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EXPONENTIAL FRAMES ON UNBOUNDED SETS

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Abstract. For every set S of finite measure in $\mathbb{R}$ we construct a discrete set of real frequencies Λ such that the exponential system $\{\exp(i\lambda t), \lambda \in \Lambda\}$ is a frame in $L^2(S)$.

1. Introduction

This note can be viewed as a continuation of our previous paper [NOU]. In [NOU] we constructed "good" sampling sets for the Paley–Wiener spaces PW_S of entire $L^2(\mathbb{R})-$functions with bounded spectrum S in $\mathbb{R}$. This construction is based on a result in [BSS] on existence of well-invertible sub-matrices of large orthogonal matrices. Recently, an important progress in the latter area has been made in [MSS]. Based on this, we prove existence of exponential frames in $L^2(S)$, for every unbounded set S in $\mathbb{R}$ of finite measure.

Recall that a system of vectors $E = \{u_j\}$ is a frame in a Hilbert space H if there are positive constants a, A such that

$$a\|h\|^2 \leq \sum_{u_j \in E} |\langle h, u_j \rangle|^2 \leq A\|h\|^2 \quad \forall h \in H.$$

The numbers a and A above are called frame bounds.

Given a discrete set Λ in $\mathbb{R}$, we denote by

$$E(\Lambda) := \{e^{i\lambda t}\}_{\lambda \in \Lambda}$$

the system of exponentials with frequencies in Λ.

Exponential frames $E(\Lambda)$ in $L^2(S)$ (equivalently, stable sampling sets Λ for PW_S) have been carefully studied from different points of view. There is a large number of results in the area. In the classical case when S is an interval, such systems were essentially characterized by Beurling [B] in terms of the so-called "lower uniform density" of Λ. A complete description of exponential frames for intervals is given by Ortega–Cerdá and Seip [OS]. However, the problem of existence of exponential frames for unbounded sets remained open. The following result fills this gap by showing that for every set S of finite measure, the space $L^2(S)$ admits an exponential frame:

Theorem 1 *There are positive constants c, C such that for every set $S \subset \mathbb{R}$ of finite measure there is a discrete set $\Lambda \subset \mathbb{R}$ such that $E(\Lambda)$ is a frame in $L^2(S)$ with frame bounds $c|S|$ and $C|S|$.*

Here by $|S|$ we denote the measure of S.

Remark 1. The frame bounds are essential in many contexts, since they characterize the "quality" of frame decompositions. Assume that an exponential system $E(\Lambda)$ forms an orthogonal basis in $L^2(S)$. One can easily check that in this case $E(\Lambda)$ is a frame in $L^2(S)$ with frame bounds $a = A = |S|$. These are, in a sense, the "optimal" frame bounds. In general, there may be no exponential orthogonal basis in $L^2(S)$. However, Theorem 1 shows that an exponential frame in $L^2(S)$ always exists with "almost" (up to fixed multiplicative constants) optimal frame bounds.

Remark 2. A similar to Theorem 1 result regarding the existence of complete exponential systems $E(\Lambda)$ in $L^2(S)$ (equivalently, existence of uniqueness sets Λ for PW_S) is obtained in [OU] by an effective direct construction. That is not the case here, since the proof of Theorem A below in [MSS] involves stochastic elements.

Remark 3. Assume that S lies on an interval of length $2\pi d, d > 0$. It follows from Lemma 10 below that a set Λ satisfying the conclusion of Theorem 1 can be chosen satisfying $\Lambda \subset (1/d)\mathbb{Z}$.

Remark 4. Assume that Λ satisfies the conclusions of Theorem 1. Then there are two absolute constants k, K such that the inequalities

$$k|S| < \frac{\#(\Lambda \cap \Omega)}{|\Omega|} < K|S|$$

hold whenever Ω is a sufficiently long interval in $\mathbb{R}$. In fact, one can choose any numbers $k < 1/2\pi$ and $K > 4C$, where C is the constant in Theorem 1. Then, as it was shown by Landau [L] (for a more elementary proof see [NO]), the left hand-side inequality above follows from the frame property of $E(\Lambda)$. The right hand-side inequality follows from Lemma 6 (ii) below.

2. Well-invertible submatrices

Our construction is based on the following result by Marcus, Spielman and Srivastava from [MSS]:

Theorem A *Let $\epsilon > 0$, and $u_1, ..., u_m \in \mathbb{C}^n$ such that $\|u_i\|^2 \le \epsilon$ for all $i = 1, ...m$, and*

$$\sum_{i=1}^{m} |\langle w, u_i\rangle|^2 = \|w\|^2 \qquad \forall w \in \mathbb{C}^n.$$

Then there exists a partition of $\{1, ..., m\}$ into S_1 and S_2, such that for each $j = 1, 2$,

$$\sum_{i\in S_j} |\langle w, u_i\rangle|^2 \le \frac{(1 + \sqrt{2\epsilon})^2}{2}\|w\|^2 \qquad \forall w \in \mathbb{C}^n. \tag{1}$$

Observe that, clearly, $(1+\sqrt{2\epsilon})^2 \leq 1+5\sqrt{\epsilon}$ when $\epsilon < 1$.

Remark 5. Let $\epsilon < 1$. Since

$$\sum_{i\in S_1} |\langle w, u_i\rangle|^2 = \|w\|^2 - \sum_{i\in S_2} |\langle w, u_i\rangle|^2,$$

estimate (1) shows that the two-sided estimate holds for each $j = 1, 2$:

$$\frac{1-5\sqrt{\epsilon}}{2}\|w\|^2 \leq \sum_{i\in S_j} |\langle w, u_i\rangle|^2 \leq \frac{1+5\sqrt{\epsilon}}{2}\|w\|^2 \qquad \forall w \in \mathbb{C}^n. \tag{2}$$

The following corollary (see Corollary F.2 in [HO]) gives a reformulation of Theorem A in a form well prepared for an induction process:

Corollary B *Let $v_1, ..., v_k \in \mathbb{C}^n$ be such that $\|v_i\|^2 \leq \delta$ for all $i = 1, ..., k$. If*

$$\alpha\|w\|^2 \leq \sum_{i=1}^{k} |\langle w, v_i\rangle|^2 \leq \beta\|w\|^2 \qquad \forall w \in \mathbb{C}^n,$$

with some numbers $\alpha > \delta$ and β, then there exists a partition of $\{1, ..., k\}$ into S_1 and S_2 such that for each $j = 1, 2$,

$$\frac{1-5\sqrt{\delta/\alpha}}{2}\alpha\|w\|^2 \leq \sum_{i\in S_j} |\langle w, v_i\rangle|^2 \leq \frac{1+5\sqrt{\delta/\alpha}}{2}\beta\|w\|^2 \qquad \forall w \in \mathbb{C}^n. \tag{3}$$

For the sake of completeness, we reproduce the proof.

Let $M : \mathbb{C}^n \to \mathbb{C}^n$ be the operator defined by $Mw = \sum_{i=1}^{k}\langle w, v_i\rangle v_i$. Observe that M is positive and that

$$\alpha\|w\|^2 \leq \|M^{1/2}w\|^2 \leq \beta\|w\|^2 \qquad \forall w \in \mathbb{C}^n.$$

Set $u_i = M^{-1/2}v_i$. Then $\|u_i\|^2 \leq \|v_i\|^2/\alpha \leq \delta/\alpha$. Further, for all $w \in \mathbb{C}^n$,

$$\sum_{i=1}^{k}\langle w, u_i\rangle u_i = M^{-1/2}\sum_{i=1}^{k}\langle M^{-1/2}w, v_i\rangle v_i = M^{-1/2}MM^{-1/2}w = w.$$

We see that u_i satisfy the assumptions of Theorem A with $m = k$ and $\epsilon = \delta/\alpha < 1$. Hence, there is a partition of $\{1, ..., k\}$ into two sets S_1 and S_2 satisfying (2). Using the right hand-side of (2) we get

$$\sum_{i\in S_j} |\langle w, v_i\rangle|^2 = \sum_{i\in S_j} |\langle M^{1/2}w, u_i\rangle|^2 \leq \frac{1+5\sqrt{\epsilon}}{2}\|M^{1/2}w\|^2 \leq$$

$$\leq \frac{1+5\sqrt{\epsilon}}{2}\beta\|w\|^2 = \frac{1+5\sqrt{\delta/\alpha}}{2}\beta\|w\|^2.$$

The proof of the left hand-side of (3) is similar.

We will use an elementary lemma:

Lemma 1 *Let $0<\delta<1/100$, and let $\alpha_j,\beta_j, j=0,1,...,$ be defined inductively*

$$\alpha_0=\beta_0=1,\ \alpha_{j+1}:=\alpha_j\frac{1-5\sqrt{\delta/\alpha_j}}{2},\quad \beta_{j+1}:=\beta_j\frac{1+5\sqrt{\delta/\alpha_j}}{2}.$$

Then there exist a positive absolute constant C and a number $L\in\mathbb{N}$ such that

$$a_j\geq 100\delta, j\leq L,\quad 25\delta\leq a_{L+1}<100\delta,\quad b_{L+1}<Ca_{L+1}.$$

Proof. Clearly, if $a_j\geq 100\delta$ then

$$\frac{\alpha_j}{4}\leq\alpha_{j+1}<\frac{\alpha_j}{2}.$$

Denote by $L\geq 1$ the greatest number such that $a_L\geq 100\delta$, and set $\gamma_j:=5\sqrt{\delta/a_j}, j\leq L$. Then $\gamma_{L-j}<2^{-1-j/2}$. It follows that

$$\prod_{j=0}^{L}\frac{1+\gamma_j}{1-\gamma_j}<C:=\prod_{j=0}^{\infty}\frac{1+2^{-1-j/2}}{1-2^{-1-j/2}}.$$

This gives $b_{L+1}<Ca_{L+1}$, and the lemma follows.

We will need the following

Lemma 2 *Assume the hypothesis of Theorem A are fulfilled and that $\|u_i\|^2=n/m, i=1,...,m$. Then there is a subset $J\subset\{1,...,m\}$ such that*

$$c_0\frac{n}{m}\|w\|^2\leq\sum_{i\in J}|\langle w,u_i\rangle|^2\leq C_0\frac{n}{m}\|w\|^2\qquad\forall w\in\mathbb{C}^n,\tag{4}$$

where c_0 and C_0 are some absolute positive constants.

Proof. If $n/m\geq 1/100$, then (4) holds with $J=\{1,...,m\}$ and $C_0=c_0=100$.

Assume $\delta:=n/m<1/100$. Let α_j and β_j be as defined in Lemma 1. Then the vectors $v_i=u_i$ satisfy the assumptions of Corollary B with $\alpha_0=\beta_0=1$. Hence, a set $J_1\subset\{1,...,m\}$ exists such that

$$\alpha_1\|w\|^2\leq\sum_{i\in J_1}|\langle w,u_i\rangle|^2\leq\beta_1\|w\|^2\quad\forall w\in\mathbb{C}^n.$$

Since $\alpha_1\geq\alpha_L>100\delta$, we may apply Corollary B the second time to get a set $J_2\subset J_1$ such that the two-sided inequality above holds with J_2,α_2 and β_2, and so on. Since $\alpha_L>100\delta$, Corollary B can be applied L times. We thus obtain a set $J_{L+1}\subset\{1,...,m\}$ for which the two-sided inequality holds with α_{L+1} and β_{L+1}. From Lemma 1 it follows that (4) is true with $J=J_{L+1}$.

We now reformulate Lemma 1 in terms more convenient for our application. Given a matrix A of order $m\times n$ and a subset $J\subseteq\{1,...,m\}$, we denote by $A(J)$ the sub-matrix of A whose rows belong to the index set J.

Lemma 3 *There exist positive constants $c_0, C_0>0$, such that whenever A is an $m\times n$ matrix which is a sub-matrix of some $m\times m$ orthonormal matrix, and such* *that all of its rows have equal l^2 norm, one can find a subset $J \subset \{1, ..., m\}$ such that*

$$c_0 \frac{n}{m} \|w\|^2 \le \|A(J)w\|^2 \le C_0 \frac{n}{m} \|w\|^2 \qquad \forall w \in \mathbb{C}^n. \tag{5}$$

3. Auxiliary results

In what follows we write $F = \hat{f}$, where f is the Fourier transform of F:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itx} F(t)\, dt.$$

Given a discrete set Λ, we denote by $d(\Lambda)$ its separation constant

$$d(\Lambda) := \inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'|.$$

Given a sequence of sets Λ_j satisfying $d(\Lambda_j) \ge d > 0$ for all j, a set Λ is called the weak limit of Λ_j if for every $\epsilon > 0$ and for every interval $\Omega = (a, b), a, b \notin \Lambda$, both inclusions $\Lambda_j \cap \Omega \subset (\Lambda \cap \Omega) + (-\epsilon, \epsilon)$ and $\Lambda \cap \Omega \subset (\Lambda_j \cap \Omega) + (-\epsilon, \epsilon)$ hold for all but a finite number of j's. The standard diagonal procedure implies that if Λ_j satisfy $d(\Lambda_j) \ge d > 0$ for all j, then there is a subsequence which weakly converges to some (maybe, empty) set Λ satisfying $d(\Lambda) \ge d$.

Recall that the Paley–Wiener space PW_S is defined as the space of all functions $f \in L^2(\mathbb{R})$ such that $\hat{f}$ vanishes a.e. outside S. When the measure of S is finite, we have

$$\int_S |F(t)|\, dt \le \|F\| \sqrt{|S|} \qquad \forall F \in L^2(S).$$

Here $\|F\|$ means the L^2-norm of F. Hence, $\hat{f} \in L^1(\mathbb{R})$ for every $f \in PW_S$, and so every function $f \in PW_S$ is continuous.

Sometimes it will be more convenient for us to work with the Paley–Winer space PW_S, rather than $L^2(S)$. In this connection we observe that by taking the Fourier transform, Theorem 1 is equivalent to the following statement:

There exist positive constants c, C such that for every set $S \subset \mathbb{R}, |S| < \infty$, there is a discrete set $\Lambda \subset \mathbb{R}$ such that

$$c|S|\|f\|^2 \le \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \le C|S|\|f\|^2 \qquad \forall f \in PW_S \tag{6}$$

We will prove (6) with the constants $C = C_0$ and $c = c_0/(36C_0)$, where c_0 and C_0 are the constants in Lemma 3.

We will need the following Bessel's inequality (see [Y], Ch. 4.3): Given a set Λ satisfying $d(\Lambda) > 0$ and a bounded set S, there is a constant K which depends only on $d(\Lambda)$ and the diameter of S such that

$$\sum_{\lambda \in \Lambda} |f(\lambda)|^2 \le K\|f\|^2 \qquad \forall f \in PW_S.$$

The proof of Theorem 1 below uses three auxiliary lemmas:

Lemma 4 *Let S be a bounded set of positive measure and let $\Lambda_k \subset \mathbb{R}$ be a sequence of sets satisfying $d(\Lambda_k) > \delta > 0, k = 1, 2, ...,$ which converges weakly to some set Λ. Then*

$$\lim_{k\to\infty} \sum_{\lambda\in\Lambda_k} |f(\lambda)|^2 = \sum_{\lambda\in\Lambda} |f(\lambda)|^2 \qquad \forall f \in PW_S.$$

Proof. Take any function $f \in PW_S$, and pick up a point $x_l \in [l\delta - \delta/2, l\delta + \delta/2]$ such that

$$|f(x_l)| = \max_{|x-l\delta|\le\delta/2} |f(x)| \qquad \forall l \in \mathbb{Z}.$$

Since $x_{l+2} - x_l \ge \delta$, the sequence x_k is a union of two sets each having separation constant $\ge \delta$. By Bessel's inequality, we see that

$$\sum_{k\in\mathbb{Z}} |f(x_k)|^2 < \infty.$$

Let $R > 0$, and write

$$\Big|\sum_{\lambda\in\Lambda_k} |f(\lambda)|^2 - \sum_{\lambda\in\Lambda} |f(\lambda)|^2\Big| \le \Big|\sum_{\lambda\in\Lambda_k,|\lambda|<R} |f(\lambda)|^2 - \sum_{\lambda\in\Lambda,|\lambda|<R} |f(\lambda)|^2\Big| + 2\sum_{|k|\ge R/\delta} |f(x_k)|^2.$$

The first term in the right hand-side tends to zero as $k \to \infty$ whenever $\pm R \notin \Lambda$, while the second one tends to zero as $R \to \infty$. This proves the lemma.

Lemma 5 *Let $S_1 \subseteq S_2 \subseteq ...$ be an increasing sequence of bounded sets in $\mathbb{R}$ with $S = \cup_k S_k$ being a set of finite measure. Let $\Lambda \subset \mathbb{R}, d(\Lambda) > 0$, and positive k, K be such that the inequalities*

$$k\|f_j\|^2 \le \sum_{\lambda\in\Lambda} |f_j(\lambda)|^2 \le K\|f_j\|^2 \qquad \forall f_j \in PW_{S_j} \tag{7}$$

hold for every j. Then

$$k\|f\|^2 \le \sum_{\lambda\in\Lambda} |f(\lambda)|^2 \le K\|f\|^2 \qquad \forall f \in PW_S. \tag{8}$$

Proof. Given a function $f \in PW_S$, let $f_j \in PW_{S_j}$ be the Fourier transform of the function $\hat{f} \cdot 1_{S_j}$, where 1_{S_j} is the indicator function of S_j. Then the L^1-norm of $\hat{f} - \hat{f}_j$ tends to zero as $j \to \infty$, and so the functions $f_j(x)$ converge uniformly to $f(x)$.

For every $R > 0$ we have,

$$\sum_{\lambda\in\Lambda,|\lambda|<R} |f_j(\lambda)|^2 \le K\|f_j\|^2.$$

Taking the limit as $j \to \infty$, we obtain

$$\sum_{\lambda\in\Lambda, |\lambda|<R} |f(\lambda)|^2 \leq K\|f\|^2.$$

By letting $R \to \infty$, we obtain the right hand-side inequality in (8). Using this inequality, we get

$$\Big(\sum_{\lambda\in\Lambda} |f(\lambda)|^2\Big)^{1/2} \geq \Big(\sum_{\lambda\in\Lambda} |f_j(\lambda)|^2\Big)^{1/2} - \Big(\sum_{\lambda\in\Lambda} |(f-f_j)(\lambda)|^2\Big)^{1/2} \geq$$

$$k^{1/2}\|f_j\| - K^{1/2}\|f-f_j\|^2.$$

Taking the limit as $j \to \infty$, we prove the left hand-side inequality in (8).

Lemma 6 *Assume that the inequality*

$$\sum_{\lambda\in\Lambda} |f(\lambda)|^2 \leq C|S|\|f\|^2 \qquad \forall f \in PW_S \tag{9}$$

is true for some $C > 0, S \subset \mathbb{R}, |S| < \infty$, *and* $\Lambda \subset \mathbb{R}$. *Then*

(i) There is a constant $\eta > 0$ *which depends only on* S *such that*

$$\#(\Lambda \cap \Omega) \leq 9C,$$

for every interval $\Omega \subset \mathbb{R}, |\Omega| = \eta$.

(ii) There is a constant $K > 0$ *which depends only on* S *such that*

$$\frac{\#(\Lambda \cap \Omega)}{|\Omega|} \leq 4C|S|,$$

for every interval $\Omega \subset \mathbb{R}, |\Omega| \geq K$.

Proof. (i) Denote by $h \in PW_S$ the Fourier transform of the indicator function 1_S. Then $h(x)$ is continuous,

$$h(0) = \frac{|S|}{\sqrt{2\pi}}, \ \|h\|^2 = \|1_S\|^2 = |S|.$$

Choose $\eta > 0$ so small that $|h(x)| > |S|/3, |x| \leq \eta/2$. Then, applying (9) for $f = h$, we see that the statement (i) of Lemma 6 holds for $\Omega = [-\eta/2, \eta/2]$. To complete the proof, it suffices to observe that every function $h(x - x_0), x_0 \in \mathbb{R}$, belongs to PW_S.

(ii) Take any function $g \in PW_S$ satisfying $\|g\| = 1$, and choose a number R such that

$$\int_{-R}^{R} |g(x)|^2\, dx \geq \frac{1}{2}.$$

Assume $K > 2R$. We now apply (9) to the function $f(x) := g(x-s)$ and integrate over $(-K, K)$ with respect to s:

$$\int_{-K}^{K} \sum_{\lambda\in\Lambda} |g(\lambda - s)|^2\, ds \leq 2KC|S|.$$

When $|\lambda| < K/2$, we have

$$\int_{-K}^{K} |g(\lambda - s)|^2\, ds \geq \int_{-R}^{R} |g(s)|^2\, ds \geq \frac{1}{2}.$$

We conclude that

$$\frac{\#(\Lambda \cap (-K/2, K/2))}{2} \leq \int_{-K}^{K} \sum_{\lambda \in \Lambda} |g(\lambda - s)|^2\, ds \leq 2KC|S|.$$

This proves statement (ii).

4. Proof of Theorem 1

The proof of Theorem 1 will consist of a series of lemmas.

Lemma 7 *Let $n, m \in \mathbb{N}, n < m$. For every set*

$$S = \bigcup_{r \in I} \left[\frac{2\pi r}{m}, \frac{2\pi(r+1)}{m}\right],\ I \subset \{0, ..., m-1\}, \#I = n,$$

there is a set $\Lambda \subset \mathbb{Z}$ such that

$$c_0|S|\|f\|^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq C_0|S|\|f\|^2 \qquad \forall f \in PW_S, \tag{10}$$

where c_0, C_0 are the constants in Lemma 3.

Proof. Observe that $|S| = 2\pi n/m$, and denote by

$$\mathscr{F}_I := (e^{i\frac{2\pi jr}{m}})_{r \in I, j = 0, ..., m-1}$$

the submatrix of the Fourier matrix $\mathscr{F}$ whose columns are indexed by I. Since the matrix $(\sqrt{m})^{-1}\mathscr{F}$ is orthonormal, by Lemma 3 there exists $J \subset \{0, ..., m-1\}$ such that

$$c_0 n\|w\|^2 \leq \|\mathscr{F}_I(J)w\|^2_{l_2(J)} \leq C_0 n\|w\|^2, \qquad w \in l_2(I). \tag{11}$$

Observe that every function $F \in L^2(S)$ can be written as

$$F(t) = \sum_{r \in I} F_r(t - \frac{2\pi r}{m}),$$

where $F_r \in L^2(0, \frac{2\pi}{m})$ is defined by

$$F_r(t) := F(t + \frac{2\pi r}{m})\mathbf{1}_{[0, \frac{2\pi}{m}]}(t).$$

Therefore, every function $f \in PW_S$ admits a representation

$$f(x) = \sum_{r \in I} e^{i\frac{2\pi r}{m}x} f_r(x), \qquad f_r \in PW_{[0, \frac{2\pi}{m}]},$$

where the functions $e^{i\frac{2\pi r}{m}x}f_r(x)$ are orthogonal in $L^2(\mathbb{R})$. We note that for every function $h \in PW_{[0,2\pi/m]}$ we have,

$$\frac{2\pi}{m}\|h\|^2 = \sum_{\lambda\in m\mathbb{Z}} |h(\lambda)|^2. \tag{12}$$

We now verify that the sequence

$$\Lambda := \{j + km : j \in J, k \in \mathbb{Z}\}$$

satisfies (10). Take any function $f \in PW_S$. Then

$$\sum_{j\in J}\sum_{k\in\mathbb{Z}} |f(j+km)|^2 = \sum_{j\in J}\sum_{k\in\mathbb{Z}} \left|\sum_{r\in I} e^{i\frac{2\pi rj}{m}} f_r(j+km)\right|^2.$$

For every $j \in J$ we apply (12) to the function $\sum_{r\in I} e^{i\frac{2\pi rj}{m}} f_r(x)$. We find that the last expression is equal to

$$\frac{2\pi}{m}\sum_{j\in J}\int_{\mathbb{R}} \left|\sum_{r\in I} e^{i\frac{2\pi rj}{m}} f_r(x)\right|^2 dx = \frac{2\pi}{m}\int_{\mathbb{R}} \|\mathscr{F}_I(J)(f_r(x))_{r\in I}\|^2_{l_2(J)} dx.$$

By inequality (11) we have on one hand,

$$\sum_{\lambda\in\Lambda} |f(\lambda)|^2 \geq c_0\frac{n}{m}\int_{\mathbb{R}}\sum_{r\in I} |f_r(x)|^2 dx =$$

$$c_0\frac{n}{m}\int_{\mathbb{R}}\sum_{r\in I} |e^{i\frac{2\pi r}{m}x} f_r(x)|^2 dx = c_0\frac{n}{m}\int_{\mathbb{R}} |\sum_{r\in I} e^{i\frac{2\pi r}{m}x} f_r(x)|^2 dx =$$

$$c_0\frac{n}{m}\int_{\mathbb{R}} |f(x)|^2 dx,$$

while on the other hand, applying the same computation, we get

$$\sum_{\lambda\in\Lambda} |f(\lambda)|^2 \leq C_0\frac{n}{m}\int_{\mathbb{R}}\sum_{r\in I} |f_r(x)|^2 dx = C_0\frac{n}{m}\int_{\mathbb{R}} |f(x)|^2 dx.$$

This completes the proof.

Lemma 8 *For every compact set $S \subset [0, 2\pi]$ of positive measure there is a set $\Lambda \subset \mathbb{Z}$ such that (10) holds.*

This follows immediately from Lemma 7, since every such set S can be covered by a set from Lemma 7 whose measure is arbitrarily close to $|S|$.

Lemma 9 *For every set $S \subset [0, 2\pi]$ of positive measure there is a set $\Lambda \subset \mathbb{Z}$ such that (10) holds.*

Proof. It suffices to prove Lemma 9 for open sets S. Let S be such a set end let $S_1 \subset S_2 \subset ...$ be an increasing sequence of compact sets such that $S = \cup_j S_j$. By Lemma 8, there exist sets $\Lambda_j \subset \mathbb{Z}$ such that

$$c_0|S|\|f_j\|^2 \leq \sum_{\lambda\in\Lambda_j} |f_j(\lambda)|^2 \leq C_0|S|\|f_j\|^2 \qquad \forall f_j \in PW_{S_j}, \tag{13}$$

where c_0, C_0 are the constants in Lemma 3. Since $PW_{S_j} \subset PW_{S_k}, k > j$, we have

$$c_0|S_k|\|f_j\|^2 \leq \sum_{\lambda\in\Lambda_k} |f_j(\lambda)|^2 \leq C_0|S_k|\|f_j\|^2 \qquad \forall f_j \in PW_{S_j} \tag{14}$$

We may assume that Λ_k converge weakly to some set $\Lambda \subset \mathbb{Z}$. Using Lemma 4, we take the limit as $k \to \infty$:

$$c_0|S|\|f_j\|^2 \leq \sum_{\lambda\in\Lambda} |f_j(\lambda)|^2 \leq C_0|S|\|f_j\|^2 \qquad \forall f_j \in PW_{S_j}. \tag{15}$$

Now, the result follows from Lemma 5.

Lemma 10 *For every bounded set S of positive measure there is a set $\Lambda \subset (1/d)\mathbb{Z}$ such that (10) holds, where d is any positive number such that S lies on an interval of length $2\pi d$.*

Observe that the translations of S change neither the frame property of $E(\Lambda)$ nor the frame constants. So, it suffices to assume that $S \subset [0, 2\pi d]$. Then the result follows from Lemma 9 by re-scaling.

Proof of Theorem 1. We may assume that S is an unbounded set of finite measure.

Let $S_1 \subset S_2 \subset ...$ be any sequence of bounded sets satisfying $S = \cup_j S_j$. By Lemma 10, there exist discrete sets Λ_j such that (13) is true. Since $PW_{S_j} \subset PW_{S_k}, j < k$, we see that (14) holds for all $j < k$.

By Lemma 6 (i), there is a number $\eta > 0$ and an integer r which depends only on the constant C_0 in (6) (it is easy to check that one may take $r \leq 36C_0$) such that every set Λ_k can be can be splitted up into r subsets $\Lambda_k^{(l)}$ satisfying $d(\Lambda_k^{(l)}) \geq \eta, l = 1, ..., r$. By taking an appropriate subsequence, we may assume that each $\Lambda_k^{(l)}$ converges weakly to some set $\Lambda^{(l)}$ as $k \to \infty$. By Lemma 4, we may take limit in (14) as $k \to \infty$:

$$c_0|S|\|f_j\|^2 \leq \sum_{k=1}^{r} \sum_{\lambda\in\Lambda^{(k)}} |f_j(\lambda)|^2 \leq C_0|S|\|f_j\|^2 \qquad \forall f_j \in PW_{S_j}.$$

Set $\Lambda := \cup_{k=1}^r \Lambda^{(k)}$. It may happen that the sets $\Lambda^{(k)}$ have common points. Anyway, we have

$$\sum_{\lambda\in\Lambda} |f_j(\lambda)|^2 \leq \sum_{k=1}^{r} \sum_{\lambda\in\Lambda^{(k)}} |f_j(\lambda)|^2 \leq r \sum_{\lambda\in\Lambda} |f_j(\lambda)|^2.$$

From the latter inequalities, it readily follows that

$$\frac{c_0}{r}|S|\|f_j\|^2 \leq \sum_{\lambda\in\Lambda} |f_j(\lambda)|^2 \leq C_0|S|\|f_j\|^2 \qquad \forall f_j \in PW_{S_j}.$$

Theorem 1 now follows easily from Lemma 5.

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