

WAVE EXTENSION PROBLEM FOR THE FRACTIONAL LAPLACIAN

MIKKO KEMPPAINEN, PETER SJÖGREN, AND JOSÉ LUIS TORREA

ABSTRACT. We show that the fractional Laplacian can be viewed as a Dirichlet-to-Neumann map for a degenerate hyperbolic problem, namely, the wave equation with an additional diffusion term that blows up at time zero. A solution to this wave extension problem is obtained from the Schrödinger group by means of an oscillatory subordination formula, which also allows us to find kernel representations for such solutions. Asymptotics of related oscillatory integrals are analysed in order to determine the correct domains for initial data in the general extension problem involving non-negative self-adjoint operators. An alternative approach using Bessel functions is also described.

1. INTRODUCTION

In the last decade a lot of attention has been devoted to fractional powers of the Laplacian $-\Delta$. The seminal paper by L. Caffarelli and L. Silvestre [CS07] showed that the operator $(-\Delta)^\sigma$, with $0 < \sigma < 1$, mediates between Dirichlet and Neumann boundary values on \mathbb{R}^d for a certain degenerate elliptic problem in the upper half-space \mathbb{R}_+^{d+1} .

Motivated by their work we show that up to a multiplicative constant the fractional Laplacian can also be viewed as a Dirichlet-to-Neumann map $f \mapsto \lim_{t \rightarrow 0} t^{1-2\sigma} \partial_t u$ for the hyperbolic problem

$$(1) \quad \begin{cases} \partial_t^2 u + \frac{1-2\sigma}{t} \partial_t u = \Delta u, \\ u(\cdot, 0) = f. \end{cases}$$

Here $0 < \sigma < 1$, and for $\sigma = \frac{1}{2}$ we have the classical wave equation.

We use the language of semigroups to study this problem in a parallel way to that used by Stinga and Torrea [ST10] for discussing the elliptic problem of Caffarelli and Silvestre in the generality of non-negative self-adjoint operators. For an extension of their method to more general classes of semigroup generators, see [GMS13]. We also wish to point out that an alternative approach to problem (1) by means of Bessel functions is available (see Section 4).

Our main result says that a solution to (1) is given by means of the ‘oscillatory’ subordination formula

$$(2) \quad u(\cdot, t) = \frac{i^\sigma t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-i\frac{t^2}{4s}} e^{is\Delta} f \frac{ds}{s^{1+\sigma}},$$

also when Δ is replaced by a more general non-negative self-adjoint operator. It allows us to make use of the Schrödinger kernel in order to find integral representations for solutions to the associated Neumann problem in different dimensions.

It is perhaps interesting to note that in the classical case when $\sigma = \frac{1}{2}$, the formula (2) leads to the identity

$$(3) \quad \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} = \sqrt{\frac{i}{\pi}} \int_0^\infty e^{-i\frac{t^2}{4s}} e^{is\Delta} \frac{ds}{\sqrt{s}}$$

2010 *Mathematics Subject Classification.* 35L80 (Primary); 35C15, 33C10 (Secondary).

Key words and phrases. fractional Laplacian, wave equation, Schrödinger group, oscillatory integrals, Bessel functions.

for the wave group $(e^{-it\sqrt{-\Delta}})_{t \in \mathbb{R}}$. Recall in particular that the solution u to the wave equation with Neumann initial data g (given by the imaginary part of the wave group)

$$(4) \quad u(\cdot, t) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} g$$

can be expressed in dimensions 3 and 2, respectively, by

$$(5) \quad u(x, t) = t \int_{\partial B(x, t)} g(y) dS(y) = \frac{c}{t} S|_{\partial B(0, t)} * g(x)$$

and

$$(6) \quad u(x, t) = \frac{t^2}{2} \int_{B(x, t)} \frac{g(y)}{(t^2 - |x - y|^2)^{\frac{1}{2}}} dy = \frac{c' 1_{B(0, t)}}{(t^2 - |\cdot|^2)^{\frac{1}{2}}} * g(x).$$

(see [Eva98, Section 2.4, Equations (22) and (27)]).¹ These formulas highlight Huygens' principle concerning the finite propagation speed of solutions to the wave equation. It would be peculiar if the smooth and fully supported Schrödinger kernel would transform by (3) into a singular and spherically supported one such as $\frac{1}{t} S|_{\partial B(0, t)}$ in (5), and indeed there are restrictions in the relation of σ and the dimension in our kernel representations.

We will now describe the content of this paper. Let L be a non-negative self-adjoint operator on a Lebesgue space \mathcal{L}^2 and consider the Schrödinger group $(e^{-isL})_{s \in \mathbb{R}}$. In Theorem 1 we show that for $0 < \sigma < 1$ the oscillatory integral

$$U_t^\sigma(f) = \frac{i^\sigma t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-i\frac{t^2}{4s}} e^{-isL} f \frac{ds}{s^{1+\sigma}},$$

when interpreted as an improper integral, converges weakly in \mathcal{L}^2 for suitable f and solves the equation

$$(7) \quad \partial_t^2 u + \frac{1-2\sigma}{t} \partial_t u = -Lu, \quad t > 0,$$

with the initial data

$$u(\cdot, 0) = f \quad \text{and} \quad \partial_t^\sigma u(\cdot, 0) = i^{-2\sigma} c_\sigma L^\sigma f.$$

where

$$\partial_t^\sigma u(\cdot, t) = \frac{1}{2^\sigma} t^{1-2\sigma} \partial_t u(\cdot, t).$$

Theorem 1 is established via the Spectral theorem (see Proposition 1); the requirement that $f \in \text{Dom}(L^{\frac{\sigma}{2} + \frac{3}{4}})$ is needed to deal with the asymptotics of oscillatory integrals in Lemmas 2 and 3. Our method for analysing these integrals and overcoming the delicate problems of convergence involves 'sidestepping' the imaginary half-axis and moving to more suitable complex paths that provide sufficient decay. In Corollary 1, a solution u to (7) with real initial data $u(\cdot, 0) = f$ and $\partial_t^\sigma u(\cdot, 0) = g$ is presented in terms of real and imaginary parts of U_t^σ as

$$u(\cdot, t) = c_\sigma \text{Re}(i^{1-2\sigma} U_t^\sigma(f)) - c'_\sigma \text{Im}(U_t^\sigma(L^{-\sigma} g)).$$

For $\sigma = \frac{1}{2}$ this reduces to

$$(8) \quad u(\cdot, t) = \text{Re}(e^{-it\sqrt{L}} f) - \text{Im}\left(\frac{e^{-it\sqrt{L}}}{\sqrt{L}} g\right) = \cos(t\sqrt{L})f + \frac{\sin(t\sqrt{L})}{\sqrt{L}} g.$$

In Section 3, we return to the case of $L = -\Delta$ and find kernel representations for solutions u to the Neumann problem

$$\begin{cases} \partial_t^2 u + \frac{1-2\sigma}{t} \partial_t u = \Delta u, \\ u(\cdot, t) = 0, \quad \partial_t^\sigma u(\cdot, t) = g \end{cases}$$

¹The positive constants c, c' appearing here and in what follows can have different values at different instances.

on \mathbb{R}^d for $d = 1, 2, \dots, 5$. From the subordination formula (2) we obtain

$$U_t^\sigma((-\Delta)^{-\sigma}g) = -\frac{i^\sigma}{\Gamma(\sigma)} \int_0^\infty e^{-i\frac{t^2}{4s}} e^{is\Delta} g \frac{ds}{s^{1-\sigma}},$$

and inserting the Schrödinger kernel

$$\frac{1}{(4\pi is)^{\frac{d}{2}}} e^{i\frac{|x|^2}{4s}},$$

we arrive at the expression

$$(9) \quad c_{d,\sigma} i^{\sigma-\frac{d}{2}} \int_0^\infty e^{-i\frac{t^2}{4s}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4s}} g(y) dy \frac{ds}{s^{1+\frac{d}{2}-\sigma}},$$

the imaginary part of which gives the solution u .

Theorem 2 concerns dimensions $d = 1, 2, 3$ and σ for which $0 < \frac{d}{2} - \sigma < 1$ and derives from (9) the formula

$$u(x, t) = c_{d,\sigma} \int_{B(x,t)} \frac{g(y)}{(t^2 - |x-y|^2)^{\frac{d}{2}-\sigma}} dy.$$

The dimensions $d = 3, 4, 5$ are considered in Theorem 3, which states that for σ such that $1 < \frac{d}{2} - \sigma < 2$ we have

$$u(x, t) = \frac{c_{d,\sigma}}{t^2} \int_{B(x,t)} \frac{2(\sigma+1)g(y) + (y-x) \cdot \nabla g(y)}{(t^2 - |x-y|^2)^{\frac{d}{2}-\sigma-1}} dy.$$

The limiting cases when $\frac{d}{2} - \sigma$ tends to either zero or one are studied in Theorem 4; in dimensions $d = 2, 3, 4$ we then have

$$u_\sigma(x, t) \longrightarrow \frac{1}{c_d t} \int_{\partial B(x,t)} g(y) dS(y).$$

In Theorem 5 we show that the solution to

$$\begin{cases} \partial_t^2 u + \frac{1-2\sigma}{t} \partial_t u = \Delta u, \\ u(\cdot, t) = f, \quad \partial_t^\sigma u(\cdot, t) = g \end{cases}$$

with Schwartz initial data f and g is unique and can be given in terms of Bessel functions $J_{\pm\sigma}$, namely,

$$u(\cdot, t) = c_\sigma (t\sqrt{-\Delta})^\sigma J_{-\sigma}(t\sqrt{-\Delta})f + c'_\sigma (t\sqrt{-\Delta})^\sigma J_\sigma(t\sqrt{-\Delta})(-\Delta)^{-\sigma}g.$$

For $\sigma = \frac{1}{2}$ this coincides with the classical formula (8) (with $L = -\Delta$); indeed

$$\sqrt{r}J_{-1/2}(r) = \sqrt{\frac{2}{\pi}} \cos r \quad \text{and} \quad \sqrt{r}J_{1/2}(r) = \sqrt{\frac{2}{\pi}} \sin r$$

(and $c_{1/2} = c'_{1/2} = \sqrt{\pi/2}$). Growth estimates for Bessel functions allow us to deduce, by means of the Fourier transform, fixed-time estimates for solutions of the equation (see Theorem 6). Finally, a classical integral representation for (modified) Bessel functions K_σ is converted into an oscillatory integral formula coinciding with (2), thus closing the circle.

In order to avoid ambiguity, let us agree that $i^\alpha = e^{i\pi\alpha/2}$. Moreover, by saying that an integral \int_0^∞ is convergent, we mean that the limit of \int_ε^R exists as $\varepsilon \rightarrow 0$ and $R \rightarrow \infty$.

By $\alpha \lesssim \beta$ we mean that there exists a constant C such that $\alpha \leq C\beta$. Two quantities α and β are comparable, $\alpha \sim \beta$, if $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$.

Acknowledgements. The first author gratefully acknowledges the financial support from the Finnish Academy of Science and Letters, Väisälä Foundation, and from the Finnish Centre of Excellence in Analysis and Dynamics Research. He is thankful for the hospitality of the Department of Mathematics at the Autonomous University of Madrid during his stay. The third author is supported by the grant MTM2011-28149-C02-01 from Spanish Government.

2. AN OSCILLATORY SUBORDINATION FORMULA

Let L be a non-negative self-adjoint operator on a Lebesgue space \mathcal{L}^2 and let $0 < \sigma < 1$. In this section, we study when and how a solution to the equation

$$(10) \quad \partial_t^2 u + \frac{1-2\sigma}{t} \partial_t u = -Lu, \quad t > 0,$$

can be obtained from the oscillatory integral

$$\frac{i^\sigma t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-i\frac{t^2}{4s}} e^{-isL} \frac{ds}{s^{1+\sigma}}.$$

The main result of the article is:

Theorem 1. *Let $0 < \sigma < 1$. The limit*

$$u(\cdot, t) = \frac{i^\sigma t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \lim_{\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_\varepsilon^R e^{-i\frac{t^2}{4s}} e^{-isL} f \frac{ds}{s^{1+\sigma}}, \quad t > 0,$$

exists weakly in \mathcal{L}^2 whenever $f \in \mathcal{L}^2$ (if $\sigma \leq \frac{1}{2}$) or $f \in \text{Dom}(L^{\frac{\sigma}{2}-\frac{1}{4}})$ (if $\sigma > \frac{1}{2}$). If $f \in \text{Dom}(L^{\frac{\sigma}{2}+\frac{3}{4}})$ then u is a weak solution to equation (10) in the sense that

$$\left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t \right) \langle u(\cdot, t), h \rangle = -\langle Lu(\cdot, t), h \rangle$$

for all $h \in \mathcal{L}^2$. Moreover, u converges to the initial data

$$u(\cdot, 0) = f \quad \text{and} \quad \partial_t^\sigma u(\cdot, 0) = -i^{2\sigma} \frac{\Gamma(1-\sigma)}{\sigma 4^\sigma \Gamma(\sigma)} L^\sigma f$$

weakly in \mathcal{L}^2 . Here $\partial_t^\sigma = \frac{1}{2^\sigma} t^{1-2\sigma} \partial_t$.

A solution for any combination of Dirichlet and Neumann initial data can be obtained in terms of real and imaginary parts:

Corollary 1. *Write*

$$U_t^\sigma = \frac{i^\sigma t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-i\frac{t^2}{4s}} e^{-isL} \frac{ds}{s^{1+\sigma}}$$

for the solution operator. Then

- *the function*

$$u(\cdot, t) = \frac{1}{\sin(\sigma\pi)} \text{Re}(i^{1-2\sigma} U_t^\sigma(f))$$

solves equation (10) with real initial data

$$u(\cdot, 0) = f, \quad \partial_t^\sigma u(\cdot, 0) = 0,$$

whenever $f \in \text{Dom}(L^{\frac{\sigma}{2}+\frac{3}{4}})$, and

- *the function*

$$u(\cdot, t) = -\frac{\sigma 4^\sigma \Gamma(\sigma)}{\sin(\sigma\pi) \Gamma(1-\sigma)} \text{Im}(U_t^\sigma(L^{-\sigma} g))$$

solves equation (10) with real initial data

$$u(\cdot, 0) = 0, \quad \partial_t^\sigma u(\cdot, 0) = g,$$

whenever $g \in \text{Dom}(L^{-\frac{\sigma}{2}+\frac{3}{4}})$.

Proof of Corollary 1. To prove the first claim, it suffices to calculate

$$\lim_{t \rightarrow 0} \text{Re}(i^{1-2\sigma} U_t^\sigma(f)) = \text{Re}(i^{1-2\sigma} \lim_{t \rightarrow 0} U_t^\sigma(f)) = \text{Re}(i^{i-2\sigma} f) = \cos((1-2\sigma)\frac{\pi}{2})f = \sin(\sigma\pi)f$$

and to note that

$$\lim_{t \rightarrow 0} \partial_t^\sigma \text{Re}(i^{1-2\sigma} U_t^\sigma(f)) = \text{Re}(i^{1-2\sigma} \lim_{t \rightarrow 0} \partial_t^\sigma U_t^\sigma(f)) = \text{Re}\left(-i \frac{\Gamma(1-\sigma)}{\sigma 4^\sigma \Gamma(\sigma)} L^\sigma f\right) = 0.$$

For the second claim we begin by noting that

$$\lim_{t \rightarrow 0} \operatorname{Im}(U_t^\sigma(L^{-\sigma}g)) = \operatorname{Im}(\lim_{t \rightarrow 0} U_t^\sigma(L^{-\sigma}g)) = \operatorname{Im}(L^{-\sigma}g) = 0,$$

and calculate

$$\lim_{t \rightarrow 0} \partial_t^\sigma \operatorname{Im}(U_t^\sigma(L^{-\sigma}g)) = \operatorname{Im}(\lim_{t \rightarrow 0} \partial_t^\sigma U_t^\sigma(L^{-\sigma}g)) = \operatorname{Im}\left(-i^{2\sigma} \frac{\Gamma(1-\sigma)}{\sigma 4^\sigma \Gamma(\sigma)} g\right) = -\sin(\sigma\pi) \frac{\Gamma(1-\sigma)}{\sigma 4^\sigma \Gamma(\sigma)} g.$$

□

The rest of the section is devoted to the proof of Theorem 1, and we start with several auxiliary results. Notice that the following simple lemma is false for $\sigma = 1$.

Lemma 1. *Let $0 < \sigma < 1$. Then*

$$\Gamma(\sigma) = i^\sigma \lim_{R \rightarrow \infty} \int_0^R e^{-is} \frac{ds}{s^{1-\sigma}}.$$

Proof. For $0 < \varepsilon < R < \infty$ we may use Cauchy's integral theorem to write

$$\int_\varepsilon^R e^{-s} \frac{ds}{s^{1-\sigma}} = \left(\int_{\operatorname{arc}_\varepsilon} + \int_{\operatorname{ray}_{\varepsilon,R}} + \int_{\operatorname{arc}_R} \right) e^{-z} \frac{dz}{z^{1-\sigma}},$$

where $\operatorname{arc}_\varepsilon$ and arc_R are

$$\theta \mapsto \varepsilon e^{i\theta} \quad \text{and} \quad \theta \mapsto R e^{i(\frac{\pi}{2}-\theta)}, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

and $\operatorname{ray}_{\varepsilon,R}$ is

$$s \mapsto is, \quad \varepsilon \leq s \leq R.$$

Now

$$\left| \int_{\operatorname{arc}_\varepsilon} e^{-z} \frac{dz}{z^{1-\sigma}} \right| = \varepsilon^\sigma \left| \int_0^{\frac{\pi}{2}} e^{-\varepsilon e^{i\theta}} e^{i\sigma\theta} d\theta \right| \leq \varepsilon^\sigma \int_0^{\frac{\pi}{2}} e^{-\varepsilon \cos\theta} d\theta \leq \frac{\pi}{2} \varepsilon^\sigma \rightarrow 0,$$

as $\varepsilon \rightarrow 0$.

Similarly,

$$\begin{aligned} \left| \int_{\operatorname{arc}_R} e^{-z} \frac{dz}{z^{1-\sigma}} \right| &= R^\sigma \left| \int_0^{\frac{\pi}{2}} e^{-R e^{i(\frac{\pi}{2}-\theta)}} e^{i\sigma(\frac{\pi}{2}-\theta)} d\theta \right| \leq R^\sigma \int_0^{\frac{\pi}{2}} e^{-R \sin\theta} d\theta \\ &\leq R^\sigma \int_0^\infty e^{-\frac{2}{\pi} R\theta} d\theta \leq R^{\sigma-1} \int_0^\infty e^{-\frac{2}{\pi}\theta'} d\theta' \rightarrow 0, \end{aligned}$$

as $R \rightarrow \infty$, where we used the inequality $\sin\theta \geq \frac{2}{\pi}\theta$, for $0 < \theta < \frac{\pi}{2}$.

Thus we have

$$\Gamma(\sigma) = \lim_{\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_\varepsilon^R e^{-s} \frac{ds}{s^{1-\sigma}} = \lim_{\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_{\operatorname{ray}_{\varepsilon,R}} e^{-z} \frac{dz}{z^{1-\sigma}} = i^\sigma \lim_{R \rightarrow \infty} \int_0^R e^{-is} \frac{ds}{s^{1-\sigma}}.$$

□

Proposition 1. *Let $0 < \sigma < 1$. For all $\lambda \geq 0$ the limit*

$$I_\sigma(\lambda, t) = \frac{i^\sigma}{\Gamma(\sigma)} \lim_{\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_\varepsilon^R e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}}, \quad t > 0,$$

exists and $I_\sigma(\lambda, \cdot)$ satisfies the equation

$$(11) \quad \left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t \right) I_\sigma(\lambda, t) = -\lambda I_\sigma(\lambda, t),$$

with the initial value $\lim_{t \rightarrow 0} I_\sigma(\lambda, t) = 1$. Moreover,

$$\partial_t^\sigma I_\sigma(\lambda, t) = -i^{2\sigma} \frac{\Gamma(1-\sigma)}{\sigma 4^\sigma \Gamma(\sigma)} \lambda^\sigma I_{1-\sigma}(\lambda, t),$$

where $\partial_t^\sigma = \frac{1}{2^\sigma} t^{1-2\sigma} \partial_t$.

Proof. The proof is split into four parts.

Part I: We consider the truncated integral

$$(12) \quad \int_{\varepsilon}^R e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}},$$

its convergence as $\varepsilon \rightarrow 0$ and $R \rightarrow \infty$, and to what extent it gives a solution to equation (11). Note first that the integral is absolutely convergent as $\varepsilon \rightarrow 0$. On the other hand, using integration by parts, we see that

$$(13) \quad \begin{aligned} \int_{\varepsilon}^R e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}} &= i \int_{\varepsilon}^R \partial_s (e^{-is}) e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}} \\ &= i \left[e^{-is} e^{-i\frac{t^2}{4s}\lambda} s^{\sigma-1} \right]_{s=\varepsilon}^R - i \int_{\varepsilon}^R e^{-is} \partial_s (e^{-i\frac{t^2}{4s}\lambda} s^{\sigma-1}) ds. \end{aligned}$$

Here

$$(14) \quad \partial_s (e^{-i\frac{t^2}{4s}\lambda} s^{\sigma-1}) = (i\frac{t^2}{4}\lambda s^{\sigma-3} - (1-\sigma)s^{\sigma-2}) e^{-i\frac{t^2}{4s}\lambda}$$

so that the last integral converges absolutely as $R \rightarrow \infty$. Since the integrated term converges as $R \rightarrow \infty$, we see that also the integral (12) converges.

We then study the convergence of the integral corresponding to the time derivatives, and calculate:

$$(15) \quad \left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t \right) e^{-i\frac{t^2}{4s}\lambda} = \left(-\frac{t^2}{4s^2} \lambda^2 - i\frac{1-\sigma}{s} \lambda \right) e^{-i\frac{t^2}{4s}\lambda},$$

so that

$$\int_{\varepsilon}^R e^{-is} \left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t \right) e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}} = -\frac{t^2}{4} \lambda^2 \int_{\varepsilon}^R e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{3-\sigma}} - i(1-\sigma)\lambda \int_{\varepsilon}^R e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{2-\sigma}},$$

which converges absolutely as $R \rightarrow \infty$, but appears problematic for $\varepsilon \rightarrow 0$ (see Remark 2). In order to see to what extent the truncated integrals (12) solve equation (11), we note from (14) and (15) that

$$(16) \quad \left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t \right) e^{-i\frac{t^2}{4s}\lambda} s^{\sigma-1} = i\lambda \partial_s (e^{-i\frac{t^2}{4s}\lambda} s^{\sigma-1}).$$

In the light of equations (13) and (16), we infer that

$$\begin{aligned} \left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t \right) \int_{\varepsilon}^R e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}} &= i\lambda \int_{\varepsilon}^R e^{-is} \partial_s (e^{-i\frac{t^2}{4s}\lambda} s^{\sigma-1}) ds \\ &= -\lambda \int_{\varepsilon}^R e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}} + i\lambda \left[e^{-is} e^{-i\frac{t^2}{4s}\lambda} s^{\sigma-1} \right]_{s=\varepsilon}^R. \end{aligned}$$

At the upper limit $R \rightarrow \infty$ the integrated term vanishes, but at the lower limit $\varepsilon \rightarrow 0$ it diverges and a careful argument will be needed. For now we record that

$$\left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t + \lambda \right) \int_1^{\infty} e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}} = -i\lambda e^{-i} e^{-i\frac{t^2}{4}\lambda}.$$

Note here that the integral (12) together with its time derivatives converges locally uniformly in t , which justifies differentiating under the integral sign.

Part II: In order to show that

$$(17) \quad \left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t + \lambda \right) \int_0^1 e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}} = i\lambda e^{-i} e^{-i\frac{t^2}{4}\lambda},$$

from which (11) follows, we view the question in terms of complex path integrals:

$$\int_0^1 e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}} = i^{-\sigma} \int_0^1 e^{-(is)} e^{\frac{t^2}{4(is)}\lambda} \frac{d(is)}{(is)^{1-\sigma}} = i^{-\sigma} \lim_{\gamma} \int_{\gamma} e^{-z} e^{\frac{t^2}{4z}\lambda} \frac{dz}{z^{1-\sigma}},$$

where the paths γ approximate the segment $[0, i]$ in a suitable way (which we elaborate on below).

Let $0 < \delta < \frac{\pi}{2}$ be fixed and consider, for $0 < \varepsilon < 1$, the paths arc_ε , arc_1 and ray_ε given by

$$\theta \mapsto \varepsilon e^{i\theta} \quad \text{and} \quad \theta \mapsto e^{i\theta}, \quad \frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} + \delta,$$

and

$$s \mapsto s e^{i(\frac{\pi}{2} + \delta)}, \quad \varepsilon \leq s \leq 1.$$

By Cauchy's integral theorem we may now write

$$\int_\varepsilon^1 e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}} = i^{-\sigma} \left(\int_{\text{arc}_\varepsilon} + \int_{\text{ray}_\varepsilon} - \int_{\text{arc}_1} \right) e^{-z} e^{\frac{t^2}{4z}\lambda} \frac{dz}{z^{1-\sigma}}.$$

To see that the first integral on the right-hand side vanishes as $\varepsilon \rightarrow 0$, it suffices to observe that $\text{Re}(e^{\pm i\theta}) = \cos \theta \leq 0$ when $\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} + \delta$ so that

$$|e^{-\varepsilon e^{i\theta}} e^{\frac{t^2}{4\varepsilon}\lambda e^{-i\theta}}| = e^{-\varepsilon \cos \theta} e^{\frac{t^2}{4\varepsilon}\lambda \cos \theta} \leq e^{-\varepsilon \cos \theta} \lesssim 1.$$

The second integral on the right-hand side is

$$\int_{\text{ray}_\varepsilon} e^{-z} e^{\frac{t^2}{4z}\lambda} \frac{dz}{z^{1-\sigma}} = e^{i\sigma(\frac{\pi}{2} + \delta)} \int_\varepsilon^1 e^{-s e^{i(\frac{\pi}{2} + \delta)}} e^{\frac{t^2}{4s}\lambda e^{-i(\frac{\pi}{2} + \delta)}} \frac{ds}{s^{1-\sigma}},$$

and it converges to a limit $J_\delta(\lambda, t)$ as $\varepsilon \rightarrow 0$, again because $\text{Re}(e^{-i(\frac{\pi}{2} + \delta)}) = \cos(\frac{\pi}{2} + \delta) < 0$.

Lastly,

$$\int_{\text{arc}_1} e^{-z} e^{\frac{t^2}{4z}\lambda} \frac{dz}{z^{1-\sigma}} = i \int_{\frac{\pi}{2}}^{\frac{\pi}{2} + \delta} e^{-e^{i\theta}} e^{\frac{t^2}{4} \lambda e^{-i\theta}} e^{i\sigma\theta} d\theta =: E_\delta(\lambda, t),$$

so that altogether we have

$$\int_0^1 e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}} = i^{-\sigma} (J_\delta(\lambda, t) - E_\delta(\lambda, t)).$$

Part III: For (17) it suffices to show that

$$\left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t + \lambda \right) E_\delta(\lambda, t) \rightarrow 0, \quad \text{as } \delta \rightarrow 0$$

and that

$$(18) \quad \left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t + \lambda \right) J_\delta(\lambda, t) \rightarrow i^{\sigma+1} \lambda e^{-i} e^{-i\frac{t^2}{4}\lambda}, \quad \text{as } \delta \rightarrow 0.$$

It is easy to see that

$$\left| \left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t + \lambda \right) E_\delta(\lambda, t) \right| \lesssim (t^2 \lambda^2 + \lambda) \int_{\frac{\pi}{2}}^{\frac{\pi}{2} + \delta} e^{-\cos \theta} e^{\frac{t^2}{4} \lambda \cos \theta} d\theta \rightarrow 0, \quad \text{as } \delta \rightarrow 0.$$

Write $\theta_\delta = \frac{\pi}{2} + \delta$ so that

$$J_\delta(\lambda, t) = e^{i\sigma\theta_\delta} \int_0^1 e^{-s e^{i\theta_\delta}} e^{\frac{t^2}{4s}\lambda e^{-i\theta_\delta}} \frac{ds}{s^{1-\sigma}}.$$

Replacing λ by $i\lambda e^{-i\theta_\delta}$ in (16) and (13), we find that

$$\begin{aligned} \left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t \right) J_\delta(\lambda, t) &= -e^{i(\sigma-1)\theta_\delta} \lambda \int_0^1 e^{-s e^{i\theta_\delta}} \partial_s (e^{\frac{t^2}{4s}\lambda e^{-i\theta_\delta}} s^{\sigma-1}) ds \\ &= -\lambda e^{i(\sigma-1)\theta_\delta} \left[e^{-s e^{i\theta_\delta}} e^{\frac{t^2}{4s}\lambda e^{-i\theta_\delta}} s^{\sigma-1} \right]_{s=0}^1 - \lambda J_\delta(\lambda, t), \end{aligned}$$

where the integrated term poses no problem at $s = 0$ because $\text{Re}(e^{-i\theta_\delta}) = \cos \theta_\delta < 0$. As $\delta \rightarrow 0$, the integrated term therefore tends to $\lambda i^{\sigma+1} e^{-i} e^{-i\frac{t^2}{4}\lambda}$, and (18) and (17) follow.

Part IV: We address the convergence to the initial value. By Lemma 1,

$$I_\sigma(\lambda, t) - 1 = \frac{i^\sigma}{\Gamma(\sigma)} \lim_{\varepsilon, R} \int_\varepsilon^R e^{-is} (e^{-i\frac{t^2}{4s}\lambda} - 1) \frac{ds}{s^{1-\sigma}}.$$

The integral above is absolutely convergent as $\varepsilon \rightarrow 0$ and therefore by dominated convergence,

$$\int_0^1 e^{-is} (e^{-i\frac{t^2}{4s}\lambda} - 1) \frac{ds}{s^{1-\sigma}} \longrightarrow 0 \quad \text{as } t \rightarrow 0.$$

Using integration by parts, we see that

$$\begin{aligned} \int_1^R e^{-is} (e^{-i\frac{t^2}{4s}\lambda} - 1) \frac{ds}{s^{1-\sigma}} &= i \int_1^R \partial_s (e^{-is}) (e^{-i\frac{t^2}{4s}\lambda} - 1) \frac{ds}{s^{1-\sigma}} \\ &= i \left[e^{-is} (e^{-i\frac{t^2}{4s}\lambda} - 1) s^{\sigma-1} \right]_{s=1}^R - i \int_1^R e^{-is} \partial_s ((e^{-i\frac{t^2}{4s}\lambda} - 1) s^{\sigma-1}) ds, \end{aligned}$$

where

$$\partial_s ((e^{-i\frac{t^2}{4s}\lambda} - 1) s^{\sigma-1}) = i \frac{t^2}{4} \lambda e^{-i\frac{t^2}{4s}\lambda} s^{\sigma-3} - (1-\sigma) (e^{-i\frac{t^2}{4s}\lambda} - 1) s^{\sigma-2},$$

so at the limit $R \rightarrow \infty$ we have

$$\begin{aligned} \int_1^\infty e^{-is} (e^{-i\frac{t^2}{4s}\lambda} - 1) \frac{ds}{s^{1-\sigma}} &= -i e^{-i} (e^{-i\frac{t^2}{4}\lambda} - 1) + \frac{t^2}{4} \lambda \int_1^\infty e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{3-\sigma}} \\ &\quad + i(1-\sigma) \int_1^\infty e^{-is} (e^{-i\frac{t^2}{4s}\lambda} - 1) \frac{ds}{s^{2-\sigma}}. \end{aligned}$$

The two integrals on the right-hand side converge absolutely and thus by dominated convergence we have

$$\int_1^\infty e^{-is} (e^{-i\frac{t^2}{4s}\lambda} - 1) \frac{ds}{s^{1-\sigma}} \longrightarrow 0 \quad \text{as } t \rightarrow 0.$$

This shows that $I_\sigma(\lambda, t) \rightarrow 1$ as $t \rightarrow 0$.

Performing a change of variables $\frac{t^2}{4s}\lambda = r$ in the truncated integral, we obtain

$$\begin{aligned} \partial_t \int_\varepsilon^R e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}} &= -i \frac{t}{2} \lambda \int_\varepsilon^R e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{2-\sigma}} \\ &= -i \frac{4^{1-\sigma}}{2} \frac{\lambda^\sigma}{t^{1-2\sigma}} \int_{t^2\lambda/4R}^{t^2\lambda/4\varepsilon} e^{-i\frac{t^2}{4r}\lambda} e^{-ir} \frac{dr}{r^{1-(1-\sigma)}}. \end{aligned}$$

Since the convergence of the last integral here is locally uniform in t , it is easy to verify that

$$\begin{aligned} \partial_t^\sigma I_\sigma(\lambda, t) &= \frac{t^{1-2\sigma}}{2\sigma} \partial_t I_\sigma(\lambda, t) \\ &= -\frac{i^{\sigma+1}}{\Gamma(\sigma)} \frac{\lambda^\sigma}{\sigma 4^\sigma} \lim_{\varepsilon', R'} \int_{\varepsilon'}^{R'} e^{-i\frac{t^2}{4r}\lambda} e^{-ir} \frac{dr}{r^{1-(1-\sigma)}} \\ &= -i^{2\sigma} \frac{\Gamma(1-\sigma)}{\sigma 4^\sigma \Gamma(\sigma)} \lambda^\sigma I_{1-\sigma}(\lambda, t). \end{aligned}$$

□

Remark. As $\varepsilon \rightarrow 0$, the integral

$$\int_\varepsilon^R e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{k-\sigma}}$$

converges for $k = 2$ but diverges for $k = 3$. In both cases, this can be seen by means of the change of variable $r = \frac{t^2}{4s}\lambda$. For $k = 2$, we also get bounds for the integral by means of Lemma 2.

Lemma 2. *Let $0 < \sigma < 1$. For every $t > 0$ we have*

$$(19) \quad \left| \int_\varepsilon^R e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}} \right| \lesssim \begin{cases} 1, & \sigma \leq \frac{1}{2}, \\ 1 + \lambda^{\frac{\sigma}{2} - \frac{1}{4}}, & \sigma > \frac{1}{2}, \end{cases}$$

uniformly in ε and R .

Proof. We denote $A = \frac{t\sqrt{\lambda}}{2}$ and show that

$$\left| \int_{\varepsilon}^R e^{-i(s+\frac{A^2}{s})} \frac{ds}{s^{1-\sigma}} \right| \lesssim \begin{cases} 1, & A < 1, \\ 1 + A^{\sigma-\frac{1}{2}}, & A \geq 1. \end{cases}$$

Observe that the phase $\varphi(s) = s + \frac{A^2}{s}$ has its critical point where $0 = \varphi'(s) = 1 - \frac{A^2}{s^2}$ at $s = A$.

Consider first the case $A < 1$. We write

$$\left(\int_{\varepsilon}^2 + \int_2^R \right) e^{-i(s+\frac{A^2}{s})} \frac{ds}{s^{1-\sigma}} =: I_1 + I_2,$$

and note that $|I_1| \lesssim 1$. Using integration by parts, we see that

$$I_2 = i \int_2^R \partial_s(e^{-i\varphi(s)}) \frac{ds}{\varphi'(s)s^{1-\sigma}} = i \left[e^{-i\varphi(s)} \frac{s^{\sigma+1}}{s^2 - A^2} \right]_{s=2}^R - i \int_2^R e^{-i\varphi(s)} \partial_s \left(\frac{s^{\sigma+1}}{s^2 - A^2} \right) ds.$$

Since $A^2 \leq 1$, we have

$$\left| \left[e^{-i\varphi(s)} \frac{s^{\sigma+1}}{s^2 - A^2} \right]_{s=2}^R \right| \leq \frac{2^{\sigma+1}}{3} + \frac{R^{\sigma+1}}{R^2 - 1} \lesssim 1$$

and upon calculating the derivative

$$\partial_s \left(\frac{s^{\sigma+1}}{s^2 - A^2} \right) = \frac{(\sigma+1)s^{\sigma}}{s^2 - A^2} - \frac{2s^{\sigma+2}}{(s^2 - A^2)^2}$$

we get

$$\left| \int_2^R e^{-i\varphi(s)} \partial_s \left(\frac{s^{\sigma+1}}{s^2 - A^2} \right) ds \right| \lesssim \int_2^{\infty} \frac{s^{\sigma}}{s^2 - A^2} ds + \int_2^{\infty} \frac{s^{\sigma+2}}{(s^2 - A^2)^2} ds \lesssim 1,$$

so that $|I_2| \lesssim 1$.

Consider then the case $A \geq 1$. We follow the principle of stationary phase and decompose the integral into three pieces

$$\left(\int_{\varepsilon}^{A-\frac{1}{2}\sqrt{A}} + \int_{A-\frac{1}{2}\sqrt{A}}^{A+\frac{1}{2}\sqrt{A}} + \int_{A+\frac{1}{2}\sqrt{A}}^R \right) e^{-i(s+\frac{A^2}{s})} \frac{ds}{s^{1-\sigma}} =: I_1 + I_2 + I_3;$$

observe that $A \pm \frac{1}{2}\sqrt{A} \sim A$.

Note first that

$$|I_2| \leq \int_{A-\frac{1}{2}\sqrt{A}}^{A+\frac{1}{2}\sqrt{A}} \frac{ds}{s^{1-\sigma}} \leq \frac{\sqrt{A}}{(A - \frac{1}{2}\sqrt{A})^{1-\sigma}} \lesssim A^{\sigma-\frac{1}{2}}.$$

For $I_1 + I_3$ we use integration by parts:

$$\begin{aligned} I_1 + I_3 &= \left(\int_{\varepsilon}^{A-\frac{1}{2}\sqrt{A}} + \int_{A+\frac{1}{2}\sqrt{A}}^R \right) e^{-i\varphi(s)} \frac{ds}{s^{1-\sigma}} \\ &= i \left(\int_{\varepsilon}^{A-\frac{1}{2}\sqrt{A}} + \int_{A+\frac{1}{2}\sqrt{A}}^R \right) \partial_s(e^{-i\varphi(s)}) \frac{ds}{\varphi'(s)s^{1-\sigma}} \\ &= i \left[e^{-i\varphi(s)} \frac{s^{\sigma+1}}{s^2 - A^2} \right]_{s=\varepsilon}^{A-\frac{1}{2}\sqrt{A}} + i \left[e^{-i\varphi(s)} \frac{s^{\sigma+1}}{s^2 - A^2} \right]_{s=A+\frac{1}{2}\sqrt{A}}^R \\ &\quad - i \left(\int_{\varepsilon}^{A-\frac{1}{2}\sqrt{A}} + \int_{A+\frac{1}{2}\sqrt{A}}^R \right) e^{-i\varphi(s)} \partial_s \left(\frac{s^{1+\sigma}}{s^2 - A^2} \right) ds. \end{aligned}$$

Here

$$\left| \left[e^{-i\varphi(s)} \frac{s^{\sigma+1}}{s^2 - A^2} \right]_{s=\varepsilon}^{A-\frac{1}{2}\sqrt{A}} \right| \leq \frac{(A - \frac{1}{2}\sqrt{A})^{\sigma+1}}{A^2 - (A - \frac{1}{2}\sqrt{A})^2} + \frac{\varepsilon^{\sigma+1}}{A^2 - \varepsilon^2} \lesssim 1 + A^{\sigma-\frac{1}{2}}.$$

and similarly

$$\left| \left[e^{-i\varphi(s)} \frac{s^{\sigma+1}}{s^2 - A^2} \right]_{s=A+\frac{1}{2}\sqrt{A}}^R \right| \leq \frac{(A + \frac{1}{2}\sqrt{A})^{\sigma+1}}{(A + \frac{1}{2}\sqrt{A})^2 - A^2} + \frac{R^{\sigma+1}}{R^2 - A^2} \lesssim 1 + A^{\sigma-\frac{1}{2}}.$$

Moreover, because

$$\partial_s \left(\frac{s^{1+\sigma}}{s^2 - A^2} \right) = \frac{(s^2 - A^2)(1 + \sigma)s^\sigma - 2s^{\sigma+2}}{(s^2 - A^2)^2} = \frac{s^{\sigma+2}(\sigma - 1) - A^2(1 + \sigma)s^\sigma}{(s^2 - A^2)^2} < 0,$$

we have

$$\begin{aligned} \left| \left(\int_\varepsilon^{A-\frac{1}{2}\sqrt{A}} + \int_{A+\frac{1}{2}\sqrt{A}}^R \right) e^{-i\varphi(s)} \partial_s \left(\frac{s^{1+\sigma}}{s^2 - A^2} \right) ds \right| &\leq - \left(\int_\varepsilon^{A-\frac{1}{2}\sqrt{A}} + \int_{A+\frac{1}{2}\sqrt{A}}^R \right) \partial_s \left(\frac{s^{1+\sigma}}{s^2 - A^2} \right) ds \\ &= - \left[\frac{s^{1+\sigma}}{s^2 - A^2} \right]_{s=\varepsilon}^{A-\frac{1}{2}\sqrt{A}} - \left[\frac{s^{1+\sigma}}{s^2 - A^2} \right]_{s=A+\frac{1}{2}\sqrt{A}}^R \\ &\lesssim 1 + A^{\sigma-\frac{1}{2}}, \end{aligned}$$

as before. Therefore

$$|I_1 + I_3| \lesssim 1 + A^{\sigma-\frac{1}{2}}.$$

□

Lemma 3. *Let $0 < \sigma < 1$. For every $t > 0$ we have*

$$(20) \quad \left| \left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t \right) I_\sigma(\lambda, t) \right| \lesssim \max\{1, \lambda^{\frac{\sigma}{2} + \frac{3}{4}}\}.$$

Proof. In order to obtain estimates for the derivatives

$$\left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t \right) \int_0^\infty e^{-i(s+\frac{t^2}{4s})\lambda} \frac{ds}{s^{1-\sigma}},$$

we now split the interval of integration in a way which does not depend on t . We shall prove the estimate for t in a small neighbourhood of a fixed point $t_0 > 0$, by moving the differentiations into the integrals. This will be uniform in $t_0 > 0$. With $A = t\sqrt{\lambda}/2$ as before, we write A_0 for the value corresponding to t_0 . Let $d = \min(A_0, \sqrt{A_0})/2$. The integral will be split at the points $a = A_0 - d$ and $b = A_0 + d$. We consider only values of t so close to t_0 that $|A - A_0| < d/2$. Thus $|s - A| > d/2 \sim \min(A, \sqrt{A})$ as soon as $s \notin (a, b)$.

For such t we write

$$\left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t \right) \left(\int_0^a + \int_a^b + \int_b^\infty \right) e^{-i(s+\frac{t^2}{4s})\lambda} \frac{ds}{s^{1-\sigma}} = I_1 + I_2 + I_3,$$

say. In I_2 and I_3 , it is clear that the derivatives can be taken inside the integrals:

$$\begin{aligned} &\left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t \right) \left(\int_a^b + \int_b^\infty \right) e^{-i(s+\frac{t^2}{4s})\lambda} \frac{ds}{s^{1-\sigma}} \\ &= -i(1-\sigma)\lambda \left(\int_a^b + \int_b^\infty \right) e^{-i(s+\frac{A^2}{s})\lambda} \frac{ds}{s^{2-\sigma}} - \frac{t^2\lambda^2}{4} \left(\int_a^b + \int_b^\infty \right) e^{-i(s+\frac{A^2}{s})\lambda} \frac{ds}{s^{3-\sigma}}. \end{aligned}$$

To estimate I_2 , we observe that $s \sim A$ for $a \leq s \leq b$ and that $b - a = 2d \lesssim \sqrt{A} \sim \lambda^{1/4}$. Thus

$$|I_2| \leq \lambda \int_a^b \frac{ds}{s^{2-\sigma}} + t^2\lambda^2 \int_a^b \frac{ds}{s^{3-\sigma}} \leq \lambda \frac{b-a}{A^{2-\sigma}} + t^2\lambda^2 \frac{b-a}{A^{3-\sigma}} \lesssim \lambda^{\frac{\sigma}{2} + \frac{1}{4}} + \lambda^{\frac{\sigma}{2} + \frac{3}{4}}.$$

For I_3 the estimate is clear when $A \leq 1$:

$$\begin{aligned} |I_3| &\leq \lambda \int_b^\infty \frac{ds}{s^{2-\sigma}} + t^2\lambda^2 \int_b^\infty \frac{ds}{s^{3-\sigma}} \\ &\lesssim \lambda A^{\sigma-1} + \lambda^2 A^{\sigma-2} \leq \lambda A^{\sigma-\frac{3}{2}} + \lambda^2 A^{\sigma-\frac{5}{2}} \\ &\lesssim \lambda^{\frac{\sigma}{2} + \frac{1}{4}} + \lambda^{\frac{\sigma}{2} + \frac{3}{4}} \lesssim \max(1, \lambda^{\frac{\sigma}{2} + \frac{3}{4}}). \end{aligned}$$

When $A > 1$ we integrate by parts, with $k = 2, 3$:

$$\begin{aligned} \int_b^\infty e^{-i\varphi(s)} \frac{ds}{s^{k-\sigma}} &= i \int_b^\infty \partial_s(e^{-i\varphi(s)}) \frac{ds}{\varphi'(s)s^{k-\sigma}} \\ &= i \left[e^{-i\varphi(s)} \frac{s^{\sigma-k+2}}{s^2 - A^2} \right]_{s=b}^\infty - i \int_b^\infty e^{-i\varphi(s)} \partial_s \left(\frac{s^{\sigma-k+2}}{s^2 - A^2} \right) ds. \end{aligned}$$

Here

$$\left| \left[e^{-i\varphi(s)} \frac{s^{\sigma-k+2}}{s^2 - A^2} \right]_{s=b}^\infty \right| = \frac{b^{\sigma-k+2}}{b^2 - A^2} \sim \frac{A^{\sigma-k+2}}{A\sqrt{A}} \lesssim A^{\sigma-k+\frac{1}{2}}.$$

Moreover, since by simple calculus,

$$\begin{aligned} \partial_s \left(\frac{s^{\sigma-k+2}}{s^2 - A^2} \right) &= \frac{(s^2 - A^2)(\sigma - k + 2)s^{\sigma-k+1} - 2s^{\sigma-k+3}}{(s^2 - A^2)^2} \\ &= \frac{(\sigma - k)s^{\sigma-k+3} - A^2(\sigma - k + 2)s^{\sigma-k+1}}{(s^2 - A^2)^2} < 0 \end{aligned}$$

for all $s > A$, we have

$$\left| \int_b^\infty e^{-i\varphi(s)} \partial_s \left(\frac{s^{\sigma-k+2}}{s^2 - A^2} \right) ds \right| \leq - \int_b^\infty \partial_s \left(\frac{s^{\sigma-k+2}}{s^2 - A^2} \right) ds = - \left[\frac{s^{\sigma-k+2}}{s^2 - A^2} \right]_{s=b}^\infty \lesssim A^{\sigma-k+\frac{1}{2}},$$

as before. Therefore

$$\begin{aligned} |I_3| &\leq \lambda \left| \int_b^\infty e^{-i\varphi(s)} \frac{ds}{s^{2-\sigma}} \right| + t^2 \lambda^2 \left| \int_b^\infty e^{-i\varphi(s)} \frac{ds}{s^{3-\sigma}} \right| \\ &\lesssim \lambda A^{\sigma-\frac{3}{2}} + \lambda^2 A^{\sigma-\frac{5}{2}} \lesssim \max(1, \lambda^{\frac{\sigma}{2} + \frac{3}{4}}), \end{aligned}$$

as before.

We handle I_1 by switching to complex path integrals as in the proof of Proposition 1, namely

$$\int_0^a e^{-i(s + \frac{t^2}{4s})\lambda} \frac{ds}{s^{1-\sigma}} = i^{-\sigma} \int_{[0, ia]} e^{-z + \frac{A^2}{z}} \frac{dz}{z^{1-\sigma}} = i^{-\sigma} \left(\int_{\text{ray}_a} - \int_{\text{arc}_a} \right) e^{-z + \frac{A^2}{z}} \frac{dz}{z^{1-\sigma}},$$

where

$$\text{ray}_a : s \mapsto se^{i\frac{2\pi}{3}}, \quad 0 < s \leq a,$$

and

$$\text{arc}_a : \theta \mapsto ae^{i(\theta + \frac{\pi}{2})}, \quad 0 \leq \theta \leq \frac{\pi}{6}.$$

In the integrals over ray_a and arc_a , we shall take the derivatives of the integrand and verify convergence of the resulting integrals. Observe first that for $0 \leq \theta \leq \pi/6$

$$\left| e^{-ae^{i(\theta + \pi/2)} + \frac{A^2}{a}} e^{-i(\theta + \pi/2)} \right| = e^{-(-a + \frac{A^2}{a}) \sin \theta},$$

and here

$$-s + \frac{A^2}{s} = 2(A - s) + \frac{(A - s)^2}{s} \gtrsim d + \frac{d^2}{s}.$$

For $\theta = \pi/6$ this implies that

$$\left| e^{-se^{i2\pi/3} + \frac{A^2}{s}} e^{-i2\pi/3} \right| = e^{-\frac{1}{2}(-s + \frac{A^2}{s})} \lesssim e^{-c\sqrt{A}} e^{-c\frac{\min(A^2, A)}{s}}.$$

With $k = 2, 3$, we get for $A \leq 1$

$$\int_{\text{ray}_a} \left| e^{-z + \frac{A^2}{z}} \frac{1}{z^{k-\sigma}} \right| dz \lesssim \int_0^a e^{-c\frac{A^2}{s}} \frac{ds}{s^{k-\sigma}} \lesssim A^{2\sigma-2k+2},$$

and for $A > 1$

$$\int_{\text{ray}_a} \left| e^{-z + \frac{A^2}{z}} \frac{1}{z^{k-\sigma}} \right| dz \leq e^{-c\sqrt{A}} \int_0^a e^{-c\frac{A}{s}} \frac{ds}{s^{k-\sigma}} \lesssim A^{-M},$$

for any M .

We also get

$$\left| \int_{\text{arc}_a} e^{-z+\frac{A^2}{z}} \frac{dz}{z^{k-\sigma}} \right| \lesssim A^{\sigma-k+1} \int_0^{\frac{\pi}{6}} e^{-c\sqrt{A}\theta} d\theta \lesssim A^{\sigma-k+\frac{1}{2}},$$

for all A .

Altogether, this implies that

$$\begin{aligned} |I_1| &\leq \lambda \left| \left(\int_{\text{ray}_a} - \int_{\text{arc}_a} \right) e^{-z+\frac{A^2}{z}} \frac{dz}{z^{2-\sigma}} \right| + t^2 \lambda^2 \left| \left(\int_{\text{ray}_a} - \int_{\text{arc}_a} \right) e^{-z+\frac{A^2}{z}} \frac{dz}{z^{3-\sigma}} \right| \\ &\lesssim \lambda \max(A^{2\sigma-2}, A^{\sigma-\frac{3}{2}}) + \lambda^2 \max(A^{2\sigma-4}, A^{\sigma-\frac{5}{2}}) \\ &\lesssim \max(1, \lambda^{\frac{\sigma}{2}+\frac{3}{4}}). \end{aligned}$$

□

Proof of Theorem 1. Assume first that

$$f \in \begin{cases} \mathcal{L}^2, & \sigma \leq \frac{1}{2}, \\ \text{Dom}(L^{\frac{\sigma}{2}-\frac{1}{4}}), & \sigma > \frac{1}{2}, \end{cases}$$

and denote by E the spectral measure of L . In order to see that the family

$$u_{\varepsilon,R}(\cdot, t) = \frac{i^\sigma t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_\varepsilon^R e^{-i\frac{t^2}{4s}} e^{-isL} f \frac{ds}{s^{1+\sigma}}$$

converges weakly in \mathcal{L}^2 as $\varepsilon \rightarrow 0$ and $R \rightarrow \infty$ note first that for any $h \in \mathcal{L}^2$,

$$\begin{aligned} \langle u_{\varepsilon,R}(\cdot, t), h \rangle &= \frac{i^\sigma t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_\varepsilon^R e^{-i\frac{t^2}{4s}} \int_0^\infty e^{-is\lambda} dE_{f,h}(\lambda) \frac{ds}{s^{1+\sigma}} \\ &= \frac{i^\sigma t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty \int_\varepsilon^R e^{-i\frac{t^2}{4s}} e^{-is\lambda} \frac{ds}{s^{1+\sigma}} dE_{f,h}(\lambda) \\ &= \frac{i^\sigma}{\Gamma(\sigma)} \int_0^\infty \int_{t^2/4R}^{t^2/4\varepsilon} e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}} dE_{f,h}(\lambda). \end{aligned}$$

Here, by Lemma 2,

$$\left| \int_{t^2/4R}^{t^2/4\varepsilon} e^{-is} e^{-i\frac{t^2}{4s}\lambda} \frac{ds}{s^{1-\sigma}} \right| \lesssim \begin{cases} 1, & \sigma \leq \frac{1}{2}, \\ 1 + \lambda^{\frac{\sigma}{2}-\frac{1}{4}}, & \sigma > \frac{1}{2}, \end{cases}$$

from which the convergence follows.

Assume then that $f \in \text{Dom}(L^{\frac{\sigma}{2}+\frac{3}{4}})$. By Lemma 3 and Proposition 1, we have

$$\begin{aligned} \left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t \right) \langle u(\cdot, t), h \rangle &= \int_0^\infty \left(\partial_t^2 + \frac{1-2\sigma}{t} \partial_t \right) I_\sigma(\lambda, t) dE_{f,h}(\lambda) \\ &= - \int_0^\infty \lambda I_\sigma(\lambda, t) dE_{f,h}(\lambda) \\ &= - \langle Lu(\cdot, t), h \rangle. \end{aligned}$$

Weak convergence to the initial data,

$$\langle u(\cdot, t), h \rangle = \int_0^\infty I_\sigma(\lambda, t) dE_{f,h}(\lambda) \longrightarrow \langle f, h \rangle, \quad \text{as } t \rightarrow 0,$$

holds by Proposition 1 and Lemma 2.

Moreover, taking into account that

$$|I_{1-\sigma}(\lambda, t)| \lesssim \begin{cases} 1, & \sigma \geq \frac{1}{2}, \\ 1 + \lambda^{\frac{1}{4}-\frac{\sigma}{2}}, & \sigma < \frac{1}{2}, \end{cases}$$

we have, by Proposition 1, that

$$\begin{aligned}\partial_t^\sigma \langle u(\cdot, t), h \rangle &= \int_0^\infty \partial_t^\sigma I_\sigma(\lambda, t) dE_{f,h}(\lambda) \\ &= -i^{2\sigma} \frac{\Gamma(1-\sigma)}{\sigma 4^\sigma \Gamma(\sigma)} \int_0^\infty \lambda^\sigma I_{1-\sigma}(\lambda, t) dE_{f,h}(\lambda) \\ &\rightarrow -i^{2\sigma} \frac{\Gamma(1-\sigma)}{\sigma 4^\sigma \Gamma(\sigma)} \langle L^\sigma f, h \rangle,\end{aligned}$$

as $t \rightarrow 0$. □

Remark. Note that the proof of Theorem 1 entails the fixed-time norm estimate

$$\|u(\cdot, t)\|_2 \lesssim \begin{cases} \|f\|_2, & \sigma \leq \frac{1}{2}, \\ \|f\|_2 + \|L^{\frac{\sigma}{2}-\frac{1}{4}} f\|_2, & \sigma > \frac{1}{2}; \end{cases}$$

cf. Theorem 6.

3. KERNEL REPRESENTATIONS IN THE CASE OF THE LAPLACIAN

Let $0 < \sigma < 1$ and consider the Neumann problem

$$(21) \quad \begin{cases} \partial_t^2 u + \frac{1-2\sigma}{t} \partial_t u = \Delta u \\ u(\cdot, 0) = 0, \quad \partial_t^\sigma u(\cdot, 0) = g, \end{cases}$$

for Schwartz initial data g on \mathbb{R}^d . In this section we prove the following three results:

Theorem 2. *Suppose that $d \in \{1, 2, 3\}$ and let $0 < \sigma < 1$ be such that $0 < \frac{d}{2} - \sigma < 1$. A solution to the Neumann problem (21) with Schwartz initial data g is given by*

$$u(x, t) = \frac{\sigma \sin((\frac{d}{2} - \sigma)\pi) \Gamma(\frac{d}{2} - \sigma)}{\pi^{\frac{d}{2}} \sin(\sigma\pi) \Gamma(1 - \sigma)} \int_{B(x,t)} \frac{g(y)}{(t^2 - |x - y|^2)^{\frac{d}{2} - \sigma}} dy.$$

Theorem 3. *Suppose that $d \in \{3, 4, 5\}$ and let $0 < \sigma < 1$ be such that $1 < \frac{d}{2} - \sigma < 2$. A solution to the Neumann problem (21) with Schwartz initial data g is given by*

$$u(x, t) = \frac{\sigma \sin((\frac{d}{2} - \sigma - 1)\pi) \Gamma(\frac{d}{2} - \sigma - 1)}{2\pi^{\frac{d}{2}} \sin(\sigma\pi) \Gamma(1 - \sigma)} \frac{1}{t^2} \int_{B(x,t)} \frac{2(\sigma + 1)g(y) + (y - x) \cdot \nabla g(y)}{(t^2 - |x - y|^2)^{\frac{d}{2} - \sigma - 1}} dy.$$

This can be continued to higher dimensions, but the formulas will be more complicated.

Theorem 4. *The solution u_σ of the Neumann problem (21) with Schwartz initial data g has the limit*

$$\lim_{\sigma} u_\sigma(x, t) = \frac{1}{c_d t} \int_{\partial B(x,t)} g(y) dS(y)$$

when

- $d = 2$ and $\sigma \searrow 0$, in which case $c_d = 2\pi$,
- $d = 3$ and $\sigma \rightarrow \frac{1}{2}$, in which case $c_d = 4\pi$,
- $d = 4$ and $\sigma \nearrow 1$, in which case $c_d = 2\pi^2$.

Before the proofs, we use the Schrödinger kernel to rewrite the solution formulas. According to Corollary 1, a solution to (21) is given by

$$u(\cdot, t) = -\frac{\sigma 4^\sigma \Gamma(\sigma)}{\sin(\sigma\pi) \Gamma(1 - \sigma)} \operatorname{Im}(U_t^\sigma((-\Delta)^{-\sigma} g)),$$

where

$$U_t^\sigma((-\Delta)^{-\sigma} g) = \frac{i^\sigma t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-i\frac{t^2}{4s}} e^{is\Delta} (-\Delta)^{-\sigma} g \frac{ds}{s^{1+\sigma}} = \frac{i^\sigma}{\Gamma(\sigma)} \int_0^\infty e^{-i\frac{t^2}{4s}} e^{is\Delta} g \frac{ds}{s^{1-\sigma}},$$

by change of variable. Hence $u(\cdot, t)$ is the imaginary part of the integral

$$(22) \quad -\frac{\sigma 4^\sigma i^\sigma}{\sin(\sigma\pi)\Gamma(1-\sigma)} \int_0^\infty e^{-i\frac{t^2}{4s}} e^{is\Delta} g \frac{ds}{s^{1-\sigma}}.$$

The Schrödinger group is given, for Schwartz functions g , by

$$e^{is\Delta} g(x) = \frac{1}{(4\pi is)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4s}} g(y) dy$$

and in

$$\int_0^\infty e^{-i\frac{t^2}{4s}} e^{is\Delta} g(x) \frac{ds}{s^{1-\sigma}} = \frac{1}{(4\pi i)^{\frac{d}{2}}} \left(\int_0^1 + \int_1^\infty \right) e^{-i\frac{t^2}{4s}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4s}} g(y) dy \frac{ds}{s^{1+\frac{d}{2}-\sigma}} =: I_1(x) + I_2(x),$$

the left-hand side converges absolutely and uniformly in x for all d and σ (restricting to $\sigma < \frac{1}{2}$ for $d = 1$). Indeed,

$$|I_2(x)| \leq \|g\|_1 \int_1^\infty \frac{ds}{s^{1+\frac{d}{2}-\sigma}} < \infty,$$

and

$$\sup_{0 < s < 1} \|e^{is\Delta} g\|_\infty \leq \sup_{0 < s < 1} \|\widehat{e^{is\Delta} g}\|_1 = \|\widehat{g}\|_1$$

so that

$$|I_1(x)| \lesssim \|\widehat{g}\|_1 \int_0^1 \frac{ds}{s^{1-\sigma}} < \infty.$$

Now the expression (22) is

$$-\frac{\sigma 4^\sigma i^{\sigma-\frac{d}{2}}}{(4\pi)^{\frac{d}{2}} \sin(\sigma\pi)\Gamma(1-\sigma)} \int_0^\infty e^{-i\frac{t^2}{4s}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4s}} g(y) dy \frac{ds}{s^{1+\frac{d}{2}-\sigma}},$$

and we define

$$V_\gamma g(x, t) = i^{-\gamma} \int_0^\infty e^{-i\frac{t^2}{4s}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4s}} g(y) dy \frac{ds}{s^{1+\gamma}}$$

for a free parameter $\gamma > 0$. Thus

$$(23) \quad u(x, t) = -\frac{\sigma 4^{\sigma-\frac{d}{2}}}{\pi^{\frac{d}{2}} \sin(\sigma\pi)\Gamma(1-\sigma)} \text{Im} V_\gamma g(x, t).$$

Observe that

$$(24) \quad \partial_t V_\gamma g(x, t) = \frac{t}{2} V_{\gamma+1} g(x, t)$$

which after iteration gives, for any $n \geq 1$, that

$$V_{\gamma+n} g(x, t) = \left(\frac{2}{t} \partial_t\right)^n V_\gamma g(x, t).$$

Cases $0 < \gamma < 1$ and $1 < \gamma < 2$. When $0 < \gamma < 1$, the kernel can be computed from

$$V_\gamma g(x, t) = i^{-\gamma} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} \int_\varepsilon^\infty e^{-i\frac{t^2-|x-y|^2}{4s}} \frac{ds}{s^{1+\gamma}} g(y) dy.$$

In the inner integral here, we make a change of variable $\frac{t^2-|x-y|^2}{4s} = \pm r$, separating the cases $t > |x-y|$ and $t < |x-y|$. Doing so we obtain, respectively,

$$\int_\varepsilon^\infty e^{-i\frac{t^2-|x-y|^2}{4s}} \frac{ds}{s^{1+\gamma}} = \frac{4^\gamma}{(t^2 - |x-y|^2)^\gamma} \int_0^{\frac{t^2-|x-y|^2}{4\varepsilon}} e^{-ir} \frac{dr}{r^{1-\gamma}} \longrightarrow \frac{4^\gamma i^{-\gamma} \Gamma(\gamma)}{(t^2 - |x-y|^2)^\gamma}$$

and

$$\int_\varepsilon^\infty e^{-i\frac{t^2-|x-y|^2}{4s}} \frac{ds}{s^{1+\gamma}} = \frac{4^\gamma}{(|x-y|^2 - t^2)^\gamma} \int_0^{\frac{|x-y|^2-t^2}{4\varepsilon}} e^{ir} \frac{dr}{r^{1-\gamma}} \longrightarrow \frac{4^\gamma i^\gamma \Gamma(\gamma)}{(|x-y|^2 - t^2)^\gamma},$$

where we also used Lemma 1. Consequently, we have by dominated convergence

$$(25) \quad V_\gamma g(x, t) = 4^\gamma \Gamma(\gamma) \int_{\mathbb{R}^d} \frac{i^{-2\gamma} 1_{B(x,t)}(y) + 1_{\mathbb{R}^d \setminus \overline{B(x,t)}}(y)}{|t^2 - |x - y|^2|^\gamma} g(y) dy.$$

Proof of Theorem 2. The formula for the solution u follows from (23) and (25) with $\gamma = \frac{d}{2} - \sigma$. \square

In order to calculate $\partial_t \text{Im} V_\beta g(x, t)$ for $0 < \beta < 1$ we argue by change of variables $y' = \frac{y-x}{t}$:

$$\begin{aligned} \partial_t \left(\int_{B(x,t)} \frac{g(y)}{(t^2 - |x - y|^2)^\beta} dy \right) &= \partial_t \left(t^{d-2\beta} \int_{B(0,1)} \frac{g(x + ty')}{(1 - |y'|^2)^\beta} dy' \right) \\ &= t^{d-2\beta-1} \int_{B(0,1)} \frac{(d-2\beta)g(x + ty') + ty' \cdot \nabla g(x + ty')}{(1 - |y'|^2)^\beta} dy' \\ &= \frac{1}{t} \int_{B(x,t)} \frac{(d-2\beta)g(y) + (y-x) \cdot \nabla g(y)}{(t^2 - |x - y|^2)^\beta} dy. \end{aligned}$$

Now (25) implies

$$(26) \quad \partial_t \text{Im} V_\beta g(x, t) = -\sin(\beta\pi) 4^\beta \Gamma(\beta) \frac{1}{t} \int_{B(x,t)} \frac{(d-2\beta)g(y) + (y-x) \cdot \nabla g(y)}{(t^2 - |x - y|^2)^\beta} dy.$$

Proof of Theorem 3. The formula for the solution follows by means of (24) from (23) with $\gamma = \frac{d}{2} - \sigma$ and (26) with $\beta = \gamma - 1$, so that

$$\begin{aligned} u(x, t) &= -\frac{\sigma 4^{\sigma-\frac{d}{2}}}{\pi^{\frac{d}{2}} \sin(\sigma\pi) \Gamma(1-\sigma)} \frac{2}{t} \partial_t \text{Im} V_{\gamma-1} g(x, t) \\ &= 2c_{d,\sigma} \frac{1}{t^2} \int_{B(x,t)} \frac{2(\sigma+1)g(y) + (y-x) \cdot \nabla g(y)}{(t^2 - |x - y|^2)^{\frac{d}{2}-\sigma-1}} dy, \end{aligned}$$

where

$$c_{d,\sigma} = \frac{\sigma 4^{\sigma-\frac{d}{2}} \sin((\gamma-1)\pi) 4^{\gamma-1} \Gamma(\gamma-1)}{\pi^{\frac{d}{2}} \sin(\sigma\pi) \Gamma(1-\sigma)} = \frac{\sigma \sin((\frac{d}{2} - \sigma - 1)\pi) \Gamma(\frac{d}{2} - \sigma - 1)}{4\pi^{\frac{d}{2}} \sin(\sigma\pi) \Gamma(1-\sigma)}.$$

\square

Limiting cases $\gamma \nearrow 1$ and $\gamma \searrow 1$. For the case $\gamma \nearrow 1$, recall first that

$$\text{Im} V_\gamma g(x, t) = -4^\gamma \Gamma(\gamma) \sin(\gamma\pi) \int_{B(x,t)} \frac{g(y)}{(t^2 - |x - y|^2)^\gamma} dy.$$

By the change of variable $z = y - x$ and writing $z' = \frac{tz}{|z|}$, the integral can be written

$$\int_{B(0,t)} \frac{g(x+z)}{(t^2 - |z|^2)^\gamma} dz = \int_{B(0,t)} \frac{g(x+z) - g(x+z')}{(t^2 - |z|^2)^\gamma} dz + \int_{B(0,t)} \frac{g(x+z')}{(t^2 - |z|^2)^\gamma} dz =: I + II.$$

Here

$$|I| \leq \|\nabla g\|_\infty \int_{B(0,t)} \frac{t - |z|}{(t^2 - |z|^2)^\gamma} dz \leq \|\nabla g\|_\infty \int_{B(0,t)} \frac{(t - |z|)^{1-\gamma}}{(t + |z|)^\gamma} dz \lesssim \|\nabla g\|_\infty$$

and, with polar coordinates,

$$II = \int_{|\omega|=1} g(x + t\omega) dS(\omega) \int_0^t \frac{r^{d-1}}{(t^2 - r^2)^\gamma} dr.$$

After the transformation $\rho = r^2/t^2$, the inner integral here will be

$$\frac{t^{d-2\gamma}}{2} \int_0^1 \frac{\rho^{\frac{d}{2}-1}}{(1-\rho)^\gamma} d\rho = \frac{t^{d-2\gamma}}{2} \frac{\Gamma(\frac{d}{2}) \Gamma(1-\gamma)}{\Gamma(\frac{d}{2} + 1 - \gamma)}.$$

Since $\Gamma(1 - \gamma) = \frac{1}{1 - \gamma} + O(1)$ and $\sin(\gamma\pi) = (1 - \gamma)\pi + O((1 - \gamma)^2)$ as $\gamma \rightarrow 1$, we conclude that

$$\operatorname{Im}V_\gamma g(x, t) \rightarrow -4\pi \frac{t^{d-2}}{2} \int_{|\omega|=1} g(x + t\omega) dS(\omega) = -\frac{2\pi}{t} \int_{\partial B(x, t)} g(y) dS(y),$$

as $\gamma \nearrow 1$.

Consider then the case $\gamma \searrow 1$ and begin by recalling that

$$\begin{aligned} \operatorname{Im}V_\gamma g(x, t) &= \frac{2}{t} \partial_t \operatorname{Im}V_{\gamma-1} g(x, t) \\ &= -\frac{2}{t^2} 4^{\gamma-1} \Gamma(\gamma-1) \sin((\gamma-1)\pi) \int_{B(0, t)} \frac{(d-2(\gamma-1))g(x+y) + y \cdot \nabla g(x+y)}{(t^2 - |y|^2)^{\gamma-1}} dy. \end{aligned}$$

Since $\Gamma(\gamma-1) \sin((\gamma-1)\pi) \rightarrow \pi$ as $\gamma \rightarrow 1$,

$$\lim_{\gamma \searrow 1} \operatorname{Im}V_\gamma g(x, t) = -\frac{2\pi}{t^2} \int_{B(0, t)} \left(dg(x+y) + y \cdot \nabla g(x+y) \right) dy.$$

Noting that

$$d = \Delta h \quad \text{and} \quad y = \nabla h \quad \text{for} \quad h(y) = \frac{|y|^2}{2},$$

we make use of Green's formula

$$\int_{B(0, t)} \left((\Delta h)g + \nabla h \cdot \nabla g \right) dy = \int_{\partial B(0, t)} \left(\nabla h \cdot \frac{y}{t} \right) g dS(y)$$

to see that also

$$(27) \quad \lim_{\gamma \searrow 1} \operatorname{Im}V_\gamma g(x, t) = -\frac{2\pi}{t} \int_{\partial B(x, t)} g(y) dS(y).$$

Proof of Theorem 4. Note first that $\gamma = \frac{d}{2} - \sigma \rightarrow 1$ is possible in dimensions $d = 2, 3$ and 4 when $\sigma \searrow 0$, $\sigma \rightarrow \frac{1}{2}$ and $\sigma \nearrow 1$, respectively. Since

$$u_\sigma(x, t) = -\frac{\sigma 4^{\sigma - \frac{d}{2}}}{\pi^{\frac{d}{2}} \sin(\sigma\pi) \Gamma(1 - \sigma)} \operatorname{Im}V_{\frac{d}{2} - \sigma} g(x, t),$$

together with the limits (3) and (27) it suffices to note that

$$\frac{\sigma 4^{\sigma - \frac{d}{2}}}{\pi^{\frac{d}{2}} \sin(\sigma\pi) \Gamma(1 - \sigma)} \rightarrow \begin{cases} \frac{1}{4\pi^2}, & \text{when } d = 2 \text{ and } \sigma \searrow 0, \\ \frac{1}{8\pi^2}, & \text{when } d = 3 \text{ and } \sigma \rightarrow \frac{1}{2}, \\ \frac{1}{4\pi^3}, & \text{when } d = 4 \text{ and } \sigma \nearrow 1. \end{cases}$$

□

Remark. The one-dimensional problem

$$(28) \quad \begin{cases} \partial_t^2 u + \frac{1-2\sigma}{t} \partial_t u = \partial_x^2 u \\ u(\cdot, 0) = 0, \quad \partial_t^\sigma u(\cdot, 0) = g \end{cases}$$

is not covered by Theorem 2 when $\sigma > \frac{1}{2}$. A solution formula can be derived by the method of descent by viewing u and g with an additional spatial variable and using Theorem 2 for $d = 2$. Doing so we obtain

$$u(x, t) = \frac{\sigma t^{2\sigma}}{\pi} \iint_{y_1^2 + y_2^2 < 1} \frac{g(x + ty_1)}{(1 - y_1^2 - y_2^2)^{1-\sigma}} dy_1 dy_2.$$

Here the double integral can be calculated as follows:

$$\int_{-1}^1 \frac{g(x + ty_1)}{(1 - y_1^2)^{1-\sigma}} \int_{y_2^2 < 1 - y_1^2} \left(1 - \frac{y_2^2}{1 - y_1^2} \right)^{\sigma-1} dy_2 dy_1 = \int_{-1}^1 \frac{g(x + ty_1)}{(1 - y_1^2)^{\frac{1}{2}-\sigma}} \int_{-1}^1 (1 - s^2)^{\sigma-1} ds dy_1,$$

where

$$\int_{-1}^1 (1-s^2)^{\sigma-1} ds = \int_0^1 (1-r)^{\sigma-1} r^{-\frac{1}{2}} dr = \frac{\Gamma(\sigma)\Gamma(\frac{1}{2})}{\Gamma(\sigma + \frac{1}{2})}.$$

Therefore

$$u(x, t) = \frac{\sigma\Gamma(\sigma)}{\pi^{\frac{1}{2}}\Gamma(\sigma + \frac{1}{2})} \int_{x-t}^{x+t} \frac{g(y)}{(t^2 - |x-y|^2)^{\frac{1}{2}-\sigma}} dy,$$

which converges to

$$\frac{1}{2} \int_{x-t}^{x+t} g(y) dy,$$

as $\sigma \searrow \frac{1}{2}$. This coincides with the limit as $\sigma \nearrow \frac{1}{2}$.

Remark. In a similar vein one may consider the modified Klein–Gordon equation arising from $L = -\Delta + m^2$ with $m > 0$. The solution to (21) is then given by

$$u(x, t) = -\frac{\sigma 4^{\sigma-\frac{d}{2}}}{\pi^{\frac{d}{2}} \sin(\sigma\pi)\Gamma(1-\sigma)} \operatorname{Im} \left(i^{\sigma-\frac{d}{2}} \int_0^\infty e^{-i\frac{t^2}{4s}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4s} - im^2s} g(y) dy \frac{ds}{s^{1+\frac{d}{2}-\sigma}} \right),$$

where the quantity inside Im coincides with

$$V_\gamma^m g(x, t) = i^{-\gamma} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} \int_\varepsilon^\infty e^{i\frac{|x-y|^2-t^2}{4s} - im^2s} \frac{ds}{s^{1+\gamma}} g(y) dy,$$

with $\gamma = \frac{d}{2} - \sigma$. Assuming that $0 < \gamma < 1$, we may use the formula (cf. Lemma 1)

$$\int_0^\infty e^{-\frac{A}{is} - im^2s} \frac{ds}{s^{1+\gamma}} = i^\gamma \int_0^\infty e^{-\frac{A}{s} - m^2s} \frac{ds}{s^{1+\gamma}}$$

when $A = \frac{|x-y|^2-t^2}{4} > 0$ to deduce finite speed of propagation:

$$u(x, t) = -\frac{\sigma 4^{\sigma-\frac{d}{2}}}{\pi^{\frac{d}{2}} \sin(\sigma\pi)\Gamma(1-\sigma)} \operatorname{Im} \left(i^{\sigma-\frac{d}{2}} \lim_{\varepsilon \rightarrow 0} \int_{B(x,t)} \int_\varepsilon^\infty e^{i\frac{|x-y|^2-t^2}{4s} - im^2s} \frac{ds}{s^{1+\frac{d}{2}-\sigma}} g(y) dy \right).$$

4. A BESSEL FUNCTION APPROACH

In this section we study the wave extension problem for the Laplacian $L = -\Delta$ on \mathbb{R}^d by means of Bessel functions and obtain some elementary estimates for the solutions. In order to simplify the presentation, we make stronger decay and regularity assumptions than necessary.

For a given $\sigma \in (0, 1)$ we consider the problem

$$(29) \quad \begin{cases} \partial_t^2 u + \frac{1-2\sigma}{t} \partial_t u = \Delta u \\ u(\cdot, 0) = f, \quad \partial_t^\sigma u(\cdot, 0) = g, \end{cases}$$

where f and g are Schwartz functions whose Fourier transforms vanish in a neighbourhood of the origin, and $\partial_t^\sigma = \frac{1}{2\sigma} t^{1-2\sigma} \partial_t$.

We interpret the problem as an evolution equation in $\mathcal{L}^2(\mathbb{R}^d)$, and call u a *Schwartz solution* if the map $(0, \infty) \rightarrow \mathcal{L}^2(\mathbb{R}^d) : t \mapsto u(\cdot, t)$ is C^2 (in the norm topology), and $u(\cdot, t)$, $\partial_t u(\cdot, t)$, and $\partial_t^\sigma u(\cdot, t)$ are Schwartz functions for every $t > 0$. The Fourier transform of such a solution u then satisfies for every $\xi \in \mathbb{R}^d$ the equation

$$(30) \quad \partial_t^2 \widehat{u}(\xi, t) + \frac{1-2\sigma}{t} \partial_t \widehat{u}(\xi, t) = -|\xi|^2 \widehat{u}(\xi, t), \quad t > 0.$$

Writing $\widehat{u}(\xi, t) = (t|\xi|)^\sigma w(t|\xi|)$ we get for w

$$r^2 w''(r) + r w'(r) + (r^2 - \sigma^2) w(r) = 0,$$

which is Bessel's differential equation. Thus two linearly independent solutions to (30) are $\widehat{u}(\xi, t) = (t|\xi|)^\sigma J_{\pm\sigma}(t|\xi|)$; see [Leb65, Sections 5.1 and 5.3].

Let us then find the coefficients that give convergence to the correct initial values. The Bessel functions satisfy for $0 < \sigma < 1$ and $r > 0$

$$J_\sigma(r) = \frac{1}{\Gamma(1+\sigma)} \left(\frac{r}{2}\right)^\sigma + O(r^{\sigma+2}) \quad \text{and} \quad J_{-\sigma}(r) = \frac{1}{\Gamma(1-\sigma)} \left(\frac{r}{2}\right)^{-\sigma} + O(r^{-\sigma+2})$$

as $r \searrow 0$, as seen from the power series expansion. Further,

$$\frac{d}{dr}(r^\sigma J_{\pm\sigma}(r)) = \pm r^\sigma J_{\pm(\sigma-1)}(r);$$

see [Leb65, Formula (5.3.5)]. This implies

$$(t|\xi|)^\sigma J_\sigma(t|\xi|) \longrightarrow 0 \quad \text{and} \quad (t|\xi|)^\sigma J_{-\sigma}(t|\xi|) \longrightarrow \frac{2^\sigma}{\Gamma(1-\sigma)}$$

as $t \rightarrow 0$. Moreover,

$$\partial_t^\sigma \left((t|\xi|)^\sigma J_\sigma(t|\xi|) \right) \longrightarrow \frac{|\xi|^{2\sigma}}{\sigma 2^\sigma \Gamma(\sigma)} \quad \text{and} \quad \partial_t^\sigma \left((t|\xi|)^\sigma J_{-\sigma}(t|\xi|) \right) \longrightarrow 0$$

as $t \rightarrow 0$.

Therefore the solution to (30) given by

$$(31) \quad \widehat{u}(\xi, t) = \frac{\Gamma(1-\sigma)}{2^\sigma} (t|\xi|)^\sigma J_{-\sigma}(t|\xi|) \widehat{f}(\xi) + \sigma 2^\sigma \Gamma(\sigma) (t|\xi|)^\sigma J_\sigma(t|\xi|) |\xi|^{-2\sigma} \widehat{g}(\xi)$$

satisfies

$$\widehat{u}(\xi, t) \longrightarrow \widehat{f}(\xi) \quad \text{and} \quad \partial_t^\sigma \widehat{u}(\xi, t) \longrightarrow \widehat{g}(\xi) \quad \text{as} \quad t \rightarrow 0.$$

Since \widehat{f} and \widehat{g} were assumed to vanish in a neighbourhood of the origin, the next result follows immediately by dominated convergence:

Theorem 5. *Let $0 < \sigma < 1$. The unique Schwartz solution u to the problem*

$$\begin{cases} \partial_t^2 u + \frac{1-2\sigma}{t} \partial_t u = \Delta u \\ u(\cdot, 0) = f, \quad \partial_t^\sigma u(\cdot, 0) = g, \end{cases}$$

with Schwartz initial data f and g whose Fourier transforms vanish in a neighbourhood of the origin, is given by

$$u(\cdot, t) = \frac{\Gamma(1-\sigma)}{2^\sigma} (t\sqrt{-\Delta})^\sigma J_{-\sigma}(t\sqrt{-\Delta}) f + \sigma 2^\sigma \Gamma(\sigma) (t\sqrt{-\Delta})^\sigma J_\sigma(t\sqrt{-\Delta}) (-\Delta)^{-\sigma} g,$$

which converges to the initial data in $\mathcal{L}^2(\mathbb{R}^d)$.

Remark. The same method can be used to prove uniqueness of solutions also in other settings, say for non-negative self-adjoint operators L with discrete spectrum (such as the Hermite operator $L = -\Delta + |x|^2$ on \mathbb{R}^d), by replacing $|\xi| = \sqrt{\lambda}$. In such a case the solution u of

$$\begin{cases} \partial_t^2 u + \frac{1-2\sigma}{t} \partial_t u = -Lu \\ u(\cdot, 0) = f, \quad \partial_t^\sigma u(\cdot, 0) = g, \end{cases}$$

is unique under the assumption that u , f and g all consist of a finite number of eigenfunctions.

Fixed-time norm estimates. For any order of smoothness $s \in \mathbb{R}$ we define the Sobolev norm of a Schwartz function f on \mathbb{R}^d by

$$\|f\|_{2,s} = \left(\int_{\mathbb{R}^d} |\langle \xi \rangle^s \widehat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}},$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$. As a consequence of Theorem 5, we derive the following fixed-time norm estimates for solutions u in the spirit of [Str70, Theorem 2]:

Theorem 6. *Let $s \geq 0$ and let u , f and g be as in Theorem 5.*

- *If $\sigma \leq \frac{1}{2}$ then $\|u(\cdot, t)\|_{2,s} \lesssim \|f\|_{2,s} + t^{2\sigma} \|g\|_2 + \|g\|_{2,s-2\sigma}$ for all $t > 0$.*

- If $\sigma \geq \frac{1}{2}$ then $\|u(\cdot, t)\|_{2,s} \lesssim (1+t)^{\sigma-\frac{1}{2}} \|f\|_{2,s+\sigma-\frac{1}{2}} + t^{2\sigma} \|g\|_2 + t^{\sigma-\frac{1}{2}} \|g\|_{2,s-\sigma-\frac{1}{2}}$ for all $t > 0$.

Proof. Let χ be a frequency cut-off with $1_{B(0,1)} \leq \chi \leq 1_{B(0,2)}$ and recall the standard estimates for Bessel functions:

$$(32) \quad |J_{\pm\sigma}(r)| \lesssim \begin{cases} r^{\pm\sigma}, & 0 < r < 1, \\ r^{-\frac{1}{2}}, & r \geq 1. \end{cases}$$

Consider first the case $0 < \sigma \leq \frac{1}{2}$. By (32) we have

$$(t|\xi|)^\sigma |J_{-\sigma}(t|\xi|)| \lesssim 1$$

and therefore

$$\|(t\sqrt{-\Delta})^\sigma J_{-\sigma}(t\sqrt{-\Delta})f\|_{2,s} \lesssim \|f\|_{2,s}.$$

On the other hand, according to (32) we have $(t|\xi|)^\sigma |J_\sigma(t|\xi|)| \lesssim \min\{(t|\xi|)^{2\sigma}, 1\}$, which implies that

$$\begin{aligned} \langle \xi \rangle^s (t|\xi|)^\sigma |J_\sigma(t|\xi|)| |\xi|^{-2\sigma} &\lesssim \chi(\xi) \langle \xi \rangle^s (t|\xi|)^{2\sigma} |\xi|^{-2\sigma} + (1 - \chi(\xi)) \langle \xi \rangle^s |\xi|^{-2\sigma} \\ &\lesssim t^{2\sigma} \chi(\xi) + (1 - \chi(\xi)) \langle \xi \rangle^{s-2\sigma}, \end{aligned}$$

and thus

$$\|(t\sqrt{-\Delta})^\sigma J_\sigma(t\sqrt{-\Delta})(-\Delta)^{-\sigma}g\|_{2,s} \lesssim t^{2\sigma} \|g\|_2 + \|g\|_{2,s-2\sigma}.$$

Assume then that $\frac{1}{2} \leq \sigma < 1$. By (32) we have $(t|\xi|)^\sigma |J_{-\sigma}(t|\xi|)| \lesssim \max\{1, (t|\xi|)^{\sigma-\frac{1}{2}}\}$ so that

$$\langle \xi \rangle^s (t|\xi|)^\sigma |J_{-\sigma}(t|\xi|)| \lesssim \chi(t\xi) \langle \xi \rangle^s + (1 - \chi(t\xi)) \langle \xi \rangle^s (t|\xi|)^{\sigma-\frac{1}{2}},$$

where $\langle \xi \rangle^s (t|\xi|)^{\sigma-\frac{1}{2}} \leq t^{\sigma-\frac{1}{2}} \langle \xi \rangle^{s+\sigma-\frac{1}{2}}$. Consequently,

$$\|(t\sqrt{-\Delta})^\sigma J_{-\sigma}(t\sqrt{-\Delta})f\|_{2,s} \lesssim \|f\|_{2,s} + t^{\sigma-\frac{1}{2}} \|f\|_{2,s+\sigma-\frac{1}{2}} \lesssim (1+t)^{\sigma-\frac{1}{2}} \|f\|_{2,s+\sigma-\frac{1}{2}}.$$

Moreover, from (32) we see that $(t|\xi|)^\sigma |J_\sigma(t|\xi|)| \lesssim \min\{(t|\xi|)^{2\sigma}, (t|\xi|)^{\sigma-\frac{1}{2}}\}$ and so

$$\begin{aligned} \langle \xi \rangle^s (t|\xi|)^\sigma |J_\sigma(t|\xi|)| |\xi|^{-2\sigma} &\lesssim \chi(\xi) \langle \xi \rangle^s (t|\xi|)^{2\sigma} |\xi|^{-2\sigma} + (1 - \chi(\xi)) (t|\xi|)^{\sigma-\frac{1}{2}} \langle \xi \rangle^s |\xi|^{-2\sigma} \\ &\lesssim t^{2\sigma} \chi(\xi) + (1 - \chi(\xi)) t^{\sigma-\frac{1}{2}} \langle \xi \rangle^{s-\sigma-\frac{1}{2}}, \end{aligned}$$

from which infer that

$$\|(t\sqrt{-\Delta})^\sigma J_\sigma(t\sqrt{-\Delta})(-\Delta)^{-\sigma}g\|_{2,s} \lesssim t^{2\sigma} \|g\|_2 + t^{\sigma-\frac{1}{2}} \|g\|_{2,s-\sigma-\frac{1}{2}}.$$

□

An integral formula. We finish by deriving the oscillatory subordination formula of Theorem 1 from a classical integral representation for modified Bessel functions

$$K_\sigma(z) = \frac{\pi I_{-\sigma}(z) - I_\sigma(z)}{2 \sin(\sigma\pi)}, \quad |\arg z| < \pi,$$

(see [Leb65, Section 5.7]). Noting that

$$K_\sigma(ir) = \frac{\pi i^{-\sigma} J_{-\sigma}(r) - i^\sigma J_\sigma(r)}{2 \sin(\sigma\pi)}, \quad r > 0,$$

we see that the solution u in Theorem 5 for real initial data can be written as

$$\begin{aligned} u(\cdot, t) &= \frac{2}{\pi} \frac{\Gamma(1-\sigma)}{2^\sigma} \operatorname{Re} \left(i^{1-2\sigma} (it\sqrt{-\Delta})^\sigma K_\sigma(it\sqrt{-\Delta})f \right) \\ &\quad - \frac{2}{\pi} \sigma 2^\sigma \Gamma(\sigma) \operatorname{Im} \left((it\sqrt{-\Delta})^\sigma K_\sigma(it\sqrt{-\Delta})(-\Delta)^{-\sigma}g \right). \end{aligned}$$

Proposition 2. Let $0 < \sigma < 1$. We have

$$(it\sqrt{-\Delta})^\sigma K_\sigma(it\sqrt{-\Delta})f = \frac{i^\sigma t^{2\sigma}}{2^{1+\sigma}} \int_0^\infty e^{-i\frac{t^2}{4s}} e^{is\Delta} f \frac{ds}{s^{1+\sigma}}$$

for Schwartz functions f .

Proof. We first show that

$$K_\sigma(ir) = \frac{r^\sigma}{2^{1+\sigma}} \int_0^\infty e^{-i\frac{r^2}{4s}} e^{-is} \frac{ds}{s^{1+\sigma}}, \quad r > 0.$$

By [Leb65, Eq. (5.10.25)] we have the integral representation

$$K_\sigma(z) = \frac{z^\sigma}{2^{1+\sigma}} \int_0^\infty e^{-\frac{z^2}{4s}} e^{-s} \frac{ds}{s^{1+\sigma}}, \quad |\arg z| < \frac{\pi}{4}.$$

Note that when $|\arg z| > \frac{\pi}{4}$, this integral diverges at zero because $\operatorname{Re} e^{-\frac{z^2}{4s}} \gtrsim 1$.

If we introduce the bianalytic function

$$(33) \quad F(z, w) = e^{-\frac{z^2}{4w}} e^{-w} w^{-1-\sigma}, \quad |\arg z| < \frac{\pi}{2}, \quad |\arg w| < \pi,$$

the integral above can be written $\int_0^\infty F(z, s) ds$.

Integrating instead $w \mapsto F(z, w)$ along the imaginary half-axis, we obtain the analytic function

$$z \mapsto i^{-\sigma} \int_0^\infty e^{i\frac{z^2}{4s}} e^{-is} \frac{ds}{s^{1+\sigma}}, \quad 0 < \arg z < \frac{\pi}{2}.$$

Indeed, for such z the integral is absolutely convergent since $|e^{i\frac{z^2}{4s}}| = e^{-c\frac{|z|^2}{4s}}$, where $c = \sin(2 \arg z) > 0$. We claim that

$$(34) \quad \int_0^\infty F(z, s) ds = i \int_0^\infty F(z, is) ds, \quad 0 < \arg z < \frac{\pi}{4}.$$

To see this, we let $\varepsilon > 0$ and $R < \infty$, and use Cauchy's integral theorem to write

$$\int_\varepsilon^R F(z, s) ds = \left(\int_{\operatorname{arc}_\varepsilon} - \int_{\operatorname{arc}_R} \right) F(z, w) dw + i \int_\varepsilon^R F(z, is) ds,$$

where $\operatorname{arc}_\varepsilon$ and arc_R are the paths

$$\theta \mapsto \varepsilon e^{i\theta} \quad \text{and} \quad \theta \mapsto R e^{i\theta}, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

Then

$$\int_{\operatorname{arc}_\varepsilon} F(z, w) dw = i \int_0^{\frac{\pi}{2}} e^{-\frac{z^2}{4\varepsilon}} e^{-i\theta} e^{-\varepsilon e^{i\theta}} \varepsilon^{-\sigma} e^{-i\sigma\theta} d\theta,$$

and with $\phi = \arg z \in (0, \frac{\pi}{4})$ we have

$$\left| e^{-\frac{z^2}{4\varepsilon}} e^{-i\theta} \right| = e^{-|z|^2 \cos(2\phi-\theta)/4\varepsilon} = e^{-|z|^2 \sin(\frac{\pi}{2}-2\phi+\theta)/4\varepsilon}.$$

Since $\sin \alpha > c \min(\alpha, \pi - \alpha)$ for $0 < \alpha < \pi$, it follows that

$$\sin\left(\frac{\pi}{2} - 2\phi + \theta\right) > c \min(\theta, \pi/2 - \theta).$$

As a consequence,

$$\left| \int_{\operatorname{arc}_\varepsilon} F(z, w) dw \right| \lesssim \varepsilon^{-\sigma} \int_0^{\frac{\pi}{2}} e^{-\frac{c|z|^2 \min(\theta, \pi/2-\theta)}{4\varepsilon}} d\theta \lesssim \varepsilon^{1-\sigma} \longrightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Likewise,

$$\int_{\operatorname{arc}_R} F(z, w) dw = i \int_0^{\frac{\pi}{2}} e^{-\frac{z^2}{4R}} e^{-i\theta} e^{-R e^{i\theta}} R^{-\sigma} e^{-i\sigma\theta} d\theta \longrightarrow 0, \quad \text{as } R \rightarrow \infty,$$

and (34) follows.

Now since the functions

$$z \mapsto \frac{z^\sigma}{2^{1+\sigma}} i \int_0^\infty F(z, is) ds$$

and $K_\sigma(z)$ are both analytic for $0 < \arg z < \frac{\pi}{2}$ and agree for $0 < \arg z < \frac{\pi}{4}$ because of (34), they must coincide, so that

$$K_\sigma(z) = \frac{i^{-\sigma} z^\sigma}{2^{1+\sigma}} \int_0^\infty e^{i\frac{z^2}{4s}} e^{-is} \frac{ds}{s^{1+\sigma}}, \quad 0 < \arg z < \frac{\pi}{2}.$$

Finally,

$$K_\sigma(ir) = \lim_{\delta \rightarrow 0} K_\sigma(ir + \delta) = \frac{r^\sigma}{2^{1+\sigma}} \lim_{\delta \rightarrow 0} \int_0^\infty e^{i\frac{(ir+\delta)^2}{4s}} e^{-is} \frac{ds}{s^{1+\sigma}} = \frac{r^\sigma}{2^{1+\sigma}} \lim_{\varepsilon, R} \int_\varepsilon^R e^{-i\frac{r^2}{4s}} e^{-is} \frac{ds}{s^{1+\sigma}},$$

where the swapping of the limits in the last equality is justified since the arguments above show that the limit in ε and R is uniform in δ .

The proposition now follows by Fourier transform from the identity

$$(it|\xi|)^\sigma K_\sigma(it|\xi|) = \frac{i^\sigma (t|\xi|)^{2\sigma}}{2^{1+\sigma}} \lim_{\varepsilon, R} \int_\varepsilon^R e^{-i\frac{t^2}{4s}|\xi|^2} e^{-is} \frac{ds}{s^{1+\sigma}} = \frac{i^\sigma t^{2\sigma}}{2^{1+\sigma}} \lim_{\varepsilon, R} \int_\varepsilon^R e^{-i\frac{t^2}{4s}} e^{-is|\xi|^2} \frac{ds}{s^{1+\sigma}}.$$

Indeed, after a change of variables $s' = \frac{t^2}{4s}$ the last integral is amenable to Lemma 2 which gives

$$\left| \int_\varepsilon^R e^{-i\frac{t^2}{4s}} e^{-is|\xi|^2} \frac{ds}{s^{1+\sigma}} \right| \lesssim \begin{cases} 1, & \sigma \leq \frac{1}{2}, \\ 1 + |\xi|^{\sigma - \frac{1}{2}}, & \sigma > \frac{1}{2}. \end{cases}$$

The functions

$$\xi \mapsto \frac{i^\sigma t^{2\sigma}}{2^{1+\sigma}} \int_\varepsilon^R e^{-i\frac{t^2}{4s}} e^{-is|\xi|^2} \widehat{f}(\xi) \frac{ds}{s^{1+\sigma}}$$

therefore converge in $\mathcal{L}^2(\mathbb{R}^d)$, as $\varepsilon \rightarrow 0$ and $R \rightarrow \infty$, which justifies the exchange of limits with the Fourier transform. \square

Remark. It is clear that Proposition 2 holds for any non-negative self-adjoint operator L in place of $-\Delta$ when f is suitably chosen (cf. Theorem 1). Moreover, recalling that

$$\sqrt{ir} K_{1/2}(ir) = \sqrt{\frac{\pi}{2}} e^{-ir}, \quad r > 0,$$

we see that in the classical case $\sigma = \frac{1}{2}$ the formula reads

$$e^{-it\sqrt{L}} f = \frac{\sqrt{it}}{2\sqrt{\pi}} \int_0^\infty e^{-i\frac{t^2}{4s}} e^{-isL} f \frac{ds}{s^{3/2}}.$$

REFERENCES

- [CS07] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.
- [Eva98] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.
- [GMS13] J. E. Galé, P. J. Miana, and P. R. Stinga. Extension problem and fractional operators: semigroups and wave equations. *J. Evol. Equ.*, 13(2):343–368, 2013.
- [Leb65] N. N. Lebedev. *Special functions and their applications*. Revised English edition. Translated and edited by Richard A. Silverman. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965.
- [ST10] P. R. Stinga and J. L. Torrea. Extension problem and Harnack's inequality for some fractional operators. *Comm. Partial Differential Equations*, 35(11):2092–2122, 2010.
- [Str70] R. S. Strichartz. Convolutions with kernels having singularities on a sphere. *Trans. Amer. Math. Soc.*, 148:461–471, 1970.

(M. Kemppainen) DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID, 28049 MADRID, SPAIN

Current address: Department of Mathematics and Statistics, University of Helsinki, FI-00014 Helsinki, Finland

E-mail address: `mikko.k.kemppainen@helsinki.fi`

(P. Sjögren) MATHEMATICAL SCIENCES, UNIVERSITY OF GOTHENBURG AND MATHEMATICAL SCIENCES, CHALMERS, SE-412 96 GÖTEBORG, SWEDEN

E-mail address: `peters@chalmers.se`

(J. L. Torrea) DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID AND ICMAT, 28049 MADRID, SPAIN

E-mail address: `joseluis.torrea@uam.es`