WEAK CONVERGENCE THEOREMS FOR EQUILIBRIUM PROBLEMS AND GENERALIZED HYBRID MAPPINGS

SATTAR ALIZADEH[1] AND FRIDOUN MORADLOU[2]

Abstract. In this paper, we introduce a new modified Ishikawa iteration for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of generalized hybrid mappings in a Hilbert space. Our results generalize, extend and enrich some existing results in the literature.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of positive integers and real numbers, respectively. Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $\|.\|$, and let E be a nonempty closed convex subset of H. Let f be a bifunction from $E \times E$ to $\mathbb{R}$. The equilibrium problem for $f : E \times E \to \mathbb{R}$ is to find $x \in E$ such that

$$f(x, y) \geq 0, \quad (y \in E). \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(f)$, i.e.,

$$EP(f) = \{x \in E : \ f(x, y) \geq 0, \ \ \forall y \in E\}.$$

A self mapping S of E is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad (x, y \in E).$$

We denote by $F(S)$ the set of fixed points of S.

Let $S : E \longrightarrow H$ be a mapping and let $f(x, y) = \langle Sx, y - x \rangle$ for all $x, y \in E$. Then $z \in EP(f)$ if only if $\langle Sz, y - z \rangle \geq 0$ for all $y \in E$, i.e., z is a solution of the variational inequality $\langle Sx, y - x \rangle \geq 0$. So, the formulation (1.1) includes variational inequalities as special cases. Also, numerous problems in physics, optimization and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem; see for instance, [4, 8, 11, 20, 16].

In the recent years, many authors studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Hilbert spaces and Banach spaces, respectively; see for instance, [3, 6, 7, 9, 14, 15, 21, 22, 24, 25] and the references therein.

2010 *Mathematics Subject Classification.* Primary 47H10, 47H09, 47J25, 47J05.

Key words and phrases. Equilibrium problems, Fixed point, Hybrid method, Hilbert space, Strong convergence, Weak convergence.

Let E be a nonempty closed convex subset of a Banach space. In 1953, for a self mapping S of E, Mann [18] defined the following iteration procedure:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Sx_n, \end{cases} \tag{1.2}$$

where $0 \leq \alpha_n \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$,

Let K be a closed convex subset of a Hilbert space H. In 1974, for a Lipschitzian pseudocontractive self mapping S of K, Ishikawa [12] defined the following iteration procedure:

$$\begin{cases} x_0 \in K \text{ chosen arbitrarily}, \\ y_n = \beta_n x_n + (1-\beta_n)Sx_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Sy_n, \end{cases} \tag{1.3}$$

where $0 \leq \beta_n \leq \alpha_n \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and he proved strong convergence of the sequence $\{x_n\}$ generated by the above iterative scheme if $\lim_{n\to\infty} \beta_n = 1$ and $\sum_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = \infty$. By taking $\beta_n = 1$ for all $n \geq 0$ in (1.3), Ishikawa iteration process reduces to Mann iteration process.

Process (1.3) is indeed more general than process (1.2). But research has been done on the latter due probably to reasons that the formulation of process (1.2) is simpler than that of (1.3) and that a convergence theorem for process (1.2) may lead to a convergence theorem for process (1.3) provided that $\{\beta_n\}$ satisfies certain appropriate conditions. On the other hand, the process (1.2) may fail to converge while process (1.3) can still converge for a Lipschitz pseudocontractive mapping in a Hilbert space [5]. Actually, Mann and Ishikawa iteration processes have only weak convergence, in general (see [10]).

In 2007, Tada and Takahashi [23] for finding an element of $EP(f) \cap F(S)$, introduced the following iterative scheme for a nonexpansive self mapping S of a nonempty, closed convex subset E in a Hilbert space H:

$$\begin{cases} x_1 = x \in H \text{ chosen arbitrarily}, \\ u_n \in E \text{ such that } \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in E \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Su_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $f : E \times E \to \mathbb{R}$ satisfies appropriate conditions, $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. They proved $\{x_n\}$ converges weakly to $w \in F(S) \cap EP(f)$, where $w = \lim_{n\to\infty} P_{F(S)\cap EP(f)}(x_n)$.

Let E be a nonempty closed convex subset of H. A self mapping S of E is called *generalized hybrid* [17] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Sx - Sy\|^2 + (1-\alpha)\|x - Sy\|^2 \leq \beta\|Sx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{1.4}$$

for all $x, y \in E$. We call such a mapping an (α, β)-*generalized hybrid* mapping.

In this paper, we modify Ishikawa iteration process for finding a common element of the set of solution of an equilibrium problem and the set of fixed points of a generalized hybrid mapping.

2. Preliminaries

A self mapping S of E is called: (i) *firmly nonexpansive*, if $\|Sx-Sy\|^2 \leq \langle x-y, Sx-Sy\rangle$ for all $x, y \in E$; (ii) *nonspreading*, if $2\|Sx-Sy\|^2 \leq \|Sx-y\|^2 + \|Sy-x\|^2$ for all $x, y \in E$; (iii) *hybrid*, if $3\|Sx-Sy\|^2 \leq \|x-y\|^2 + \|Sx-y\|^2 + \|Sy-x\|^2$ for all $x, y \in E$. Also, a self mapping S of E with $F(S) \neq \emptyset$ is called *quasi-nonexpansive* if $\|x-Sy\| \leq \|x-y\|$ for all $x \in F(S)$ and $y \in E$. It is well-known that for a *quasi-nonexpansive* mapping S, $F(S)$ is closed and convex [13].

It easy to see that

- $(1,0)$-generalized hybrid mapping is nonexpansive mapping;
- $(2,1)$-generalized hybrid mapping is nonspreading mapping;
- $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mapping is hybrid mapping.

Let S be a generalized hybrid mapping. If $F(S) \neq \phi$, then there exists $x \in E$ such that $x = Sx$, so for all $y \in E$ we have

$$\alpha\|x-Sy\|^2 + (1-\alpha)\|x-Sy\|^2 \leq \beta\|x-y\|^2 + (1-\beta)\|x-y\|^2$$

and this yields that $\|x-Sy\| \leq \|x-y\|$, i.e., an (α, β)-generalized hybrid mapping with $F(S) \neq \phi$, is quasi-nonexpansive.

We denote the weak convergence and the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively and denote $\omega_\omega(x_n)$ the weak ω-limit set of the sequence $\{x_n\}$, i.e., $\omega_\omega(x_n) := \{x \in H : \exists\{x_{n_k}\} \subset \{x_n\}; \quad x_{n_k} \rightharpoonup x\}$.

Now, we recall some basic properties of Hilbert spaces which we will use in next section. For $x, y \in H$, we have

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \quad \forall \alpha \in \mathbb{R}, \tag{2.1}$$

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \tag{2.2}$$

and

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle. \tag{2.3}$$

Let K be a closed convex subset of H and let P_K be metric (or nearest point) projection from H onto K (i.e., for $x \in H$, $P_K x$ is the only point in K such that $\|x - P_K x\| = inf\{\|x-z\| : z \in H\}$). Let $x \in H$ and $z \in K$, then $z = P_K x$ if and only if:

$$\langle x-z, y-z\rangle \leq 0, \tag{2.4}$$

for all $y \in K$. For more details we refer readers to [1, 26].

Lemma 2.1. [28] *Let H be a Hilbert space and $\{x_n\}$ be a sequence in H such that there exists a nonempty subset $E \subset H$ satisfying*

(i) *For every $u \in E$, $lim_{n\to\infty}\|x_n - u\|$ exists.*
(ii) *If a subsequence $\{x_{n_j}\} \subset \{x_n\}$ converges weakly to u, then $u \in E$,*

then there exists $x_0 \in E$ such that $x_n \rightharpoonup x_0$.

We will use the following lemmas in the proof of our main results in next section.

Lemma 2.2. [27] *Let H be a Hilbert space and E be a nonempty, closed and convex subset of H and $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - x\| \leq \|x_n - x\|$ for all $n \in \mathbb{N}$ and $x \in E$, then $\{P_E(x_n)\}$ converges strongly to some $z \in E$, where P_E stands for the metric projection on H onto E.*

To study the equilibrium problem, for the bifunction $f : E \times E \longrightarrow \mathbb{R}$, we assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in E$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in E$;
(A3) for each $x, y, z \in E$,

$$\lim_{\downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y);$$

(A4) for each $x \in E$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

The following lemma can be found in [4].

Lemma 2.3. *Let E be a nonempty closed convex subset of H, let f be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and let $r > 0$ and $x \in H$. Then, there exists $z \in E$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,$$

for all $y \in E$.

The following lemma is established in [8].

Lemma 2.4. *For $r > 0$, $x \in H$, define a mapping $T_r : H \longrightarrow E$ as follows:*

$$T_r(x) = \{z \in E : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in E\},$$

for all $x \in H$. Then, the following statements hold:

(i) *T_r is singel-valued;*
(ii) *T_r is firmly nonexpansive, i.e., for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(iii) $F(T_r) = EP(f)$;
(iv) *EP(f) is closed and convex.*

3. Main Results

In this section, we prove weak the convergence theorems for finding a common element of the set of solution of an equilibrium problem and the set of fixed points of a generalized hybrid mapping.

Theorem 3.1. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying $(A1)-(A4)$ and S be a generalized hybrid mapping of E to H with $F(S) \cap EP(f) \neq \phi$. Assume that $0 < \alpha \leq \alpha_n \leq 1$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$ and $\{\beta_n\}$ is sequence in $[b, 1]$ for some $b \in (0, 1)$ such that $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. If $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x = x_1 \in H$ and*

$$\begin{cases} u_n \in E \text{ such that } \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in E \\ y_n = (1-\beta_n)x_n + \beta_n S u_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n, \end{cases}$$

for all $n \in \mathbb{N}$. Then $x_n \rightharpoonup v \in F(S) \cap EP(f)$, where $v = lim_{n\to\infty} P_{F(S)\cap EP(f)}(x_n)$.

Proof. By Lemma 2.3, $\{u_n\}, \{y_n\}$ and $\{x_n\}$ are well defined. Since S is a generalized hybrid mapping such that $F(S) \neq \phi$, S is quasi-nonexpansive. So $F(S)$ is closed and convex. Also by hypothesis $EP(f) \neq \phi$. Set $q \in F(S) \cap EP(f)$.

From $u_n = T_{r_n} x_n$, we get

$$\|u_n - q\| = \|T_{r_n} x_n - T_{r_n} q\| \leq \|x_n - q\|. \tag{3.1}$$

On the other hand

$$\begin{aligned} \|y_n - q\|^2 &= (1-\beta_n)\|x_n - q\|^2 + \beta_n\|Su_n - q\|^2 - \beta_n(1-\beta_n)\|x_n - Su_n\|^2 \\ &\leq (1-\beta_n)\|x_n - q\|^2 + \beta_n\|x_n - q\|^2 - \beta_n(1-\beta_n)\|x_n - Su_n\|^2 \\ &= \|x_n - q\|^2 - \beta_n(1-\beta_n)\|x_n - Su_n\|^2, \end{aligned} \tag{3.2}$$

and hence

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|(1-\alpha_n)x_n + \alpha_n S y_n - q\|^2 \\ &= (1-\alpha_n)\|x_n - q\|^2 + \alpha_n\|Sy_n - q\|^2 - \alpha_n(1-\alpha_n)\|x_n - Sy_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - q\|^2 + \alpha_n\|y_n - q\|^2 - \alpha_n(1-\alpha_n)\|x_n - Sy_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - q\|^2 + \alpha_n\|x_n - q\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n - Su_n\|^2 \\ &\qquad\qquad - \alpha_n(1-\alpha_n)\|x_n - Sy_n\|^2 \\ &\leq \|x_n - q\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n - Su_n\|^2 \\ &\leq \|x_n - q\|^2. \end{aligned} \tag{3.3}$$

So, we can conclude that $lim_{n\to\infty}\|x_n - q\|$ exists. This yields that $\{x_n\}$ and $\{y_n\}$ are bounded. It follows from (3.3) that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n - Su_n\|^2.$$

Using $0 < \alpha \leq \alpha_n \leq 1$, it is easy to see that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha\beta_n(1-\beta_n)\|x_n - Su_n\|^2.$$

Also, we have

$$0 \leq \alpha\beta_n(1-\beta_n)\|x_n - Su_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \to 0,$$

as $n \to \infty$, since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Therefore

$$\|x_n - Su_n\| \longrightarrow 0. \tag{3.4}$$

This yields that

$$\|y_n - x_n\| = \beta_n\|x_n - Su_n\| \longrightarrow 0. \tag{3.5}$$

Using (2.3) and Lemma 2.4 , we get

$$\begin{aligned}\|u_n - q\|^2 &= \|T_{r_n}x_n - T_{r_n}q\|^2\\ &\le \langle T_{r_n}x_n - T_{r_n}q, x_n - q\rangle\\ &= \langle u_n - q, x_n - q\rangle\\ &= \frac{1}{2}(\|u_n - q\|^2 + \|x_n - q\|^2 - \|x_n - u_n\|^2),\end{aligned}$$

hence

$$\|u_n - q\|^2 \le \|x_n - q\|^2 - \|x_n - u_n\|^2.$$

Then, by the convexity of $\|.\|^2$, we have

$$\begin{aligned}\|y_n - q\|^2 &= \|(1-\beta_n)(x_n - q) + \beta_n(Su_n - q)\|^2\\ &\le (1-\beta_n)\|x_n - q\|^2 + \beta_n\|Su_n - q\|^2\\ &\le (1-\beta_n)\|x_n - q\|^2 + \beta_n\|u_n - q\|^2\\ &\le (1-\beta_n)\|x_n - q\|^2 + \beta_n(\|x_n - q\|^2 - \|x_n - u_n\|^2)\\ &= \|x_n - q\|^2 - \beta_n\|x_n - u_n\|^2.\end{aligned}$$

Therefore

$$\beta_n\|x_n - u_n\|^2 \le \|x_n - q\|^2 - \|y_n - q\|^2 \tag{3.6}$$

Since $\{\beta_n\} \subset [b, 1]$, it follows from (3.6) that

$$\begin{aligned}b\|x_n - u_n\|^2 &\le \beta_n\|x_n - u_n\|^2\\ &\le \|x_n - q\|^2 - \|y_n - q\|^2\\ &= (\|x_n - q\| - \|y_n - q\|)(\|x_n - q\| + \|y_n - q\|)\\ &\le \|y_n - x_n\|(\|x_n - q\| + \|y_n - q\|)\end{aligned}$$

Using the boundedness of $\{x_n\}$ and $\{y_n\}$, it follows from (3.5) and the above inequality that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.7}$$

Since $\liminf_{n\to\infty} r_n > 0$, we get

$$\lim_{n\to\infty} \left\|\frac{x_n - u_n}{r_n}\right\| = \lim_{n\to\infty} \frac{1}{r_n}\|x_n - u_n\| = 0. \tag{3.8}$$

As $\beta_n Su_n = y_n - (1-\beta_n)x_n$, we have

$$\begin{aligned}b\|u_n - Su_n\| &\le \beta_n\|u_n - Su_n\| = \|y_n - (1-\beta_n)x_n - \beta_n u_n\|\\ &\le \|y_n - x_n\| + \beta_n\|x_n - u_n\|\\ &\le \|y_n - x_n\| + \|x_n - u_n\|\end{aligned}$$

From (3.5) and (3.7), we obtain

$$\lim_{n\to\infty} \|u_n - Su_n\| = 0. \tag{3.9}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$. By (3.7) we obtain $u_{n_i} \rightharpoonup u$. We know that E is closed and convex and $\{u_{n_i}\} \subset E$, therefore $u \in E$.

Now, we show that $u \in F(S) \cap EP(f)$. Since $u_n = T_{r_n} x_n$, we get

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0,$$

for all $y \in E$. From the condition $(A2)$, we obtain

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq f(y, u_n),$$

for all $y \in E$, therefore

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq f(y, u_{n_i}), \tag{3.10}$$

for all $y \in E$. It follows from (3.8), (3.10) and condition $(A4)$ that

$$0 \geq f(y, u),$$

for all $y \in E$. Suppose that $t \in (0,1]$, $y \in E$ and $y_t = ty + (1-t)u$. Therefore, $y_t \in E$ and so $f(y_t, u) \leq 0$. Hence

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, u) \leq tf(y_t, y),$$

and dividing by t, we have $f(y_t, y) \geq 0$, for all $y \in E$. By taking the limit as $t \downarrow 0$ and using $(A3)$, we get $u \in EP(f)$.

Next we show that $u \in F(S)$. Since S is a generalized hybrid mapping, then

$$\gamma\|Sx - Sy\|^2 + (1-\gamma)\|x - Sy\|^2 \leq \lambda\|Sx - y\|^2 + (1-\lambda)\|x - y\|^2$$

hence

$$0 \leq \lambda\|Sx - y\|^2 + (1-\lambda)\|x - y\|^2 - \gamma\|Sx - Sy\|^2 - (1-\gamma)\|x - Sy\|^2$$

replacing x and y by u_n and u in above inequality, respectively, we get

$$\begin{aligned}
0 &\leq \lambda(\|Su_n\|^2 - 2\langle Su_n, u\rangle + \|u\|^2) + (1-\lambda)(\|u_n\|^2 - 2\langle u_n, u\rangle + \|u\|^2) \\
&\quad - \gamma(\|Su_n\|^2 - 2\langle Su_n, Su\rangle + \|Su\|^2) - (1-\gamma)(\|u_n\|^2 - 2\langle u_n, Su\rangle + \|Su\|^2) \\
&= \|u\|^2 - \|Su\|^2 + (\lambda - \gamma)(\|Su_n\|^2 - \|u_n\|^2) \\
&\quad + 2\gamma\langle Su_n - u_n, Su\rangle - 2\lambda\langle Su_n - u_n, u\rangle + 2\langle u_n, Su - u\rangle \\
&\leq \|u\|^2 - \|Su\|^2 + (\lambda - \gamma)(\|Su_n\| + \|u_n\|)(\|Su_n - u_n\|) \\
&\quad + 2\gamma\langle Su_n - u_n, Su\rangle - 2\lambda\langle Su_n - u_n, u\rangle + 2\langle u_n, Su - u\rangle.
\end{aligned}$$

Now, substituting n by n_i, we have

$$\begin{aligned}
0 &\leq \|u\|^2 - \|Su\|^2 + (\lambda - \gamma)(\|Su_{n_i}\| + \|u_{n_i}\|)(\|Su_{n_i} - u_{n_i}\|) \\
&\quad + 2\gamma\langle Su_{n_i} - u_{n_i}, Su\rangle - 2\lambda\langle Su_{n_i} - u_{n_i}, u\rangle + 2\langle u_{n_i}, Su - u\rangle.
\end{aligned} \tag{3.11}$$

for all $i \in \mathbb{N}$. Since $u_{n_i} \rightharpoonup u$ as $i \to \infty$, it follows from (3.9) and (3.11) that

$$\begin{aligned} 0 &\leq \|u\|^2 - \|Su\|^2 + 2\langle u, Su - u\rangle \\ &= 2\langle u, Su\rangle - \|u\|^2 - \|Su\|^2 \\ &= -\|u - Su\|^2. \end{aligned}$$

So, we have $Su = u$, i.e., $u \in F(S)$. Therefore the condition (ii) of Lemma 2.1 satisfies for $E = F(S) \cap EP(f)$. On the other hand, we see that $\lim_{n\to\infty} \|x_n - q\|$ exists for $q \in F(S) \cap EP(f)$. Hence, it follows from Lemma 2.1 that there exists $v \in F(S) \cap EP(f)$ such that $x_n \rightharpoonup v$. In addition, for all $q \in F(S) \cap EP(f)$, we have

$$\|x_{n+1} - q\| \leq \|x_n - q\|, \qquad \forall\, n \in \mathbb{N},$$

so, Lemma 2.2 implies that there exists some $w \in F(S)\cap EP(f)$ such that $P_{F(T)\cap EP(f)}(x_n) \to w$. Then

$$\Big\langle v - P_{F(T)\cap EP(f)}(x_n), x_n - P_{F(T)\cap EP(f)}(x_n)\Big\rangle \leq 0.$$

Hence, we get

$$\langle v - w, v - w\rangle = \|v - w\|^2 \leq 0.$$

Therefore $v = w$, i.e., $x_n \rightharpoonup v = lim_{n\to\infty} P_{F(T)\cap EP(f)}(u_n)$. □

Corollary 3.2. *Let E be a nonempty closed convex subset of a real Hilbert space H and S be a generalized hybrid mapping of E to H with $F(S) \neq \phi$. Assume that $0 < \alpha \leq \alpha_n \leq 1$ and $\{\beta_n\}$ is sequence in $[b, 1]$ for some $b \in (0, 1)$ such that $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. If $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x = x_1 \in H$ and*

$$\begin{cases} u_n \in E \text{ such that } \quad \langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in E \\ y_n = (1 - \beta_n)x_n + \beta_n S u_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n, \end{cases}$$

for all $n \in \mathbb{N}$. Then $x_n \rightharpoonup v \in F(S)$, where $v = lim_{n\to\infty} P_{F(S)}(x_n)$.

Proof. Letting $f(x, y) = 0$ for all $x, y \in E$ and $r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.1, we get the desired result. □

Corollary 3.3. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and S be a generalized hybrid mapping of E to H with $F(S) \cap EP(f) \neq \phi$. Assume that $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$ and $\{\beta_n\}$ is sequence in $[b, 1]$ for some $b \in (0, 1)$ such that $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. If $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x = x_1 \in H$ and*

$$\begin{cases} u_n \in E \text{ such that } \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in E \\ x_{n+1} = S((1 - \beta_n)x_n + \beta_n S u_n), \end{cases}$$

for all $n \in \mathbb{N}$. Then $x_n \rightharpoonup v \in F(S) \cap EP(f)$, where $v = lim_{n\to\infty} P_{F(S)\cap EP(f)}(x_n)$.

Proof. Letting $\alpha_n = 1$ for all $n \in \mathbb{N}$, in Theorem 3.1, we get the desired result. □

Theorem 3.4. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and S be a hybrid mapping of E to H with $F(S) \cap EP(f) \neq \phi$. Assume that $0 < \alpha \leq \alpha_n \leq 1$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$ and $\{\beta_n\}$ is sequence in $[b, 1]$ for some $b \in (0, 1)$ such that $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. If $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x = x_1 \in H$ and*

$$\begin{cases} u_n \in E \text{ such that } \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in E \\ y_n = (1 - \beta_n)x_n + \beta_n S u_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n, \end{cases}$$

for all $n \in \mathbb{N}$. Then $x_n \rightharpoonup v \in F(S) \cap EP(f)$, where $v = lim_{n\to\infty} P_{F(S)\cap EP(f)}(x_n)$.

Proof. Since S is a hybrid mapping, hence S is a $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mapping. Therefore by Theorem 3.1, we get the desired result. □

Remark 3.5. Since nonexpansive mappings are $(1, 0)$- generalized hybrid mappings and nonspreading mappings are $(2, 1)$-generalized hybrid mappings, then the Theorem 3.1 holds for these mappings.

References

[1] R. P. Agarwal, Donal O'Regan and D. R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer 2009.

[2] S. Alizadeh and F. Moradlou, *Strong convergence theorems for m-generalized hybrid mappings in Hilbert spaces,* to appear in Topol. Methods Nonlinear Anal.

[3] S. Alizadeh and F. Moradlou, *A strong convergence theorem for equilibrium problems and generalized hybrid mappings,* Mediterr. J. Math. DOI: 10.1007/s00009-014-0462-6.

[4] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems, Mathematics Students,* **63** (1994), 123–145.

[5] C. E. Chidume and S. A. Mutangadura, *An example on the Mann iteration method for Lipschitz pseudocontractions,* Proc. Am. Math. Soc. **129** (2001) 2359-2363.

[6] L. C. Ceng and J. C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems,* J. Comput. Appl. Math. **214** (2008) 186–201.

[7] L. C. Ceng, S. Al-Homidan and Q. H. Ansari, J. C. Yao, *An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings,* J. Comput. Appl. Math. **223** (2009) 967–974.

[8] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces,* J. Nonlinear Convex Anal. **6** (2005) 117–136.

[9] S. D. Flam and A. S. Antipin, *Equilibrium progamming using proximal-link algolithms,* Math. Program. **78** (1997) 29–41.

[10] A. Genel and J. Lindenstrass, *An example concerning fixed points,* Israel J. Math. **22** (1975) 81-86.

[11] F. Giannessi, A. Maugeri and P. M. Pardalos, *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models,* Kluwer Academics Publishers, Dordrecht, Holland, 2001.

[12] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **40** (1974) 147–150.

[13] S. Itoh and W. Takahashi, *The common fixed point theory of single-valued mappings and multi-valued mappings,* Pacific J. Math. **79** (1978) 493-508.

[14] C. Jaiboon and P. Kumam, *A hybrid extragradient viscosity approximation method for solving equilibrium problems and fixed point problems of infinitely many nonexpansive mappings,* Fixed Point Theory Appl. (2009) Article ID 374815, 32 pages.

[15] C. Jaiboon and P. Kumam, H. W. Humphries, *Weak convergence theorem by an extragradient method for variational inequality, equilibrium and fixed point problems,* Bull. Malays. Math. Sci. Soc. (2) **32** (2) (2009) 173–185.

[16] D. Kinderlehrer and G. Stampacchia, *An Introduction to Varational Inequalities and Their Applications,* Academic Press, New york, 1980.

[17] P. Kocourek, W. Takahashi and J. -C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mapping in Hilbert spaces* , Taiwanese J. Math. **6** (2010) 2497–2511.

[18] W. R. Mann, *Mean value methods in iteration,* Proc. Amer. Math. Soc. **4** (1953) 506–510.

[19] C. Martinez-Yanes and H. K. Xu, *Strong convergence of CQ method fixed point biteration processes,* Nonlinear Anal. **64** (2006) 2400–2411.

[20] A. Moudafi and M. Thera, *Proximal and dynamical approaches to equilibrium problems, in: Lecture Note in Economics and Mathematical Systems,* vol. **477**, Springer-Verlag, New York, (1999), pp. 187–201.

[21] S. Plubtieng and P. Kumam, *Weak convergence theorem for monotone mappings and a countable family of nonexpansive mappings,* J. Comput. Appl. Math. **224** (2009) 614–621.

[22] X. Qin, Y. J. Cho and S. M. Kang, *Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces,* J. Comput. Appl. Math. **225** (2009) 20–30.

[23] A. Tada and W. Takahashi, *Weak and strong convergence theorems for a nonexpansive mappings and an equilibrium problem,* J. Optim. Theory Appl. **133** (2007) 359–370.

[24] S. Takahashi and W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces,* J. Math. Anal. Appl. **331** (2007) 506–515.

[25] W. Takahashi and K. Zembayashi, *Strong convergence theorems by hybrid methods for equilibrium problems and relatively nonexpansive mapping,* Fixed Point Theory Appl. (2008) Article ID 528476, 11 pages.

[26] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohoma Publishers, Yokohoma, 2009.

[27] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings,* J. Optim. Theory Appl. **118** (2003) 417-428.

[28] F. Yan, Y. Su, And Q. Feng, *Convergence of Ishikawa method for generalized hybrid mappings,* commu. Korean Math. Soc. **28** (2013), No. 1 135–141.

[1,2] Department of Mathematics
Sahand University of Technology
Tabriz, Iran
E-mail address: [1] sa_alizadeh@sut.ac.ir
E-mail address: [2] moradlou@sut.ac.ir & fridoun.moradlou@gmail.com