ADDENDUM TO THE PAPER "HYPERSURFACES WITH ISOMETRIC REEB FLOW IN COMPLEX HYPERBOLIC TWO-PLANE GRASSMANNIANS"

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Abstract. We classify all of real hypersurfaces M with Reeb invariant shape operator in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 2$. Then it becomes a tube over a totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or a horosphere whose center at infinity is singular and of type $JN{\in}\mathfrak{J}N$ for a unit normal vector field N of M.

Introduction

Let us introduce a paper due to Suh [9] for the classification of all real hypersurfaces with isometric *Reeb flow* in complex hyperbolic two-plane Grassmann manifold $SU_{2,m}/S(U_2{\cdot}U_m)$ as follows:

Theorem A. *Let M be a connected orientable real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or a horosphere whose center at infinity is singular and of type $JN{\in}\mathfrak{J}N$ for a unit normal vector field N of M.*

A tube around $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ is a principal orbit of the isometric action of the maximal compact subgroup $SU_{1,m+1}$ of SU_{m+2}, and the orbits of the *Reeb flow* corresponding to the orbits of the action of U_1. The action of $SU_{1,m+1}$ has two kinds of singular orbits. One is a totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ and the other is a totally geodesic $\mathbb{C}H^m$ in $SU_{2,m}/S(U_2{\cdot}U_m)$.

When the shape operator A of M in $SU_{2,m}/S(U_2{\cdot}U_m)$ is Lie-parallel along the *Reeb flow* of the Reeb vector field ξ, that is $\mathcal{L}_\xi A = 0$, we say that the shape opearator is *Reeb invariant*. The purpose of this addendum is, by Theorem A, to give a complete classification of real hypersurfaces in $SU_{2,m}/S(U_2{\cdot}U_m)$ with *Reeb invariant* shape operator as follows:

Main Theorem. *Let M be a connected orientable real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$. Then the shape operator on M is Reeb invariant if and only if M is an open part of a tube around*

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some totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or a horosphere whose center at infinity is singular and of type $JN{\in}\mathfrak{J}N$ for a unit normal vector field N of M.

Moreover, related to the invariancy of shape operator, by using the result of Main Theorem, we have the following two corollaries.

Corollary 1. *There does not exist any connected orientable real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$, with $\mathcal{F}$-invariant shape operator.*

Corollary 2. *There does not exist any connected orientable real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$, with invariant shape operator.*

In previous corollaries, if the shape operator A of M in $SU_{2,m}/S(U_2{\cdot}U_m)$ satisfies a property of $\mathcal{L}_X A = 0$ on a distribution $\mathcal{F}$ defined by $\mathcal{F} = \mathcal{C}^{\perp} \cup \mathcal{Q}^{\perp}$ (or for any tangent vector field X on M, resp.), then it is said to be *$\mathcal{F}$-invariant* (or *invariant*, resp.).

We use some references [1], [2], [3], [4], and [5] to recall the Riemannian geometry of complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$. And some fundamental formulas related to the Codazzi and Gauss equations from the curvature tensor of complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$ will be recalled (see [6], [7], and [8]). In this addendum we give an important Proposition 1.1 and prove our Main Theorem in section 1. Lastly, we give a brief proof for Corollaries 1 and 2 by using our Main Theorem.

1. Proof of the Main Theorem

In order to give a complete proof of our Main Theorem in the introduction, we need the following Key Proposition. Then by virtue of Theorem A we give a complete proof of our main theorem.

Proposition 1.1. *Let M be a real hypersurface in noncompact complex two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$. If the shape operator on M is Reeb invariant, then the shape operator A commutes with the structure tensor ϕ.*

Proof. First note that

$$\begin{aligned}(\mathcal{L}_\xi A)X &= \mathcal{L}_\xi(AX) - A\mathcal{L}_\xi X \\ &= \nabla_\xi(AX) - \nabla_{AX}\xi - A(\nabla_\xi X - \nabla_X \xi) \\ &= (\nabla_\xi A)X - \nabla_{AX}\xi + A\nabla_X\xi \\ &= (\nabla_\xi A)X - \phi A^2 X + A\phi AX\end{aligned}$$

for any vector field X on M. Then the assumption $\mathcal{L}_\xi A = 0$, that is, the shape operator is Reeb invariant if and only if $(\nabla_\xi A)X = \phi A^2 X - A\phi AX$.

On the other hand, by the equation of Codazzi in [9] and the assumption of Reeb invariant, we have

$$(1.1)\quad \begin{aligned}(\nabla_X A)\xi &= \phi A^2 X - A\phi AX\\ &\quad + \frac{1}{2}\Big[\phi X + \sum_{\nu=1}^{3}\big\{\eta_\nu(\xi)\phi_\nu X - \eta_\nu(X)\phi_\nu\xi + 3\eta_\nu(\phi X)\xi_\nu\big\}\Big].\end{aligned}$$

Now, let us take an orthonormal basis $\{e_1, e_2, \cdots, e_{4m-1}\}$ for the tangent space T_xM, $x \in M$, for M in $SU_{2,m}/S(U_2{\cdot}U_m)$. Then the equation of Codazzi gives

$$(1.2)\quad \begin{aligned}(\nabla_{e_i}A)X - (\nabla_X A)e_i = -\frac{1}{2}\Big[&\eta(e_i)\phi X - \eta(X)\phi e_i - 2g(\phi e_i, X)\xi\\ &+ \sum_{\nu=1}^{3}\big\{\eta_\nu(e_i)\phi_\nu X - \eta_\nu(X)\phi_\nu e_i - 2g(\phi_\nu e_i, X)\xi_\nu\big\}\\ &+ \sum_{\nu=1}^{3}\big\{\eta_\nu(\phi e_i)\phi_\nu\phi X - \eta_\nu(\phi X)\phi_\nu\phi e_i\big\}\\ &+ \sum_{\nu=1}^{3}\big\{\eta(e_i)\eta_\nu(\phi X) - \eta(X)\eta_\nu(\phi e_i)\big\}\xi_\nu\Big],\end{aligned}$$

from which, together with the fundamental formulas mentioned in [9], we know that

$$(1.3)\quad \begin{aligned}&\sum_{i=1}^{4m-1} g((\nabla_{e_i}A)X, \phi e_i)\\ &= (2m-1)\eta(X) + \frac{1}{2}\sum_{\nu=1}^{3}\big\{g(\phi_\nu\xi, \phi_\nu X) - \eta_\nu(X)\mathrm{Tr}(\phi\phi_\nu)\big\}\\ &\quad + \frac{1}{2}\sum_{\nu=1}^{3}\big\{g(\phi_\nu\phi X, \phi\phi_\nu\xi) + \eta(X)g(\phi\xi_\nu, \phi\xi_\nu)\big\}\\ &= (2m+1)\eta(X) - \frac{1}{2}\sum_{\nu=1}^{3}\eta_\nu(X)\mathrm{Tr}\phi\phi_\nu - \sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(X),\end{aligned}$$

where the following formulas are used in the second equality

$$\begin{aligned}&\sum_{\nu=1}^{3} g(\phi_\nu\xi, \phi_\nu X) = 3\eta(X) - \sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(X),\\ &\sum_{\nu=1}^{3} g(\phi_\nu\phi X, \phi\phi_\nu\xi) = \sum_{\nu=1}^{3}\eta(X)\eta_\nu^2(\xi) - \sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(X),\\ &\sum_{\nu=1}^{3}\eta(X)g(\phi\xi_\nu, \phi\xi_\nu) = 3\eta(X) - \sum_{\nu=1}^{3}\eta_\nu^2(\xi)\eta(X).\end{aligned}$$

Now let us denote by U the vector $\nabla_\xi\xi = \phi A\xi$. Then using the equation $(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi$ given in [9] and taking the derivative to the vector field U gives

$$\nabla_{e_i}U = \eta(A\xi)Ae_i - g(Ae_i, A\xi)\xi + \phi(\nabla_{e_i}A)\xi + \phi A(\nabla_{e_i}\xi).$$

Then naturally its divergence can be given by

$$
\begin{aligned}
\operatorname{div} U &= \sum_{i=1}^{4m-1} g(\nabla_{e_i} U, e_i) \\
&= h\eta(A\xi) - \eta(A^2\xi) - \sum_{i=1}^{4m-1} g\big((\nabla_{e_i} A)\xi, \phi e_i\big) - \sum_{i=1}^{4m-1} g\big(\phi A e_i, A\phi e_i\big),
\end{aligned} \tag{1.4}
$$

where h denotes the trace of the shape operator of M in $SU_{2,m}/S(U_2{\cdot}U_m)$.

Now we calculate the squared norm of the tensor $\phi A - A\phi$ as follows:

$$
\begin{aligned}
\| \phi A - A\phi \|^2 &= \sum_i g\big((\phi A - A\phi)e_i, (\phi A - A\phi)e_i\big) \\
&= \sum_{i,j} g\big((\phi A - A\phi)e_i, e_j\big) g\big((\phi A - A\phi)e_i, e_j\big) \\
&= \sum_{i,j} \Big\{ g(\phi A e_j, e_i) + g(\phi A e_i, e_j) \Big\} \Big\{ g(\phi A e_j, e_i) + g(\phi A e_i, e_j) \Big\} \\
&= 2\sum_{i,j} g(\phi A e_j, e_i) g(\phi A e_j, e_i) + 2\sum_{i,j} g(\phi A e_j, e_i) g(\phi A e_i, e_j) \\
&= 2\sum_j g(\phi A e_j, \phi A e_j) - 2\sum_j g(\phi A e_j, A\phi e_j) \\
&= 2\operatorname{div} U - 2h\eta(A\xi) + 2\sum_j g\big((\nabla_{e_j} A)\xi, \phi e_j\big) + 2\mathrm{TrA}^2,
\end{aligned} \tag{1.5}
$$

where $\sum_i$ (respectively, $\sum_{i,j}$) denotes the summation from $i = 1$ to $4m - 1$ (respectively, from $i, j = 1$ to $4m - 1$) and in the final equality we have used (1.4). From this, together with the formula (1.3), it follows that

$$
\begin{aligned}
\operatorname{div} U =& \frac{1}{2} \| \phi A - A\phi \|^2 - \mathrm{Tr}A^2 \\
&+ h\eta(A\xi) - 2(m+1) + \frac{1}{2}\sum_{\nu=1}^{3} \eta_\nu(\xi)\mathrm{Tr}\phi\phi_\nu + \sum_{\nu=1}^{3} \eta_\nu^2(\xi).
\end{aligned} \tag{1.6}
$$

From (1.6), together with the assumption of Reeb invariant shape operator, we want to show that the structure tensor ϕ commutes with the shape operator A, that is, $\phi A = A\phi$.

Let us take the inner product (1.1) with the Reeb vector field ξ. Then we have

$$
\begin{aligned}
g\big((\nabla_X A)\xi, \xi\big) &= -g(A\phi A X, \xi) + \frac{1}{2}\sum_{\nu=1}^{3} \Big\{ \eta_\nu(\xi) g(\phi_\nu X, \xi) + 3\eta_\nu(\phi X) g(\xi_\nu, \xi) \Big\} \\
&= -g(A\phi A X, \xi) + 2\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi X) \\
&= g(AX, U) + 2\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi X).
\end{aligned} \tag{1.7}
$$

On the other hand, by applying the structure tensor ϕ to the vector field U, we have

$$\phi U = \phi^2 A\xi = -A\xi + \eta(A\xi)\xi = -A\xi + \alpha\xi,$$

where the function α denotes $\eta(A\xi)$. From this, differentiating and using the formula $(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi$ gives

$$(\nabla_X A)\xi = g(AX,U)\xi - \phi(\nabla_X U) - A\phi AX + (X\alpha)\xi + \alpha\phi AX. \tag{1.8}$$

Taking the inner product (1.8) with ξ and using $U = \phi A\xi$ gives

$$\begin{aligned} g\big((\nabla_X A)\xi,\xi\big) &= g(AX,U) - g(A\phi AX,\xi) + (X\alpha) \\ &= 2g(AX,U) + (X\alpha). \end{aligned}$$

Then, together with (1.7), it follows that

$$g(AX,U) - 2\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi X) + (X\alpha) = 0. \tag{1.9}$$

Substituting (1.1) and (1.9) into (1.8) gives

$$\begin{aligned} &2\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi X)\xi - \phi(\nabla_X U) \\ &\quad= \frac{1}{2}\phi X + \phi A^2 X - \alpha\phi AX \\ &\qquad+ \frac{1}{2}\sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)\phi_\nu X - \eta_\nu(X)\phi_\nu\xi + 3\eta_\nu(\phi X)\xi_\nu\Big\}. \end{aligned} \tag{1.10}$$

Then the above equation can be arranged as follows:

$$\begin{aligned} \phi\nabla_X U = &-\frac{1}{2}\phi X - \phi A^2 X + \alpha\phi AX + 2\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi X)\xi \\ &+ \frac{1}{2}\sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3\eta_\nu(\phi X)\xi_\nu\Big\}. \end{aligned}$$

From this, summing up from 1 to $4m-1$ for an orthonormal basis of T_xM, $x \in M$, we have

(1.11)

$$\begin{aligned} &\sum_i g(\phi\nabla_{e_i}U,\phi e_i) = \operatorname{div} U + \| U \|^2 \\ &= -\frac{1}{2}\sum_i g(\phi e_i,\phi e_i) - \sum_i g(\phi A^2 e_i,\phi e_i) + \alpha\sum_i g(\phi A e_i,\phi e_i) \\ &\quad+ \frac{1}{2}\sum_{\nu=1}^{3}\sum_i\Big\{\eta_\nu(e_i)g(\phi_\nu\xi,\phi e_i) - \eta_\nu(\xi)g(\phi_\nu e_i,\phi e_i) - 3\eta_\nu(\phi e_i)g(\phi e_i,\xi_\nu)\Big\} \\ &= -2(m+1) - \operatorname{Tr}A^2 + \eta(A^2\xi) + \alpha h - \alpha^2 + \sum_{\nu=1}^{3}\eta_\nu^2(\xi) + \frac{1}{2}\sum_{\nu=1}^{3}\eta_\nu(\xi)\operatorname{Tr}(\phi\phi_\nu), \end{aligned}$$

where in the first equality we have used the notion of div U. Then it follows that

$$\text{div}\ U = -2(m+1) - \text{Tr}A^2 + \alpha h + \sum_{\nu=1}^{3} \eta_\nu^2(\xi) + \frac{1}{2}\sum_{\nu=1}^{3} \eta_\nu(\xi)\text{Tr}(\phi\phi_\nu), \tag{1.12}$$

where we have used $\| U \|^2 = \eta(A^2\xi) - \alpha^2$ in (1.11).

Now if we compare (1.6) with the formula (1.12), we can assert that the squared norm $\| \phi A - A\phi \|^2$ vanishes, that is, the structure tensor ϕ commutes with the shape operator A. This completes the proof of our proposition. □

Hence by Proposition 1.1 we know that the Reeb flow on M is isometric. From this, together with Theorem A we give a complete proof of our Main Theorem in the introduction. □

Remark 1.1. *It can be easily checked that the shape operator of real hypersurfaces M in $SU_{2,m}/S(U_2{\cdot}U_m)$ is Reeb invariant, that is, $\mathcal{L}_\xi A = 0$ when M is locally congruent to a tube around some totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or a horosphere whose center at infinity is singular and of type $JN \in \mathfrak{J}N$ for a unit normal vector field N of M. So the converse of our main theorem naturally holds.*

2. Proof of Corollaries

From the definitions of three kinds of the invariancy of the shape operator A defined on M in the Introduction, namely *invariant*, *$\mathcal{F}$-invariant* and *Reeb invariant* shape operator, the notion of *Reeb invariant* is the most weakest condition. Thus from our Main Theorem, we assert that *if a real hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$, has $\mathcal{F}$-invariant (or invariant) shape operator, then M is locally congruent to a tube around some totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or a horosphere whose center at infinity is singular.*

Conversely, if we check whether a tube M_r of radius r around the totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ and a horosphere $\mathcal{H}$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ whose center at infinity is singular have the $\mathcal{F}$-invariant (or invariant) shape operator, then it does not hold. In fact, we get a contradiction for the case $(\mathcal{L}_{\xi_2}A)\xi_3$. From such a view point, we can assert that *the shape operator A of M_r (or $\mathcal{H}$, respectively) satisfy neither the property of $\mathcal{F}$-invariant nor invariant shape operator.*

Summing up these discussion, we give a complete proof of our Corollaries in the introduction. □

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