{\mathcal{F}} = \bigcup_{t\geq 0} \widetilde{\mathcal{F}}_t$. From the spine construction, we know that

$$\text{Prob}(u \in \xi) = \prod_{\nu<u} \frac{1}{r_\nu}.$$

It is easy to see that

$$\sum_{u\in L_t} \prod_{\nu<u} \frac{1}{r_\nu} = 1. \tag{1.1}$$

where L_t is the set of particles that are alive at time t. For the definition of $\widetilde{\mathbb{P}}_{\delta_x}$ and the relations of $\mathbb{P}_{\delta_x}$ with $\widetilde{\mathbb{P}}_{\delta_x}$, see [11] for details.

Next we define a probability measure $\widetilde{\mathbb{Q}}_{\delta_x}$ on the branching Hunt process with a spine. Before that, we need to give some facts concerning change of measures.

Girsanov change of measure Let $\mathcal{G}_t = \sigma(Y_s; s \leq t)$. Note that

$$\frac{\phi(Y_t)}{\phi(x)} e^{-\lambda_1 t} e_{(1-A)\beta}(t)$$

is a martingale under Π_x, and so we can define a martingale change of measure by

$$\left.\frac{d\Pi_x^\phi}{d\Pi_x}\right|_{\mathcal{G}_t} = \frac{\phi(Y_t)}{\phi(x)} e^{-\lambda_1 t} e_{(1-A)\beta}(t).$$

Then $\{Y,\Pi_x^\phi\}$ is a conservative Markov process, and $\phi\widetilde{\phi}$ is a unique invariant probability measure for the semigroup $\{P_t^{(1-A)\beta}: t\geq 0\}$, that is, for any $f\in\mathcal{B}^+(E)$,

$$\int_E \phi(x)\widetilde{\phi}(x)P_t^{(1-A)\beta}f(x)m(dx)=\int_E f(x)\phi(x)\widetilde{\phi}(x)m(dx).$$

Let $p^\phi(t,x,y)$ be the transition density of Y in E under Π_x^ϕ. Then

$$p^\phi(t,x,y)=\frac{e^{-\lambda_1 t}}{\phi(x)}p^{(1-A)\beta}(t,x,y)\phi(y).$$

It follows from Theorem 2.8 in [13] that, if Assumption 1.3 holds, there exist constants $c>0$ and $\nu>0$ such that

$$\left|\frac{e^{-\lambda_1 t}p^{(1-A)\beta}(t,x,y)}{\phi(x)\widetilde{\phi}(y)}-1\right|\leq ce^{-\nu t},\ x\in E. \tag{1.2}$$

which is equivalent to

$$\sup_{x\in E}\left|\frac{p^\phi(t,x,y)}{\phi(y)\widetilde{\phi}(y)}-1\right|\leq ce^{-\nu t}. \tag{1.3}$$

Change of measure for Possion process Suppose that given a nonnegative measurable function $\beta(Y_t)$, $t\geq 0$, the Possion process (n,L^β) where $n=\{\{\sigma_i: i=1,2,\ldots,n_t\}: t\geq 0\}$ has instantaneous rate $\beta(Y_t)$. Further, assume that n is adapted to $\{\mathcal{L}_t: t\geq 0\}$. Then under the change of measure

$$\frac{dL^{A\beta}}{dL^\beta}\bigg|_{\mathcal{L}_t}=\prod_{i\leq n_t}A(Y_t)\cdot\exp\left(-\int_0^t((A-1)\beta)(Y_s)ds\right)$$

the process $(n,L^{A\beta})$ is also a Possion process with rate $A\beta$. See, Chapter 3 in [14].

The spine construction Let $\{\mathcal{F}_t: t\geq 0\}$ be the natural filtration generated by X. For any $x\in E$, we define

$$M_t(\phi)=e^{-\lambda_1 t}\frac{\langle X_t,\phi\rangle}{\phi(x)}.$$

Then $\{M_t(\phi),t\geq 0\}$ is a nonnegative martingale with respect to $\{\mathcal{F}_t: t\geq 0\}$. Define the change of measure

$$\frac{d\widetilde{\mathbb{Q}}_{\delta_x}}{d\widetilde{\mathbb{P}}_{\delta_x}}\bigg|_{\mathcal{F}_t}=M_t(\phi).$$

Then, under $\widetilde{\mathbb{Q}}_{\delta_x}$, X can be constructed as follows:

1. a single particle, $\widetilde{Y}=\{\widetilde{Y}_t\}_{t\geq 0}$, referred to as the spine, initially starts at x moves according to the measure Π_x^ϕ;

2. The spine undergoes fission into particles at an accelerated intensity $(A\beta)(\widetilde{Y}_t)dt$;

3. At the fission time ζ_u of node u in the spine, it give birth to r_u particles with size-biased offspring distribution $\widehat{P}(\widetilde{Y}_{\zeta_u}):=(\widehat{P}_k(\widetilde{Y}_{\zeta_u}))_{k\geq 1}$, where $\widehat{P}_k(y):=\frac{kp_k(y)}{A(y)}$, $k=1,2,\ldots$, $y\in E$.

4. The spine is chosen uniformly from the r_u particles at the fission time of u.
5. Each of the remaining $r_u - 1$ particles gives rise to independent copys of a P-branching Hunt process started at its space-time point of creation.

Theorem 1.1 *([11], Theorem 2.9) (Spine decomposition) We have the following spine decomposition for the martingale $M_t(\phi)$*

$$\phi(x)\widetilde{\mathbb{Q}}_{\delta_x}[M_t(\phi)|\mathcal{G}] = e^{-\lambda_1 t}\phi(\widetilde{Y}_t) + \sum_{u<\xi_{N_t}} (r_u - 1)\phi(\widetilde{Y}_{\zeta_u})e^{-\lambda_1 \zeta_u}$$

Denote by $M_\infty(\phi)$ the almost sure limit of $M_t(\phi)$ as $t \to \infty$. In [11], the author studied the relationship between the degeneracy property of $M_\infty(\phi)$ and the function l:

$$l(x) = \sum_{k=2}^{\infty} k\phi(x)\log^+(k\phi(x))p_k(x),\ x \in E. \tag{1.4}$$

Theorem 1.2 *([11], Theorem 1.6) Suppose that $\{X_t : t \geq 0\}$ is a (Y, β, ψ)-branching Hunt process and the Assumptions 1.1-1.3 are satisfied. Then $M_\infty(\phi)$ is a non-degenerate under $\mathbb{P}_\mu$ for any nonzero measure $\mu \in M_p(E)$ if and only if*

$$\int_E \widetilde{\phi}(x)\beta(x)l(x)m(dx) < \infty,$$

where l is defined by (1.4).

1.4 Main Result

Define

$$W_t(\phi) := e^{-\lambda_1 t}\langle X_t, \phi\rangle.$$

The main goal of this paper is to establish the following almost sure convergence result.

Theorem 1.3 *Suppose that $\{X_t : t \geq 0\}$ is a (Y, β, ψ)-branching Hunt process and the Assumptions 1.1-1.3 are satisfied. If $\int_E \widetilde{\phi}(x)\beta(x)l(x)m(dx) < \infty$, then there exists $\Omega_0 \subset \Omega$ with full probability (that is, $\mathbb{P}_{\delta_x}(\Omega_0) = 1$ for every $x \in E$) such that, for every $\omega \in \Omega_0$ and for every bounded Borel measurable function f on E with compact support whose set of discontinuous points has zero m-measure, we have*

$$\lim_{t\to\infty} e^{-\lambda_1 t}\langle X_t, f\rangle = W_\infty(\phi)\int_E \widetilde{\phi}(x)f(x)m(dx), \tag{1.5}$$

where $W_\infty(\phi)$ is the $\mathbb{P}_{\delta_x}$-almost sure limit of $e^{-\lambda_1 t}\langle X_t, \phi\rangle$.

As a consequence of this theorem we immediately get the following

Corollary 1.1 *(Strong law of large numbers) Suppose that $\{X_t : t \geq 0\}$ is a (Y, β, ψ)-branching Hunt process and the Assumptions 1.1-1.3 are satisfied. If $\int_E \widetilde{\phi}(x)\beta(x)l(x)m(dx) < \infty$, then there exists $\Omega_0 \subset \Omega$ with full probability such that, for every $\omega \in \Omega_0$ and for every relatively compact Borel subset B in E having $m(B) > 0$ and $m(\partial B) = 0$, we have $\mathbb{P}_{\delta_x}$-almost surely,*

$$\lim_{t\to\infty} \frac{X_t(B)(\omega)}{\mathbb{P}_{\delta_x}[X_t(B)]} = \frac{W_\infty(\phi)(\omega)}{\phi(x)}.$$

2 Proof of Theorem 1.3

We will prove the theorem by the following steps.

Proposition 2.1 *If $\int_E \widetilde{\phi}(x)\beta(x)l(x)m(dx) < \infty$, then for any $m \in \mathbb{N}, \sigma > 0$,*

$$\lim_{n\to\infty} |U_{(m+n)\sigma}(\phi f) - \mathbb{E}_{\delta_x}(U_{(m+n)\sigma}(\phi f)|\mathcal{F}_{n\sigma})| = 0, \ \mathbb{P}_{\delta_x}\text{-a.s.}$$

where $U_t(f\phi) := e^{-\lambda_1 t}\langle X_t, f\phi\rangle$ for $f \in \mathcal{B}_b^+(E)$.

We will prove this result later. According to the spine construction, if a particle $u \in \xi$, then at the fission time ζ_u, it give birth to r_u offspring, one of which continues the spine while the other $r_u - 1$ individuals go off to create subtrees which are copies of the original branching Hunt process, we write them by $(\tau, M)_j^u$, $j = 1, \ldots, r_u - 1$. Put

$$X^j_{t-\zeta_u} = \sum_{\nu\in L_t, \nu\in(\tau,M)^u_j} \delta_{Y_\nu(t)}(\cdot), \ t \geq \zeta_u,$$

where $\{Y_\nu : \nu \in (\tau, M)^u_j\}$ are independent copies of Y. $(X^j_{t-\zeta_u}, t \geq \zeta_u)$ is a (Y, β, ψ)-branching Hunt process with birth time ζ_u and starting point $\widetilde{Y}_{\zeta_u}$. Then we can write

$$U_t(f\phi) = e^{-\lambda_1 t}(f\phi)(\widetilde{Y}_t) + e^{-\lambda_1 t}\sum_{u<\xi_{N_t}}\sum_{j=1}^{r_u-1}\langle f\phi, X^j_{t-\zeta_u}\rangle.$$

Define

$$\widetilde{U}_t(f\phi) = e^{-\lambda_1 t}(f\phi)(\widetilde{Y}_t) + e^{-\lambda_1 t}\sum_{u<\xi_{N_t}}\sum_{j=1}^{r_u-1}\langle f\phi, X^j_{t-\zeta_u}\rangle I_{\{r_u\phi(\widetilde{Y}_{\zeta_u})\leq e^{\lambda_1\zeta_u}\}}.$$

and

$$M^{u,j}_t(\phi) := e^{-\lambda_1(t-\zeta_u)}\frac{\langle\phi, X^j_{t-\zeta_u}\rangle}{\phi(\widetilde{Y}_{\zeta_u})}, \quad t \geq \zeta_u.$$

Then $\{M^{u,j}_t(\phi), t \geq \zeta_u\}$ is, conditional on $\mathcal{G}$, a nonnegative $\widetilde{\mathbb{P}}_{\delta_x}$-martingale on the subtree $(\tau, M)^u_j$, and therefore

$$\widetilde{\mathbb{Q}}_{\delta_x}[M^{u,j}_t(\phi)|\mathcal{G}] = \widetilde{\mathbb{P}}_{\delta_x}[M^{u,j}_t(\phi)|\mathcal{G}] = 1. \tag{2.1}$$

Suppose that $\{Y_i : i = 1, \ldots, L_{n\sigma}\}$ describes the path of particles alive at time $n\sigma$. Note that we may always write

$$U_{(m+n)\sigma}(f\phi) = \sum_{i=1}^{L_{n\sigma}} e^{-\lambda_1 n\sigma}U^{(i)}_{m\sigma}(f\phi)$$

where given $\mathcal{F}_{n\sigma}$, the collection $\{U^{(i)}_{m\sigma}(f\phi) : i = 1, \ldots, L_{n\sigma}\}$ are mutually independent and equal in distribution to $U_{m\sigma}(f\phi)$ under $\mathbb{P}_{\delta_{Y_i}}$. Then we can write

$$\begin{aligned} U_{(m+n)\sigma}(f\phi) &= \sum_{i=1}^{L_{n\sigma}} e^{-\lambda_1 n\sigma}\widetilde{U}^{(i)}_{m\sigma}(f\phi) + \sum_{i=1}^{L_{n\sigma}} e^{-\lambda_1 n\sigma}\left(U^{(i)}_{m\sigma}(f\phi) - \widetilde{U}^{(i)}_{m\sigma}(f\phi)\right) \\ &:= U^{[1]}_{(m+n)\sigma}(f\phi) + U^{[2]}_{(m+n)\sigma}(f\phi), \end{aligned} \tag{2.2}$$

where $U^{[1]}_{(m+n)\sigma}(f\phi)$ and $U^{[2]}_{(m+n)\sigma}(f\phi)$ stand for the first term and the second term on the right hand respectively.

Lemma 2.1 *If $\int_E \widetilde{\phi}(x)\beta(x)l(x)m(dx) < \infty$ and $\int_E \phi^2(y)\widetilde{\phi}(y)m(dy) < \infty$. Then, for $f \in \mathcal{B}_b^+(E)$ and $x \in E$,*

$$\widetilde{\mathbb{E}}_{\delta_x}[\widetilde{U}_t(\phi f)]^2 < \infty$$

Proof. First, we rewrite $\widetilde{U}_t(f\phi)$ into a new form and take the conditional expectation,

$$\begin{aligned}
&\widetilde{\mathbb{E}}_{\delta_x}[\widetilde{U}_t(\phi f)|\mathcal{F}_t]\\
=\ & \sum_{u\in L_t}\left(e^{-\lambda_1 t}(f\phi)(\widetilde{Y}_u(t)) + e^{-\lambda_1 t}\sum_{\nu<u}\sum_{j=1}^{r_\nu-1}\langle f\phi, X^j_{t-\zeta_\nu}\rangle I_{\{r_\nu\phi(\widetilde{Y}_{\zeta_\nu})\le e^{\lambda_1\zeta_\nu}\}}\right)\widetilde{\mathbb{E}}_{\delta_x}(I_{\{u\in\xi\}}|\mathcal{F}_t)\\
=\ & \sum_{u\in L_t}\left(e^{-\lambda_1 t}(f\phi)(\widetilde{Y}_u(t)) + e^{-\lambda_1 t}\sum_{\nu<u}\sum_{j=1}^{r_\nu-1}\langle f\phi, X^j_{t-\zeta_\nu}\rangle I_{\{r_\nu\phi(\widetilde{Y}_{\zeta_\nu})\le e^{\lambda_1\zeta_\nu}\}}\right)\prod_{\nu<u}\frac{1}{r_\nu}\\
\overset{d}{=}\ & e^{-\lambda_1 t}(f\phi)(\widetilde{Y}_t) + e^{-\lambda_1 t}\sum_{\nu<\xi_{N_t}}\sum_{j=1}^{r_\nu-1}\langle f\phi, X^j_{t-\zeta_\nu}\rangle I_{\{r_\nu\phi(\widetilde{Y}_{\zeta_\nu})\le e^{\lambda_1\zeta_\nu}\}},
\end{aligned}$$

where in the last equation, in order not to introduce another symbol, we still use ξ_{N_t} to denote one of the particles alive at time t, "$\overset{d}{=}$" means equal in distribution under $\mathbb{P}_{\delta_x}$ and the equality (1.1) was used. Using (2.1) and $\widetilde{U}_t(\phi f) \le \|f\|_\infty \cdot W_t(\phi)$, we have

$$\begin{aligned}
&\phi(x)^{-1}\widetilde{\mathbb{E}}_{\delta_x}[\widetilde{U}_t(\phi f)]^2\\
\le\ & \|f\|_\infty\cdot\phi(x)^{-1}\widetilde{\mathbb{E}}_{\delta_x}[W_t(\phi)\widetilde{U}_t(\phi f)]\\
=\ & \|f\|_\infty\cdot\phi(x)^{-1}\widetilde{\mathbb{E}}_{\delta_x}\left[W_t(\phi)\widetilde{\mathbb{E}}_{\delta_x}[\widetilde{U}_t(\phi f)|\mathcal{F}_t]\right]\\
=\ & \|f\|_\infty\cdot\widetilde{\mathbb{Q}}_{\delta_x}\left[e^{-\lambda_1 t}(f\phi)(\widetilde{Y}_t) + e^{-\lambda_1 t}\sum_{\nu<\xi_{N_t}}\sum_{j=1}^{r_\nu-1}\langle f\phi, X^j_{t-\zeta_\nu}\rangle I_{\{r_\nu\phi(\widetilde{Y}_{\zeta_\nu})\le e^{\lambda_1\zeta_\nu}\}}\right]\\
\le\ & \|f\|^2_\infty\cdot\widetilde{\mathbb{Q}}_{\delta_x}\left[e^{-\lambda_1 t}\phi(\widetilde{Y}_t) + \sum_{\nu<\xi_{N_t}}(r_\nu-1)\phi(\widetilde{Y}_{\zeta_\nu})e^{-\lambda_1\zeta_\nu}I_{\{r_\nu\phi(\widetilde{Y}_{\zeta_\nu})\le e^{\lambda_1\zeta_\nu}\}}\right]\\
\le\ & \|f\|^2_\infty\left(\Pi^\phi_x[e^{-\lambda_1 t}\phi(\widetilde{Y}_t)] + \Pi^\phi_x\int_0^t e^{-\lambda_1 s}\phi(\widetilde{Y}_s)\beta(\widetilde{Y}_s)A(\widetilde{Y}_s)\sum_{k=2}^\infty k\widehat{p}_k(\widetilde{Y}_s)I_{\{k\phi(\widetilde{Y}_s)\le e^{\lambda_1 s}\}}ds\right).
\end{aligned}$$

where $\|\cdot\|_\infty$ means the supremum norm here and in the paper. Call the two expressions in bracket on the right hand side the *spine term* $A(x,t)$ and the *sum term* $B(x,t)$ respectively. Note that (1.3) implies that

$$\left|\int_E p^\phi(t,x,y)\phi(y)m(dy) - \int_E \phi^2(y)\widetilde{\phi}(y)m(dy)\right| \le ce^{-\nu t}\int_E \phi^2(y)\widetilde{\phi}(y)m(dy).$$

Therefore,

$$e^{\lambda_1 t}A(x,t) = \Pi^\phi_x(\phi(\widetilde{Y}_t)) \le (c+1)\int_E \phi^2(y)\widetilde{\phi}(y)m(dy) < \infty. \tag{2.3}$$

For the sum term, using the assumption that A and β are bounded, we get

$$\begin{aligned}
B(x,t) &= \Pi_x^\phi \int_0^t e^{-\lambda_1 s}\phi(\widetilde{Y}_s)\beta(\widetilde{Y}_s)A(\widetilde{Y}_s)\sum_{k=2}^\infty k\widehat{p}_k(\widetilde{Y}_s)I_{\{k\phi(\widetilde{Y}_s)\le e^{\lambda_1 s}\}}ds \\
&= \Pi_x^\phi \int_0^t e^{-\lambda_1 s}\phi(\widetilde{Y}_s)\beta(\widetilde{Y}_s)A(\widetilde{Y}_s)\sum_{k=2}^\infty k\frac{k}{A(\widetilde{Y}_s)}p_k(\widetilde{Y}_s)I_{\{k\phi(\widetilde{Y}_s)\le e^{\lambda_1 s}\}}ds \\
&\le \Pi_x^\phi \int_0^t e^{-(\lambda_1-\lambda_1)s}\beta(\widetilde{Y}_s)\sum_{k=2}^\infty kp_k(\widetilde{Y}_s)I_{\{k\phi(\widetilde{Y}_s)\le e^{\lambda_1 s}\}}ds \\
&\le \|\beta A\|_\infty \cdot t < \infty, \qquad (2.4)
\end{aligned}$$

for all $x \in E$, then the conclusion follows. □

Lemma 2.2 *If $\int_E \widetilde{\phi}(x)\beta(x)l(x)m(dx) < \infty$, then for any $m \in \mathbb{N}, \sigma > 0$,*

$$\sum_{n=1}^\infty \widetilde{\mathbb{P}}_{\delta_x}\{U_{(n+m)\sigma}(f\phi) \ne U^{[1]}_{(n+m)\sigma}(f\phi)\} < \infty,$$

$$\sum_{n=1}^\infty \widetilde{\mathbb{E}}_{\delta_x}\left[U^{[1]}_{(m+n)\sigma}(\phi f) - \widetilde{\mathbb{E}}_{\delta_x}(U^{[1]}_{(m+n)\sigma}(\phi f)|\widetilde{\mathcal{F}}_{n\sigma})\right]^2 < \infty.$$

where $U^{[1]}_{(m+n)\sigma}(f\phi)$ was defined in (2.2). In particular

$$\lim_{n\to\infty}\left|U^{[1]}_{(m+n)\sigma}(\phi f) - \widetilde{\mathbb{E}}_{\delta_x}(U^{[1]}_{(m+n)\sigma}(\phi f)|\widetilde{\mathcal{F}}_{n\sigma})\right| = 0,\ \widetilde{\mathbb{P}}_{\delta_x}\text{-a.s.}$$

Proof. Note that (1.2) implies that for any $s \in [0, m\sigma]$, there is a constant $C_{m\sigma}$ such that

$$p^{(1-A)\beta}(s,x,y) \le C_{m\sigma}\phi(x)\widetilde{\phi}(y), \quad x,y \in E.$$

Then by the spine construction and Fubini theorem, we get

$$\begin{aligned}
&\sum_{n=1}^\infty \widetilde{\mathbb{P}}_{\delta_x}\{U_{(n+m)\sigma}(f\phi) \ne U^{[1]}_{(n+m)\sigma}(f\phi)\} \\
\le\ & \sum_{n=1}^\infty \widetilde{\mathbb{E}}_{\delta_x}\left[\sum_{i=1}^{L_{n\sigma}} \widetilde{\mathbb{P}}_{\delta_x}\left(U^{(i)}_{m\sigma}(f\phi) \ne \widetilde{U}^{(i)}_{m\sigma}(f\phi)|\widetilde{\mathcal{F}}_{n\sigma}\right)\right] \\
\le\ & \sum_{n=1}^\infty \widetilde{\mathbb{E}}_{\delta_x}\left(\sum_{i=1}^{L_{n\sigma}}\int_0^{m\sigma} ds \int_E p^{(1-A)\beta}(s,Y_i,y)\beta(y)\sum_{k=2}^\infty p_k(y)I_{\{k\phi(y)>e^{\lambda_1(s+n\sigma)}\}}m(dy)\right) \\
\le\ & C_{m\sigma}\sum_{n=1}^\infty \widetilde{\mathbb{E}}_{\delta_x}\left(\sum_{i=1}^{L_{n\sigma}}\int_0^{m\sigma} ds \int_E \phi(Y_i)\widetilde{\phi}(y)\beta(y)\sum_{k=2}^\infty p_k(y)I_{\{k\phi(y)>e^{\lambda_1(s+n\sigma)}\}}m(dy)\right) \\
=\ & C_{m\sigma}\sum_{n=1}^\infty e^{\lambda_1 n\sigma}\phi(x)\left(\int_0^{m\sigma} ds \int_E \widetilde{\phi}(y)\beta(y)\sum_{k=2}^\infty p_k(y)I_{\{k\phi(y)>e^{\lambda_1(s+n\sigma)}\}}m(dy)\right) \\
=\ & C_{m\sigma}\phi(x)\left(\int_0^{m\sigma} ds \int_E \widetilde{\phi}(y)\beta(y)\sum_{n=1}^\infty e^{\lambda_1 n\sigma}\sum_{k=2}^\infty p_k(y)I_{\{k\phi(y)>e^{\lambda_1(s+n\sigma)}\}}m(dy)\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq C_{m\sigma}\phi(x)\left(\int_0^{m\sigma}ds\int_E\widetilde{\phi}(y)\beta(y)\sum_{k=2}^{\infty}\sum_{n=1}^{\frac{1}{\lambda_1\sigma}log^+[k\phi(y)]}e^{\lambda_1 n\sigma}p_k(y)I_{\{k\phi(y)>e^{\lambda_1(s+n\sigma)}\}}m(dy)\right)\\
&\leq \frac{C_{m\sigma}}{\lambda_1\sigma}\phi(x)\left(\int_0^{m\sigma}ds\int_E\widetilde{\phi}(y)\beta(y)\sum_{k=2}^{\infty}log^+[k\phi(y)]k\phi(y)p_k(y)m(dy)\right)\\
&= \frac{C_{m\sigma}m}{\lambda_1}\phi(x)\int_E\widetilde{\phi}(y)\beta(y)l(y)m(dy)<\infty.
\end{aligned}
$$

For the second inequality, recall that, if X_i are independent random variables with $E(X_i)=0$ or they are martingale difference, then

$$
E\left|\sum_{i=1}^n X_i\right|^p\leq 2^p\sum_{i=1}^n E|X_i|^p.
$$

Jensen's inequality also implies that $|u+v|^p\leq 2^{p-1}(|u|^p+|v|^p)$ for $p\in(1,2]$. Then we have

$$
\begin{aligned}
&\mathbb{E}(|U_{s+t}-\mathbb{E}(U_{s+t}|\mathcal{F}_t)|^p|\mathcal{F}_t))\\
\leq\;& 2^pe^{-\lambda_1pt}\sum_{i=1}^{L_t}\mathbb{E}\left(|U_s^{(i)}-\mathbb{E}(U_s^{(i)}|\mathcal{F}_t)|^p\middle|\mathcal{F}_t\right)\\
\leq\;& 2^pe^{-\lambda_1pt}\sum_{i=1}^{L_t}\mathbb{E}\left(2^{p-1}(|U_s^{(i)}|^p+|\mathbb{E}(U_s^{(i)}|\mathcal{F}_t)|^p)\middle|\mathcal{F}_t\right)\\
\leq\;& 2^pe^{-\lambda_1pt}\sum_{i=1}^{L_t}2^{p-1}\mathbb{E}\left(|U_s^{(i)}|^p+|\mathbb{E}(U_s^{(i)}|\mathcal{F}_t)|^p)\middle|\mathcal{F}_t\right)\\
\leq\;& 2^{2p}e^{-\lambda_1pt}\sum_{i=1}^{L_t}\mathbb{E}(|U_s^{(i)}|^p|\mathcal{F}_t).
\end{aligned}
$$

Note that for any $f\in\mathcal{B}_b^+(E)$, $U_t(f\phi)\leq\|f\|_\infty\cdot W_t(\phi)$, we have that

$$
\begin{aligned}
&\sum_{n\geq1}\widetilde{\mathbb{E}}_{\delta_x}\left(\left|U^{[1]}_{(m+n)\sigma}-\mathbb{E}\left(\widetilde{U}^{[1]}_{(m+n)\sigma}\middle|\widetilde{\mathcal{F}}_{n\sigma}\right)\right|^2\right)\\
\leq\;& 2^4\sum_{n\geq1}e^{-2\lambda_1n\sigma}\widetilde{\mathbb{E}}_{\delta_x}\left(\sum_{i=1}^{L_{n\sigma}}\widetilde{\mathbb{E}}_{\delta_{Y_i}}[\widetilde{U}^{(i)}_{m\sigma}(\phi f)]^2\right)\\
\leq\;& 2^4\sum_{n\geq1}\mathbb{E}_{\delta_x}\left(\sum_{i=1}^{L_{n\sigma}}e^{-2\lambda_1n\sigma}\phi(Y_i)(A(Y_i,m\sigma)+B(Y_i,m\sigma))\right)\\
=\;& 2^4\sum_{n\geq1}e^{-\lambda_1n\sigma}\phi(x)\Pi_x^\phi[A(Y_{n\sigma},m\sigma)+B(Y_{n\sigma},m\sigma)]
\end{aligned}
$$

where $A(x,t)$ and $B(x,t)$ were defined in Lemma 2.1. Then as a consequence of the previous estimates (2.3) and (2.4), we conclude that the last sum remains finite. □

Lemma 2.3 *If* $\int_E \widetilde{\phi}(x)\beta(x)l(x)m(dx) < \infty$*, then for any* $m \in \mathbb{N}, \sigma > 0$*,*

$$\sum_{n=0}^{\infty} \widetilde{\mathbb{E}}_{\delta_x} \left[\left(U_{(m+n)\sigma}(\phi f) - U^{[1]}_{(m+n)\sigma}(\phi f) \right) \Big| \widetilde{\mathcal{F}}_{n\sigma} \right] \text{ converges } \widetilde{\mathbb{P}}_{\delta_x}\text{-a.s.}$$

Proof. Take $f = 1$ in Lemma 2.2, then $\{U_t(\phi) : t \geq 0\}$ is a nonnegative martingale. By Lemma 2.2 we have

$$\sum_{n=0}^{\infty} \widetilde{\mathbb{P}}_{\delta_x} \left\{ U_{(n+m)\sigma}(\phi) \neq U^{[1]}_{(n+m)\sigma}(\phi) \right\} < \infty, \tag{2.5}$$

$$\sum_{n=0}^{\infty} \widetilde{\mathbb{E}}_{\delta_x} \left[U^{[1]}_{(m+n)\sigma}(\phi) - \widetilde{\mathbb{E}}_{\delta_x} \left(U^{[1]}_{(m+n)\sigma}(\phi) \Big| \widetilde{\mathcal{F}}_{n\sigma} \right) \right]^2 < \infty. \tag{2.6}$$

Note that

$$\begin{aligned} \widetilde{\mathbb{E}}_{\delta_x} \left[U^{[1]}_{(m+n)\sigma}(\phi) \Big| \widetilde{\mathcal{F}}_{n\sigma} \right] &= \widetilde{\mathbb{E}}_{\delta_x} \left[\left(U_{(m+n)\sigma}(\phi) - U^{[2]}_{(m+n)\sigma}(\phi) \right) \Big| \widetilde{\mathcal{F}}_{n\sigma} \right] \\ &= U_{n\sigma}(\phi) - \widetilde{\mathbb{E}}_{\delta_x} \left[U^{[2]}_{(m+n)\sigma}(\phi) \Big| \widetilde{\mathcal{F}}_{n\sigma} \right] \end{aligned}$$

By (2.5) and (2.6), we have

$$\sum_{n=0}^{\infty} \left(U_{(m+n)\sigma}(\phi) - U_{n\sigma}(\phi) + \widetilde{\mathbb{E}}_{\delta_x} \left[U^{[2]}_{(m+n)\sigma}(\phi) \Big| \widetilde{\mathcal{F}}_{n\sigma} \right] \right) \text{converges } \widetilde{\mathbb{P}}_{\delta_x}\text{-a.s.}$$

since $U_t(\phi)$ converges almost surely as $t \to \infty$, we have

$$\sum_{n=0}^{\infty} \widetilde{\mathbb{E}}_{\delta_x} \left[U^{[2]}_{(m+n)\sigma}(\phi) \Big| \widetilde{\mathcal{F}}_{n\sigma} \right] \text{ converges } \widetilde{\mathbb{P}}_{\delta_x}\text{-a.s.}$$

So we have

$$\sum_{n=0}^{\infty} \widetilde{\mathbb{E}}_{\delta_x} \left[\left(U_{(m+n)\sigma}(\phi f) - U^{[1]}_{(m+n)\sigma}(\phi f) \right) \Big| \widetilde{\mathcal{F}}_{n\sigma} \right] \leq \|f\|_\infty \sum_{n=0}^{\infty} \widetilde{\mathbb{E}}_{\delta_x} \left[U^{[2]}_{(m+n)\sigma}(\phi) \Big| \widetilde{\mathcal{F}}_{n\sigma}) \right]$$

converges $\widetilde{\mathbb{P}}_{\delta_x}$-a.s. □

Proof of Porposition 2.1. From the decomposition (2.2), we have

$$\begin{aligned} & U_{(m+n)\sigma}(\phi f) - \mathbb{E}_{\delta_x}(U_{(m+n)\sigma}(\phi f)|\mathcal{F}_{n\sigma}) \\ = \; & U_{(m+n)\sigma}(\phi f) - \widetilde{\mathbb{E}}_{\delta_x}(U_{(m+n)\sigma}(\phi f)|\widetilde{\mathcal{F}}_{n\sigma}) \\ = \; & U_{(m+n)\sigma}(\phi f) - U^{[1]}_{(m+n)\sigma}(\phi f) + U^{[1]}_{(m+n)\sigma}(\phi f) - \widetilde{\mathbb{E}}_{\delta_x} \left(U^{[1]}_{(m+n)\sigma}(\phi f) \Big| \widetilde{\mathcal{F}}_{n\sigma} \right) \\ & - \widetilde{\mathbb{E}}_{\delta_x} \left[\left(U_{(m+n)\sigma}(\phi f) - U^{[1]}_{(m+n)\sigma}(\phi f) \right) \Big| \widetilde{\mathcal{F}}_{n\sigma} \right] \end{aligned}$$

Now the conclusion of this proposition follows immediately form Lemma 2.1, Lemma2.2 and Lemma 2.3. □

Theorem 2.1 *If $\int_E \widetilde{\phi}(x)\beta(x)l(x)m(dx) < \infty$, then for any $\sigma > 0$ and $f \in \mathcal{B}_b^+(E)$,*

$$\lim_{n\to\infty} e^{-\lambda_1 n\sigma}\langle \phi f, X_{n\sigma}\rangle = W_\infty(\phi)\int_E \widetilde{\phi}(x)\phi(x)f(x)m(dx),\ \mathbb{P}_{\delta_x}\text{-a.s.}$$

Proof. By Markov property of branching processes we have

$$\mathbb{E}_\mu[e^{-\lambda_1(m+n)\sigma}\langle \phi f, X_{(m+n)\sigma}\rangle|\mathcal{F}_{n\sigma}] = e^{-\lambda_1 n\sigma}\langle e^{-\lambda_1 m\sigma}P_{m\sigma}^{(1-A)\beta}(\phi f), X_{n\sigma}\rangle.$$

Note that (1.2) implies that, for any $f \in \mathcal{B}_b^+(E)$,

$$\left|\frac{e^{-\lambda_1 m\sigma}P_{m\sigma}^{(1-A)\beta}(\phi f)(x)}{\phi(x)} - \int_E \phi(y)\widetilde{\phi}(y)f(y)m(dy)\right| \le ce^{-\nu m\sigma}\int_E \phi(y)\widetilde{\phi}(y)f(y)m(dy),$$

which is equivalent to

$$\left|\frac{e^{-\lambda_1 m\sigma}P_{m\sigma}^{(1-A)\beta}(\phi f)(x)}{\phi(x)\int_E \phi(y)\widetilde{\phi}(y)f(y)m(dy)} - 1\right| \le ce^{-\nu m\sigma}.$$

Thus there exist positive constants $c_m \le 1$ and $C_m \ge 1$ such that

$$c_m\phi(x)\int_E \phi(y)\widetilde{\phi}(y)f(y)m(dy) \le e^{-\lambda_1 m\sigma}P_{m\sigma}^{(1-A)\beta}(\phi f)(x) \le C_m\phi(x)\int_E \phi(y)\widetilde{\phi}(y)f(y)m(dy),$$

and that $\lim_{m\to\infty} c_m = \lim_{m\to\infty} C_m = 1$. Hence,

$$\begin{aligned} e^{-\lambda_1 n\sigma}\langle e^{-\lambda_1 m\sigma}P_{m\sigma}^{(1-A)\beta}(\phi f), X_{n\sigma}\rangle &\ge c_m e^{-\lambda_1 n\sigma}\langle \phi, X_{n\sigma}\rangle\int_E \phi(y)\widetilde{\phi}(y)f(y)m(dy) \\ &= c_m W_{n\sigma}(\phi)\int_E \phi(y)\widetilde{\phi}(y)f(y)m(dy), \end{aligned}$$

and

$$\begin{aligned} e^{-\lambda_1 n\sigma}\langle e^{-\lambda_1 m\sigma}P_{m\sigma}^{(1-A)\beta}(\phi f), X_{n\sigma}\rangle &\le C_m e^{-\lambda_1 n\sigma}\langle \phi, X_{n\sigma}\rangle\int_E \phi(y)\widetilde{\phi}(y)f(y)m(dy) \\ &= C_m W_{n\sigma}(\phi)\int_E \phi(y)\widetilde{\phi}(y)f(y)m(dy). \end{aligned}$$

Those two inequalities and Proposition 2.1 imply that

$$\begin{aligned} \limsup_{n\to\infty} e^{-\lambda_1 n\sigma}\langle \phi f, X_{n\sigma}\rangle &= \limsup_{n\to\infty} e^{-\lambda_1(m+n)\sigma}\langle \phi f, X_{(m+n)\sigma}\rangle \\ &= \limsup_{n\to\infty} e^{-\lambda_1 n\sigma}\langle e^{-\lambda_1 m\sigma}P_{m\sigma}^{(1-A)\beta}(\phi f), X_{n\sigma}\rangle \\ &\le \limsup_{n\to\infty} C_m W_{n\sigma}(\phi)\int_E \phi(y)\widetilde{\phi}(y)f(y)m(dy) \\ &= C_m W_\infty(\phi)\int_E \phi(y)\widetilde{\phi}(y)f(y)m(dy),\quad \mathbb{P}_{\delta_x}\text{-a.s.} \end{aligned}$$

and that

$$\liminf_{n\to\infty} e^{-\lambda_1 n\sigma}\langle \phi f, X_{n\sigma}\rangle \ge c_m W_\infty(\phi)\int_E \phi(y)\widetilde{\phi}(y)f(y)m(dy),\quad \mathbb{P}_{\delta_x}\text{-a.s.}$$

Letting $m \to \infty$, we get

$$\lim_{n\to\infty} e^{-\lambda_1 n\sigma}\langle \phi f, X_{n\sigma}\rangle = W_\infty(\phi)\int_E \phi(y)\widetilde{\phi}(y)f(y)m(dy), \quad \mathbb{P}_{\delta_x}\text{-a.s.}$$

The proof is now complete. □

Lemma 2.4 *If $\int_E \widetilde{\phi}(x)\beta(x)l(x)m(dx) < \infty$, then for any open subset U in E and $x \in E$, we have*

$$\liminf_{t\to\infty} e^{-\lambda_1 t}\langle \phi I_U, X_t\rangle \geq W_\infty(\phi)\int_E \phi(y)\widetilde{\phi}(y)I_U(y)m(dy), \quad \mathbb{P}_{\delta_x}\text{-a.s.}$$

Proof. For $x \in E$ and $\varepsilon > 0$, let

$$U^\varepsilon(x) := \left\{ y \in U : \phi(y) \geq \frac{1}{1+\varepsilon}\phi(x) \right\}.$$

Define

$$Z^{\sigma,\varepsilon}_{n,\nu} = \frac{1}{1+\varepsilon}\,\phi(Y_\nu(n\sigma))1_{\{Y_\nu(t)\in U^\varepsilon(Y_\nu(n\sigma)) \text{ for every } t\in[n\sigma,(n+1)\sigma)\}}$$

where each Y_ν describes the motion of particle ν in the branching particle system. Let

$$S^{\sigma,\varepsilon}_n = e^{-\lambda_1 n\sigma}\sum_{u<\xi_{N_{n\sigma}}}\sum_{j=1}^{r_u-1}\sum_{\nu\in(\tau,M)^u_j} Z^{\sigma,\varepsilon}_{n,\nu}$$

where the subtrees $\{(\tau, M)^u_j\}$ were defined below Proposition 2.1. For $t \in [n\sigma,(n+1)\sigma)$, we have

$$\begin{aligned} e^{-\lambda_1 t}\langle \phi I_U, X_t\rangle &= e^{-\lambda_1 t}(\phi I_U)(\widetilde{Y}_t) + e^{-\lambda_1 t}\sum_{u<\xi_{N_t}}\sum_{j=1}^{r_u-1}\sum_{\nu\in(\tau,M)^u_j}\phi(Y_\nu(t))1_{\{Y_\nu(t)\in U\}} \\ &\geq e^{-\lambda_1 t}(\phi I_U)(\widetilde{Y}_t) + \frac{e^{-\lambda_1 n\sigma}}{1+\varepsilon}\sum_{u<\xi_{N_{n\sigma}}}\sum_{j=1}^{r_u-1}\sum_{\nu\in(\tau,M)^u_j} e^{-\lambda_1\sigma}\phi(Y_\nu(n\sigma))1_{\{Y_\nu(t)\in U^\varepsilon(Y_\nu(n\sigma))\}}. \end{aligned}$$

Applying Proposition 2.1 with $\langle f\phi, X^j_{(m+n)\sigma-\zeta_u}\rangle \Big(= \sum_{\nu\in(\tau,M)^u_j}(f\phi)(Y_\nu((n+m)\sigma))\Big)$ replaced by

$$\sum_{\nu\in(\tau,M)^u_j} e^{-\lambda_1\sigma}\phi(Y_\nu(n\sigma))1_{\{Y_\nu(t)\in U^\varepsilon(Y_\nu(n\sigma)), \text{ for every } t\in[n\sigma,(n+1)\sigma)\}},$$

similar estimates to those found in Proposition 2.1 show us that

$$\lim_{n\to\infty}|S^{\sigma,\varepsilon}_n - E(S^{\sigma,\varepsilon}_n|\mathcal{F}_{n\sigma})| = 0, \ \mathbb{P}_{\delta_x}\text{-a.s.}$$

Then we have

$$\begin{aligned} \liminf_{t\to\infty} e^{-\lambda_1 t}\langle \phi I_U, X_t\rangle &\geq e^{-\lambda_1\sigma}\liminf_{n\to\infty} S^{\sigma,\varepsilon}_n \\ &= e^{-\lambda_1\sigma}\liminf_{n\to\infty} e^{-\lambda_1 n\sigma}\sum_{i=1}^{L_{n\sigma}} E_{Y^i_{n\sigma}}[S^{\sigma,\varepsilon}_0] \\ &= e^{-\lambda_1\sigma}W_\infty(\phi)\int_E \widetilde{\phi}(y)\xi^{\sigma,\varepsilon}_U(y)m(dy), \end{aligned}$$

where we have used Theorem 2.1 in the last equality and

$$\xi_U^{\sigma,\varepsilon}(Y_{n\sigma}^i) = E_{Y_{n\sigma}^i}\left[S_0^{\sigma,\varepsilon}\right] = \frac{\phi(Y_{n\sigma}^i)}{1+\varepsilon} P_{Y_{n\sigma}^i}\Big(Y(t) \in U^\varepsilon(Y_{n\sigma}^i) \text{ for all } t \in [0,\sigma)\Big).$$

Taking $\sigma \downarrow 0$, we get that $\int_E \widetilde{\phi}(y)\xi_U^{\sigma,\varepsilon}(y)m(dy) \to \frac{1}{1+\varepsilon}\int_E \phi(y)\widetilde{\phi}(y)1_U(y)m(dy)$; hence subsequently taking $\varepsilon \downarrow 0$ gives us

$$\liminf_{t\to\infty} e^{-\lambda_1 t}\langle \phi I_U, X_t\rangle \geq W_\infty(\phi)\int_E \phi(y)\widetilde{\phi}(y)1_U(y)m(dy). \tag{2.7}$$

□

Proof of Theorem 1.3 Since E is a locally compact separable metric space, there exists a countable base $\mathcal{U}$ of open set $\{U_k, k \geq 1\}$ that is closed under finite union. By Lemma 2.4, there exists $\Omega_0 \subset \Omega$ of full probability so that for every $\omega \in \Omega_0$,

$$\liminf_{t\to\infty} e^{-\lambda_1 t}\langle \phi I_{U_k}, X_t\rangle \geq W_\infty(\phi)\int_{U_k} \phi(y)\widetilde{\phi}(y)m(dy).$$

For any open set U, there exists a sequence of increasing open sets $\{U_{n_k}, k \geq 1\}$ in $\mathcal{U}$ so that $\bigcup_{k=1}^\infty U_{n_k} = U$. We have for every $\omega \in \Omega_0$,

$$\begin{aligned}\liminf_{t\to\infty} e^{-\lambda_1 t}\langle \phi I_U, X_t\rangle &\geq \liminf_{t\to\infty} e^{-\lambda_1 t}\langle \phi I_{U_{n_k}}, X_t\rangle \\ &\geq W_\infty(\phi)\int_{U_{n_k}} \phi(y)\widetilde{\phi}(y)m(dy) \text{ for every } k \geq 1.\end{aligned}$$

Passing $k \to \infty$ yields that

$$\liminf_{t\to\infty} e^{-\lambda_1 t}\langle \phi I_U, X_t\rangle \geq W_\infty(\phi)\int_U \phi(y)\widetilde{\phi}(y)m(dy).$$

We consider (1.5) on $\{W_\infty(\phi) > 0\}$. For each fixed $\omega \in \Omega_0 \cap \{W_\infty(\phi) > 0\}$, define the probability measure μ_t and μ on E respectively, by

$$\mu_t(A)(\omega) = \frac{e^{-\lambda_1 t}\langle \phi I_A, X_t\rangle}{W_t(\phi)(\omega)} \quad \text{and} \quad \mu(A) = \int_A \phi(x)\widetilde{\phi}(x)m(dx), \quad A \in \mathcal{B}(E)$$

for every $t \geq 0$. Note that the measure μ_t is well defined for every $t \geq 0$. The inequality (2.7) tell us that μ_t converges weakly to μ. Since ϕ is strictly positive and continuous on E, for every function f on E with compact support whose discontinuity set has zero m-measure (equivalently zero μ-measure), $h := \frac{f}{\phi}$ is a bounded function having compact support with the same set of discontinuity with f. We thus have

$$\lim_{t\to\infty}\int_E h\,d\mu_t = \int_E h\,d\mu$$

which is equivalent to say that

$$\lim_{t\to\infty} e^{-\lambda_1 t}\langle X_t, f\rangle(\omega) = W_\infty(\phi)(\omega)\int_E \widetilde{\phi}(x)f(x)m(dx) \quad \text{for every } \omega \in \Omega_0 \cap \{W_\infty(\phi) > 0\}.$$

Since, for every function f on E such that $|f|$ is bounded by $c\phi$ for some $c > 0$,

$$e^{-\lambda_1 t}|\langle X_t, f\rangle| \le e^{-\lambda_1 t}\langle X_t, |f|\rangle \le cW_t,$$

(1.5) holds automatically on $\{W_\infty(\phi) = 0\}$. This complete the proof of the theorem. $\square$

Proof of corollary 1.1 It is enough if we can prove that

$$\lim_{t\to\infty} e^{-\lambda_1 t}\mathbb{E}_{\delta_x}\langle X_t, f\rangle = \phi(x)\int_E \widetilde{\phi}(x) f(x) m(dx), \quad f \in \mathcal{B}_b^+(E).$$

Note that, (1.3) implies that $\Pi_x^\phi(f(Y_t)) \to \langle f\phi, \widetilde{\phi}\rangle$ for every measurable function f satisfying $\langle f\phi, \widetilde{\phi}\rangle < \infty$. Then we have

$$\begin{aligned} e^{-\lambda_1 t}\mathbb{E}_{\delta_x}\langle X_t, f\rangle &= e^{-\lambda_1 t} P_t^{(1-A)\beta} f(x) \\ &= e^{-\lambda_1 t}\int_E p_t^{(1-A)\beta}(t,x,y) f(y) m(dy) \\ &= \phi(x)\int_E p_t^\phi(t,x,y)\frac{f(y)}{\phi(y)} m(dy) \\ &\to \phi(x)\int_E \widetilde{\phi}(x) f(x) m(dx). \end{aligned}$$

Combining with Theorem 1.3, we get the desired result. $\square$

Acknowledgement. I would like to give my sincere thanks to my supervisor Professor Zenghu Li for his encouragement and helpful discussions. Thanks are also given to the referee for her or his valuable comments and suggestions.

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