The Gindikin-Karpelevich Formula and Constant Terms of Eisenstein Series for Brylinski-Deligne Extensions

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Declaration

I hereby declare that the thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

Fan Gao
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Notations and Terminology

F: a number field or a local field with finite residue field of size q in the nonarchimedean case.

Frob or Frob_v: the geometric Frobenius class of a local field.

I or I_v: the inertia group of the absolute Galois group of a local field.

$\mathcal{O}_F$: the ring of integers of F.

$\mathbb{G}_{\mathrm{add}}$ and $\mathbb{G}_{\mathrm{mul}}$: the additive and multiplicative group over F respectively.

$\mathbb{G}$: a general split reductive group (over F) with root datum $(X, \Psi, Y, \Psi^\vee)$. We fix a set positive roots $\Psi^+ \subseteq \Psi$ and thus also a set of simple roots $\Delta \subseteq \Psi$. Let $\mathbb{G}^{sc}$ be the simply connected cover of the derived subgroup $\mathbb{G}^{sc}$ of $\mathbb{G}$ with the natural map denoted by $\Phi : \mathbb{G}^{sc} \longrightarrow \mathbb{G}$

We fix a Borel subgroup $\mathbb{B} = \mathbb{T}\mathbb{U}$ of $\mathbb{G}$ and also a Chevalley system of épinglage for $(\mathbb{G}, \mathbb{T}, \mathbb{B})$ (cf. [BrTi84, §3.2.1-2]), from which we have an isomorphism $\mathbb{e}_\alpha : \mathbb{G}_{\mathrm{add}} \to \mathbb{U}_\alpha$ for each $\alpha \in \Psi$ with associated root subgroup $\mathbb{U}_\alpha$. Moreover, for each $\alpha \in \Psi$, there is the induced morphism $\varphi_\alpha : \mathbb{SL}_2 \longrightarrow \mathbb{G}$ which restricts to $\mathbb{e}_{\pm\alpha}$ on the upper and lower triangular subgroup of unipotent matrices of $\mathbb{SL}_2$.

$\mathbb{T}$: a maximally split torus of $\mathbb{G}$ with character group X and cocharacter group Y.

Q: an integer-valued Weyl-invariant quadratic form on Y with associated symmetric bilinear form

$$B_Q(y_1, y_2) := Q(y_1 + y_2) - Q(y_1) - Q(y_2).$$

In general, notations will be explained the first time they appear in the text.

"character": by a *character* of a group we just mean a continuous homomorphism valued in $\mathbf{C}^\times$, while a *unitary character* refers to a character with absolute value 1.

"section" and "splitting": for an exact sequence $A \hookrightarrow B \twoheadrightarrow C$ of groups we call any map $s : C \longrightarrow B$ a *section* if its post composition with the last projection map on C is the identity map on C. We call s a *splitting* if it is a homomorphism, and write $\mathfrak{S}(B, C)$ for all splittings of B over C, which is a torsor over $\mathrm{Hom}(C, A)$ when the extension is central.

"push-out": for a group extension $A \hookrightarrow B \twoheadrightarrow C$ and a homomorphism $f : A \to A'$ whose image is a normal subgroup of A', the push-out f_*B (as a group extension of

C by A') is given by

$$f_*B := \frac{A' \times B}{\langle (f(a), i^{-1}(a)) : a \in A \rangle},$$

whenever it is well-defined. Here $i : A \hookrightarrow B$ is the inclusion in the extension. For example, if f is trivial or both i and f are central, i.e. the image of the map lies in the center of B and A' respectively, then f_*B is well-defined.

"pull-back": for a group extension $A \hookrightarrow B \twoheadrightarrow C$ and a homomorphism $h : C' \to C$, the pull-back h^*B is the group

$$h^*B := \big\{ (b, c') : q(b) = h(c') \big\} \subseteq B \times C',$$

where $q : B \twoheadrightarrow C$ is the quotient map. The group h^*B is an extension of C' by A.

Summary

We work in the framework of the Brylinski-Deligne (BD) central covers of general split reductive groups. To facilitate the computation, we use an incarnation category initially given by M. Weissman which is equivalent to that of Brylinski-Deligne.

Let F be a number field containing n-th root of unity, and let v be an arbitrary place of F. The objects of main interest will be the topological covering groups of finite degree arising from the BD framework, which are denoted by $\overline{G}_v$ and $\overline{\mathbb{G}}(\mathbb{A}_F)$ in the local and global situations respectively. The aim of the dissertation is to compute the Gindikin-Karpelevich (GK) coefficient which appears in the intertwining operator for global induced representations from parabolic subgroups $\overline{\mathbb{P}}(\mathbb{A}_F) = \overline{\mathbb{M}}(\mathbb{A}_F)\mathbb{U}(\mathbb{A}_F)$ of general BD-type covering groups $\overline{\mathbb{G}}(\mathbb{A}_F)$. The result is expressed in terms of naturally defined elements without assuming $\mu_{2n} \subseteq F^\times$, and thus could be considered as a refinement of that given by McNamara etc.

Moreover, using the construction of the L-group ${}^L\overline{G}$ by Weissman for the global covering $\overline{\mathbb{G}}(\mathbb{A}_F)$, we define partial automorphic L-functions for covers $\overline{\mathbb{G}}(\mathbb{A}_F)$ of BD type. We show that the GK coefficient computed can be interpreted as Langlands-Shahidi type partial L-functions associated to the adjoint representation of ${}^L\overline{M}$ on a certain subspace $\overline{\mathfrak{u}}^\vee \subset \overline{\mathfrak{g}}^\vee$ of the Lie algebra of ${}^L\overline{G}$. Consequently, we are able to express the constant term of Eisenstein series of BD covers, which relies on the induction from parabolic subgroups as above, in terms of certain partial L-functions of Langlands-Shahidi type.

The interpretation relies crucially on the local consideration. Therefore, along the way, we discuss properties of the local L-group ${}^L\overline{G}_v$ for $\overline{G}_v$, which by the construction of Weissman sits in an exact sequence $\overline{G}^\vee \hookrightarrow {}^L\overline{G}_v \twoheadrightarrow \mathrm{W}_{F_v}$. For instance, in general ${}^L\overline{G}_v$ is not isomorphic to the direct product $\overline{G}^\vee \times \mathrm{W}_{F_v}$ of the complex dual group $\overline{G}^\vee$ and the Weil group W_{F_v}. There is a close link between splittings of ${}^L\overline{G}_v$ over W_{F_v} which realize such a direct product and Weyl-group invariant genuine characters of the center $Z(\overline{T}_v)$ of the covering torus $\overline{T}_v$ of $\overline{G}_v$. In particular, for $\overline{G}_v$ a cover of a simply-connected group there always exist Weyl-invariant genuine characters of $Z(\overline{T}_v)$. We give a construction for general BD coverings with certain constraints. In the case of BD coverings of simply-laced simply-connected groups, our construction agrees with that given by G. Savin. It also agrees with the classical double cover $\overline{\mathbb{Sp}}_{2r}(F_v)$ of $\mathbb{Sp}_{2r}(F_v)$. Moreover, the discussion for the splitting of ${}^L\overline{G}_v$ in the local situation could be carried over parallel for the global ${}^L\overline{G}$ as well.

In the end, for illustration purpose we determine the residual spectrum of general BD coverings of $\mathbb{SL}_2(\mathbb{A}_F)$ and $\mathbb{GL}_2(\mathbb{A}_F)$. In the case of the classical double cover $\overline{\mathbb{Sp}}_4(\mathbb{A}_F)$ of $\mathbb{Sp}_4(\mathbb{A}_F)$, it is also shown that the partial Langlands-Shahidi type L-functions obtained here agree with what we computed before in another work, where the residual spectrum for $\overline{\mathbb{Sp}}_4(\mathbb{A}_F)$ is determined completely.

Contents

Chapter 1

Introduction

It has been one of the central themes in the theory of automorphic forms for a *split* reductive group $\mathbb{G}$ to determine completely the spectral decomposition of $L^2(\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A}_F))$, where F is a number field (or in general a global field) and $\mathbb{A}_F$ its adele ring. In rough terms, the space $L^2(\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A}_F))$ carries a $\mathbb{G}(\mathbb{A}_F)$ action and is endowed with a representation of the group, thus the notion of automorphic representations as its constituents. It is important to be able to construct automorphic representations. Moreover, one would like to give arithmetic parametrization of such automorphic representations. All these are integrated in the enterprise of the Langlands program, which has successfully weaved different disciplines of mathematics together and proved to be a cornerstone of modern number theory (cf. [Gel84], [BaKn97]).

The profound theory of Eisenstein series as developed by Langlands in [Lan71] is a fundamental tool for the study of the above problem regarding the spectral decomposition of $\mathbb{G}(\mathbb{A}_F)$. It enables us to answer part of the above question and provides an inductive machinery that reduces the question to the understanding of the subset of so-called cuspidal automorphic representations. More precisely, using Eisenstein series, the continuous and residual spectrum in the spectral decomposition of $L^2(\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A}_F))$ could be understood in terms of cuspidal representations of Levi subgroups $\mathbb{M}$ of $\mathbb{G}$.

The residual spectrum arises from taking residues of Eisenstein series. In this way, L-functions appear naturally in determining the residual spectrum, as observed by Langlands ([Lan71]). The poles of such L-functions, which are further determined by the inducing cuspidal representation on the Levi, give precise information on the location and space of such desired residues. It is in this sense that L-functions play an essential role in determining the residual spectrum of $\mathbb{G}(\mathbb{A}_F)$. The properties of such L-functions could also in turn be derived from those of the Eisenstein series formed, e.g. meromorphic continuation and crude functional equation. The theory is developed and completed to some extent by various mathematicians, notably Langlands and Shahidi, and thus bears the name Langlands-Shahidi method (cf. [CKM04], [Sha10]).

Moreover, to determine completely the residual spectrum, there are local considerations and thus a good understanding of local representation theory is necessary. Such interplay between global and local problems is not surprising at all. It should be mentioned that for the parametrization problem, J. Arthur (cf. [Art89]) has proposed a conjectural classification of $L^2(\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A}_F))$, which could be viewed as a refined Langlands parametrization for automorphic representation in the view of the spectral decomposition. The conjecture could also be formulated for the double covering of $\mathbb{S}\mathbb{p}_{2r}(\mathbb{A}_F)$ as in [GGP13].

From the spectral theory of automorphic forms for linear algebraic groups, it is natural to wonder about what could be an analogous theory for covering groups. To start with, we will concentrate on the Brylinski-Deligne type coverings $\overline{\mathbb{G}}$ (cf. [BD01]) of general reductive group $\mathbb{G}$ and determine the Langlands-Shahidi type partial L-functions which appear naturally in the course of determining the residues of Eisenstein series.

1.1 Covering groups and L-groups

Covering groups of linear algebraic groups, especially those of algebraic nature, arise naturally. For example, the beautiful construction of Steinberg (cf. [Ste62]) dated back to 1962 gives a simple description of the universal coverings of certain simply-connected groups. Since then, there have been investigations of covering groups by many mathematicians such as Moore, Matsumoto and Deligne, to mention a few. Connections with arithmetic have been discovered and developed. There have also been close relations between automorphic forms on covering groups and those on linear groups since the seminal paper of Shimura ([Shi73]), which concerns automorphic representation of the double cover $\overline{\mathbb{S}\mathbb{p}}_2(\mathbb{A}_F)$ of $\mathbb{S}\mathbb{p}_2(\mathbb{A}_F)$.

In some sense, these could be viewed as efforts to establish a Langlands program for covering groups. As for more examples, we can mention the works by Flicker-Kazhdan (cf.[FlKa86]), Kazhdan-Patterson ([KaPa84], [KaPa86]), Savin ([Sav04]) and many others. Such works, despite their success in treating aspects of the theory, usually focus on particular coverings rather than a general theory.

However, recently Brylinski-Deligne has developed quite a general theory of covering groups of algebraic nature in their influential paper [BD01]. In particular, they classified multiplicative $\mathbb{K}_2$-torsors $\overline{\mathbb{G}}$ (equivalently in another language, central extensions by $\mathbb{K}_2$) over an algebraic group $\mathbb{G}$ in the Zariski site of Spec(F):

$$\mathbb{K}_2 \hookrightarrow \overline{\mathbb{G}} \twoheadrightarrow \mathbb{G}\,.$$

The extension has kernel the sheaf $\mathbb{K}_2$ defined by Quillen. In fact, they actually work over general schemes and not necessarily Spec(F), but for our purpose we take this more restrictive consideration.

There are two features among others which make the Brylinski-Deligne extension distinct:

1. The classification of the $\mathbb{K}_2$-torsors above is functorial in terms of combinatorial data. Thus, it could be viewed as a generalization of the classification of connected reductive group by root data.

2. The category is encompassing. From $\overline{\mathbb{G}}$, we obtain the local topological extension $\overline{G}_v := \overline{\mathbb{G}}(F_v)$
$$\mu_n \hookrightarrow \overline{G}_v \twoheadrightarrow \mathbb{G}(F_v)$$
as well as global
$$\mu_n \hookrightarrow \overline{\mathbb{G}}(\mathbb{A}_F) \twoheadrightarrow \mathbb{G}(\mathbb{A}_F)\ ,$$
where we have assumed $\mu_n \subseteq F$. Though topological coverings which arise in this way do not exhaust all existing ones, such Brylinski-Deligne type does contain all classically interesting examples which are of concern to us. We could mention for example coverings for split and simply-connected $\mathbb{G}$ and the Kazhdan-Patterson type extension for $\mathbb{G} = \mathbb{GL}_n$ (cf. [GaG14, §13.2]).

In the arithmetic classification of automorphic representations and local representations in terms of Galois representations, and more generally in the formation of Langlands functoriality which has been established for several cases, a crucial role is played by the L-group ${}^L\mathbb{G}$ of $\mathbb{G}$. However, the construction of L-group classically is restricted only to connected reductive linear algebraic groups (cf. [Bor79]).

Due to the algebraic nature of BD extensions, it is expected that the theory of automorphic forms and representations of such coverings could be developed in line with the linear algebraic case. For this purpose, a global L-group ${}^L\overline{G}$ and more importantly for our purpose its local analog ${}^L\overline{G}_v$ are indispensable. The latter should fit in the exact sequence
$$\overline{G}^\vee \hookrightarrow {}^L\overline{G}_v \twoheadrightarrow \mathrm{W}_{F_v}\ ,$$
where $\overline{G}^\vee$ is the pinned complex dual group of $\overline{G}_v$ and W_{F_v} the Weil group (cf. [Tat79]) of F_v.

There has already been a series of work in this direction starting with P. McNamara and M. Weissman (cf. [McN12], [We09], [We13], [We14]). In the geometric setting, one may refer to the work of Reich ([Re11]) and Finkelberg-Lysenko ([FiLy10]). In particular, McNamara gave the definition of the root data of $\overline{G}^\vee$ in order to interpret the established Satake isomorphism for $\overline{G}_v$. The root data of $\overline{G}^\vee$ rely on the degree n and the root data of $\mathbb{G}$, modified using the combinatorics associated with $\overline{\mathbb{G}}$ in the BD classification. Therefore it is independent of the place $v \in |F|$, and this justifies the absence of v in the notation $\overline{G}^\vee$ we use.

Since we assume $\mathbb{G}$ split, it is inclined to take ${}^L\overline{G}_v$ to be just the product of $\overline{G}^\vee \times \mathrm{W}_{F_v}$. However, Weissman firstly realized that such approach could be insufficient, especially in view of the role that ${}^L\overline{G}_v$ should play in the parametrization of genuine representations of $\overline{G}_v$.

In his Crelle's paper [We13], Weissman gave a construction of the L-group ${}^L\overline{G}_v$ of certain BD covers using the language of Hopf algebras. Later in a letter to Deligne ([We12]), he gave a simple construction for all BD covers utilizing the combinatorial data associated with the $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}$. In a recent work [We14], the construction is realized for covering of not necessarily split $\mathbb{G}$.

The insight of [We13] is that an L-parameter is just a splitting of ${}^L\overline{G}_v$. More generally, a Weil-Deligne parameter is just a continuous group homomorphism $\mathrm{WD}_{F_v} \longrightarrow {}^L\overline{G}_v$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{WD}_{F_v} \arrow[r] \arrow[dr] & {}^L\overline{G}_v \arrow[d] \\ & \mathrm{W}_{F_v}. \end{tikzcd}$$

Here $\mathrm{WD}_{F_v} := \mathbb{SL}_2(\mathbf{C}) \times \mathrm{W}_{F_v}$ is the Weil-Deligne group and the diagonal map the projection onto its second component. Moreover, the key is that *even if* the group ${}^L\overline{G}_v$ as an extension

$$\overline{G}^\vee \hookrightarrow {}^L\overline{G}_v \twoheadrightarrow \mathrm{W}_{F_v}$$

is isomorphic to the direct product $\overline{G}^\vee \times \mathrm{W}_{F_v}$, it is *not canonically* so. This reflects the fact, locally for instance, that there is no canonical genuine representation of $\overline{G}_v$. In fact, one could show by examples that in general ${}^L\overline{G}_v$ is only a semidirect product of $\overline{G}^\vee$ and W_{F_v}, see [GaG14].

To be brief, the work of Weissman has supplied us the indispensable local ${}^L\overline{G}_v$ and global ${}^L\overline{G}$ for any further development of the theory of automorphic forms on Brylinski-Deligne covers.

1.2 Main results

We assume only (the necessary) $\mu_n \subseteq F^\times$ as opposed to $\mu_{2n} \subseteq F^\times$ in most literature on covering groups, and consider covering groups $\overline{\mathbb{G}}(\mathbb{A}_F)$ for $\overline{\mathbb{G}}$ arising from the Brylinski-Deligne framework and Eisenstein series induced from genuine cuspidal representation on parabolic $\overline{\mathbb{P}} = \overline{\mathbb{M}}\mathbb{U}$. The global analysis for the spectral decomposition for more general central covering groups is carried out in the book [MW95], which also contains details of how Eisenstein series play the fundamental role in the spectral decomposition.

In order to carry out the computation, we first introduce an incarnation category which is equivalent to the Brylinski-Delgine category of multiplicative $\mathbb{K}_2$-torsors over $\mathbb{G}$. The definition is motivated from Weissman's paper and generalized properly here.

The aim is to compute the GK formula and interpret it as partial L-functions appearing in the constant term of such Eisenstein series. The knowledge of poles of the completed L-functions, which is yet to be fully understood even in the linear algebraic case, together with local analysis determine completely the residual spectrum $L^2_{\mathrm{res}}(\mathbb{G}(F)\backslash\overline{\mathbb{G}}(\mathbb{A}_F))$.

To explain the idea which is essentially the classical one, we note that the constant term of Eisenstein series can be written as global intertwining operators which decompose into local ones. These local intertwining operators enjoy the cocycle relation, which enables us to compute by reduction to the rank one case. The outcome is the analogous Gindikin-Karpelevich formula for intertwining operators at unramified places. The GK formula gives the coefficient in terms of the inducing unramified characters, and therefore the constant term takes a form involving the global inducing data.

We note that the GK formula has been computed in [McN11] using crystal basis decomposition of the integration domain. Recently, as a consequence of the computation of the Casselman-Shalika formula, McNamara also computed the GK formula in [McN14]. However, our computation is carried along the classical line and removes the condition that $2n$-th root of unity lies in the field. More importantly, our GK formula is expressed in naturally defined elements. It is precisely this fact which enables us to give an interpretation in terms local Langlands-Shahidi L-functions.

Now we explain more on the dual side. The construction of ${}^L\overline{G}_v$ in [We14] could be recast using the languages in the incarnation category. It is important that the construction is functorial with respect to Levi subgroups of $\overline{G}_v$. In particular, if $\overline{M}_v$ is a Levi of $\overline{G}_v$, there is a natural map from ${}^L\varphi : {}^L\overline{M}_v \longrightarrow {}^L\overline{G}_v$ such that the diagram

$$\begin{tikzcd}
\overline{M}^{\vee} \arrow[r, hook] \arrow[d, hook, "\varphi^{\vee}"] & {}^L\overline{M}_v \arrow[r, two heads] \arrow[d, "{}^L\varphi"] & \mathrm{W}_{F_v} \arrow[d, equal] \\
\overline{G}^{\vee} \arrow[r, hook] & {}^L\overline{G}_v \arrow[r, two heads] & \mathrm{W}_{F_v}
\end{tikzcd}$$

commutes, where $\varphi^{\vee}$ is the natural inclusion by construction of the dual group. In the case $\overline{M}_v = \overline{T}_v$ is the covering torus of $\overline{G}_v$, we have an explicit description of the map in terms of the incarnation language. This turns out to be essential for our interpretation of GK formula as local Langlands-Shahidi type L-functions later.

After recalling the construction of L-groups, we discuss the problem whether ${}^L\overline{G}_v$ is isomorphic to the direct product $\overline{G}^{\vee} \times \mathrm{W}_{F_v}$, and refer to [GaG14] for a discussion of more properties of the L-group. Thus here the question is equivalent to whether there exist splittings of ${}^L\overline{G}_v$ over W_{F_v} which take values in the centralizer $Z_{{}^L\overline{G}_v}(\overline{G}^{\vee})$ of $\overline{G}^{\vee}$ in ${}^L\overline{G}_v$. It is shown that such splittings, which we call admissible, could arise from certain characters of $Z(\overline{T}_v)$, which we call *qualified*. There is even a subclass of qualified characters of $Z(\overline{T}_v)$ which we name as *distinguished* characters. In the simply-connected case, there is no obstruction to the existence of distinguished characters, while in general there is. One property of qualified characters is that they are Weyl-invariant, which holds in particular for distinguished characters. This could be considered as a generalization of [LoSa10, Cor. 5.2], where the authors use global methods to show the Weyl-invarance of certain unramified principal series for degree two covers of simply-connected groups. We give an explicit construction of distinguished characters and show that they agree with those in the case of double cover $\overline{\mathbb{Sp}}_{2r}(F_v)$ (cf. [Rao93] [Kud96]) and simply-laced simply-connected case treated by Savin (cf. [Sav04]).

As a consequence of the discussion above on the admissible splittings of ${}^L\overline{G}_v$ applied to the case $\overline{G}_v = \overline{T}_v$, we obtain a local Langlands correspondence (LLC) for covering tori. More precisely, any genuine character $\overline{\chi}$ of $Z(\overline{T}_v)$ is qualified and gives rise to a splitting $\rho_{\overline{\chi}}$ of ${}^L\overline{T}_v$ over W_{F_v}. Coupled with the Stone von-Neumann theorem which gives a bijection between isomorphism classes of irreducible genuine characters $\mathrm{Hom}_\epsilon(Z(\overline{T}_v), \mathbf{C}^\times)$ of $Z(\overline{T}_v)$ and irreducible representations $\mathrm{Irr}_\epsilon(\overline{T}_v)$ of $\overline{T}_v$, this correspondence could be viewed with $\mathrm{Hom}_\epsilon(Z(\overline{T}_v), \mathbf{C}^\times)$ replaced by $\mathrm{Irr}_\epsilon(\overline{T}_v)$. In the case $n = 1$, it recovers the LLC for linear tori.

Back to the case of general $\overline{G}_v$, because of the compatibility between ${}^L\overline{T}_v$ and ${}^L\overline{G}_v$ above given by ${}^L\varphi$, the splitting $\rho_{\overline{\chi}}$ could be viewed as a splitting of ${}^L\overline{G}_v$ over W_{F_v}. On the L-group ${}^L\overline{G}_v$ we could define the adjoint representation

$$Ad: \quad {}^L\overline{G}_v \longrightarrow GL(\overline{\mathfrak{g}}^\vee).$$

Our local Langlands correspondence for representations of covering tori gives locally a splitting of ${}^L\overline{G}_v$ over W_{F_v} which arises from a splitting ${}^L\overline{T}_v \twoheadrightarrow \mathrm{W}_{F_v}$. We express the GK formula for unramified principal series in terms of the composition $Ad \circ {}^L\varphi \circ \rho_{\overline{\chi}}$.

The discussion can be carried in parallel for the global setting; more importantly, we have local and global compatibility. For example, one can consider similarly admissible splittings of the global ${}^L\overline{G}$; there is the adjoint representation of the ${}^L\overline{G}$, which by restriction to ${}^L\overline{G}_v$ is just the adjoint representation of ${}^L\overline{G}_v$ above.

We will define automorphic (partial) L-function of an automorphic representation $\overline{\sigma}$ of $\overline{\mathbb{H}}(\mathbb{A}_F)$ of BD type associated with a finite dimensional representation $R: {}^L\overline{H} \longrightarrow GL(V)$. In particular, we are interested in the case where $\overline{\mathbb{H}} = \overline{\mathbb{M}}$ is a Levi of $\overline{\mathbb{G}}$ and that R is the adjoint representation of ${}^L\overline{M}$ on a certain subspace $\overline{\mathfrak{u}}^\vee$ of the Lie algebra $\overline{\mathfrak{g}}^\vee$.

In view of this, the constant term of Eisenstein series for induction from general parabolics can be expressed in terms of certain Langlands-Shahidi type L-functions, by combining the formula from the unramified places. We work out the case for maximal parabolic, and the general case is similar despite the complication in notations.

As simple examples, we will determine the residual spectra of arbitrary degree BD covers $\overline{\mathbb{SL}}_2(\mathbb{A}_F)$ and $\overline{\mathbb{GL}}_2(\mathbb{A}_F)$ of $\mathbb{SL}_2(\mathbb{A}_F)$ and $\mathbb{GL}_2(\mathbb{A}_F)$ respectively. We also compute the partial L-functions appearing in the constant terms of Eisenstein series for induction from maximal parabolic of the double cover $\overline{\mathbb{Sp}}_4(\mathbb{A}_F)$. It is shown to agree with that given in [Gao12].

In the end, we give brief discussions on immediate follow-up or future work that we would like to carry out. For instance, we would like to explore in details the Kazhdan-Patterson covers (cf. [KaPa84]) from the BD-perspective. Also since the construction of ${}^L\overline{G}$ by Weissman is actually for $\mathbb{G}$ not necessarily split, one can readily implement the computation here with proper modifications and expect same Langlands-Shahidi L-

function appears. Moreover, to determine the residual spectrum of $\overline{\mathbb{G}}(\mathbb{A}_F)$, a natural step in the sequel would be to develop a theory of local L-functions, which in the case of metaplectic extension $\overline{\mathbb{Sp}}_{2r}(F_v)$ has been covered by the work of Szpruch. In his thesis, the Langlands-Shahidi method is extended to such groups for generic representations. Moreover, such a theory of local L-functions would lay foundations for the theory of converse theorems, which perhaps could be used to provide links between these completed Langlands-Shahidi L-functions arising from BD covering groups and those from linear algebraic groups.

Chapter 2

The Brylinski-Deligne extensions and their L-groups

2.1 The Brylinski-Deligne extensions and basic properties

In this section, let F be a number field or its localization. We will be more specific when the context requires so. Let $\mathbb{G}$ be a split reductive group over F with root data $(X, \Psi, Y, \Psi^\vee)$. We also fix a set of simple roots $\Delta \subseteq \Psi$.

In their seminal paper [BD01], Brylinski and Degline have studied a certain category of central extensions of $\mathbb{G}$ and given a classification of such objects in terms of combinatorial data. We will recall in this section the main results of that paper and state some properties which are important for our consideration later.

A central extension $\overline{\mathbb{G}}$ of $\mathbb{G}$ by $\mathbb{K}_2$ is an extension in the category of sheaves of groups on the big Zariski site over $\mathrm{Spec}(F)$. It is written in the form

$$\mathbb{K}_2 \hookrightarrow \overline{\mathbb{G}} \twoheadrightarrow \mathbb{G}.$$

The category of such central extensions of $\mathbb{G}$ is denoted by $\mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$.

Any $\overline{\mathbb{G}} \in \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ gives an exact sequence of F'-rational points for any field extension F' of F:

$$\mathbb{K}_2(F') \hookrightarrow \overline{\mathbb{G}}(F') \twoheadrightarrow \mathbb{G}(F').$$

The left exactness follows from the the fact that the extension $\overline{\mathbb{G}}$ is an extension of sheaves, while the right exactness at last term is due to the vanishing of $\mathrm{H}^1_{\mathrm{Zar}}(F', \mathbb{K}_2)$, an analogue of Hilbert Theorem 90.

We will recall the classification of such extensions for $\mathbb{G}$ being a torus, a semi-simple simply-connected group and a general reductive group in the sequel.

2.1.1 Central extensions of tori

Let $\mathbb{T}$ be a split torus with character group $X = X(\mathbb{T})$ and cocharacter group $Y = Y(\mathbb{T})$. The category $\mathsf{CExt}(\mathbb{T}, \mathbb{K}_2)$ of central extensions of $\mathbb{T}$ by $\mathbb{K}_2$ is described as follows.

Theorem 2.1.1. *Let $\mathbb{T}$ be a split torus over F. The category of central extensions $\mathsf{CExt}(\mathbb{T}, \mathbb{K}_2)$ is equivalent to the category of pairs $(Q, \mathcal{E})$, where Q is a quadratic form on Y and $\mathcal{E}$ is a central extension of Y by $F^\times$*

$$F^\times \hookrightarrow \mathcal{E} \twoheadrightarrow Y$$

such that the commutator $Y \times Y \longrightarrow F^\times$ is given by

$$[-,-]: \quad (y_1, y_2) \longmapsto (-1)^{B_Q(y_1,y_2)}.$$

Here B_Q is the symmetric bilinear form associated with Q, i.e. $B_Q(y_1, y_2) = Q(y_1+y_2) - Q(y_1) - Q(y_2)$.

Note that the commutator, which is defined on the group $\mathcal{E}$, descends to Y since the extension is central. For any two pairs $(Q, \mathcal{E})$ and $(Q', \mathcal{E}')$, the group of morphisms exists if and only if $Q = Q'$, in which case it is defined to consist of the isomorphisms between the two extensions $\mathcal{E}$ and $\mathcal{E}'$.

To recall the functor $\mathsf{CExt}(\mathbb{T}, \mathbb{K}_2) \rightsquigarrow \{(Q, \mathcal{E})\}$, assume we are given with $\overline{\mathbb{T}} \in \mathsf{CExt}(\mathbb{T}, \mathbb{K}_2)$. The quadratic from Q thus obtained does not allow for a simple description, and we refer to [BD01, §3.9-3.11] for the details. However, the description of $\mathcal{E}$ is relatively simple and we reproduce it here.

Start with $\overline{\mathbb{T}} \in \mathsf{CExt}(\mathbb{T}, \mathbb{K}_2)$ over F. Taking the rational points of the Laurent field $F(\!(\tau)\!)$ gives

$$\mathbb{K}_2(F(\!(\tau)\!)) \hookrightarrow \overline{\mathbb{T}}(F(\!(\tau)\!)) \twoheadrightarrow \mathbb{T}(F(\!(\tau)\!)) .$$

Pull-back by $Y \longrightarrow \mathbb{T}(F(\!(\tau)\!))$ which sends $y \in Y$ to $y \otimes \tau \in \mathbb{T}(F(\!(\tau)\!))$, and then push-out by the tame symbol $\mathbb{K}_2(F(\!(\tau)\!)) \longrightarrow F^\times$ give the extension $\mathcal{E}$ over Y by $F^\times$. Here the tame symbol is defined to be

$$\{f, g\} \mapsto (-1)^{\mathrm{val}(f)\mathrm{val}(g)} \frac{f^{\mathrm{val}(g)}}{g^{\mathrm{val}(f)}}(0).$$

In particular, $\{a, \tau\} \in \mathbb{K}_2(F(\!(\tau)\!))$ is sent to a for all $a \in F^\times$. This process describes the construction of $\mathcal{E}$.

For convenience, for any lifting $\overline{y \otimes \tau} \in \overline{\mathbb{T}}(F(\!(\tau)\!))$ of $y \otimes \tau \in \mathbb{T}(F(\!(\tau)\!))$, we write

$$\left[\overline{y \otimes \tau}\right] := \text{ the image of } \overline{y \otimes \tau} \text{ in } \mathcal{E}. \tag{2.1}$$

Moreover, any $\overline{\mathbb{T}} \in \mathsf{CExt}(\mathbb{T}, \mathbb{K}_2)$ is isomorphic to $\mathbb{K}_2 \times_D \mathbb{T}$, where D is a (not necessarily symmetric) bilinear form on Y such that $D(y_1, y_2) + D(y_2, y_1) = B_Q(y_1, y_2)$. The trivialized torsor $\mathbb{K}_2 \times_D \mathbb{T}$ is thus endowed with a multiplicative structure described as follows.

Write $D = \sum_i x_1^i \otimes x_2^i \in X \otimes_{\mathbf{Z}} X$. Then the cocycle of $\mathbb{K}_2(F') \times_D \mathbb{T}(F')$, for F' any field extension of F, is given by

$$\sigma_D(t_1, t_2) = \prod_i \{x_1^i(t_1), x_2^i(t_2)\}, \quad t_1, t_2 \in \mathbb{T}(F'). \tag{2.2}$$

Now it is easy to check that the commutator of $\mathcal{E}$ is given by the formula $[y_1, y_2] = (-1)^{B_Q(y_1,y_2)}$.

2.1.2 Central extensions of semi-simple simply-connected groups

Let $\mathbb{G}$ be a split semi-simple simply-connected group over F with root data $(X, \Psi, Y, \Psi^\vee)$. Let $\mathbb{T}$ be a maximal split torus of $\mathbb{G}$ with character group X and cocharacter group Y. By the perfect pairing of X and Y, $\mathrm{Sym}^2(X)$ is identified with integer-valued quadratic forms on Y. Let W be the Weyl-group of $\mathbb{G}$. We have the following classification theorem for $\mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$.

Theorem 2.1.2. *The category $\mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ is rigid, i.e., any two objects have at most one morphism between them. The set of isomorphism classes is classified by W-invariant integer-valued quadratic forms $Q : Y \longrightarrow \mathbf{Z}$, i.e., by $Q \in \mathrm{Sym}^2(X)^W$.*

A special case of $\mathbb{G}$ is when it is almost simple. In this case we can identify

$$\mathrm{Sym}^2(X)^W \longrightarrow \mathbf{Z}, \quad Q \longmapsto Q(\alpha^\vee),$$

where $\alpha^\vee \in \Psi^\vee$ is the short coroot associated to any long root. The fact that $Q(\alpha^\vee)$ for short coroot uniquely determines the quadratic form Q follows from the following easy fact.

Lemma 2.1.3. *For any $\alpha^\vee \in \Psi^\vee$ and $y \in Y$,*

$$B_Q(\alpha^\vee, y) = Q(\alpha^\vee) \cdot \langle \alpha, y \rangle,$$

where $\langle -, - \rangle$ denotes the paring between X and Y.

Proof. The Weyl invariance property of B_Q follows from that of Q, and it gives

$$\begin{aligned} B_Q(\alpha^\vee, y) &= B_Q(s_{\alpha^\vee}(\alpha^\vee), s_{\alpha^\vee}(y)) \\ &= B_Q(-\alpha^\vee, y - \langle \alpha, y \rangle \alpha^\vee) \\ &= -B_Q(\alpha^\vee, y) + 2\langle \alpha, y \rangle \cdot Q(\alpha^\vee). \end{aligned}$$

The claim follows. □

Example 2.1.4. The classical metaplectic double cover arises from a central extension $\overline{\mathbb{Sp}}_{2r}$ over $\mathbb{Sp}_{2r}$ of this type. Let $\alpha_1^\vee, \alpha_2^\vee, ..., \alpha_r^\vee$ be the simple coroots of $\mathbb{Sp}_{2r}$ with $\alpha_1^\vee$ the unique short one. Let Q be the unique Weyl invariant quadratic form on Y with $Q(\alpha_1^\vee) = 1$, see also [BD01, pg. 7-8]. This gives the desired $\overline{\mathbb{Sp}}_{2r}$ according to the above classification theorem.

2.1.3 Central extensions of general split reductive groups

Now we fix a split reductive group $\mathbb{G}$ and a maximal split torus $\mathbb{T}$ over F with root data $(X, \Psi, Y, \Psi^\vee)$. The classification of $\mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ relies on more data in the description. It is a combined result from both the classifications of $\mathbb{K}_2$-torsors over tori and of semisimple simply-connected groups in 2.1.1 and 2.1.2 respectively. The following is the main result by Brylinski and Deligne in the split case.

Theorem 2.1.5. *Let $\mathbb{G}$ be a split connected reductive group over F with maximal split torus $\mathbb{T}$. Let X and Y be the character and cocharacter groups of $\mathbb{T}$ respectively. The category $\mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ is equivalent to the category specified by the triples $(Q, \mathcal{E}, \phi)$ with the following properties:*

The Q is a Weyl invariant quadratic form on Y and $\mathcal{E}$ a central extension

$$F^\times \hookrightarrow \mathcal{E} \twoheadrightarrow Y\ ,$$

such that the commutator $[-,-] : Y \times Y \longrightarrow F^\times$ is given by

$$[y_1, y_2] = (-1)^{B_Q(y_1, y_2)}.$$

Let $\Phi : \mathbb{G}^{sc} \longrightarrow \mathbb{G}^{der} \longrightarrow \mathbb{G}$ be the natural composition, where $\mathbb{G}^{sc}$ is the simply connected cover of the derived group $\mathbb{G}^{der}$ of $\mathbb{G}$. Let $\mathbb{T}^{sc} = \Phi^{-1}(\mathbb{T})$ be a maximal split torus of $\mathbb{G}^{sc}$ with cocharacter group $Y^{sc} \subseteq Y$. The restriction $Q|_{Y^{sc}}$ gives an element $\overline{\mathbb{G}}^{sc} \in \mathsf{CExt}(\mathbb{G}^{sc}, \mathbb{K}_2)$ unique up to unique isomorphism by Theorem 2.1.2, which by further pull-back to the torus $\mathbb{T}^{sc}$ gives a central extension $\overline{\mathbb{T}}^{sc}$ by $\mathbb{K}_2$. Therefore, we have from Theorem 2.1.1 a corresponding central extension

$$F^\times \hookrightarrow \mathcal{E}^{sc} \twoheadrightarrow Y^{sc}\ .$$

The requirement on ϕ is that it is a morphism from $\mathcal{E}^{sc}$ to $\mathcal{E}$ such that the following diagram commute:

$$\begin{tikzcd} F^\times \arrow[r, hook] \arrow[d, equal] & \mathcal{E}^{sc} \arrow[r, two heads] \arrow[d, "\phi"] & Y^{sc} \arrow[d, hook] \\ F^\times \arrow[r, hook] & \mathcal{E} \arrow[r, two heads] & Y. \end{tikzcd}$$

Homomorphisms between two triples $(Q_1, \mathcal{E}_1, \phi_1)$ and $(Q_2, \mathcal{E}_2, \phi_2)$ exist only for $Q_1 = Q_2$, in which case they are defined to be the homomorphisms between $\mathcal{E}_1$ and $\mathcal{E}_2$ which respect the above commutative diagram.

In fact, the theorem stated here could be strengthened, since the category $\mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ and the category consisting of $(Q, \mathcal{E}, \phi)$ are both commutative Picard categories with respect to the Baer sum operation.

In general, for any group C and an abelian group A, we view A as a trivial C-module. Then, the second cohomology group $H^2(C, A)$ classifies the isomorphism classes of central extensions of C by A. The group law on $H^2(C, A)$ is then realized as the Baer sum.

Recall the definition of Baer sum. Let $A \hookrightarrow E_i \twoheadrightarrow C$ be two central extensions with A an abelian group (written multiplicatively say). Let

$$\delta : C \longrightarrow C \times C$$

be the diagonal map and let

$$m : A \times A \longrightarrow A$$

be the multiplication map. From E_1 and E_2, we obtain the extension $E_1 \times E_2$ of $C \times C$ by $A \times A$ by forming the Cartesian product. Then by definition the Baer sum of E_1 and E_2 is given by

$$E_1 \oplus_B E_2 := \Big(\delta^* \circ m_*(E_1 \times E_2) \simeq m_* \circ \delta^*(E_1 \times E_2)\Big).$$

Thus for example, the additive structure for the category $\{(Q, \mathcal{E}, \phi)\}$ is given such that the sum of two $(Q_i, \mathcal{E}_i, \phi_i)$ for $i = 1, 2$ is by definition

$$(Q_1 + Q_2,\ \mathcal{E}_1 \oplus_B \mathcal{E}_2,\ \phi_1 \oplus_B \phi_2),$$

where $\phi_1 \oplus_B \phi_2$ is the obvious induced map.

Theorem 2.1.6 ([BD01, §7])**.** *The equivalence of the two categories in Theorem 2.1.5 respects the Picard structure, i.e., it establishes an equivalence between the two commutative Picard categories.*

2.1.4 The Brylinski-Deligne section

Assume $\mathbb{G}$ reductive and $\mathbb{G}^{sc}$ the semisimple simply-connected group as before. From the $\mathbb{K}_2$-torsor $\overline{\mathbb{T}}$ over $\mathbb{T}$ we have constructed the extension $\mathcal{E}$. By pull-back, any $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}$ gives a $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}^{sc}$ over $\mathbb{G}^{sc}$. Further restriction gives the covering $\overline{\mathbb{T}}^{sc}$ of $\mathbb{T}^{sc} \subseteq \mathbb{G}^{sc}$. Similarly one obtains $\mathcal{E}^{sc}$ in the same way starting from $\overline{\mathbb{T}}^{sc}$. It is possible to characterize $\overline{\mathbb{T}}^{sc}$ and the corresponding extension

$$F^\times \hookrightarrow \mathcal{E}^{sc} \twoheadrightarrow Y^{sc}$$

given by Theorem 2.1.1 which arise in this way, among general $\mathbb{K}_2$-torsors over $\mathbb{T}^{sc}$ associated with the same Q (cf. [BD01, §11]).

For the time being, we also denote by $\Phi : \overline{\mathbb{G}}^{sc} \longrightarrow \overline{\mathbb{G}}$ the natural pull-back map, and which restricts to the tori to give the middle map of the following diagram

$$\begin{tikzcd}
\mathbb{K}_2 \arrow[r, hook] \arrow[d, equal] & \overline{\mathbb{T}}^{sc} \arrow[r, two heads] \arrow[d, "\Phi"] & \mathbb{T}^{sc} \arrow[d, "\Phi"] \\
\mathbb{K}_2 \arrow[r, hook] & \overline{\mathbb{T}} \arrow[r, two heads] & \mathbb{T}.
\end{tikzcd}$$

Let $\alpha \in \Psi$, and let $\overline{\mathbb{T}}^{sc}_\alpha$ be the pull-back of $\overline{\mathbb{G}}^{sc}$ to the one-dimensional torus $\mathbb{T}^{sc}_\alpha \subseteq \mathbb{T}^{sc}$. What is important to us is that $\overline{\mathbb{T}}^{sc}_\alpha$ is endowed with a natural section over $\mathbb{T}^{sc}_\alpha$, which depends on the épinglage we fix for $\mathbb{G}$.

The Bylinski-Deligne section of $\overline{\mathbb{T}}^{sc}_\alpha$

Recall that the extension $\overline{\mathbb{G}}$ splits uniquely over the unipotent subgroup $\mathbb{U} \subseteq \mathbb{G}$, and this splitting is $\mathbb{B} = \mathbb{T}\mathbb{U}$-equivariant (cf. [BD01, Prop. 11.3]). Here $\mathbb{U}$ could be viewed as the unipotent radical of the Borel subgroup $\mathbb{B}^{sc} = \mathbb{T}^{sc}\mathbb{U}$ of $\overline{\mathbb{G}}^{sc}$, which splits uniquely in $\overline{\mathbb{G}}^{sc}$ and thus is compatible with the map $\Phi : \overline{\mathbb{G}}^{sc} \longrightarrow \overline{\mathbb{G}}$. We denote by $\overline{\mathbb{e}} \in \overline{\mathbb{U}}$ the image of this splitting of any element $\mathbb{e} \in \mathbb{U}$, and no confusion will arise on the context of such definition, i.e. with respect to $\mathbb{G}$ or $\mathbb{G}^{sc}$.

We fix a Chevalley system of eṕinglage for $(\mathbb{G}, \mathbb{T}, \mathbb{B})$ (cf. [BrTi84, §3.2.1-2]). In particular, for each $\alpha \in \Psi$ with associated root subgroup $\mathbb{U}_\alpha$, we have a fixed isomorphism $\mathbb{e}_\alpha : \mathbb{G}_{\text{add}} \longrightarrow \mathbb{U}_\alpha$. Also, there is the induced morphism $\varphi_\alpha : \mathbb{SL}_2 \longrightarrow \mathbb{G}$. In fact, the data for the eṕinglage above gives an eṕinglage of $(\mathbb{G}^{sc}, \mathbb{T}^{sc}, \mathbb{B}^{sc})$, and we still denoted by $\varphi_\alpha : \mathbb{SL}_2 \hookrightarrow \mathbb{G}^{sc}$ the induced morphism which is an injection in this case.

Let $a \in \mathbb{G}_{\text{mul}}$, consider $\mathbb{e}_+(a), \mathbb{e}_-(a), \mathbb{w}_o(a)$ of $\mathbb{SL}_2$ as follows:

$$\mathbb{e}_+(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad \mathbb{e}_-(a) = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix},$$

$$\mathbb{w}_o(a) = \mathbb{e}_+(a)\mathbb{e}_-(a^{-1})\mathbb{e}_+(a) = \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}, \quad \mathbb{h}_o(a) = \mathbb{w}_o(a)\mathbb{w}_o(-1) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

By the Tits trijection (cf. [BD01, §11]) we mean the triple $\mathbb{e}_\alpha(a), \mathbb{e}_{-\alpha}(a^{-1}), \mathbb{w}_\alpha(a) \in \mathbb{G}^{sc}$ given by

$$\mathbb{e}_\alpha(a) = \varphi_\alpha(\mathbb{e}_+(a)), \quad \mathbb{e}_{-\alpha}(a^{-1}) = \varphi_\alpha(\mathbb{e}_-(a^{-1})), \quad \mathbb{w}_\alpha(a) := \varphi_\alpha(\mathbb{w}_o(a)).$$

We also write $\mathbb{h}_\alpha(a) := \varphi_\alpha(\mathbb{h}_o(a))$ and thus $\mathbb{h}_\alpha(a) = \mathbb{w}_\alpha(a)\mathbb{w}_\alpha(-1)$.

Now we can proceed to describe the Brylinski-Deligne (BD) section $\overline{\mathbb{h}}^{[b]}_\alpha$ (which depends on $b \in \mathbb{G}_{\text{mul}}$) of $\overline{\mathbb{T}}^{sc}_\alpha$ over $\mathbb{T}^{sc}_\alpha \simeq \mathbb{G}_{\text{mul}}$.

In particular, we describe the BD section at the level of F'-rational points, where F'/F is a field extension. That is, for any $b \in \mathbb{G}_{\text{mul}}(F') = (F')^\times$, we have the BD section $\overline{\mathbb{h}}^{[b]}_\alpha$:

$$\begin{tikzcd} \mathbb{K}_2(F') \arrow[r, hook] & \overline{\mathbb{T}}^{sc}_\alpha(F') \arrow[r, two heads] & \mathbb{T}^{sc}_\alpha(F') \arrow[l, bend right, "\overline{\mathbb{h}}^{[b]}_\alpha"'] \end{tikzcd}.$$

Recall the definition of $\overline{\mathbb{h}}^{[b]}_\alpha$ as follows. For any $a \in (F')^\times$, first define a lifting $\overline{\mathbb{w}}_\alpha(a) \in \overline{\mathbb{G}}^{sc}(F')$ of the element $\mathbb{w}_\alpha(a) \in N(\mathbb{T})(F')$ by

$$\mathbb{w}_\alpha(a) \longmapsto \overline{\mathbb{w}}_\alpha(a) := \overline{\mathbb{e}}_\alpha(a) \cdot \overline{\mathbb{e}}_{-\alpha}(a^{-1}) \cdot \overline{\mathbb{e}}_\alpha(a).$$

The BD section $\overline{\mathbb{h}}^{[b]}_\alpha(a)$ of $\overline{\mathbb{T}}^{sc}(F')$ over $\mathbb{T}^{sc}(F')$ is then by definition (cf. [BD01, §11.1])

$$\overline{\mathbb{h}}^{[b]}_\alpha(a) := \overline{\mathbb{w}}_\alpha(ab) \cdot \overline{\mathbb{w}}_\alpha(b)^{-1}.$$

Two important properties of this section are

$$\overline{\mathbb{h}}_\alpha^{[b]}(a)\cdot\overline{\mathbb{h}}_\alpha^{[b]}(c)=\overline{\mathbb{h}}_\alpha^{[b]}(ac)\cdot\big\{a,c\big\}^{Q(\alpha^\vee)}, \tag{2.3}$$

$$\overline{\mathbb{h}}_\alpha^{[db]}(a)=\overline{\mathbb{h}}_\alpha^{[b]}(a)\cdot\big\{d,a\big\}^{Q(\alpha^\vee)}. \tag{2.4}$$

This section described above gives rise to an inherited lifting into $\mathcal{E}^{sc}$ of the one-dimension lattice $Y_\alpha^{sc}\subseteq Y^{sc}$ spanned by $\alpha^\vee\in\Psi^\vee$. More precisely, apply the case $F'=F(\!(\tau)\!)$ and pick any nonzero $f\in F(\!(\tau)\!)$. Let $\overline{\mathbb{h}}_\alpha^{[f]}$ be the BD section $\overline{\mathbb{T}}_\alpha^{sc}(F(\!(\tau)\!))$ over $\mathbb{T}_\alpha^{sc}(F(\!(\tau)\!))$. It induces a section over $Y_\alpha^{sc}(\tau)\subseteq\mathbb{T}^{sc}(F(\!(\tau)\!))$. Thus, we obtain an inherited section of $\mathcal{E}^{sc}$ over Y_α^{sc}, which is still denoted by $\overline{\mathbb{h}}_\alpha^{[f]}$. Write $\mathcal{E}_\alpha^{sc}$ for the pull-back of $\mathcal{E}^{sc}$ via $Y_\alpha^{sc}\subseteq Y^{sc}$, then we have the section

$$\begin{tikzcd} F^\times \arrow[r, hook] & \mathcal{E}_\alpha^{sc} \arrow[r, two heads] & Y_\alpha^{sc} \arrow[l, bend right, "\overline{\mathbb{h}}_\alpha^{[f]}"'] \end{tikzcd}.$$

Recall that for any lifting $\overline{y\otimes\tau}\in\overline{\mathbb{T}}^{sc}(F(\!(\tau)\!))$ of $y\otimes\tau\in\mathbb{T}^{sc}(F(\!(\tau)\!))$, we have denoted by $\big[\overline{y\otimes\tau}\big]\in\mathcal{E}^{sc}$ its image in $\mathcal{E}^{sc}$, as in (2.1). In particular for any $k\in\mathbf{Z}$,

$$\big[\overline{\mathbb{h}}_\alpha^{[f]}(\tau^k)\big]:=\text{ the image of }\overline{\mathbb{h}}_\alpha^{[f]}(\tau^k)\in\overline{\mathbb{T}}^{sc}(F(\!(\tau)\!))\text{ in }\mathcal{E}^{sc}. \tag{2.5}$$

Definition 2.1.7. Consider

$$\overline{\mathbb{h}}_\alpha^{[f]}(k\cdot\alpha^\vee):=\big[\overline{\mathbb{h}}_\alpha^{[f]}(\tau^k)\big],$$

which is a lifting of $k\alpha^\vee\in Y_\alpha^{sc}$. We call it the Brylinski-Deligne section of $\mathcal{E}_\alpha^{sc}$ over Y_α^{sc}.

Rigidifying $\mathcal{E}^{sc}$

Let $\mathcal{E}_Q^{sc}$ be the abstract group generated by $\big\{a\big\}_{a\in F^\times}\cup\big\{\gamma_\alpha\big\}_{\alpha^\vee\in\Delta^\vee}$ subject to the conditions:

(*i*) $F^\times$ is contained in the center of $\mathcal{E}_Q^{sc}$,

(*ii*) $[\gamma_\alpha,\gamma_\beta]=(-1)^{B_Q(\alpha^\vee,\beta^\vee)}$ for any $\alpha^\vee,\beta^\vee\in\Delta^\vee$.

We obtain the exact sequence $F^\times\hookrightarrow\mathcal{E}_Q^{sc}\twoheadrightarrow Y$, which is uniquely determined by requiring that $a\in F^\times$ sent to the generator a of $\mathcal{E}_Q^{sc}$ and γ_α to $\alpha^\vee$.

Thus it is possible to rigidify the extension $\mathcal{E}^{sc}$ obtained above by using the unique isomorphism given by

$$\mathcal{E}^{sc}\xrightarrow{\simeq}\mathcal{E}_Q^{sc},\quad \overline{\mathbb{h}}_\alpha^{[1]}(\alpha^\vee)\longmapsto\gamma_\alpha,$$

where $\overline{\mathbb{h}}_\alpha^{[1]}(\alpha^\vee)$ is just $\overline{\mathbb{h}}_\alpha^{[f]}(\alpha^\vee)$ for $f=1$.

2.2 Incarnation functor and an equivalent category

2.2.1 Equivalence between the incarnation category and the BD category

By adapting and modifying the definition in [We13], one can define a Picard category $\mathsf{Bis}_{\mathbb{G}} = \bigsqcup_Q \mathsf{Bis}_{\mathbb{G}}^Q$ and an incarnation functor

$$\mathsf{Inc}_{\mathbb{G}} : \quad \mathsf{Bis}_{\mathbb{G}} \rightsquigarrow \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2),$$

which gives an equivalence of Picard categories. That is, $\mathsf{Inc}_{\mathbb{G}}$ is fully faithful and essentially surjective, namely, surjective onto the isomorphism classes of $\mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$. Moreover, it respects the Picard structures on both categories. Because of this equivalence, we can concentrate and work in the category $\mathsf{Bis}_{\mathbb{G}}$.

It is important for us to have a description of $\mathsf{Bis}_{\mathbb{G}}^Q$, and so we recall the definition here.

Definition 2.2.1. The category $\mathsf{Bis}_{\mathbb{G}}^Q$ consists of pairs (D, η), where D is a $\mathbf{Z}$-valued bilinear (not necessarily symmetric) form on Y such that $D(y_1, y_2) + D(y_2, y_1) = B_Q(y_1, y_2)$ and $\eta : Y^{sc} \longrightarrow F^\times$ a group homomorphism. In particular,

$$D(y, y) = Q(y).$$

We call D a bisector of Q. Morphisms of pairs (D_i, η_i) for $i = 1, 2$ consist of maps $H : Y \longrightarrow F^\times$ (not necessarily a homomorphism) such that

$$\begin{aligned} &(i) \quad (-1)^{D_2(y_1,y_2) - D_1(y_1,y_2)} = H(y_1 + y_2) \cdot H(y_1)^{-1} \cdot H(y_2)^{-1}, \\ &(ii) \quad \eta_2(\alpha^\vee)/\eta_1(\alpha^\vee) = H(\alpha^\vee) \text{ for all } \alpha^\vee \in \Delta^\vee. \end{aligned}$$

The composition of two morphisms is given by multiplication, i.e., $H_1 \circ H_2(y) = H_1(y) \cdot H_2(y)$.

Remark 2.2.2. Note that in (ii) we do not require $\eta_2/\eta_1 = H|_{Y^{sc}}$ which will enforce $H|_{Y^{sc}}$ to be a homomorphism, which is unnecessary and in fact not true for the desired equivalence of categories mentioned. Also the definition given above is not exactly the same as in [We13], where D is assumed with some fairness condition. The morphism between two objects here is also defined in a less restrictive way.

It is shown in [We13, Prop. 2.4] that for $(D_i, \eta_i), i = 1, 2$ associated with the same Q, there always exists H satisfying (i). Consequently, we see that up to isomorphism we could always fix a base D and allow η to be varied. More precisely, we have the following.

Example 2.2.3. Let D_1, D_2 be two bisectors of Q. Then (D_1, η_1) for any η_1 is isomorphic to (D_2, η_2) for some η_2. We explain how the η_2 can be obtained. Pick $H : Y \longrightarrow F^\times$

such that the property (i) is satisfied with respect to D_1 and D_2. Define η_2 to be such that

$$\eta_2(\alpha^\vee)/\eta_1(\alpha^\vee) = H(\alpha^\vee) \text{ for all } \alpha^\vee \in \Delta^\vee.$$

Then $(D_1, \eta_1) \simeq (D_2, \eta_2)$ for η_2 obtained in this way.

On the other hand, we observe that not every (D_2, η) is isomorphic to $(D_1, \mathbb{1})$ for some D_1. Suppose on the contrary there is a H which realizes an isomorphism from $(D_1, \mathbb{1})$ to (D_2, η). Then property (i) implies $H(ky) = H(y)^k$ for all $k \in \mathbf{Z}, y \in Y$. If $\alpha^\vee = ky$ for some $y \in Y$, this implies necessarily by (b)

$$\eta(\alpha^\vee) = H(y)^k \in (F^\times)^k.$$

However, in general η may not satisfy such a condition. A concrete example is for $\mathbb{G} = \mathbb{PGL}_2$ with $Q = 0$ with $\eta(\alpha^\vee) \in F^\times \backslash (F^\times)^2$ where $\alpha^\vee \in Y$ is the coroot.

Example 2.2.4. Assume that $\mathbb{G}$ has a simply-connected derived group $\mathbb{G}^{\rm der}$. Then Y/Y^{sc} is a free $\mathbf{Z}$-module. Let $(D, \eta) \in \mathsf{Bis}_{\mathbb{G}}^Q$, then $(D, \eta) \simeq (D, \mathbb{1})$. In fact, any $H \in \mathrm{Hom}(Y, F^\times)$ extending η will provide a morphism from $(D, \mathbb{1})$ to (D, η).

We also claim that for such $\mathbb{G}$ any two $(D_i, \eta_i) \in \mathsf{Bis}_{\mathbb{G}}^Q, i = 1, 2$ are isomorphic. This can be seen from the composition of isomorphisms:

$$(D_1, \eta_1) \longrightarrow (D_1, \mathbb{1}) \xrightarrow{H} (D_2, \eta_H) \longrightarrow (D_2, \mathbb{1}) \longrightarrow (D_2, \eta_2), \tag{2.6}$$

where the second isomorphism exists and depends on H as in previous example.

The incarnation functor from $\mathsf{Bis}_{\mathbb{G}}^Q$ to $\mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ is realized by first defining a functor from $\mathsf{Bis}_{\mathbb{G}}^Q$ to the category of $\big\{(Q, \mathcal{E}, \phi)\big\}$, in which the target object of (D, η) is denoted as $(Q, \mathcal{E}_D, \phi_{D,\eta})$.

The extension $\mathcal{E}_D$ is described as $F^\times \times_D Y$ with group law given by

$$(a, y_1) \cdot (b, y_2) = \big(ab \cdot (-1)^{D(y_1, y_2)}, y_1 + y_2\big).$$

The map $\phi_{D,\eta} : \mathcal{E}_Q^{sc} \longrightarrow \mathcal{E}$ is the one uniquely determined by

$$\gamma_\alpha \mapsto (\eta(\alpha^\vee), \alpha^\vee) \text{ for all } \alpha^\vee \in \Delta^\vee.$$

Lemma 2.2.5. *The map $\phi_{D,\eta}$ is a homomorphism.*

Proof. By the definition of $\mathcal{E}_Q^{sc}$ using generators and relations, it suffices to check that for $\alpha, \beta \in \Delta$ the equality

$$\phi_{D,\eta}([\gamma_\alpha, \gamma_\beta]) = [\phi_{D,\eta}(\gamma_\alpha), \phi_{D,\eta}(\gamma_\beta)].$$

Now the left hand side is $\phi_{D,\eta}\big((-1)^{B_Q(\alpha^\vee, \beta^\vee)}\big) = (-1)^{B_Q(\alpha^\vee, \beta^\vee)}$. On the other hand, the right hand side is $[\alpha^\vee, \beta^\vee] = (-1)^{B_Q(\alpha^\vee, \beta^\vee)}$.

This completes the proof. □

Hence, the incarnation functor is well-defined. Now we show it gives an equivalence of categories. Any morphism $H \in \mathrm{Hom}\big((D_1, \eta_1), (D_2, \eta_2)\big)$ gives a map $\mathsf{Inc}_{\mathbb{G}}(H)$ from $\mathcal{E}_{D_1}$ to $\mathcal{E}_{D_2}$ given by

$$\mathsf{Inc}_{\mathbb{G}}(H): \quad (b, y)_{\mathcal{E}_{D_1}} \longmapsto (b \cdot H(y), y)_{\mathcal{E}_{D_2}}.$$

It is easy to see that $\mathsf{Inc}_{\mathbb{G}}$ is essentially sujective. To show the equivalence between the category of $\mathsf{Bis}_{\mathbb{G}}^{Q}$ and $\big\{(Q, \mathcal{E}, \phi)\big\}$, it suffices to prove the following.

Proposition 2.2.6. *The functor* $H \rightsquigarrow \mathsf{Inc}_{\mathbb{G}}(H)$ *gives an isomorphism*

$$Hom\big((D_1, \eta_1), (D_2, \eta_2)\big) \simeq Hom\big((\mathcal{E}_{D_1}, \phi_{D_1,\eta_1}), (\mathcal{E}_{D_2}, \phi_{D_2,\eta_2})\big).$$

Proof. Property (i) of H implies that $\mathsf{Inc}_{\mathbb{G}}(H)$ is a homomorphism. Now, we check

$$\phi_{D_2,\eta_2} = \mathsf{Inc}_{\mathbb{G}}(H) \circ \phi_{D_1,\eta_1}.$$

For any $\gamma_\alpha \in \mathcal{E}_Q$,

$$\begin{aligned}
\mathsf{Inc}_{\mathbb{G}}(H) \circ \phi_{D_1,\eta_1}(\gamma_\alpha) =& \mathsf{Inc}_{\mathbb{G}}(H)\big((\eta_1(\alpha^\vee), \alpha^\vee)_{\mathcal{E}_{D_1}}\big) \\
=& (\eta_1(\alpha^\vee) \cdot H(\alpha^\vee), \alpha^\vee)_{\mathcal{E}_{D_2}} \\
=& (\eta_2(\alpha^\vee), \alpha^\vee)_{\mathcal{E}_{D_2}} \\
=& \phi_{D_2,\eta_2}(\gamma_\alpha).
\end{aligned}$$

It is easy to see from definition that if $\mathsf{Inc}_{\mathbb{G}}(H_1) = \mathsf{Inc}_{\mathbb{G}}(H_2)$, then $H_1 = H_2$. Moreover, one can check easily that any morphism in $\mathrm{Hom}\big((\mathcal{E}_{D_1}, \phi_{D_1,\eta_1}), (\mathcal{E}_{D_2}, \phi_{D_2,\eta_2})\big)$ arises in this way, i.e., is equal to $\mathsf{Inc}_{\mathbb{G}}(H)$ for some H. ☐

On the category $\mathsf{Bis}_{\mathbb{G}} = \bigsqcup_Q \mathsf{Bis}_{\mathbb{G}}^{Q}$ there is a commutative Picard structure. For $(D_i, \eta_i) \in \mathsf{Bis}_{\mathbb{G}}^{Q_i}$, $i = 1, 2$, define the sum of the two to be

$$(D_1 + D_2, \eta_1 \cdot \eta_2).$$

With respect to this structure on $\mathsf{Bis}_{\mathbb{G}}$, the functor $(D, \eta) \rightsquigarrow (Q, \mathcal{E}_D, \phi_{D,\eta})$ gives an equivalence between the two Picard categories. We consider the following composition (still denoted by $\mathsf{Inc}_{\mathbb{G}}$)

$$\mathsf{Inc}_{\mathbb{G}}: \quad \mathsf{Bis}_{\mathbb{G}} \rightsquigarrow \big\{(Q, \mathcal{E}, \phi)\big\} \rightsquigarrow \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2), \tag{2.7}$$

where the second functor is a quasi-inverse of the Brylinski-Deligne classification functor, well-defined up to natural equivalence.

To summarize,

Proposition 2.2.7. *The incarnation functor* $\mathsf{Inc}_{\mathbb{G}}$ *establishes an equivalence of commutative Picard categories between* $\mathsf{Bis}_{\mathbb{G}}$ *and* $\mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$.

2.2.2 Description of $\overline{\mathbb{G}}_{D,\eta}$

It follows from the above equivalence of categories that no information is lost when working with (D,η). From now, we fix a quasi-inverse functor $\{(Q,\mathcal{E},\phi)\} \rightsquigarrow \mathsf{CExt}(\mathbb{G},\mathbb{K}_2)$ in (2.7). We will work with $\overline{\mathbb{G}}$ which is incarnated by (D,η), and write it as $\overline{\mathbb{G}}_{D,\eta}$.

It is desirable and in fact crucial to have a more precise description of $\overline{\mathbb{G}}_{D,\eta}$, including some structure facts useful for later considerations. Note that the following diagram of functors commutes (upto to equivalence of categories):

$$\begin{tikzcd}(D,\eta) \arrow[rr, rightsquigarrow, mapsto] \arrow[dr, rightsquigarrow, mapsto] & & \overline{\mathbb{G}}_{D,\eta} := \mathsf{Inc}_{\mathbb{G}}\big((D,\eta)\big) \arrow[dl, rightsquigarrow, mapsto, "\text{BD classif.}"] \\ & (Q,\mathcal{E}_D,\phi_{D,\eta}). & \end{tikzcd} \tag{2.8}$$

That is, given (D,η), we may assume that $\overline{\mathbb{G}}_{D,\eta}$ gives rise to $(Q,\mathcal{E}_D,\phi_{D,\eta})$ from the BD classification.

First of all, we consider the case $\mathbb{G}=\mathbb{T}$ and a $\mathbb{K}_2$-torsor $\overline{\mathbb{T}}_D$ which is incarnated by D (there is no η for torus). Equivalently, there is a section of $\overline{\mathbb{T}}_D$ over $\mathbb{T}$ and therefore a bisector D as above, with respect to which we can write $\overline{\mathbb{T}}_D = \mathbb{K}_2 \times_D \mathbb{T}$ with the group law given by (2.2). In particular, for any field extension F' of F, one can write $\overline{\mathbb{T}}_D(F') = \mathbb{K}_2(F') \times_D \mathbb{T}(F')$ with the group law given by:

$$i)\quad \big[y_1\otimes a_1, y_2\otimes a_2\big] = \big\{a_1,a_2\big\}^{B_Q(y_1,y_2)}, \tag{2.9}$$

$$ii)\quad (1,y_1\otimes a)\cdot(1,y_2\otimes a) = \big(\big\{a,a\big\}^{D(y_1,y_2)}, (y_1+y_2)\otimes a\big), \tag{2.10}$$

$$iii)\quad (1,y\otimes a_1)\cdot(1,y\otimes a_2) = \big(\big\{a_1,a_2\big\}^{Q(y)}, y\otimes(a_1a_2)\big). \tag{2.11}$$

Now back to the case of a general reductive group $\mathbb{G}$. We will describe properties of $\overline{\mathbb{G}}_{D,\eta}$.

The covering torus of $\overline{\mathbb{G}}_{D,\eta}$ is incarnated by D as above, for which we have written as $\overline{\mathbb{T}}_D$. Note that implicitly the incarnation depends on a certain section over $\mathbb{T}$, although notationally we are only using the resulting bilinear from D. We have assumed that the extension $\overline{\mathbb{G}}_{D,\eta}$ gives rise to the Brylinski-Deligne data Q and $\mathcal{E}_D$.

Recall the pull-back $\overline{\mathbb{T}}^{sc}$ of $\overline{\mathbb{T}}_D$ to $\mathbb{T}^{sc}$ and the natural pull-back map Φ from $\overline{\mathbb{T}}^{sc}$ to $\overline{\mathbb{T}}_D$. Here $\overline{\mathbb{T}}^{sc}$ inherits a description in terms of D; however, since this fact is never used, we will refrain from considering it. For the same reason, we use the notation $\overline{\mathbb{T}}^{sc}$ without the subscript D.

We have the sheaves of extensions

$$\begin{tikzcd}\mathbb{K}_2 \arrow[r, hook] \arrow[d, equal] & \overline{\mathbb{T}}^{sc} \arrow[r, two heads] \arrow[d, "\Phi"] & \mathbb{T}^{sc} \arrow[d, "\Phi"] \\ \mathbb{K}_2 \arrow[r, hook] & \overline{\mathbb{T}}_D \arrow[r, two heads] & \mathbb{T}.\end{tikzcd}$$

Consider the Brylinski-Deligne lifting $\overline{\mathbb{h}}_\alpha^{[b]}(a) \in \overline{\mathbb{T}}^{sc}(F')$ of $\mathbb{h}_\alpha(a) \in \mathbb{T}^{sc}(F')$ as in section 2.1.4, which depends on $b \in (F')^\times$. We are interested in the element $\Phi\big(\overline{\mathbb{h}}_\alpha^{[b]}(a)\big)$ for $\alpha \in \Delta$ expressed in terms of $\mathbb{K}_2(F') \times_D \mathbb{T}_D(F')$.

We have the following explicit description for Φ at the level of F'-rational points.

Proposition 2.2.8. *Keep notations as above. Then for all $\alpha \in \Delta$, we have*

$$\Phi\big(\overline{\mathbb{h}}_\alpha^{[b]}(a)\big) = \big(\big\{b^{Q(\alpha^\vee)} \cdot \eta(\alpha^\vee), a\big\}, \mathbb{h}_\alpha(a)\big), \tag{2.12}$$

where $a, b \in (F')^\times$ are nonzero elements of F'. Here the right hand side is written in terms of $\mathbb{K}_2(F') \times_D \mathbb{T}(F')$.

Proof. Fix $\alpha^\vee \in \Delta^\vee$. Recall the property $\overline{\mathbb{h}}_\alpha^{[b]}(a) = \overline{\mathbb{h}}_\alpha^{[1]}(a) \cdot \big\{b, a\big\}^{Q(\alpha^\vee)}$ from (2.4). It follows that it suffices to show (2.12) for $b = 1$ i.e. $\Phi(\overline{\mathbb{h}}_\alpha^{[1]}(a)) = \big(\big\{\eta(\alpha^\vee), a\big\}, \mathbb{h}_\alpha(a)\big)$.

Note that for fixed $\alpha \in \Delta$, Φ takes a general form

$$\Phi(\overline{\mathbb{h}}_\alpha^{[1]}(a)) = \big(\aleph_\alpha(a), \mathbb{h}_\alpha(a)\big),$$

where $\aleph_\alpha \in \mathsf{Hom}_{\mathrm{Zar}}(\mathbb{G}_{\mathrm{mul}}, \mathbb{K}_2)$ is a (sheaf) homomorphism of abelian sheaves for the big Zariski site, by property (2.3). In particular,

$$a \in (F')^\times \longmapsto \aleph_\alpha(a) \in \mathbb{K}_2(F')$$

is a homomorphism from $\mathbb{G}_{\mathrm{mul}}(F')$ to $\mathbb{K}_2(F')$ for any field extension F'/F. Note that since $\mathbb{G}_{\mathrm{mul}}$ is represented by $F[t, t^{-1}]$, we have $\mathsf{Mor}_{\mathrm{Zar}}(\mathbb{G}_{\mathrm{mul}}, \mathbb{K}_2) \simeq \mathbb{K}_2\big(F[t, t^{-1}]\big)$ (the algebraic-geometric morphisms which do not necessarily respect the group structure) by Yoneda's lemma.

However, by [BD01, §3.7-3.8], or in more details [Blo78, Thm 1.1], we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{K}_1(F) \arrow[r, "\simeq"] \arrow[d, hook, "-\otimes t"'] & \mathsf{Hom}_{\mathrm{Zar}}(\mathbb{G}_{\mathrm{mul}}, \mathbb{K}_2) \arrow[d, hook] \\
\mathbb{K}_2\big(F[t, t^{-1}]\big) \arrow[r, "\simeq"] & \mathsf{Mor}_{\mathrm{Zar}}(\mathbb{G}_{\mathrm{mul}}, \mathbb{K}_2).
\end{tikzcd}$$

That is, the left vertical cup product with $t \in \mathbb{K}_1\big(F[t, t^{-1}]\big)$ induces an isomorphism of the top row. More precisely, every $\aleph_\alpha \in \mathsf{Hom}_{\mathrm{Zar}}(\mathbb{G}_{\mathrm{mul}}, \mathbb{K}_2)$ arises from a certain $\lambda_\alpha \in F^\times = \mathbb{K}_1(F)$, which at the level F'-points is given by

$$\aleph_\alpha(a) = \big\{\lambda_\alpha, a\big\}, a \in F'.$$

However, since we have assumed that $\overline{\mathbb{G}}_{D,\eta}$ gives rise to $\phi_{D,\eta}$ from the Brylinski-Deligne classification, we have that Φ realizes $\phi_{D,\eta}$ by passing to $\mathcal{E}^{sc}$ and $\mathcal{E}$. More precisely, once we identify γ_α with $\big[\overline{\mathbb{h}}_\alpha^{[1]}(\tau)\big]$, we must have

$$\phi_{D,\eta}\big(\big[\overline{\mathbb{h}}_\alpha^{[1]}(\tau)\big]\big) = \big[\Phi\big(\overline{\mathbb{h}}_\alpha^{[1]}(\tau)\big)\big] \in \mathcal{E}. \tag{2.13}$$

Tracing through the definition of tame symbols, it gives $\phi_{D,\eta}(\gamma_\alpha) = (\lambda_\alpha, \alpha^\vee) \in \mathcal{E}$ for $\alpha \in \Delta$. But by definition $\phi_{D,\eta}(\gamma_\alpha) = (\eta(\alpha^\vee), \alpha^\vee)$. Therefore, $\lambda_\alpha = \eta(\alpha^\vee)$ for all simple root $\alpha \in \Delta$, and this completes the proof. □

To summarize, for $\overline{\mathbb{G}}_{D,\eta}$ incarnated by (D, η), its covering torus is incarnated by D with the group law given by (2.9)-(2.11). Moreover, if we write $\Phi_{D,\eta}$ for Φ instead to emphasize the dependence of the target expressed using (D, η) , then we have

$$\Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha^{[b]}(a)\big) = \big(\{b^{Q(\alpha^\vee)} \cdot \eta(\alpha^\vee), a\}, \mathbb{h}_\alpha(a)\big) \text{ for all } \alpha^\vee \in \Delta^\vee. \tag{2.14}$$

2.3 Finite degree topological covers: local and global

The goal of this section is to introduce the local and global covering groups which arise from BD framework and to give a brief discussion on some of their properties.

2.3.1 Local topological central extensions of finite degree

Assume first that F is a local field with residual characteristic p. Let $n \in \mathbf{N}_{\geq 1}$ be a natural number and assume $\mu_n \subseteq F$. The Hilbert symbol

$$(-,-)_n : \quad F^\times \times F^\times \longrightarrow \mu_n$$

descends to a bilinear from on $F^\times/(F^\times)^n$ and factors through $\mathbb{K}_2(F)$. In the tame case, i.e. when $\gcd(n, p) = 1$, we have the formula

$$(x, y)_n = \kappa(x, y)^{\frac{q-1}{n}}, \quad \kappa(x, y) = \overline{(-1)}^{\mathrm{val}(x)\mathrm{val}(y)} \overline{\left(\frac{x^{\mathrm{val}(y)}}{y^{\mathrm{val}(x)}}\right)} \in \mathbf{f},$$

where $\overline{(-)}$ denotes the reduction of $\mathcal{O}_F$ into the residue field $\mathbf{f}$ of F.

Let $\overline{\mathbb{G}} \in \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ be a central extension over F. It gives a central extension of F-groups. From the push-out by the n-th Hilbert symbol we get a central extension of $G = \mathbb{G}(F)$:

$$\begin{tikzcd}
\mathbb{K}_2(F) \arrow[r, hook] \arrow[d, "(-,-)_n"] & \overline{\mathbb{G}}(F) \arrow[r, two heads] \arrow[d] & \mathbb{G}(F) \arrow[d, equal] \\
\mu_n \arrow[r, hook] & \overline{G} \arrow[r, two heads] & G.
\end{tikzcd}$$

Consider the category $\mathsf{CExt}(\mathbb{G}(F), \mu_n)$ of central extensions of $\mathbb{G}(F)$ by μ_n. Then we have defined a functor

$$\mathsf{Hs} : \quad \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2) \rightsquigarrow \mathsf{CExt}(\mathbb{G}(F), \mu_n).$$

For any morphism $\mathbb{G}' \longrightarrow \mathbb{G}$, let $\overline{\mathbb{G}}'$ be the pull-back of $\overline{\mathbb{G}}$ to $\mathbb{G}'$ and write $r : \overline{\mathbb{G}} \longmapsto \overline{\mathbb{G}}'$ for the map. Then the following diagram commutes

$$\begin{tikzcd}
\mathsf{CExt}(\mathbb{G}, \mathbb{K}_2) \arrow[r, squiggly, "\mathsf{Hs}"] \arrow[d, squiggly, "r"] & \mathsf{CExt}(\mathbb{G}(F), \mu_n) \arrow[d, squiggly, "\mathsf{Hs}(r)"] \\
\mathsf{CExt}(\mathbb{G}', \mathbb{K}_2) \arrow[r, squiggly, "\mathsf{Hs}"] & \mathsf{CExt}(\mathbb{G}'(F), \mu_n),
\end{tikzcd}$$

where $\mathsf{Hs}(r)$ is the induced map on the central extension of F-points.

When r is induced from the inclusion $\mathbb{T} \to \mathbb{G}$

Let $\overline{\mathbb{G}} = \overline{\mathbb{G}}_{D,\eta} \in \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ be incarnated by $(D,\eta) \in \mathsf{Bis}^Q_{\mathbb{G}}$, we call $\overline{G} = \mathsf{Hs}(\overline{\mathbb{G}})$ incarnated by (D,η) also. Let $\overline{\mathbb{T}}$ be the restriction (i.e. pull-back) of $\overline{\mathbb{G}}$ to $\mathbb{T}$.

Now consider the torus $T := \mathbb{T}(F) \subseteq G$, the group $\overline{G}$ arising from $\overline{\mathbb{G}}$ gives a central extension $\overline{T}$ of T by pull-back:

$$\mu_n \hookrightarrow \overline{T} \twoheadrightarrow T .$$

Equivalently, this $\overline{T}$ is also the F-points of $\overline{\mathbb{T}} = r(\overline{\mathbb{G}}) \in \mathsf{CExt}(\mathbb{T}, \mathbb{K}_2)$. Note $T = Y \bigotimes_{\mathbf{Z}} F^\times$. Since $\overline{\mathbb{T}}$ is incarnated by D, it follows that $\overline{T}$ can be described as $\mu_n \times_D T$ with the group law inherited from (2.9)-(2.10):

$$(i) \quad \big[(\zeta_1, y_1 \otimes a), (\zeta_2, y_2 \otimes b)\big] = (a,b)_n^{B_Q(y_1,y_2)}, \tag{2.15}$$

$$(ii) \quad (1, y_1 \otimes a) \cdot (1, y_2 \otimes a) = \big((a,a)_n^{D(y_1,y_2)}, (y_1 + y_2) \otimes a\big), \tag{2.16}$$

$$(iii) \quad (1, y \otimes a) \cdot (1, y \otimes b) = \big((a,b)_n^{Q(y)}, y \otimes (ab)\big). \tag{2.17}$$

When r is induced from $\varphi_\alpha : \mathbb{SL}_2 \to \mathbb{G}^{sc}$

Let $\alpha \in \Psi$ be any root of $\mathbb{G}$ and $\varphi_\alpha : \mathbb{SL}_2 \longrightarrow \mathbb{G}^{sc}$ the map in section 2.1.4. We may also denote by φ_α the composition $\mathbb{SL}_2 \longrightarrow \mathbb{G}^{sc} \longrightarrow \mathbb{G}$. For any $\overline{\mathbb{G}} \in \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$, we get $\overline{\mathbb{SL}}_2^\alpha = \varphi_\alpha^*(\overline{\mathbb{G}})$. By Brylinski-Deligne classification as in Theorem 2.1.2, the extension $\overline{\mathbb{SL}}_2^\alpha$ is determined up to a unique isomorphism by the quadratic form $Q_{\alpha^\vee} := Q|_{\mathbf{Z}\alpha^\vee}$, or equivalently by $Q(\alpha^\vee) \in \mathbf{Z}$. The map φ_α induces a central extension $\overline{SL}_2^\alpha := \overline{\mathbb{SL}}_2^\alpha(F)$ of $SL_2 := \mathbb{SL}_2(F)$ by μ_n and we have

$$\begin{tikzcd}
\mu_n \arrow[r, hook] & \overline{G} \arrow[r, two heads] & G \\
\mu_n \arrow[r, hook] \arrow[u, equal] & \overline{SL}_2^\alpha \arrow[r, two heads] \arrow[u, "\mathsf{Hs}(r)"] & SL_2 . \arrow[u]
\end{tikzcd}$$

If $\overline{G}$ is incarnated by (D,η), then $\overline{SL}_2^\alpha$ is incarnated by $D|_{\mathbf{Z}\alpha^\vee}$, i.e., by $Q_{\alpha^\vee}$. Now the central extension $\overline{SL}_2^\alpha$ is isomorphic by a unique isomorphism to the extension described by Matsumuto (cf. [Mat69]). We can write $\overline{SL}_2^\alpha \simeq \mu_n \times_{Q_{\alpha^\vee}} SL_2$ with respect to the cocycle given by

$$\sigma(g_1, g_2) = \left(\frac{\mathrm{x}(g_1 g_2)}{\mathrm{x}(g_2)}, \frac{\mathrm{x}(g_1 g_2)}{\mathrm{x}(g_1)} \right)_n^{Q(\alpha^\vee)},$$

where

$$\mathrm{x}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{else.} \end{cases}$$

This cocycle, when restricted to the one dimensional torus T of SL_2, gives a description of the covering torus $\overline{T}^\alpha \subseteq \overline{SL}_2^\alpha$ as $\mu_n \times_{Q_{\alpha^\vee}} F^\times$. Such description agrees with the group law given by (2.15)-(2.17), which states exactly as

$$\big(\zeta_1, \mathbb{h}_\alpha(a)\big) \cdot \big(\zeta_2, \mathbb{h}_\alpha(c)\big) = \big(\zeta_1\zeta_2(a,c)_n^{Q(\alpha^\vee)}, \mathbb{h}_\alpha(ac)\big).$$

That is, the cocycle σ on SL_2 given by Matsumuto above is an extension to the whole group SL_2 of the cocycle on T specified from incarnation.

2.3.2 Local splitting properties

Splitting over a maximal compact group

Continue to assume F a local field, and we fix the epinglage for $(\mathbb{G}, \mathbb{T}, \mathbb{B})$ as before. The Bruhat-Tits building of $G = \mathbb{G}(F)$ over F has a hyperspecial point determined by the epinglage, which gives an associated group scheme $\mathbf{G}$ over $\mathcal{O}_F$ with generic fibre $\mathbb{G}$ via the Bruhat-Tits theory, i.e., we have $\mathbb{G} = \mathbf{G} \times_{\mathcal{O}_F} F$.

Let $K = \mathbf{G}(\mathcal{O}_F)$, which is a maximal compact subgroup of G. We are interested in the case when K splits into $\overline{G}$.

More precisely, assume n prime to the residue characteristic of F, the Hilbert symbol $(-,-)_n$ becomes a power of the tame symbol, and it gives a degree n central cover $\overline{G}$ of G. We are interested in the case when there exists a splitting s_K of K into $\overline{G}$:

$$\begin{tikzcd} \mu_n \arrow[r, hook] & \overline{G} \arrow[r, two heads] & G \\ & & K \arrow[u, hook] \arrow[ul, dashed, "s_K"] \end{tikzcd} .$$

Definition 2.3.1. The group $\overline{G}$ called s_K-unramified (or simply unramified) if $\gcd(n,p) = 1$ and there exists a splitting s_K of K into $\overline{G}$.

Note that the $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}$ defined over F may not be the base change of some $\mathbf{K}_2$-torsor $\overline{\mathbf{G}}$ over $\mathrm{Spec}(\mathcal{O}_F)$. Otherwise suppose $\overline{\mathbb{G}} = \overline{\mathbf{G}} \times_{\mathcal{O}_F} F$, since the tame Hilbert symbol (if we assume $\gcd(n,d) = 1$) vanishes on $\mathbf{K}_2(\mathcal{O}_F)$, the existence of the splitting of K into $\overline{G}$ is then automatic (cf. [BD01, §10.7]). This suggest that the existence of splitting of K, relies on other data besides the condition $\gcd(n,d) = 1$. We refer to the recent work of Weissman (cf. [We11], [We14-1]) on integral models for Brylinski-Deligne covering groups over F.

In the language of incarnations, if one starts in general with $\overline{\mathbb{G}}_{D,\eta}$, then the existence of splitting s_K as above holds only conditionally. In fact, one sufficient condition is that the homomorphism η has image in the units $\mathcal{O}_F^\times$ of $F^\times$:

$$\eta : \qquad \begin{tikzcd} Y^{sc} \arrow[r] \arrow[d, dashed] & F^\times \\ \mathcal{O}_F^\times \arrow[ur, hook] & \end{tikzcd}$$

In particular, the condition $\gcd(n, q) = 1$ is not sufficient to guarantee the splitting of K in general. For more detailed discussions, see [GaG14, §4].

Unipotent splitting of $\overline{G}$

In section 2.1.4, we have seen that the unipotent subgroup $\mathbb{U}$ of the Borel subgroup $\mathbb{B}^{sc}$ of $\mathbb{G}^{sc}$ splits in $\overline{\mathbb{G}}^{sc}$, and the splitting is $\mathbb{B}^{sc}$-equivariant. In fact, the same holds with $\mathbb{G}^{sc}$ replaced by $\mathbb{G}$ (cf. [BD01, Prop. 11.3]): there exists a unique $\mathbb{B}$-equivariant splitting of $\mathbb{U}$ into $\overline{\mathbb{G}}$. Taking F-rational points, we obtain a B-equivariant splitting of $\mathbb{U}(F)$ into $\overline{G}$. It is important that such splitting is unique, see also [MW95, App. A].

In fact, the above splitting on the unipotent radical of the Borel subgroup could be extended to a section of the set G_u of unipotent elements of G with certain properties. As before, the group $\overline{G}$ acts on G_u by conjugation, which descends to G. We have the following useful fact, whose proof we refer to [Li12, Prop. 2.2.1].

Proposition 2.3.2. *There exists uniquely a continuous set section* $i_u : G_u \longrightarrow \overline{G}$ *such that*

(i) for all unipotent subgroup U of G, the restriction $i_u|_U$ is a homomorphism (i.e. a splitting), and

(ii) i_u is G-conjugation invariant. That is, $i_u(g^{-1}ug) = g^{-1}i_u(u)g$ for all $g \in G$.

Note that in section 2.1.4, we have used the notation $\overline{\mathbb{e}}_\alpha(-)$ for the splitting of $\mathbb{U}(F)$ into $\overline{\mathbb{G}}^{sc}(F)$:

$$\mathbb{e}_\alpha(a) \longmapsto \overline{\mathbb{e}}_\alpha(a).$$

We also recall that we have denoted by Φ for the natural map $\mathbb{G}^{sc}(F) \longrightarrow \mathbb{G}(F)$, and also $\Phi_{D,\eta}$ for the induced natural pull-back map $\overline{\mathbb{G}}^{sc}(F) \longrightarrow \overline{\mathbb{G}}(F)$. Write

$$e_\alpha(a) := \Phi(\mathbb{e}_\alpha(a)), \quad \overline{e}_\alpha(a) := \Phi_{D,\eta}(\overline{\mathbb{e}}_\alpha(a)).$$

Since Φ is injective on the unipotent elements, the map

$$e_\alpha(a) \longmapsto \overline{e}_\alpha(a),$$

is a splitting of $\mathbb{U}(F)$ into G.

As a consequence of above proposition,

Corollary 2.3.3. *For $\alpha \in \Psi$, the unique splitting i_u of the unipotent subgroup U_α is just given by*

$$e_\alpha(a) \longmapsto \overline{e}_\alpha(a),$$

which is G-equivariant.

Moreover, assume $\overline{G}$ s_K-unramified. Let $U := \mathbb{U}(F)$ be the unipotent radical of some parabolic $P \subseteq G$, then

$$s_K|_{K\cap U} = i_u|_{K\cap U}.$$

Proof. We only need to show the second assertion. Note that $K \cap U$ is a pro-p group. The two splittings $s_K|_{K\cap U}$ and $i_u|_{K\cap U}$ differ by a homomorphism from $K \cap U$ to μ_n, which has to be trivial since n is prime to p. □

2.3.3 Global topological central extensions of finite degree

Back to the global situation now and assume F is a number field with $\mu_n \subseteq F^\times$. Write F_v for the completion of F with respect to any place $v \in |F|$. Let $\mathbb{A}_F$ be the adele ring of F. Let $\overline{\mathbb{G}}_{D,\eta} \in \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ be a $\mathbb{K}_2$-torsor over F.

In [BD01], it is shown that one has the following inherited data:

- For all v, a central extension $\mu_n \hookrightarrow \overline{G}_v \twoheadrightarrow G_v$. For almost all v, there is a splitting s_{K_v} of the group $K_v := \mathbb{G}(\mathcal{O}_v)$, which is well-defined, into $\overline{G}_v$. In this case, K_v is equal to $\mathbf{G}_v(\mathcal{O}_v)$, where $\mathbf{G}_v$ is the integral model of $\mathbb{G}_v := \mathbb{G} \times_F F_v$ given by the Bruhat-Tits theory.

- An adelic group $\overline{\mathbb{G}}(\mathbb{A}_F) := \prod'_v \overline{G}_v \big/ (\bigoplus_v \mu_n)^o$, where $\prod'_v \overline{G}_v$ is the restricted product with respect to the K_v's and $(\bigoplus_v \mu_n)^o := \left\{\bigoplus \xi_v : \prod_v \xi_v = 1\right\}$. For $v \in |F|$, there is the natural inclusion

$$\begin{tikzcd} \mu_n \arrow[r, hook] & \overline{\mathbb{G}}(\mathbb{A}_F) \arrow[r, two heads] & \mathbb{G}(\mathbb{A}_F) \\ \mu_n \arrow[u, equal] \arrow[r, hook] & \overline{G}_v \arrow[u, hook] \arrow[r, two heads] & G_v \arrow[u, hook] \end{tikzcd} .$$

- There is a natural splitting of $\mathbb{G}(F)$ into $\overline{\mathbb{G}}(\mathbb{A}_F)$,

$$\begin{tikzcd} \mu_n \arrow[r, hook] & \overline{\mathbb{G}}(\mathbb{A}_F) \arrow[r, two heads] & \mathbb{G}(\mathbb{A}_F) \\ & & \mathbb{G}(F) \arrow[u, hook] \arrow[ul, hook] \end{tikzcd}$$

 which allows us to define the notion of automorphic forms on $\overline{\mathbb{G}}(\mathbb{A}_F)$.

- For any unipotent subgroup $\mathbb{U}$ of a parabolic $\mathbb{P} = \mathbb{M}\mathbb{U}$ of $\mathbb{G}$, there is a unique $\mathbb{P}(F)$-equivariant splitting of $\mathbb{U}(\mathbb{A}_F)$ into $\overline{\mathbb{G}}(\mathbb{A}_F)$

$$\begin{tikzcd} \mu_n \arrow[r, hook] & \overline{\mathbb{G}}(\mathbb{A}_F) \arrow[r, two heads] & \mathbb{G}(\mathbb{A}_F) \\ & & \mathbb{U}(\mathbb{A}_F) \arrow[u, hook] \arrow[ul, hook] \end{tikzcd}$$

 such that its restriction to $\mathbb{U}(F)$ is the fixed splitting of $\mathbb{G}(F)$ restricted to $\mathbb{U}(F)$.

2.4 Dual groups and L-groups for topological extensions

The references are [FiLy10], [McN12], [We13] and [We14]. We will only recall here some facts important to us and refer to the original papers for details.

2.4.1 The dual group $\overline{\mathbb{G}}^\vee$ à la Finkelberg-Lysenko-McNamara-Reich

Let $\mathbb{G}$ be a split reductive group over F with root datum $(X, \Psi, Y, \Psi^\vee)$. We have also fixed $\Delta \subseteq \Psi$, a set of simple roots. For any $\overline{\mathbb{G}}$ in the Brylinski-Deligne framework, it gives rise to local and global topological degree n covers, depending on whether F is local or global. Several authors have defined a group scheme $\overline{\mathbb{G}}^\vee$ over F, whose complex points $\overline{G}^\vee = \overline{\mathbb{G}}^\vee(\mathbf{C})$ would be called the complex dual group for the degree n topological covers. In fact $\overline{\mathbb{G}}^\vee$ could be defined over a much smaller ring (cf. [We13] [We14]); but for simplicity here, we will refrain from stating the construction in its most general form.

First we describe the root datum of $\overline{\mathbb{G}}^\vee$. Let Q be the quadratic forms associated with $\overline{\mathbb{G}}$. Define

$$Y_{Q,n} = \left\{ y \in Y \mid B_Q(y, y') \in n\mathbf{Z} \text{ for all } y' \in Y \right\}.$$

It is a sublattice of Y and clearly contains nY. For any $\alpha \in \Psi$, we also write

$$n_\alpha = \frac{n}{\gcd(Q(\alpha^\vee), n)}, \quad \alpha^\vee_{[n]} = n_\alpha \alpha^\vee.$$

Let $Y^{sc}_{Q,n}$ be the lattice generated by $\alpha^\vee_{[n]}, \alpha^\vee \in \Psi^\vee$ with relations inherited from Y (cf. [BD01, Lm. 11.5]). Now consider the quadruple given by

$$\begin{aligned}
Y_1 &= Y_{Q,n}, \\
\Psi_1^\vee &= \left\{ \alpha^\vee_{[n]} : \alpha^\vee \in \Psi^\vee \right\}, \\
X_1 &= \mathrm{Hom}(Y_1, \mathbf{Z}) \subseteq X \otimes \mathbf{Q}, \\
\Psi_1 &= \left\{ n_\alpha^{-1} \alpha : \alpha \in \Psi \right\}.
\end{aligned}$$

Theorem 2.4.1. *The quadruple $(Y_1, \Psi_1^\vee, X_1, \Psi_1)$ forms a root datum.*

Proof. See [McN12, §13.11]. □

Define the dual group $\overline{\mathbb{G}}^\vee$ to be the split pinned reductive group associated with this root datum. Let $\overline{G}^\vee = \overline{\mathbb{G}}^\vee(\mathbf{C})$. In particular, if $\mathbb{G} = \mathbb{T}$ is a torus, then $\overline{T}^\vee = X_1 \otimes \mathbf{C}$.

2.4.2 Local L-group à la Weissman

Let F be a number field or a local field obtained as the localization of a number field. Let $(Q, \mathcal{E}, \phi)$ be the BD classification data associated with $\overline{\mathbb{G}} \in \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$, then the construction of $\overline{\mathbb{G}}^\vee$ uses the data n and B_Q alone. The construction of the L-group for covering groups is due to M. Weissman and utilizes the full data $(Q, \mathcal{E}, \phi)$. In [We13] and moreover [We14], he has constructed a proalgebraic group scheme ${}^L\overline{\mathbb{G}}$ over $\mathbf{Z}[\mu_n]$. For our purpose, it suffices to have ${}^L\overline{\mathbb{G}}$ defined over certain subfield of $\mathbf{C}$ which sits in the exact sequence

$$\overline{\mathbb{G}}^\vee \hookrightarrow {}^L\overline{\mathbb{G}} \twoheadrightarrow \mathbb{W}_F .$$

Here $\mathbb{W}_F$ is the constant group scheme over the field of definition whose complex point is the global Weil group W_F if F is a number field; and it could be the local Weil group W_F or local Weil-Deligne group $\mathrm{WD}_F = \mathbb{SL}_2(\mathbf{C}) \times \mathrm{W}_F$ if F is a local field.

The freedom for different candidates in the place of $\mathbb{W}_F$ in the local case is due to the fact that the essential part of the construction of ${}^L\overline{\mathbb{G}}$ is a fundamental extension over $F^\times$. However, for our interest we will concentrate on $\mathbb{W}_F$ associated with the local and global Weil group respectively.

The constructions for the dual group $\overline{\mathbb{G}}^\vee$ and the L-group ${}^L\overline{\mathbb{G}}$ are both compatible with Levi subgroups. More precisely, consider any Levi subgroup $\mathbb{H}$ of $\mathbb{G}$ embedded in the latter by φ. It induces a morphism $\mathsf{Hs}(\varphi) : \overline{H} \longrightarrow \overline{G}$. Then the construction of dual group and L-group gives the commutative diagram

$$\begin{tikzcd}
\overline{\mathbb{G}}^\vee \arrow[r, hook] & {}^L\overline{\mathbb{G}} \arrow[r, two heads] & \mathbb{W}_F \arrow[d, equal] \\
\overline{\mathbb{H}}^\vee \arrow[r, hook] \arrow[u, hook, "\varphi^\vee"] & {}^L\overline{\mathbb{H}} \arrow[r, two heads] \arrow[u, hook, "{}^L\varphi"] & \mathbb{W}_F
\end{tikzcd}$$

In the case of $\mathbb{H} = \mathbb{T} \subseteq \mathbb{G}$, by taking $\mathbf{C}$-point we obtain

$$\begin{tikzcd}
\overline{G}^\vee \arrow[r, hook] & {}^L\overline{G} \arrow[r, two heads] & \mathbb{W}_F(\mathbf{C}) \arrow[d, equal] \\
\overline{T}^\vee \arrow[r, hook] \arrow[u, hook, "\varphi^\vee"] & {}^L\overline{T} \arrow[r, two heads] \arrow[u, hook, "{}^L\varphi"] & \mathbb{W}_F(\mathbf{C}) .
\end{tikzcd}$$

In the remaining part of this section, we assume F a local field. We will recall in details the construction of the extension of ${}^L\overline{G}$ in [We14] for general BD type extension $\overline{G}$:

$$\overline{G}^\vee \hookrightarrow {}^L\overline{G} \twoheadrightarrow \mathrm{W}_F .$$

Since we work with $\overline{G}$ which is incarnated by (D, η), the description in [We14] can be carried out in this language. The construction of ${}^L\overline{G}$ relies on two abelian extensions E_i for $i = 1, 2$ by the center $Z(\overline{G}^\vee)$:

$$Z(\overline{G}^\vee) \hookrightarrow E_i \twoheadrightarrow F^\times.$$

Now we describe the two extensions E_i. Note $Z(\overline{G}^\vee) = \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{sc}, \mathbf{C}^\times)$.

The extension E_1

The extension E_1 is given by the cocycle

$$\begin{aligned} F^\times \times F^\times &\longrightarrow \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{sc}, \mathbf{C}^\times) \\ (a,b) &\longmapsto \big(y \mapsto (a,b)_n^{Q(y)}\big). \end{aligned} \tag{2.18}$$

Thus, we write $E_1 = Z(\overline{G}) \times_Q F^\times$ with the group law given above. Note that since $(a,b)_n^{Q(y)} \in \mu_2$ since $2n|Q(y)$ for $y \in Y_{Q,n}$, the cocycle actually takes values in $\mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{sc}, \mu_2)$.

The extension E_2

For E_2, consider the extension $\mathcal{E}$ and $\mathcal{E}^{sc}$ associated to the covering tori $\overline{\mathbb{T}}$ and $\overline{\mathbb{T}}^{sc}$ of $\overline{\mathbb{G}}$ and $\overline{\mathbb{G}}^{sc}$ respectively.

For convenience, we write $F^\times/n = F^\times/(F^\times)^n$. Consider the pull-back of $\mathcal{E}^{sc}$ by $Y_{Q,n}^{sc}$ and the followed-up push-out by the quotient map $F^\times \longrightarrow F^\times/n$, we obtain a group $\mathcal{E}_{Q,n}^{sc}$ which sits in an exact sequence

$$F^\times/n \hookrightarrow \mathcal{E}_{Q,n}^{sc} \twoheadrightarrow Y_{Q,n}^{sc}.$$

Fix an arbitrary nonzero $f \in F(\!(\tau)\!)$. We define a section $\mathbf{s}$ of $\mathcal{E}_{Q,n}^{sc}$ over $Y_{Q,n}^{sc}$ as follows. For every element $\alpha_{[n]}^\vee \in Y_{Q,n}^{sc}$ with $\alpha \in \Psi$, it is given by

$$\mathbf{s}: \quad \alpha_{[n]}^\vee \longmapsto \overline{\mathbb{h}}_\alpha^{[f]}(n_\alpha \alpha^\vee), \quad \alpha \in \Psi.$$

Recall from Definition 2.1.7 $\overline{\mathbb{h}}_\alpha^{[f]}(n_\alpha \alpha^\vee) = \big[\overline{\mathbb{h}}_\alpha^{[f]}(\tau^{n_\alpha})\big] \in \mathcal{E}^{sc}$ and it clearly lies over $\alpha_{[n]}^\vee$, and by abuse of notation we also use it to denote its image in $\mathcal{E}_{Q,n}^{sc}$. Because of the property (cf. (2.4))

$$\overline{\mathbb{h}}_\alpha^{[f]}(\tau^{n_\alpha}) = \overline{\mathbb{h}}_\alpha^{[1]}(\tau^{n_\alpha}) \cdot \big\{f, \tau^{n_\alpha}\big\}^{Q(\alpha^\vee)},$$

it follows that $\big[\overline{\mathbb{h}}_\alpha^{[f]}(\tau^{n_\alpha})\big] \in \mathcal{E}_{Q,n}^{sc}$ is independent of f, which justifies the absence of f in the notation $\mathbf{s}$ we use. Therefore we may omit the superscript in both $\overline{\mathbb{h}}_\alpha^{[f]}(n_\alpha \alpha^\vee)$ and $\big[\overline{\mathbb{h}}_\alpha^{[f]}(\tau^{n_\alpha})\big]$, and instead just write $\overline{\mathbb{h}}_\alpha(n_\alpha \alpha^\vee)$, $\big[\overline{\mathbb{h}}_\alpha(\tau^{n_\alpha})\big]$ respectively. For computational convenience, one may take $f = 1$ and there is no loss of generality because of the independence of f in general.

What is more important is the following

Proposition 2.4.2. *The section* $\mathbf{s}: \alpha_{[n]}^\vee \longmapsto \overline{\mathbb{h}}_\alpha(n_\alpha \alpha^\vee)$ *for all* $\alpha \in \Psi$ *above gives a well-defined splitting (i.e. a homomorphism) of the sequence:*

$$\begin{tikzcd} F^\times/n \arrow[r, hook] & \mathcal{E}_{Q,n}^{sc} \arrow[r, two heads] & Y_{Q,n}^{sc} \arrow[l, bend right, "\mathbf{s}"'] \end{tikzcd}.$$

Proof. For all $\alpha \in \Psi$, write $\alpha_{[n]} := \alpha/n_\alpha$. The lattice $Y_{Q,n}^{sc}$ is generated by $\alpha_{[n]}^\vee$ for all $\alpha^\vee \in \Psi^\vee$ with the following relation (cf. [BD01, §11.5]):

$$s_{\alpha_{[n]}}(\beta_{[n]})^\vee = \beta_{[n]}^\vee - \langle \alpha_{[n]}, \beta_{[n]}^\vee \rangle \alpha_{[n]}^\vee,$$

which is equivalent to $n_\beta s_\alpha(\beta)^\vee = n_\beta \beta^\vee - n_\beta \langle \alpha, \beta^\vee \rangle \alpha^\vee$. Note that Lemma 2.1.3 is valid for general $\overline{\mathbb{G}}$ of BD type, and it gives

$$\begin{aligned} n_\beta \cdot \langle \alpha, \beta^\vee \rangle Q(\beta^\vee) &= n \cdot \langle \alpha, \beta^\vee \rangle \\ &= n_\alpha \cdot Q(\alpha^\vee) \langle \alpha, \beta^\vee \rangle \\ &= n_\alpha \cdot B(\alpha^\vee, \beta^\vee) \quad \text{ by Lemma 2.1.3} \\ &= n_\alpha \cdot \langle \beta, \alpha^\vee \rangle Q(\beta^\vee). \end{aligned}$$

Thus it follows

$$n_\beta \cdot \langle \alpha, \beta^\vee \rangle = n_\alpha \cdot \langle \beta, \alpha^\vee \rangle. \tag{2.19}$$

We see that it suffices to show

$$\mathbf{s}\big(n_\beta s_\alpha(\beta)^\vee\big) = \mathbf{s}\big(n_\beta \beta^\vee\big) \cdot \mathbf{s}\big(n_\alpha \alpha^\vee\big)^{-\langle \beta, \alpha^\vee \rangle}.$$

Write $\gamma = s_\alpha(\beta)$, then $Q(\gamma^\vee) = Q(\beta^\vee)$ and $n_\gamma = n_\beta$. We have $\mathbf{s}\big(n_\beta s_\alpha(\beta)^\vee\big) = \mathbf{s}\big(n_\gamma \gamma^\vee\big) = \big[\mathbb{h}_\gamma(\tau^{n_\gamma})\big]$ which by [BD01, §11.3] is equal to

$$\begin{aligned} &\big[\mathbb{h}_\beta(\tau^{n_\beta})\big] \cdot \big[\mathbb{h}_\alpha(\tau^{-n_\beta \langle \alpha, \beta^\vee \rangle})\big] \\ =&\big[\mathbb{h}_\beta(\tau^{n_\beta})\big] \cdot \big[\mathbb{h}_\alpha(\tau^{-n_\alpha \langle \beta, \alpha^\vee \rangle})\big] \\ =&\big[\mathbb{h}_\beta(\tau^{n_\beta})\big] \cdot \big[\mathbb{h}_\alpha(\tau^{n_\alpha})\big]^{-\langle \beta, \alpha^\vee \rangle} \cdot (-1)^{n_\alpha^2 \varepsilon(-\langle \beta, \alpha^\vee \rangle) Q(\alpha^\vee)} \quad \text{ by [BD01, (11.6.1)]}, \end{aligned}$$

where $\varepsilon(N) = N(N-1)/2$. Note $n | n_\alpha Q(\alpha^\vee)$, and hence $(-1)^{n_\alpha^2 \varepsilon(-\langle \beta, \alpha^\vee \rangle) Q(\alpha^\vee)} \in (F^\times)^n$. It follows $\big[\mathbb{h}_\gamma(\tau^{n_\gamma})\big] = \big[\mathbb{h}_\beta(\tau^{n_\beta})\big] \cdot \big[\mathbb{h}_\alpha(\tau^{n_\alpha})\big]^{-\langle \beta, \alpha^\vee \rangle}$ in $\mathcal{E}_{Q,n}^{sc}$, which is an extension with kernel $F^\times/n$. This exactly shows that $\mathbf{s}$ is a homomorphism and the proof is completed. ☐

Back to the main discussion, we could obtain on the other hand a group $\mathcal{E}_{Q,n}$ as composition of the pull-back of $\mathcal{E}$ by $Y_{Q,n} \hookrightarrow Y$ and the push-out by the quotient map $F^\times \longrightarrow F^\times/n$. It is an extension of $Y_{Q,n}$ by $F^\times/n$:

$$F^\times/n \hookrightarrow \mathcal{E}_{Q,n} \twoheadrightarrow Y_{Q,n}.$$

There is the inherited map $\mathcal{E}_{Q,n}^{sc} \longrightarrow \mathcal{E}_{Q,n}$ still denoted by $\phi_{D,\eta}$.

To summarize, we have the following commutative diagram with a canonical splitting $\mathbf{s}$ for the top row.

$$\begin{tikzcd} F^\times/n \arrow[r, hook] \arrow[d, equal] & \mathcal{E}_{Q,n}^{sc} \arrow[r, two heads] \arrow[d, "\phi_{D,\eta}"] & Y_{Q,n}^{sc} \arrow[l, bend right, "\mathbf{s}"'] \arrow[d, hook] \\ F^\times/n \arrow[r, hook] & \mathcal{E}_{Q,n} \arrow[r, two heads] & Y_{Q,n}. \end{tikzcd} \tag{2.20}$$

As a simple consequence of above discussion

Corollary 2.4.3. *The splitting* $\mathbf{s} : Y_{Q,n}^{sc} \longrightarrow \mathcal{E}_{Q,n}^{sc}$ *gives rise to a splitting of* $Y_{Q,n}^{sc}$ *into* $\mathcal{E}_{Q,n}$ *which takes the explicit form for* $\alpha \in \Delta$ *as*

$$\phi_{D,\eta} \circ \mathbf{s}(\alpha_{[n]}^\vee) = \big(\eta(\alpha_{[n]}^\vee), \alpha_{[n]}^\vee\big), \quad \alpha^\vee \in \Delta^\vee,$$

where the right hand side is written in the form with respect to $\mathcal{E}_{Q,n} = F^\times \times_D Y_{Q,n}$.

Proof. For $\alpha^\vee \in \Delta^\vee$, direct computation gives

$$\begin{aligned}\phi_{D,\eta} \circ \mathbf{s}(\alpha_{[n]}^\vee) =& \phi_{D,\eta}\big(\big[\overline{\mathbb{h}}_\alpha(\tau^{n_\alpha})\big]\big)\\ =& \phi_{D,\eta}\big(\big[\overline{\mathbb{h}}_\alpha(\tau)\big]^{n_\alpha} \cdot (-1)^{\varepsilon(n_\alpha)Q(\alpha^\vee)}\big) \text{ by [BD01, (11.6.1)]}\\ =& \big((\eta(\alpha^\vee), \alpha^\vee)\big)^{n_\alpha} \cdot (-1)^{\varepsilon(n_\alpha)Q(\alpha^\vee)}\\ =& \big(\eta(n_\alpha \alpha^\vee), n_\alpha \alpha^\vee\big),\end{aligned}$$

which is the desired result. $\square$

The splitting of the row over $Y_{Q,n}^{sc}$ gives

$$\begin{tikzcd} F^\times/n \arrow[r, hook] \arrow[d, equal] & \mathcal{E}_{Q,n} \arrow[r, two heads] \arrow[d] & Y_{Q,n} \arrow[d] \\ F^\times/n \arrow[r, hook] & \mathcal{E}_{Q,n}/\phi_{D,\eta} \circ \mathbf{s}(Y_{Q,n}^{sc}) \arrow[r, two heads] & Y_{Q,n}/Y_{Q,n}^{sc}. \end{tikzcd} \tag{2.21}$$

By taking $\mathrm{Hom}(-, \mathbf{C}^\times)$, we further consider the pull-back by the map $h : F^\times \to \mathrm{Hom}(F^\times/n, \mathbf{C}^\times)$ given by the Hilbert symbol $a \mapsto h_a$ with $h_a(b) = (b, a)_n$. Then by definition E_2 is the pull-back of $\mathrm{Hom}(\mathcal{E}_{Q,n}/\phi_{D,\eta} \circ \mathbf{s}(Y_{Q,n}^{sc}), \mathbf{C}^\times)$ by h:

$$\begin{tikzcd} Z(\overline{G}^\vee) \arrow[r, hook] \arrow[d, equal] & \mathrm{Hom}(\mathcal{E}_{Q,n}/\phi_{D,\eta} \circ \mathbf{s}(Y_{Q,n}^{sc}), \mathbf{C}^\times) \arrow[r, two heads] & \mathrm{Hom}(F^\times/n, \mathbf{C}^\times) \\ Z(\overline{G}^\vee) \arrow[r, hook] & E_2 \arrow[r, two heads] \arrow[u] & F^\times. \arrow[u, "h"] \end{tikzcd}$$

The fundamental extension $E_{\overline{G}}$ and the group ${}^L\overline{G}$

Definition 2.4.4. Consider the Baer sum $E_1 \oplus_B E_2$, which we denote by $E_{\overline{G}}$. We call $E_{\overline{G}}$ the fundamental extension associated with $\overline{G}$.

For fixed n, the construction $\overline{G} \rightsquigarrow E_{\overline{G}}$ could be viewed as a construction of the abelian $E_{\overline{G}}$ in the category $\mathsf{Ab}(F^\times, Z(\overline{G}^\vee))$ of abelian extensions of $F^\times$ by $Z(\overline{G})$, which starts with the category of incarnations $\mathsf{CExt}(\mathbb{G}, \mathsf{K}_2)$. The category $\mathsf{Ab}(F^\times, Z(\overline{G}^\vee))$ is clearly a commutative Picard category with respect to the Baer sum.

Proposition 2.4.5 ([We14])**.** *Fix n. The construction of* $E_{\overline{G}}$ *from* $\overline{\mathbb{G}} \in \mathsf{CExt}(\mathbb{G}, \mathsf{K}_2)$ *is a functor of Picard categories*

$$\mathsf{CExt}(\overline{\mathbb{G}}, \mathsf{K}_2) \rightsquigarrow \mathsf{Ab}(F^\times, Z(\overline{G}^\vee)).$$

Therefore, the composition $(D,\eta) \mapsto \overline{G}_{D,\eta} \mapsto E_{\overline{G}_{D,\eta}}$ *also gives a functor of Picard categories:*

$$\mathsf{Bis}_{\overline{\mathbb{G}}} \rightsquigarrow \mathsf{Ab}(F^\times, Z(\overline{G}^\vee)).$$

Recall that W_F is the Weil group of F. Denote by Rec the composition

$$\mathrm{Rec}: \quad \mathrm{W}_F \longrightarrow F^\times,$$

which is induced from the Artin recirpocity map $\mathrm{W}_F^{ab} \longrightarrow F^\times$ which sending the class of geometric Frobenius to the class of uniformizer (modulo $\mathcal{O}_F^\times$) of $F^\times$.

We obtain the pull-back $\mathrm{Rec}^*(E_{\overline{G}})$:

$$Z(\overline{G}) \hookrightarrow \mathrm{Rec}^*(E_{\overline{G}}) \twoheadrightarrow \mathrm{W}_F .$$

Then,

Definition 2.4.6. Let $\overline{\mathbb{G}} \in \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ be a BD extension, and let $\overline{G} \in \mathsf{CExt}(G, \mu_n)$ be the resulting topological extension. Consider the natural inclusion $j^{\overline{G}^\vee}: Z(\overline{G}^\vee) \hookrightarrow \overline{G}^\vee$. The L-group ${}^L\overline{G}$ of $\overline{G}$ is defined to be the push-out $j_*^{\overline{G}}$ of $\mathrm{Rec}^*(E_{\overline{G}})$:

$${}^L\overline{G} := j_*^{\overline{G}^\vee} \circ \mathrm{Rec}^*(E_{\overline{G}}) = \frac{\overline{G}^\vee \times \mathrm{Rec}^*(E_{\overline{G}})}{\nabla Z(\overline{G}^\vee)},$$

where $\nabla: Z(\overline{G}^\vee) \hookrightarrow \overline{G}^\vee \times Z(\overline{G}^\vee)$ is the anti-diagonal embedding. Also, we may use the equivalent $\mathrm{Rec}^* \circ j_*^{\overline{G}^\vee}(E_{\overline{G}})$ for the definition of ${}^L\overline{G}$.

Remark 2.4.7. The definition of of $E_{2,\overline{G}}$ here differs slightly from that of [We14], where one considers the section of $\mathcal{E}_{Q,n}^{sc}$ over $Y_{Q,n}^{sc}$ given by

$$\mathbf{s}': \quad \alpha_{[n]}^\vee \longmapsto \overline{\mathbb{h}}_\alpha^{[f]}(\alpha^\vee)^{n_\alpha}.$$

Similar proof shows that this gives a splitting, and in particular is independent of f as well. However, the splitting $\mathbf{s}$ we defined gives exactly the expected match-up between the eigenvalue of adjoint action and the GK coefficient, whereas for $\mathbf{s}'$ there would be an additional term of (-1)-powers in the expression.

Functoriality for Levi subgroups

The group $\overline{G}$ gives rise to the Levi $\overline{M}$ by restriction of the Brylinski-Deligne data, and we obtain ${}^L\overline{G}$ and ${}^L\overline{M}$. It is shown in [We14] (see also [GaG14, §5.5]) that there is a canonical morphism ${}^L\varphi$ extending the inclusion $\varphi^\vee: {}^L\overline{M} \longrightarrow {}^L\overline{G}$ such that the diagram commutes:

$$\begin{tikzcd}
\overline{G}^\vee \arrow[r, hook] & {}^L\overline{G} \arrow[r, two heads] & \mathrm{W}_F \\
\overline{M}^\vee \arrow[r, hook] \arrow[u, hook, "\varphi^\vee"] & {}^L\overline{M} \arrow[r, two heads] \arrow[u, hook, "{}^L\varphi"] & \mathrm{W}_F. \arrow[u, equal]
\end{tikzcd}$$

The case $\overline{M} = \overline{T}$ is easy and crucial to us, and therefore we give the proof. Note ${}^L\overline{T} = \mathrm{Rec}^*(E_{\overline{T}})$ since $Z(\overline{T}^\vee) = \overline{T}^\vee$.

Proposition 2.4.8. *Consider $\overline{G}$ and $\overline{T}$ the covering torus of $\overline{G}$. Let $j^{\overline{T}^\vee} : Z(\overline{G}^\vee) \hookrightarrow \overline{T}^\vee$ be the inclusion. Then we have a canonical isomorphism*

$$ {}^L\overline{T} \simeq j_*^{\overline{T}^\vee} \circ Rec^*(E_{\overline{G}}), $$

where ${}^L\overline{T} = Rec^*(E_{\overline{T}})$ by definition. Therefore it induces a canonical ${}^L\varphi$ from ${}^L\overline{T}$ to ${}^L\overline{G}$ giving the desired commutative diagram above.

Proof. It suffices to show

$$ E_{\overline{T}} \simeq j_*^{\overline{T}^\vee}(E_{\overline{G}}). $$

Since $E_{\overline{T}} = E_{1,\overline{T}} \oplus_B E_{2,\overline{T}}$ and $E_{\overline{G}} = E_{1,\overline{G}} \oplus_B E_{2,\overline{G}}$, we are reduced to show

$$ E_{i,\overline{T}} \simeq j_*^{\overline{T}^\vee}(E_{i,\overline{G}}), i = 1, 2. $$

The case $i = 1$ can be easily checked to be true. Consider the case $i = 2$ and the diagram as in (2.21):

$$ \begin{tikzcd} F^\times/n \arrow[r, hook] \arrow[d, equal] & \mathcal{E}_{Q,n} \arrow[r, two heads] \arrow[d] & Y_{Q,n} \arrow[d] \\ F^\times/n \arrow[r, hook] & \mathcal{E}_{Q,n}/\phi_{D,\eta} \circ \mathbf{s}(Y_{Q,n}^{sc}) \arrow[r, two heads] & Y_{Q,n}/Y_{Q,n}^{sc}. \end{tikzcd} $$

Taking $\mathrm{Hom}(-, \mathbf{C}^\times)$ and pull-back by $F^\times \longrightarrow F^\times/n$, we obtain

$$ \begin{tikzcd} \overline{T}^\vee \arrow[r, hook] & E_{2,\overline{T}} \arrow[r, two heads] & F^\times \arrow[d, equal] \\ Z(\overline{G}^\vee) \arrow[r, hook] \arrow[u, hook, "j^{\overline{T}^\vee}"] & E_{2,\overline{G}} \arrow[r, two heads] \arrow[u, "\phi_{D,\eta}^*"'] & F^\times \end{tikzcd} $$

The universal property of push-out $j_*^{\overline{T}^\vee}$ gives the middle map of the following commutative diagram

$$ \begin{tikzcd} \overline{T}^\vee \arrow[r, hook] \arrow[d, equal] & j_*^{\overline{T}^\vee}(E_{2,\overline{G}}) \arrow[r, two heads] \arrow[d] & F^\times \arrow[d, equal] \\ \overline{T}^\vee \arrow[r, hook] & E_{2,\overline{T}} \arrow[r, two heads] & F^\times \end{tikzcd}, $$

which has identity maps on both $\overline{T}^\vee$ and $F^\times$. Therefore the middle map is a canonical isomorphism $j_*^{\overline{T}^\vee}(E_{2,\overline{G}}) \simeq E_{2,\overline{T}}$. ∎

Recall by definition $j_*^{\overline{T}^\vee}(E_{\overline{G}}) = \overline{T}^\vee \times E_{\overline{G}}/\nabla Z(\overline{G}^\vee)$. Write $\mathcal{L}$ for the canonical isomorphism from above proof

$$ \mathcal{L} : \quad \overline{T}^\vee \times E_{\overline{G}}/\nabla Z(\overline{G}^\vee) \longrightarrow E_{\overline{T}}. $$

What is important for us is the explicit form of $\mathcal{L}$. From the construction we may identify

$$E_{2,\overline{T}} = \big\{([P_a], a) \in \mathrm{Hom}(\mathcal{E}_{Q,n}, \mathbf{C}^\times) \times F^\times : [P_a]|_{F^\times/n} = h_a\big\},$$

where as before $h_a(-) = (-, a)_n$ denotes the n-th Hilbert symbol. Here $[P_a]\big((b, y)\big) = h_a(b) \cdot P_a(y)$ is a homomorphism of $\mathcal{E}_{Q,n}$ with respect to a certain map (not necessarily a morphism) $P_a : Y_{Q,n} \longrightarrow \mathbf{C}^\times$.

Similarly $E_{2,\overline{G}}$ could be identified with the collection

$$\big\{([P_a], a) \in \mathrm{Hom}(\mathcal{E}_{Q,n}/\phi_{D,\eta} \circ \mathbf{s}(Y_{Q,n}^{sc}), \mathbf{C}^\times) \times F^\times : [P_a]|_{F^\times/n} = h_a\big\}.$$

Let $\overline{t}^\vee \in \overline{T}^\vee = \mathrm{Hom}(Y_{Q,n}, \mathbf{C}^\times)$. It gives rise to an element $[\overline{t}^\vee] \in \mathrm{Hom}(\mathcal{E}_{Q,n}, \mathbf{C}^\times)$ via inflation. Tracing through the proof of Proposition 2.4.8, we see that the canonical isomorphism takes the explicit form:

$$\begin{aligned} \overline{T}^\vee \times E_{2,\overline{G}}/\nabla Z(\overline{G}^\vee) &\longrightarrow E_{2,\overline{T}} \\ \big(\overline{t}^\vee, ([P_a], a)\big) &\longmapsto \big([\overline{t}^\vee] \cdot [P_a], a\big). \end{aligned}$$

Thus the desired isomorphism $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L} :\ \overline{T}^\vee \times E_{\overline{G}}/\nabla Z(\overline{G}^\vee) &\longrightarrow E_{\overline{T}} \\ \big(\overline{t}^\vee, ([P_a], a) \oplus_B (1, a)\big) &\longmapsto ([\overline{t}^\vee] \cdot [P_a], a) \oplus_B (1, a). \end{aligned}$$

It is important for our purpose of the Gindikin-Karpelevich formula latter to determine the element $\overline{t}^\vee_{P_a} \in \overline{T}$ associated with $\mathcal{L}^{-1}\big(([P_a], a) \oplus_B (1, a)\big)$, where $([P_a], a) \oplus_B (1, a) \in E_{\overline{T}}$. Moreover, it is easy to see that $\overline{t}^\vee_{P_a}$ could be only determined up to a class modulo $Z(\overline{G}^\vee)$. That is, only the image of $\overline{t}^\vee_{P_a}$ in the quotient map $\overline{T}^\vee \longrightarrow \overline{T}^\vee/Z(\overline{G}^\vee)$ is uniquely determined.

For this purpose, consider the trivial extension

$$\overline{T}^\vee/Z(\overline{G}^\vee) \hookrightarrow \overline{T}^\vee/Z(\overline{G}^\vee) \times F^\times \twoheadrightarrow F^\times,$$

which has a canonical splitting s^{Tr} of the first map. There is a natural map q_* from $j_*^{\overline{T}^\vee}(E_{\overline{G}})$ to $\overline{T}^\vee/Z(\overline{G}^\vee) \times F^\times$ such that the following diagram commutes

$$\begin{tikzcd} \overline{T}^\vee \arrow[r, hook] \arrow[d, "q"'] & j_*^{\overline{T}^\vee}(E_{\overline{G}}) \arrow[r, two heads] \arrow[d, "q_*"] & F^\times \arrow[d, equal] \\ \overline{T}^\vee/Z(\overline{G}^\vee) \arrow[r, hook] & \overline{T}^\vee/Z(\overline{G}^\vee) \times F^\times \arrow[r, two heads] \arrow[l, bend left, "s^{\mathrm{Tr}}"] & F^\times. \end{tikzcd}$$

Take the composition

$$\mathcal{C} = s^{\mathrm{Tr}} \circ q_* \circ \mathcal{L}^{-1} : \quad E_{\overline{T}} \longrightarrow j_*^{\overline{T}^\vee}(E_{\overline{G}}) \longrightarrow \overline{T}^\vee/Z(\overline{G}^\vee) \times F^\times \longrightarrow \overline{T}^\vee/Z(\overline{G}^\vee).$$

Now we can describe explicitly the image of this map $\mathcal{C}$.

Corollary 2.4.9. *Let* $\mathcal{P}_a = ([P_a], a) \oplus_B (1, a) \in E_{\overline{T}}$*, and identify* $Hom(Y^{sc}_{Q,n}, \mathbf{C}^\times) = \overline{T}^\vee / Z(\overline{G}^\vee)$*. Then* $\mathcal{C}(\mathcal{P}_a)(\alpha^\vee_{[n]}) = [P_a] \circ \phi_{D,\eta}\big(\mathbf{s}(\alpha^\vee_{[n]})\big)$ *for all* $\alpha \in \Psi$*. That is, the image* $\bar{t}^\vee_{P_a}$ *of* $\mathcal{L}^{-1}(\mathcal{P}_a)$ *in* $\overline{T}^\vee / Z(\overline{G}^\vee)$ *is uniquely determined by*

$$\bar{t}^\vee_{P_a}(\alpha^\vee_{[n]}) = [P_a] \circ \phi_{D,\eta}\big(\mathbf{s}(\alpha^\vee_{[n]})\big), \alpha \in \Psi. \tag{2.22}$$

Proof. Clearly $\bar{t}^\vee_{P_a}(\alpha^\vee_{[n]}) = \mathcal{C}(\mathcal{P}_a)(\alpha^\vee_{[n]})$. Now write

$$\mathcal{L}^{-1}(\mathcal{P}_a) = \big(\bar{t}^\vee_{P_a}, ([P_a]/[\bar{t}^\vee_{P_a}], a) \oplus_B (1, a)\big) \in j_*^{\overline{T}^\vee}(E_{\overline{G}}),$$

where $[P_a]/[\bar{t}^\vee_{P_a}] \in \mathrm{Hom}(\mathcal{E}_{Q,n}/\phi_{D,\eta} \circ \mathbf{s}(Y^{sc}_{Q,n}), \mathbf{C}^\times)$. In particular, it vanishes on $\phi_{D,\eta} \circ \mathbf{s}(Y^{sc}_{Q,n})$, and therefore

$$[\bar{t}^\vee_{P_a}] \circ \phi_{D,\eta}\big(\mathbf{s}(\alpha^\vee_{[n]})\big) = [P_a] \circ \phi_{D,\eta}\big(\mathbf{s}(\alpha^\vee_{[n]})\big), \ \alpha \in \Psi.$$

But we have $\bar{t}^\vee_{P_a}(\alpha^\vee_{[n]}) = [\bar{t}^\vee_{P_a}] \circ \phi_{D,\eta}\big(\mathbf{s}(\alpha^\vee_{[n]})\big)$ from the definition of $[\bar{t}^\vee_{P_a}]$. This gives the desired result and completes the proof. □

2.4.3 Global L-group

Let $\overline{\mathbb{G}} \in \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ be defined over a number field F. We may assume that it is incarnated by (D, η). Assume $\mu_n \subseteq F^\times$, one obtains a global covering group $\overline{\mathbb{G}}(\mathbb{A}_F)$ in section 2.3.3:

$$\mu_n \hookrightarrow \overline{\mathbb{G}}(\mathbb{A}_F) \twoheadrightarrow \mathbb{G}(\mathbb{A}_F)\ .$$

As in the local case, it is essential to have the global L-group for the purpose of arithmetic parametrization of automorphic forms on $\overline{\mathbb{G}}(\mathbb{A}_F)$ and problems alike. Though we are largely dealing with local problems, we include the construction of global L-group here. For more details, see [We14]. One benefit we gain is that it will enable us to define (partial) automorphic L-functions later.

The construction of global ${}^L\overline{G}$ relies on a global fundamental extension

$$Z(\overline{G}^\vee) \hookrightarrow E_{\mathbb{A}_F} \twoheadrightarrow F^\times \backslash \mathbb{A}_F^\times\ ,$$

from which one may define the global L-group associated with W_F as $\pi^* \circ j_*^{\overline{G}^\vee}(E_{\mathbb{A}_F})$. Here $j^{\overline{G}^\vee} : Z(\overline{G}) \hookrightarrow \overline{G}^\vee$ is the natural inclusion and W_F the global Weil group with the natural map (cf. [Tat79]) $\pi : W_F \longrightarrow F^\times \backslash \mathbb{A}_F^\times$ induced from the reciprocity map. In principle one may also replace W_F by any arithmetic group for F with a natural map to $F^\times \backslash \mathbb{A}_F^\times$, for example the conjectural Langlands automorphic L-group.

We recall the construction of $E_{\mathbb{A}_F}$ (cf. [We14]), which follows from similar consideration of the local case. From now, assume we are given with the data $(n, Q, \mathcal{E}, \phi)$.

The group $E_{\mathbb{A}_F}$ is constructed as the Baer sum of two extensions $E_{1,\mathbb{A}_F}$ and $E_{2,\mathbb{A}_F}$:

$$Z(\overline{G}^\vee) \hookrightarrow E_{i,\mathbb{A}_F} \twoheadrightarrow F^\times \backslash \mathbb{A}_F^\times \ \text{ for } i = 1, 2.$$

First of all, the complex dual group $\overline{G}^\vee$ is the one with root data as before in the local case:

$$\big(Y_{Q,n},\ \{\alpha_{[n]}^\vee\}_{\alpha\in\Psi},\ \mathrm{Hom}(Y_{Q,n},\mathbf{Z}),\ \{n_\alpha^{-1}\alpha\}_{\alpha\in\Psi}\big).$$

The modification here depends on the data (n,Q). As usual, identify $Z(\overline{G}^\vee)$ with $\mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{sc},\mathbf{C}^\times)$.

The construction of $E_{1,\mathbb{A}_F}$

First, we have an extension

$$\mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{sc},\mathbf{C}^\times) \hookrightarrow E_{0,\mathbb{A}_F} \twoheadrightarrow \mathbb{A}_F^\times$$

which is given by the cocycle

$$\begin{aligned} \mathbb{A}_F^\times\times\mathbb{A}_F^\times &\longrightarrow \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{sc},\mathbf{C}^\times)\\ (a,b) &\longmapsto (y\mapsto (a,b)_n^{Q(y)}), \end{aligned}$$

where $a,b\in\mathbb{A}_F^\times$ and $(-,-)_n$ denotes the global n-th Hilbert symbol. Since the Hilbert symbol is trivial on $F^\times\times F^\times$, the extension $E_{0,\mathbb{A}_F}$ splits over $F^\times$. The quotient of $E_{0,\mathbb{A}_F}$ by the splitting image of $F^\times$ gives us the extension

$$Z(\overline{G}^\vee) \hookrightarrow E_{1,\mathbb{A}_F} \twoheadrightarrow F^\times\backslash\mathbb{A}_F^\times .$$

The construction of $E_{2,\mathbb{A}_F}$

Start with $(n,Q,\mathcal{E},\phi)$, we have the diagram with a canonical splitting $\mathbf{s}$ as in the local case:

$$\begin{tikzcd} F^\times/n \arrow[r, hook] \arrow[d, equal] & \mathcal{E}_{Q,n}^{sc} \arrow[r, two heads] \arrow[d, "\phi_{D,\eta}"] & Y_{Q,n}^{sc} \arrow[d, hook] \arrow[l, bend right, "\mathbf{s}"'] \\ F^\times/n \arrow[r, hook] & \mathcal{E}_{Q,n} \arrow[r, two heads] & Y_{Q,n}, \end{tikzcd}$$

from which we have

$$F^\times/n \hookrightarrow \mathcal{E}_{Q,n}/\phi_{D,\eta}\circ\mathbf{s}(Y_{Q,n}^{sc}) \twoheadrightarrow Y_{Q,n}/Y_{Q,n}^{sc}.$$

Apply $\mathrm{Hom}(-,\mathbf{C}^\times)$ and pull-back by the global Hilbert symbol

$$h_{\mathbb{A}_F}: F^\times\backslash\mathbb{A}_F^\times \longrightarrow \mathrm{Hom}(F^\times/n,\mathbf{C}^\times)$$

given by $h_{\mathbb{A}_F}(b)=(b,a)_n$, we obtain our definition of $E_{2,\mathbb{A}_F}$:

$$Z(\overline{G}^\vee) \hookrightarrow E_{2,\mathbb{A}_F} \twoheadrightarrow F^\times\backslash\mathbb{A}_F^\times .$$

Local-global compatibility

For each place v of F, use the inclusion $i_v : F \hookrightarrow F_v$ we could push-out the data $(Q, \mathcal{E}, \phi)$ to $(Q, \mathcal{E}_v, \phi_v)$, which is then the Brylinski-Deligne data associated with $\overline{\mathbb{G}}_{/F_v}$, obtained from the base change of $\overline{\mathbb{G}}$ via i_v. Here, $\mathcal{E}_v = (i_v)_*(\mathcal{E})$ and $\phi_v = i_v \circ \phi$.

Associated with $(Q, \mathcal{E}_v, \phi_v)$ there is the local fundamental extension $E_{\overline{G}_v}$ which sits in

$$Z(\overline{G}^\vee) \hookrightarrow E_{\overline{G}_v} \twoheadrightarrow F_v^\times .$$

The construction of $E_{\mathbb{A}_F}$ above gives a canonical map $E_{\overline{G}_v} \longrightarrow E_{\mathbb{A}_F}$ such that the following diagram commutes:

$$\begin{tikzcd}
Z(\overline{G}^\vee) \arrow[r, hook] \arrow[d, equal] & E_{\overline{G}_v} \arrow[r, two heads] \arrow[d] & F_v^\times \arrow[d, hook] \\
Z(\overline{G}^\vee) \arrow[r, hook] & E_{\mathbb{A}_F} \arrow[r, two heads] & F^\times \backslash \mathbb{A}_F^\times,
\end{tikzcd} \tag{2.23}$$

where the right hand side map is the natural inclusion. Clearly, instead of for the fundamental extensions, the local and global compatibility could be stated at the level of local and global L-groups. That is, we define the global L group with respect to W_F to be

$${}^L\overline{G} := \pi^* \circ j_*^{\overline{G}^\vee}(E_{\mathbb{A}_F}),$$

where π and $j^{\overline{G}^\vee}$ are the aforementioned maps from the beginning of this section. For all v, the following diagram commutes:

$$\begin{tikzcd}
\overline{G}^\vee \arrow[r, hook] \arrow[d, equal] & {}^L\overline{G}_v \arrow[r, two heads] \arrow[d] & \mathrm{W}_{F_v} \arrow[d] \\
\overline{G}^\vee \arrow[r, hook] & {}^L\overline{G} \arrow[r, two heads] & W_F,
\end{tikzcd} \tag{2.24}$$

where the right hand side is a certain natural map $\mathrm{W}_{F_v} \longrightarrow \mathrm{W}_F$, see [Tat79].

Chapter 3

Admissible splittings of the L-group

3.1 Subgroups of $\overline{G}$

Assume that F is a local field, unless stated otherwise. Let $\overline{G}$ be incarnated by (D, η), and let $\overline{T}$ be the covering torus of $\overline{G}$. Then the center $Z(\overline{T})$ allows for a simple description as follows.

Let $Y_{Q,n} \hookrightarrow Y$ be the inclusion, and let $\mathbb{T}_{Q,n}$ be the torus associated with $Y_{Q,n}$ whose F-rational point is denoted by $T_{Q,n}$. Thus we have a natural map $i_{Q,n} : T_{Q,n} \longrightarrow T$ whose image we denote by $T^\dagger$.

Lemma 3.1.1 ([We09]). *The center $Z(\overline{T})$ is the preimage of $T^\dagger$ in $\overline{T}$:*

$$\begin{tikzcd}
\mu_n \arrow[r, hook] \arrow[d, equal] & \overline{T} \arrow[r, two heads] & T \\
\mu_n \arrow[r, hook] & Z(\overline{T}) \arrow[r, two heads] \arrow[u, hook] & T^\dagger \arrow[u, hook] .
\end{tikzcd}$$

Now we define $\overline{T}_{Q,n} := i^*_{Q,n}(Z(\overline{T}))$ to be the pull-back of $Z(\overline{T})$ via $i_{Q,n}$:

$$\begin{tikzcd}
\mu_n \arrow[r, hook] \arrow[d, equal] & Z(\overline{T}) \arrow[r, two heads] & T^\dagger \\
\mu_n \arrow[r, hook] & \overline{T}_{Q,n} \arrow[r, two heads] \arrow[u, two heads, "\bar{i}_{Q,n}"'] & T_{Q,n}. \arrow[u, two heads, "i_{Q,n}"']
\end{tikzcd}$$

That is, elements of $\overline{T}_{Q,n}$ are of the form $\big((\zeta, i_{Q,n}(t)), t\big) \in Z(\overline{T}) \times T_{Q,n}$. Since $\overline{T}$ is assumed to be incarnated by D, we can write $\overline{T} = \mu_n \times_D T$ and therefore also $Z(\overline{T}) = \mu_n \times_D T^\dagger$.

From now, we will write (ζ, t) for $\big((\zeta, i_{Q,n}(t)), t\big) \in \overline{T}_{Q,n}$. The group law on $\overline{T}_{Q,n} = \mu_n \times_D T_{Q,n}$ inherited from $Z(\overline{T})$ is thus given by

$$(i) \quad \big[(\zeta_1, y_1 \otimes a), (\zeta_2, y_2 \otimes b)\big] = 1 \text{ for } y_1, y_2 \in Y_{Q,n}; \tag{3.1}$$

$$(ii) \quad (\zeta_1, y_1 \otimes a) \cdot (\zeta_2, y_2 \otimes a) = \big(\zeta_1\zeta_2 (a,a)_n^{D(y_1,y_2)}, (y_1 + y_2) \otimes a\big); \tag{3.2}$$

$$(iii) \quad (\zeta_1, y \otimes a) \cdot (\zeta_2, y \otimes b) = \big(\zeta_1\zeta_2 (a,b)_n^{Q(y)}, y \otimes (ab)\big). \tag{3.3}$$

Clearly, there is a canonical isomorphism $\mathrm{Ker}(i_{Q,n}) \longrightarrow \mathrm{Ker}(\bar{i}_{Q,n})$ given by

$$t \in \mathrm{Ker}(i_{Q,n}) \longmapsto (1,t) \in \mathrm{Ker}(\bar{i}_{Q,n}). \tag{3.4}$$

It is desirable to have a simple description of $\mathrm{Ker}(\bar{i}_{Q,n})$. However, we do not have an explicit one and the following consideration would be helpful, at least for discussions in next section.

Consider the three lattices

$$nY \hookrightarrow Y_{Q,n} \hookrightarrow Y$$

and the induced isogenies

$$T_{nY} \xrightarrow{i_{nY}} T_{Q,n} \xrightarrow{i_{Q,n}} T,$$

where $\mathbb{T}_{nY}$ is the torus defined over F associated with nY and $T_{nY} := \mathbb{T}_{nY}(F)$.

Lemma 3.1.2. *The following inclusion holds:*

$$Ker(i_{Q,n}) \subseteq Im(i_{nY}).$$

Proof. Note that the composition $i_{Q,n} \circ i_{nY}$ is the n-th power map of T. By elementary divisor theorem, let $\{e_i\}$ be a basis of Y such that $\{m_i e_i\}$ is a basis of $Y_{Q,n}$ where $i = 1, 2, ..., r$. Then $\{ne_i\}$ is a basis of nY, and $m_i | n$. Then the isogenies $T_{nY} \xrightarrow{i_{nY}} T_{Q,n} \xrightarrow{i_{Q,n}} T$ could be identified with $\prod_i F^\times \xrightarrow{i_{nY}} \prod_i F^\times \xrightarrow{i_{Q,n}} \prod_i F^\times$, whose i-th component maps are given by

$$a_i \longmapsto a_i^{n/m_i} \longmapsto a_i^n, \quad a_i \in F^\times.$$

So $\mathrm{Ker}(i_{Q,n})$ is generated by $(m_i e_i) \otimes \zeta_{m_i} \in Y_{Q,n} \otimes F^\times$, where ζ_{m_i} is some m_i-th root of unity. But we see that there always exists n-th root of unity $\zeta_n \in F^\times$ such that $(ne_i) \otimes \zeta_n \in T_{nY}$ is mapped to $(m_i e_i) \otimes \zeta_{m_i}$. It follows $\mathrm{Ker}(i_{Q,n}) \subseteq \mathrm{Im}(i_{nY})$. $\square$

Define a map s_n on the generators of T_{nY} by

$$s_n : \quad T_{nY} \longrightarrow \overline{T}_{Q,n}, \quad (ny) \otimes a \longmapsto (1, (ny) \otimes a) \in \overline{T}_{Q,n}, \quad y \in Y.$$

By checking the relations in (3.1)-(3.3), it follows from the observation $\mathrm{Ker}(s_n) = \mathrm{Ker}(i_{nY})$ that we have the following.

Lemma 3.1.3. *The map s_n is a homomorphism, and therefore gives a splitting of $Im(i_{nY})$ into $\overline{T}_{Q,n}$, which extends the canonical isomorphism $Ker(i_{Q,n}) \longrightarrow Ker(\bar{i}_{Q,n})$ given by (3.4). That is, the following diagram commutes:*

$$\begin{tikzcd}
\mu_n \arrow[r, hook] & \overline{T}_{Q,n} \arrow[r, two heads] & T_{Q,n} \\
& Ker(\bar{i}_{Q,n}) \arrow[u, hook] & Im(i_{nY}) \arrow[u, hook] \arrow[ul, dashed, "s_n"'] \\
& & Ker(i_{Q,n}) \arrow[u, hook] \arrow[ul, "can\ iso"]
\end{tikzcd}$$

Thus the homomorphism s_n also induces a splitting of $\mathrm{Im}(i_{Q,n} \circ i_{nY}) \subseteq T^\dagger$ into $Z(\overline{T})$, which by abuse of notation is also written as s_n. Note $\mathrm{Im}(i_{Q,n} \circ i_{nY}) = \{t^n : t \in T\}$.

3.2 Admissible splittings of the L-group

We are interested in splittings of ${}^L\overline{G}$ over W_F, with respect to which we have ${}^L\overline{G} \simeq \overline{G}^\vee \times \mathrm{W}_F$. For example, we will see that ${}^L\overline{T}$ always splits and we have a local Langlands correspondence between splittings of ${}^L\overline{T}$ and genuine characters of the center $Z(\overline{T})$.

Let $\mathrm{WD}_F := \mathbb{SL}_2(\mathbf{C}) \times \mathrm{W}_F$ be the Weil-Deligne group. For splittings of ${}^L\overline{G}$ over W_F, we have

Definition 3.2.1. An L-parameter is just a splitting ρ of ${}^L\overline{G}$ over W_F:

$$\begin{tikzcd} \overline{G}^\vee \arrow[r, hook] & {}^L\overline{G} \arrow[r, two heads] & \mathrm{W}_F, \arrow[l, bend left, "\rho"] \end{tikzcd}$$

while a Weil-Deligne parameter is a homomorphism $\mathrm{WD}_F \longrightarrow {}^L\overline{G}$ such that the following diagram commutes

$$\begin{tikzcd} \overline{G}^\vee \arrow[r, hook] & {}^L\overline{G} \arrow[r, two heads] & \mathrm{W}_F \\ & & \mathrm{WD}_F \arrow[ul] \arrow[u, two heads] \end{tikzcd}$$

The vertical surjection is the projection of WD_F onto its second component. Any L-parameter ρ is called admissible if and only if it gives an isomorphism

$${}^L\overline{G} \simeq \overline{G}^\vee \times_\rho \mathrm{W}_F$$

with the inverse map given by $(g^\vee, x) \longmapsto g^\vee \cdot \rho(x)$ for $g^\vee \in \overline{G}^\vee, x \in \mathrm{W}_F$.

Write $\mathfrak{S}^a({}^L\overline{G}, \mathrm{W}_F)$ for the set of admissible splittings. Then

Lemma 3.2.2. *A splitting ρ lies in $\mathfrak{S}^a({}^L\overline{G}, W_F)$ if and only if its image lies in the centralizer $Z_{{}^L\overline{G}}(\overline{G}^\vee) \simeq Rec^*(E_{\overline{G}})$ of $\overline{G}^\vee$ in ${}^L\overline{G}$. That is, we have the isomorphism $\mathfrak{S}^a({}^L\overline{G}, W_F) \simeq \mathfrak{S}(E_{\overline{G}}, F^\times)$ consisting of splittings of $E_{\overline{G}}$ over $F^\times$. It is clearly a $Hom(W_F, Z(\overline{G}^\vee))$-torsor.*

Proof. The assignment $(g^\vee, x) \longmapsto g^\vee \cdot \rho(x)$ is a homomorphism, then one must have

$$g_1^\vee \cdot \rho(x_1) g_2^\vee \cdot \rho(x_2) = g_1^\vee g_2^\vee \cdot \rho(x_1)\rho(x_2),$$

where $g_i^\vee \in \overline{G}^\vee$ and $x_i \in \mathrm{W}_F$ are arbitrary elements. In particular, $\rho(x)$ commutes with arbitrary element of $\overline{G}^\vee$. The construction of ${}^L\overline{G}$ is:

$$\begin{tikzcd} Z(\overline{G}^\vee) \arrow[r, hook] \arrow[d, equal] & E_{\overline{G}} \arrow[r, two heads] & F^\times \\ Z(\overline{G}^\vee) \arrow[r, hook] \arrow[d, hook, "j^{\overline{G}^\vee}"'] & \mathrm{Rec}^*(E_{\overline{G}}) \arrow[r, two heads] \arrow[u] \arrow[d] & \mathrm{W}_F \arrow[u, "\mathrm{Rec}"'] \arrow[d, equal] \\ \overline{G}^\vee \arrow[r, hook] & {}^L\overline{G} \arrow[r, two heads] & \mathrm{W}_F. \end{tikzcd}$$

So admissible splittings are valued in $\mathrm{Rec}^*(E_{\overline{G}})$. Tracing through the diagrams, it is easy to see that we have the canonical isomorphisms

$$\mathfrak{S}^a({}^L\overline{G}, \mathrm{W}_F) \simeq \mathfrak{S}(\mathrm{Rec}^*(E_{\overline{G}}), \mathrm{W}_F) \simeq \mathfrak{S}(E_{\overline{G}}, F^\times).$$

□

3.2.1 Conditions on the existence of admissible splittings

The goal of this section is on some subsets of $\mathrm{Hom}_\epsilon(Z(\overline{T}), \mathbf{C}^\times) \subseteq \mathrm{Hom}_\epsilon(\overline{T}_{Q,n}, \mathbf{C}^\times)$, which are called qualified or distinguished characters (to be defined later) and which give rise to admissible splittings of $\mathfrak{S}^a({}^L\overline{G}, \mathrm{W}_F)$.

Before we proceed, we give some general discussion on the conditions for the existence of admissible splittings. Recall the construction of $E_{\overline{G}}$ as

$$E_{\overline{G}} = E_{2,\overline{G}} \oplus_B E_{1,\overline{G}}.$$

We have identified $E_{2,\overline{G}}$ with

$$\big\{([P_a], a) \in \mathrm{Hom}\big(\mathcal{E}_{Q,n}/\phi_{D,\eta} \circ \mathbf{s}(Y^{sc}_{Q,n}), \mathbf{C}^\times\big) \times F^\times : [P_a]|_{F^\times/n} = h_a\big\},$$

where $[P_a]\big((b, y)\big) = (b, a)_n \cdot P_a(y)$ for some $\mathbf{C}^\times$-valued function P_a on $Y_{Q,n}$. The conditions on $[P_a]$ translate into

$$(c1) \quad [P_a] \circ \phi_{D,\eta}\big(\mathbf{s}(\alpha^\vee_{[n]})\big) = 1, \alpha \in \Psi, \tag{3.5}$$

$$(c2) \quad P_a(y_1 + y_2) = P_a(y_1) \cdot P_a(y_2) \cdot (a, a)_n^{D(y_1,y_2)}. \tag{3.6}$$

The cocycle of $E_{1,\overline{G}}$ actually takes value in the 2-torsion of $Z(\overline{G})$, i.e. $\mathrm{Hom}(Y_{Q,n}/Y^{sc}_{Q,n}, \mu_2)$. Therefore, any section of $E_{\overline{G}}$ over $F^\times$ given by

$$a \longmapsto ([P_a], a) \oplus_B (1, a) \in E_{\overline{G}}$$

is a splitting if and only if the section $a \mapsto ([P_a], a)$ of $E_{2,\overline{G}}$ over $F^\times$ produces a cocycle the same as that on $E_{1,\overline{G}}$. That is, to translate this into P_a, we must have

$$(c3) \quad P_{ac}(y) = P_a(y) \cdot P_c(y) \cdot (a, c)_n^{Q(y)}, \quad y \in Y_{Q,n}. \tag{3.7}$$

Note that in particular $(c2)$ and $(c3)$ imply $P_{a^m}(y) = P_a(y^m)$. Consider the covering group $\overline{T}_{Q,n}$ and define a map $\overline{\chi}_P : \overline{T}_{Q,n} \longrightarrow \mathbf{C}^\times$ on generators of $\overline{T}_{Q,n}$ by

$$\overline{\chi}_P : \quad (\xi, y \otimes a) \longmapsto \xi \cdot [P_a]\big((1, y)\big) = \xi \cdot P_a(y).$$

Then $\overline{\chi}_P \in \mathrm{Hom}_\epsilon(\overline{T}_{Q,n}, \mathbf{C}^\times)$ is a well-defined genuine character from $(c2)$ and $(c3)$. That is, admissible splittings of ${}^L\overline{G}$ over W_F correspond to genuine characters of $\mathrm{Hom}_\epsilon(\overline{T}_{Q,n}, \mathbf{C}^\times)$ satisfying a certain condition $(C1)$ derived from $(c1)$. We will state the condition $(C1)$ explicitly later.

Before we proceed, we observe that in general characters defined on the external $\overline{T}_{Q,n}$ do not reflect the ambient group $\overline{G}$. Recall the definition of $\overline{T}_{Q,n}$ as the pull-back:

$$\begin{tikzcd} \mu_n \arrow[r, hook] \arrow[d, equal] & Z(\overline{T}) \arrow[r, two heads] & T^{\dagger} \\ \mu_n \arrow[r, hook] & \overline{T}_{Q,n} \arrow[r, two heads] \arrow[u, "\bar{i}_{Q,n}"'] & T_{Q,n}. \arrow[u, "i_{Q,n}"'] \end{tikzcd}$$

If one reverses the discussion above and starts with a character on $\overline{T}_{Q,n}$ which gives rise to an admissible splitting of ${}^L\overline{G}$, it is natural to require that it descends to $Z(\overline{T})$. That is, we are interested in characters $\overline{\chi}_P \in \mathrm{Hom}_\epsilon(\overline{T}_{Q,n}, \mathbf{C}^\times)$ which satisfy the following condition in addition to $(C1)$:

$$(C0) \quad \overline{\chi}_P\big((1,t)\big) = 1 \text{ for any } t \in \mathrm{Ker}(i_{Q,n}). \tag{3.8}$$

To proceed, we first state a very useful result for explicating the condition $(C1)$ and also for the GK formula later. Let $\overline{\chi} \in \mathrm{Hom}_\epsilon(Z(\overline{T}), \mathbf{C}^\times)$ be an arbitrary genuine character of $Z(\overline{T})$. For any $a \in F^\times$, it gives rise to an element $([P_a], a) \oplus_B (1, a) \in E_{\overline{T}}$ where $[P_a] \in \mathrm{Hom}(\mathcal{E}_{Q,n}, \mathbf{C}^\times)$ is given by

$$[P_a]\big((1,y)\big) := \overline{\chi}\big((1, y\otimes a)\big). \tag{3.9}$$

Recall the natural map $\Phi_{D,\eta} : \overline{\mathbb{T}}^{sc} \longrightarrow \overline{\mathbb{T}}$, which by abuse of notation is also used to denote the induced map $\overline{T}^{sc} \longrightarrow \overline{T}$. For any $a \in F^\times$ and any $\alpha \in \Psi$, let $\overline{\mathbb{h}}_\alpha^{[b]}(a) \in \overline{T}^{sc}$ be the element of the canonical section which a priori depends on $b \in F^\times$.

Then we have

Proposition 3.2.3. *The element $\overline{\mathbb{h}}_\alpha^{[b]}(a^{n_\alpha}) \in \overline{T}^{sc}$ is independent of $b \in F^\times$, and thus we could omit b for notational simplicity and just write $\overline{\mathbb{h}}_\alpha(a^{n_\alpha})$ for it. More importantly, for any $\alpha \in \Psi$ and any $a \in F^\times$,*

$$\overline{\chi} \circ \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(a^{n_\alpha})\big) = [P_a] \circ \phi_{D,\eta}\big(\mathbf{s}(\alpha^\vee_{[n]})\big),$$

where $[P_a]$ is associated with $\overline{\chi}$ by (3.9).

Proof. First it follows from (2.4) that for all $\alpha \in \Psi$,

$$\overline{\mathbb{h}}_\alpha^{[db]}(a^{n_\alpha}) = \overline{\mathbb{h}}_\alpha^{[b]}(a^{n_\alpha}) \cdot (d, a^{n_\alpha})_n^{Q(\alpha^\vee)} = \overline{\mathbb{h}}_\alpha^{[b]}(a^{n_\alpha}) \in \overline{T}^{sc}.$$

Thus the first assertion is proved.

It is clear that $\Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(a^{n_\alpha})\big) \in Z(\overline{T})$. Now we show the desired equality by first treating the case $\alpha \in \Delta$. If $\alpha \in \Delta$, by (2.12),

$$\Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(a^{n_\alpha})\big) = \big((\eta(\alpha^\vee), a^{n_\alpha})_n, \mathbb{h}_\alpha(a^{n_\alpha})\big) \in \overline{T}.$$

On the other hand, by (2.4.3), $\phi_{D,\eta}\big(\mathbf{s}(\alpha^\vee_{[n]})\big) = \big(\eta(n_\alpha\alpha^\vee), n_\alpha\alpha^\vee\big) \in \mathcal{E}_{Q,n}$. Thus,

$$\begin{aligned}\overline{\chi}\circ\Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(a^{n_\alpha})\big) &= (\eta(n_\alpha\alpha^\vee), a)_n \cdot \overline{\chi}\big((1, \alpha^\vee_{[n]}\otimes a)\big) \\ &= (\eta(n_\alpha\alpha^\vee), a)_n \cdot [P_a]\big((1, \alpha^\vee_{[n]})\big) \\ &= [P_a]\big((\eta(\alpha^\vee_{[n]}), \alpha^\vee_{[n]})\big) \\ &= [P_a]\circ\phi_{D,\eta}\big(\mathbf{s}(\alpha^\vee_{[n]})\big).\end{aligned}$$

In general, let $\gamma \in \Psi$, we may use induction on the minimum of the lengths of $w \in W$ such that $w(\gamma) \in \Delta$. Thus assume $\gamma = s_\alpha(\beta)$ with $\alpha \in \Delta$ and the equality hold for β, i.e. $\overline{\chi}\circ\Phi_{D,\eta}\big(\overline{\mathbb{h}}_\beta(a^{n_\beta})\big) = [P_a]\circ\phi_{D,\eta}\big(\mathbf{s}(\beta^\vee_{[n]})\big)$.

Note $Q(\gamma^\vee) = Q(\beta^\vee)$ and $n_\gamma = n_\beta$. By [BD01, §11.3], we have the following equalities in $\overline{T}^{sc}$

$$\begin{aligned}\overline{\mathbb{h}}_\gamma(a^{n_\gamma}) &= \overline{\mathbb{h}}_\beta(a^{n_\beta})\cdot\overline{\mathbb{h}}_\alpha(a^{-n_\beta\langle\alpha,\beta^\vee\rangle}) \\ &= \overline{\mathbb{h}}_\beta(a^{n_\beta})\cdot\overline{\mathbb{h}}_\alpha(a^{-n_\alpha\langle\beta,\alpha^\vee\rangle}) \\ &= \overline{\mathbb{h}}_\beta(a^{n_\beta})\cdot\overline{\mathbb{h}}_\alpha(a^{n_\alpha})^{-\langle\beta,\alpha^\vee\rangle}\cdot(a^{n_\alpha}, a^{n_\alpha})_n^{\varepsilon(-\langle\beta,\alpha^\vee\rangle)Q(\alpha^\vee)} \\ &= \overline{\mathbb{h}}_\beta(a^{n_\beta})\cdot\overline{\mathbb{h}}_\alpha(a^{n_\alpha})^{-\langle\beta,\alpha^\vee\rangle}.\end{aligned}$$

On the other hand, $\mathbf{s}(\gamma^\vee_{[n]}) = \mathbf{s}(\beta^\vee_{[n]})\cdot\mathbf{s}(\alpha^\vee_{[n]})^{-\langle\beta,\alpha^\vee\rangle} \in \mathcal{E}_{Q,n}$. By the induction hypothesis on β and the proved equality for $\alpha \in \Delta$, we see $\overline{\chi}\circ\Phi_{D,\eta}\big(\overline{\mathbb{h}}_\gamma(a^{n_\alpha})\big) = [P_a]\circ\phi_{D,\eta}\big(\mathbf{s}(\gamma^\vee_{[n]})\big)$. That is, the assertion holds for γ, and the proof is completed. □

Now we explicate or state the conditions $(C0), (C1)$ in equivalent or stronger forms.

The condition $(C0)$

In view of the inclusion $\mathrm{Ker}(\overline{i}_{Q,n}) \subset \mathrm{Im}(s_n)$ of Lemma 3.1.3, there is the stronger condition that could be imposed on $\overline{\chi}_P$:

$$(C0)_+ \quad \overline{\chi}_P\circ s_n = \mathbb{1}, \text{ or equivalently } \overline{\chi}_P(\overline{t}) = 1 \text{ for all } \overline{t}\in\mathrm{Im}(s_n)\subset\overline{T}_{Q,n}. \tag{3.10}$$

It takes an explicit form

$$(C0)_+^{\mathrm{eq}} \quad \overline{\chi}_P\big((1, (ny)\otimes a)\big) = 1 \text{ for all } (ny)\otimes a\in T_{Q,n} \text{ with } y\in Y. \tag{3.11}$$

The condition $(C1)$

To write $(C1)$ in explicit form, we would like to impose that $\overline{\chi}_P$ satisfies $(C0)$ in the first place. Then we could consider $\overline{\chi}_P$ as a genuine character of $Z(\overline{T})$.

Thus assuming $(C0)$ of $\overline{\chi}_P$, the condition $(c1)$ on $[P_a]$ translates into

$$(C1) \quad \overline{\chi}_P\circ\Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(a^{n_\alpha})\big) = 1, \text{ for all } \alpha\in\Psi, \tag{3.12}$$

or equivalently (from above proof) the same statement but only for $\alpha \in \Delta$, which takes the explicit form

$$(C1)^{\mathrm{eq}} \quad \overline{\chi}_P\big((1, \alpha_{[n]}^\vee \otimes a)\big) = \big(a, \eta(\alpha_{[n]}^\vee)\big)_n, \text{ for all } \alpha \in \Delta. \tag{3.13}$$

Here $(1, \alpha_{[n]}^\vee \otimes a)$ could be viewed either in $\overline{T}_{Q,n}$ or $Z(\overline{T})$ as we have assumed $(C0)$.

Definition 3.2.4. A genuine character $\overline{\chi}$ of $\overline{T}_{Q,n}$ is called *qualified* if it satisfies the two conditions

$$(C0) \quad \overline{\chi}\big((1, t)\big) = 1 \text{ for any } t \in \mathrm{Ker}(i_{Q,n}), \tag{3.14}$$

$$(C1)^{\mathrm{eq}} \quad \overline{\chi}\big((1, \alpha_{[n]}^\vee \otimes a)\big) = \big(a, \eta(\alpha_{[n]}^\vee)\big)_n, \text{ for all } \alpha \in \Delta. \tag{3.15}$$

Since $(C0)$ implies that $\overline{\chi}$ descends to $Z(\overline{T})$, thus an admissible character is a genuine character $\overline{\chi}$ of $Z(\overline{T})$ satisfying $(C1)^{\mathrm{eq}}$. In this case, $(1, y \otimes a)$ and $(1, \alpha_{[n]}^\vee \otimes a)$ above are then viewed as elements in $Z(\overline{T})$.

A genuine character $\overline{\chi}$ of $\overline{T}_{Q,n}$ is called *distinguished* if it satisfies

$$(C0)_+^{\mathrm{eq}} \quad \overline{\chi}\big((1, (ny) \otimes a)\big) = 1 \text{ for all } (ny) \otimes a \in T_{Q,n} \text{ with } y \in Y, \tag{3.16}$$

$$(C1)^{\mathrm{eq}} \quad \overline{\chi}\big((1, \alpha_{[n]}^\vee \otimes a)\big) = \big(a, \eta(\alpha_{[n]}^\vee)\big)_n, \text{ for all } \alpha \in \Delta. \tag{3.17}$$

Equivalently, a distinguished character is a genuine character of $Z(\overline{T})$ satisfying the same conditions $(C0)_+^{\mathrm{eq}}$ and $(C1)^{\mathrm{eq}}$, where $(1, y \otimes a)$ and $(1, \alpha_{[n]}^\vee \otimes a)$ above are then viewed as elements in $Z(\overline{T})$.

Write $\mathrm{Hom}_\epsilon^q(Z(\overline{T}), \mathbf{C}^\times)$ and $\mathrm{Hom}_\epsilon^d(Z(\overline{T}), \mathbf{C}^\times)$ for the set of qualified and distinguished characters respectively. Clearly, any distinguished character is qualified, and qualified characters give rise to admissible splittings of ${}^L\overline{G}$. We have the following relations:

$$\mathrm{Hom}_\epsilon^d(Z(\overline{T}), \mathbf{C}^\times) \hookrightarrow \mathrm{Hom}_\epsilon^q(Z(\overline{T}), \mathbf{C}^\times) \hookrightarrow \mathfrak{S}^a({}^L\overline{G}, \mathrm{W}_F) \hookrightarrow \mathrm{Hom}_\epsilon(\overline{T}_{Q,n}, \mathbf{C}^\times).$$

There are obstructions to the existence of qualified or distinguished characters. For example, for the existence of distinguished characters one has the necessary condition that for all $\alpha \in \Delta$,

$$\begin{cases} (a, \eta(\alpha_{[n]}^\vee))_n = 1 \text{ if } \alpha_{[n]}^\vee \otimes a = 1 \in T_{Q,n}, \\ (a, \eta(\alpha_{[n]}^\vee))_n = 1 \text{ if } \alpha_{[n]}^\vee \otimes a = (ny) \otimes b \in T_{Q,n} \text{ for some } y \in Y, b \in F^\times. \end{cases} \tag{3.18}$$

However, these conditions may not be sufficient, and in general we do not have an explicit description of the necessary and sufficient condition for the existence of distinguished characters.

Nevertheless, we could restrict to look at certain coverings $\overline{G}_{D,\eta}$ which arise from nicer incarnation object.

Definition 3.2.5. A bisector D is called fair if for all $\alpha^\vee \in \Delta^\vee$,

$$2|Q(\alpha^\vee) \text{ implies } 2|D(\alpha^\vee, y) \text{ for all } y \in Y.$$

We also call (D, η) fair if D is fair.

Regarding the existence, we have

Proposition 3.2.6 ([We13, §2.5])**.** *Any Weyl-invariant quadratic form Q on Y possesses a fair bisector defined above.*

Fix a fair D^{fair} for Q. In view of Example 2.2.3, any general (D, η) is isomorphic to (D^{fair}, η') for some η'. Therefore, there is no loss of generalities to assume that the bisector D is always fair and allow η to vary.

Example 3.2.7. Let $\mathbb{G}$ be a reductive group with simply-connected derived group $\mathbb{G}^{der}$. The quotient Y/Y^{sc} is a free $\mathbf{Z}$-module. A fair bisector D could be defined in the following way.

Let $\{\mathfrak{b}_1, \mathfrak{b}_2, ..., \mathfrak{b}_r\} \cup \{\mathfrak{b}_{r+1}, ..., \mathfrak{b}_k\}$ be a basis of Y such that $\mathfrak{b}_i = \alpha_i^\vee, 1 \le i \le r$ for a certain order of $\alpha_i^\vee \in \Delta^\vee$, and $\mathfrak{b}_i$ for $r+1 \le i \le k$ belongs to Y. Then define D^{fair} by

$$D^{\text{fair}}(\mathfrak{b}_i, \mathfrak{b}_j) = \begin{cases} 0 & \text{if } i < j, \\ Q(\mathfrak{b}_i) & \text{if } i = j, \\ B(\mathfrak{b}_i, \mathfrak{b}_j) & \text{if } i > j. \end{cases}$$

Clearly D^{fair} defined in this manner is fair. For general $\overline{G}_{D,\eta}$ associated to $\overline{\mathbb{G}}$ of $\mathbb{G}$ of this type, it follows from (2.6) that $(D, \eta) \simeq (D^{\text{fair}}, \mathbb{1})$ for arbitrary (D, η). Therefore, for such groups, there is no loss of generalities (up to isomorphism classes) in considering fair incarnation objects of the form $(D^{\text{fair}}, \mathbb{1})$.

Let $T_{Q,n}^{sc}$ be the torus associated with $Y_{Q,n}^{sc}$ and $i_{Q,n}^{sc} : T_{Q,n}^{sc} \longrightarrow T_{Q,n}$ the isogeny induced from $Y_{Q,n}^{sc} \hookrightarrow Y_{Q,n}$. Here $T_{Q,n}^{sc}$ is generated by $\alpha_{[n]}^\vee \otimes a$, $\alpha^\vee \in \Delta^\vee, a \in F^\times$. The fairness enables us to have the following

Lemma 3.2.8. *If D is fair, then the map s_ϕ generated by*

$$s_\phi : \quad T_{Q,n}^{sc} \longrightarrow \overline{T}_{Q,n}, \quad \alpha_{[n]}^\vee \otimes a \longmapsto \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(a^{n_\alpha})\big), \alpha^\vee \in \Delta^\vee,$$

where $\Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(a^{n_\alpha})\big) = \big((\eta(\alpha_{[n]}^\vee), a)_n, \alpha_{[n]}^\vee \otimes a\big)$ for $\alpha^\vee \in \Delta^\vee$, is a well-defined homomorphism.

Proof. Note that $T_{Q,n}^{sc}$ is generated by $\alpha_{[n]}^\vee \otimes a$, $\alpha \in \Delta^\vee, a \in F^\times$. Thus by considering the group law on $\overline{T}_{Q,n}^{sc}$ it suffices to check the following. First we check for $a, b \in F^\times$

$$\begin{aligned}(1, \alpha_{[n]}^\vee \otimes a) \cdot (1, \alpha_{[n]}^\vee \otimes b) =& \big((a, b)_n^{Q(n_\alpha \alpha^\vee)}, \alpha_{[n]}^\vee \otimes (ab)\big) \\ =& \big(1, \alpha_{[n]}^\vee \otimes (ab)\big).\end{aligned}$$

Second, for $\alpha^\vee$ and $\beta^\vee$ in $\Delta^\vee$, we need to check

$$(1, \alpha^\vee_{[n]} \otimes a) \cdot (1, \beta^\vee_{[n]} \otimes a) = \big(1, (\alpha^\vee_{[n]} + \beta^\vee_{[n]}) \otimes a\big).$$

However, direct computation gives

$$\begin{aligned}(1, \alpha^\vee_{[n]} \otimes a) \cdot (1, \beta^\vee_{[n]} \otimes a) =& \big((a,a)_n^{D(\alpha^\vee_{[n]}, \beta^\vee_{[n]})}, (\alpha^\vee_{[n]} + \beta^\vee_{[n]}) \otimes a\big) \\ =& \big((a,a)_n^{n_\alpha n_\beta \cdot D(\alpha^\vee, \beta^\vee)}, (\alpha^\vee_{[n]} + \beta^\vee_{[n]}) \otimes a\big).\end{aligned}$$

If n is odd $(a,a)_n = 1$. If n is even then either $2|n_\alpha$ or $2|Q(\alpha^\vee)$, and in the latter case $2|D(\alpha^\vee, \beta^\vee)$ since we have assumed D fair. That is, in any case we have

$$(a,a)_n^{n_\alpha n_\beta \cdot D(\alpha^\vee, \beta^\vee)} = 1.$$

The proof is completed. $\square$

By using the induction as in the proof of Proposition 3.2.3 and simple computation, we have:

Corollary 3.2.9. *For D fair, the homomorphism s_ϕ has the property that*

$$s_\phi\big(\alpha^\vee_{[n]} \otimes a\big) = \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(a^{n_\alpha})\big) \text{ for all } \alpha^\vee \in \Psi^\vee.$$

Also in this case s_ϕ takes the explicit form

$$s_\phi\big(y \otimes a\big) = \big((\eta(y), a)_n, y \otimes a\big), \quad y \in Y^{sc}_{Q,n}. \tag{3.19}$$

It follows that for D fair, $(C1)^{eq}$ is equivalent to

$$\overline{\chi} \circ s_\phi = \mathbb{1}.$$

Thus, in this case a character $\overline{\chi}$ of $\overline{T}_{Q,n}$ is distinguished if and only if the following two conditions hold:

$$(C0)_+ \quad \overline{\chi} \circ s_n = \mathbb{1}, \tag{3.20}$$

$$(C1)^{eq} \quad \overline{\chi} \circ s_\phi = \mathbb{1}. \tag{3.21}$$

A character satisfying $(C1)^{eq}$ can exist if and only if

$$(\text{Obs1}) \quad \mathrm{Ker}(s_\phi) = \mathrm{Ker}(i^{sc}_{Q,n}),$$

where the inclusion $\mathrm{Ker}(s_\phi) \subseteq \mathrm{Ker}(i^{sc}_{Q,n})$ is automatic. If there exists $t \in \mathrm{Ker}(i^{sc}_{Q,n}) \backslash \mathrm{Ker}(s_\phi)$, then $s_\phi(t) \in \mu_n$ and $s_\phi(t) \neq 1$, in which case there does not exist $\overline{\chi}$ such that $\overline{\chi} \circ s_\phi(t) = 1$. Conversely, if the equality in (Obs1) is satisfied, then s_ϕ gives a splitting of $\overline{T}_{Q,n}$ over the image $\mathrm{Im}(i^{sc}_{Q,n}) \subseteq T_{Q,n}$. By Pontrjagin duality, there exist $\overline{\chi}$ satisfying $(C1)^{eq}$.

We could also rephrase the condition $(C1)^{eq}$ in explicit terms. By elementary divisor theorem, $\mathrm{Ker}(i^{sc}_{Q,n})$ is generated by pure tensors $y \otimes a \in T^{sc}_{Q,n}$ such that $y \otimes a = 1 \in T_{Q,n}$. Thus in view of (3.19), an equivalent formation for (Obs1) is

$$(\mathrm{Obs1})^{eq} \quad (\eta(y), a)_n = 1 \text{ for any } y \otimes a = 1 \in T_{Q,n},\ y \in Y^{sc}_{Q,n}, a \in F^\times.$$

Moreover, characters satisfying both $(C0)_+$ and $(C1)^{eq}$ can exist if and only if

$$(\mathrm{Obs2}) \quad (\eta(y), a)_n = 1 \text{ for any } y \in nY \cap Y^{sc}_{Q,n}, a \in F^\times.$$

These obstructions can not be removed automatically. However, they can be removed if the character $\eta_n : Y^{sc} \xrightarrow{\eta} F^\times \twoheadrightarrow F^\times/n$ is extendable to Y.

To summarize, we have

Proposition 3.2.10. *Suppose D is fair and the conditions in (Obs1) and (Obs2) hold, in particular when η_n is extendable to Y. Let $J = nY + Y^{sc}_{Q,n}$, and*

$$Z(\overline{G}^\vee)[J] = Hom(Y_{Q,n}/J, \mathbf{C}^\times) \subseteq Z(\overline{G}^\vee).$$

Then

1) The set of distinguished genuine characters of $\overline{T}_{Q,n}$ is nonempty, and is a torsor over $Hom(W_F, Z(\overline{G}^\vee)[J])$.

2) Each distinguished character $\overline{\chi}$ gives an admissible splitting $\rho_{\overline{\chi}}$ in $\mathfrak{S}^a({}^L\overline{G}, W_F)$, with respect to which we have

$${}^L\overline{G} \simeq_{\rho_{\overline{\chi}}} \overline{G}^\vee \times W_F.$$

Proof. It suffices to prove 1).

The absence of (Obs1) and (Obs2) implies that the two conditions $\overline{\chi} \circ s_\phi = \mathbb{1}$ and $\overline{\chi} \circ s_n = \mathbb{1}$ are compatible. Thus, by the Pontrjagin duality, there exists $\overline{\chi} \in \mathrm{Hom}_\epsilon(Z(\overline{T}), \mathbf{C}^\times)$ satisfying both conditions.

Finally, let $\overline{\chi}_i, i = 1, 2$ be two distinguished characters which give rise to $\rho_i \in \mathfrak{S}(E_{\overline{G}}, F^\times)$. For $a \in F^\times$, $\rho_1/\rho_2 \in \mathrm{Hom}(F^\times, Z(\overline{G}))$. As before let $[P_a]_i \in \mathrm{Hom}(\mathcal{E}_{Q,n}, \mathbf{C}^\times)$ be given by

$$[P_a]_i\big((1, y)\big) = \overline{\chi}_i\big((1, y \otimes a)\big).$$

With $Z(\overline{G}^\vee) = \mathrm{Hom}(Y_{Q,n}/Y^{sc}_{Q,n}, \mathbf{C}^\times)$, then in fact

$$\Big(\frac{\rho_1}{\rho_2}(a)\Big)(y) = [P_a]_1\big((1, y)\big)/[P_a]_2\big((1, y)\big), \quad y \in Y_{Q,n}.$$

Since $\overline{\chi}_i$ both satisfy $(C0)_+$ and $(C1)^{eq}$, $[\rho_1/\rho_2](a)$ vanishes on J and the result follows. $\square$

In general, there may exist no qualified or distinguished characters, and above obstructions to their existence do exist.

Example 3.2.11. Consider $\mathbb{PGL}_2$ with cocharacter $Y = \mathbf{Z} \cdot e$ and coroot $\alpha^\vee = 2e$. Let Q be the unique quadratic form such that

$$Q(e) = 1.$$

Let $n = 2$. Then $Q(\alpha^\vee) = 4$ and $n_\alpha = 1$, which gives

$$Y_{Q,2} = Y, \quad Y_{Q,2}^{sc} = Y^{sc} = 2Y.$$

For $(C1)^{\mathrm{eq}}$ to be satisfied, one necessarily has

$$\text{(Obs1)} \quad \big(-1, \eta(\alpha^\vee)\big)_2 = 1 \text{ since } \alpha^\vee \otimes (-1) = 1 \in T_{Q,n} = T,$$

which is an obstruction to the existence of both qualified and distinguished characters. This obstruction can be removed if and only if $-1 \in (F^\times)^2$, which in general may not be satisfied.

This shows that obstruction does exist.

Remark 3.2.12. In general, the condition $(C0)_+$ or its equivalent $(C0)_+^{\mathrm{eq}}$ is certainly not the most minimum requirement and it may be an overkill. However, we do not know any simple characterization of the condition $(C0)$. Thus to replace $(C0)$ by $(C0)_+^{\mathrm{eq}}$ does not seem too restrictive. In particular, if η_n is extendable to Y, there is no problem.

Moreover, we may similarly define extensions $E_i^{[n]}, i = 1, 2$ of $F^\times/n$ by $Z(\overline{G})$ whose pull-back via the quotient map $F^\times \longrightarrow F^\times/n$ are just E_i. For $E_1^{[n]}$, the definition is clear. For $E_2^{[n]}$, it is just the pull-back of $\mathrm{Hom}(\mathcal{E}_{Q,n}/\phi_\eta \circ \mathbf{s}(Y_{Q,n}^{sc}), \mathbf{C}^\times)$ via the Hilbert symbol $h : F^\times/n \longrightarrow \mathrm{Hom}(F^\times, \mathbf{C}^\times)$.

It is also natural to ask for the splitting of the Baer sum of $E_1^{[n]} \oplus_B E_2^{[n]}$ over $F^\times/n$:

$$Z(\overline{G}^\vee) \hookrightarrow E_1^{[n]} \oplus_B E_2^{[n]} \twoheadrightarrow F^\times/n \ .$$

The previous argument for the splitting of $E_{\overline{G}}$ could be applied, provided that there is an additional condition besides $C(0)$ and $C(1)$:

$$\overline{\chi}(1, y \otimes a) = 1, \text{ for all } y \in nY_{Q,n}, a \in F^\times.$$

Since this condition is subsumed by $(C0)_+^{eq}$ in the definition of distinguished characters, we see that any distinguished character actually gives a splitting of $E_1^{[n]} \oplus_B E_2^{[n]}$ over $F^\times/n$. However, in general a qualified character may not give rise to a splitting of $E_1^{[n]} \oplus_B E_2^{[n]}$ over $F^\times/n$.

3.2.2 The case for $\overline{G} = \overline{T}$: the local Langlands correspondence

In the case $\overline{G} = \overline{T}$, any splitting of ${}^L\overline{T}$ is admissible, i.e. $\mathfrak{S}^a({}^L\overline{T}, \mathrm{W}_F) = \mathfrak{S}({}^L\overline{T}, \mathrm{W}_F)$. The condition $(C1)^{\mathrm{eq}}$ for qualified character $\overline{\chi}$ of $\overline{T}_{Q,n}$ is vacuous. That is, any genuine character $\overline{\chi}$ of $Z(\overline{T})$ is qualified. Also $\overline{\chi} \in \mathrm{Hom}_\epsilon(Z(\overline{T}), \mathbf{C}^\times)$ is distinguished if and only if it satisfies $(C0)_+^{\mathrm{eq}}$.

Thus,

Proposition 3.2.13. *There is a natural injective homomorphism as compositions*

$$Hom_\epsilon(Z(\overline{T}), \mathbf{C}^\times) \hookrightarrow \mathfrak{S}(E_{\overline{T}}, F^\times) \xrightarrow{\simeq} \mathfrak{S}^a({}^L\overline{T}, W_F),$$

where the first map is explicitly given by

$$\overline{\chi} \longmapsto \rho_{\overline{\chi}} \text{ with } \rho_{\overline{\chi}}(a) = \big([\overline{\chi}(1, -\otimes a)], a) \oplus_B (1, a)\big).$$

We call it the local Langlands correspondence (LLC).

Now back to the case of general $\overline{G}$ with covering torus $\overline{T}$. Recall we have the canonical homomorphism $\mathcal{C} : E_{\overline{T}} \longrightarrow \overline{T}^\vee / Z(\overline{G}^\vee)$ defined right before Corollary 2.4.9. It gives an induced map $\mathcal{C}_* : \mathfrak{S}(E_{\overline{T}}, F^\times) \longrightarrow \mathrm{Hom}(F^\times, \overline{T}^\vee / Z(\overline{G}^\vee))$ by post composition with $\mathcal{C}$.

Consider the composition

$$\mathrm{Hom}_\epsilon(Z(\overline{T}), \mathbf{C}^\times) \xhookrightarrow{\mathrm{LLC}} \mathfrak{S}(E_{\overline{T}}, F^\times) \xrightarrow{\mathcal{C}_*} \mathrm{Hom}(F^\times, \overline{T}^\vee / Z(\overline{G}^\vee))$$

which is given by

$$\overline{\chi} \longmapsto \rho_{\overline{\chi}} \longmapsto \mathcal{C} \circ \rho_{\overline{\chi}}.$$

The following result is of fundamental importance to the GK formula.

Corollary 3.2.14. *With notations as above, identify $\overline{T}^\vee / Z(\overline{G}^\vee)$ with $Hom(Y_{Q,n}^{sc}, \mathbf{C}^\times)$. Then for any $a \in F^\times$,*

$$\big(\mathcal{C} \circ \rho_{\overline{\chi}}(a)\big)(\alpha_{[n]}^\vee) = \overline{\chi} \circ \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(a^{n_\alpha})\big) \text{ for all } \alpha \in \Psi.$$

Proof. Just combine Corollary 2.4.9 and Proposition 3.2.3. ☐

3.2.3 Weyl group invariance for qualified characters

The normalizer $N(\mathbb{T})$ acts on $\overline{\mathbb{T}}$, which gives rise to an action of $N(\mathbb{T})$ on $\overline{T}$. The action does not descend to $N(\mathbb{T})/\mathbb{T}$ in general.

However, on the other hand $N(\mathbb{T})$ preserves the center $Z(\overline{T})$ since it preserves $T^\dagger$. Also $\mathbb{T}$ acts trivially on $Z(\overline{T})$. Therefore we obtain a well-defined action of the Weyl group $W = N(\mathbb{T})/\mathbb{T}$ on $Z(\overline{T})$.

Let $\alpha \in \Delta$. For any genuine character $\overline{\chi}$ of $Z(\overline{T})$, we have an action of $\mathbf{w}_\alpha \in W$ on $\overline{\chi}$ given by

$${}^{\mathbf{w}_\alpha}\overline{\chi}(\overline{t}) := \overline{\chi}(\mathbf{w}_\alpha^{-1}\overline{t}\mathbf{w}_\alpha).$$

We may ask for whether $\overline{\chi}$ is W-invariant, for which it is sufficient to check for the simple reflections $\mathbf{w}_\alpha$ for $\alpha \in \Delta$. For this purpose, we have the following useful result.

Lemma 3.2.15. *For any root $\alpha \in \Psi$ and any $y \in Y_{Q,n}$, write $\langle \alpha, y \rangle$ for the pairing between X and Y. Then*

$$n_\alpha \text{ divides } \langle \alpha, y \rangle.$$

Proof. Using the Weyl-invariance of B_Q we have shown (cf. Lemma 2.1.3) $\langle \alpha, y\rangle \cdot Q(\alpha^\vee) = B_Q(y, \alpha^\vee)$ which is divisible by n with $y \in Y_{Q,n}$. It follows $n_\alpha | \langle \alpha, y\rangle$. □

Proposition 3.2.16. *Let $\overline{\chi}$ be a qualified genuine character of $Z(\overline{T})$, then it is W-invariant. That is, for all $\mathbf{w}_\alpha \in W$ with $\alpha \in \Delta$,*

$$^{\mathbf{w}_\alpha}\overline{\chi} = \overline{\chi}.$$

Proof. We use the notation $\mathbb{h}_\alpha(a)$ for $\alpha^\vee(a), a \in F^\times$. Let t be the image of $\bar{t}$ in $T^\dagger$. It suffices to show $^{\mathbf{w}_\alpha}\overline{\chi}(\bar{t}) = \overline{\chi}(\bar{t})$ on the generators of $Z(\overline{T})$. Hence, we may assume $t = y \otimes b$ with $y \in Y_{Q,n}$ and $b \in F^\times$. From [BD01, (11.9.1)] we have

$$\begin{aligned} \mathbf{w}_\alpha^{-1}\bar{t}\mathbf{w}_\alpha =& \bar{t} \cdot \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(\alpha(t)^{-1})\big) \\ =& \bar{t} \cdot \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(b^{-\langle \alpha, y\rangle})\big). \end{aligned}$$

We need to show $\overline{\chi} \circ \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(b^{-\langle \alpha, y\rangle})\big) = 1$. However, from [BD01, 11.1.5], we have

$$\begin{aligned} \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(b^{-\langle \alpha, y\rangle})\big) &= \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(b^{n_\alpha})\big)^{-\langle \alpha, y\rangle/n_\alpha} \cdot (b^{n_\alpha}, b^{n_\alpha})_n^{\varepsilon(-\langle \alpha, y\rangle/n_\alpha)Q(\alpha^\vee)} \\ &= \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(b^{n_\alpha})\big)^{-\langle \alpha, y\rangle/n_\alpha} \end{aligned}$$

Since $\overline{\chi}$ is a qualified character, we have $\overline{\chi} \circ \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(b^{n_\alpha})\big) = 1$ by the condition $(C1)$ (which is equivalent to $(C1)^{eq}$). It follows $\overline{\chi} \circ \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(b^{-\langle \alpha, y\rangle})\big) = 1$. Therefore, $^{\mathbf{w}_\alpha}\overline{\chi} = \overline{\chi}$ and this concludes the proof. □

3.3 Construction of distinguished characters for fair $(D, \mathbb{1})$

Consider fair $(D, \mathbb{1})$. By Proposition 3.2.10, there exist distinguished characters of $\overline{T}_{Q,n}$. In this section, we will give an explicit construction.

Recall $J = nY + Y_{Q,n}^{sc}$. Using the fairness of D, it is easy to see that the map given by its image of the generators of $J \otimes F^\times$ as

$$s_J: \quad J \otimes F^\times \longrightarrow \overline{T}_{Q,n}, \quad y \otimes a \longmapsto \big(1, y \otimes a\big) \in \overline{T}_{Q,n} \tag{3.22}$$

is a well-defined homomorphism.

Let $\overline{\chi}$ be a genuine character of $\overline{T}_{Q,n}$, then the condition $\overline{\chi} \circ s_J = \mathbb{1}$ implies both $(C0)_+$ and $(C1)^{\mathrm{eq}}$. Now we give an explicit construction of genuine characters of $\overline{T}_{Q,n}$ which satisfy $\overline{\chi} \circ s_J = \mathbb{1}$ using the Weil index and Hilbert symbol. These characters will be distinguished characters.

Recall for any additive character ψ of F, the Weil index $\gamma_\psi : F^\times \longrightarrow \mu_4 \subseteq \mathbf{C}^\times$ is a map satisfying the following properties:

$$\gamma_\psi(a) \cdot \gamma_\psi(b) = \gamma_\psi(ab) \cdot (a, b)_2 \tag{3.23}$$

$$\gamma_{\psi_{c^2}} = \gamma_\psi, \tag{3.24}$$

where for any $c \in F^\times$ we define $\psi_c(x) := \psi(cx), x \in F$. The Weil index plays an important role in describing the representations of the classical double cover of $\mathbb{S}\mathrm{p}_{2r}(F)$, in particular the construction of genuine characters of its abelian covering torus. Our construction below will recover this.

Reduction to dimension one tori

First of all, by elementary divisor theorem, let $\{e_i\}$ for $1 \le i \le r$ be a basis of $Y_{Q,n}$ such that $\{k_i e_i\}$ is a basis of the lattice $J = nY + Y_{Q,n}^{sc}$.

Let $\mathbb{T}_J$ be the torus defined over F associated with J, and let $T_J := \mathbb{T}_J(F)$. We may write $T_J = J \otimes F^\times \simeq \prod_i F^\times$ and $T_{Q,n} = Y_{Q,n} \otimes F^\times \simeq \prod_i F^\times$. Thus we obtain the map

$$\begin{array}{ccc} T_J & \longrightarrow & T_{Q,n} \\ \prod_i (k_i e_i) \otimes a & \longmapsto & \prod_i e_i \otimes a^{k_i}. \end{array}$$

On the i-th component of the product, the map is the k_i-power. Write $T_{Q,n,i}$ and $T_{J,i}$ for the one-dimensional tori for the lattices generated by e_i and $k_i e_i$ respectively. Since $\overline{T}_{Q,n}$ is abelian, we have

$$\overline{T}_{Q,n} = \overline{T}_{Q,n,1} \times ... \times \overline{T}_{Q,n,r}/Z,$$

where $\overline{T}_{Q,n,i}$ is the preimage of $T_{Q,n,i}$ in $\overline{T}_{Q,n}$ and

$$Z = \{(\zeta_i) \in \prod_i \mu_n : \prod_i \zeta_i = 1\}.$$

To construct a genuine character on $\overline{T}_{Q,n}$ such that $\overline{\chi} \circ s_J = \mathbb{1}$, it suffices to do so for each $\overline{T}_{Q,n,i}$ by requiring that it is trivial on the image of the splitting

$$\begin{array}{rccc} s_{J,i} : & T_{J,i} & \longrightarrow & \overline{T}_{Q,n,i} \\ & (k_i e_i) \otimes a & \longmapsto & (1, e_i \otimes a^{k_i}). \end{array}$$

Note the group law on $\overline{T}_{Q,n,i}$ is given by

$$(1, y_i \otimes a) \cdot (1, y_i \otimes b) = ((a,b)_n^{Q(y_i)}, y_i \otimes (ab)).$$

The definition of $\overline{\chi}$

We now attempt to define a character $\overline{\chi}_i$ of $\overline{T}_{Q,n,i}$ by

$$\overline{\chi}_i(1, e_i \otimes a) = \gamma_\psi(a)^{f_i},$$

where f_i is to be determined. There are several requirements on f_i:

1). First, the relation

$$\overline{\chi}_i((1, e_i \otimes a)) \cdot \overline{\chi}_i((1, e_i \otimes b)) = (a,b)_n^{Q(e_i)} \cdot \overline{\chi}_i((1, e_i \otimes (ab)))$$

gives

$$\text{(R1)} \quad f_i = \frac{2Q(e_i)}{n} \mod 2.$$

2). Write $A_i = 2Q(e_i)/n \in \mathbf{Z}$. Simple computation gives

$$\overline{\chi}_i((1, e_i \otimes a^{k_i})) = \gamma_\psi(a)^{k_i f_i + k_i(k_i-1)A_i}.$$

We require this to be trivial, which gives

$$\text{(R2)} \quad k_i f_i + k_i(k_i - 1)A_i = 0 \mod 4.$$

Lemma 3.3.1. *There exists f_i such that both (R1) and (R2) hold. In fact, the assignment $f_i = \pm(k_i - 1)A_i$ works.*

Proof. Clearly, if $f_i = -(k_i - 1)A_i$, then R2 is satisfied. If $f_i = (k_i - 1)A_i$, then we have

$$k_i f_i + k_i(k_i - 1)A_i = 2k_i(k_i - 1)A_i = 0 \mod 4,$$

which follows from the fact $k_i(k_i - 1) = 0 \mod 2$.

Now it suffices to show

$$\pm(k_i - 1)A_i = A_i \mod 2.$$

Equivalently,

$$k_i A_i = 0 \mod 2.$$

If k_i is even we are done. Now assume k_i is odd.

We have $k_i e_i \in J = nY + Y^{sc}_{Q,n}$, and thus clearly $n | Q(k_i e_i)$. Since k_i is odd,

$$k_i A_i = \frac{2k_i^2 Q(e_i)}{n} = \frac{2Q(k_i e_i)}{n} = 0 \mod 2.$$

This completes the proof. □

Let $y = \sum_i n_i e_i \in Y_{Q,n}$ and $a \in F^\times$ be arbitrary. Then we define

$$\overline{\chi}_\psi\big((1, y \otimes a)\big) = \prod_i \overline{\chi}_i(1, e_i \otimes a^{n_i}) \cdot (a, a)_n^{\sum_{j<j'} n_j n_{j'} D(e_j, e_{j'})} \tag{3.25}$$

$$= \prod_i \gamma_\psi(a^{n_i})^{f_i} \cdot (a, a)_n^{\sum_{j<j'} n_j n_{j'} D(e_j, e_{j'})}, \tag{3.26}$$

where $f_i = (k_i - 1)A_i$. Then $\overline{\chi}_\psi$ is a distinguished character of $\overline{T}_{Q,n}$.

3.4 Explicit distinguished characters and compatibility

In this section, we consider a simply-connected group $\mathbb{G}$ of arbitrary type. By Theorem 2.1.2, there is up to unique isomorphism a $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}$ associated to a Weyl-invariant quadratic form on $Y^{sc} = Y$. Consider $\overline{G}$ incarnated by (D, η), then there is no loss of generality in assuming D fair and $\eta = \mathbb{1}$. We will do so in the following.

We apply the construction in previous section and explicate the distinguished characters case by case.

For simplicity we assume $n = 2$ except for the case of the exceptional G_2 where the computation is very simple for general n. We also assume that Q is the unique Weyl-invariant quadratic form which takes value 1 on the short coroots of $\mathbb{G}$. The general case follows from similar computations.

In the simply-laced case and the case C_r for the Dynkin diagram, we shall see that these explicit distinguished characters are compatible with those given by Savin (cf. [Sav04]) and those for the classical double cover of $\mathbb{Sp}_{2r}(F)$ respectively (cf. [Kud96] [Rao93]).

Note that since we have assume $n = 2$, we will write $Y_{Q,2}$ and $Y_{Q,2}^{sc}$ for the lattices $Y_{Q,n}$ and $Y_{Q,n}^{sc}$ which are of interest. We also have $J = 2Y + Y_{Q,2}^{sc} = Y_{Q,2}^{sc}$ since $Y = Y^{sc}$.

3.4.1 The simply-laced case A_r, D_r, E_6, E_7, E_8 and compatibility

Now let $\mathbb{G}$ be a simply-laced simply-connected group of type A_r for $r \geq 1$, D_r for $r \geq 3$, and E_6, E_7, E_8. Let $\Delta = \{\alpha_1, ..., \alpha_r\}$ be a fixed set of simple roots of $\mathbb{G}$. Let $\overline{\mathbb{G}}$ be the extension of $\mathbb{G}$ determined by the quadratic form Q with $Q(\alpha_i^\vee) = 1$ for all coroots $\alpha_i^\vee$.

We obtain the two-fold cover $\overline{G}$ of G. We show that our distinguished genuine character in previous section agrees with the one given by Savin.

Clearly we have $n_\alpha = 2$ for all $\alpha \in \Psi$ in this case. As mentioned, we also have

$$J = 2Y^{sc} + Y_{Q,2}^{sc} = 2Y^{sc} = Y_{Q,2}^{sc}.$$

Let $\alpha_i^\vee \in \Delta^\vee$ for $i = 1, ..., r$ be the simple coroots of $\mathbb{G}$. It is easy to compute the bilinear form B_Q associated with Q:

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} -1 & \text{if } \alpha_i \text{ and } \alpha_j \text{ connected in the Dynkin diagram,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.27}$$

In order to show compatibility with Savin, we may further assume that $\overline{G}$ is incarnated by the following fair bisector D associated with B_Q as given in [Sav04],

$$D(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 0 & \text{if } i < j, \\ Q(\alpha_i^\vee) & \text{if } i = j, \\ B_Q(\alpha_i^\vee, \alpha_j^\vee) & \text{if } i > j. \end{cases} \tag{3.28}$$

The following lemma is in [Sav04] and reproduced here for convenience. The stated result can also be checked by straightforward computation.

Lemma 3.4.1. *Let Ω be a subset of the vertices in the Dynkin diagram of $\mathbb{G}$ satisfying:*

(i) No two vertices in Ω are adjacent,

(ii) Every vertex not in Ω is adjacent to an even number of vertices in Ω.

Then the map given by $\Omega \longmapsto e_\Omega$ with $e_\Omega := \sum_{\alpha_i \in \Omega} \alpha_i^\vee$ gives a well-defined correspondence between such sets Ω and the cosets of $Y_{Q,2}/J$. In particular, the empty set $\varnothing$ corresponds to the trivial coset J.

By properties of B_Q and (i) of Ω above, it follows that

$$Q(e_\Omega) = |\Omega|.$$

We now give a brief case by case discussion.

The A_r case.

There are two situations according to the parity of r.

Case 1: r is even. As an illustration, we first do the straightforward computation. Let $\sum_i k_i \alpha_i^\vee \in Y_{Q,2}$ for proper $k_i \in \mathbf{Z}$. Then $B_Q(\sum_i k_i \alpha_i^\vee, \alpha_j^\vee) \in 2\mathbf{Z}$ for all $1 \le j \le r$ by the definition of $Y_{Q,2}$. In view of (3.27), it is equivalent to

$$\begin{cases} 2k_1 + (-1)k_2 & \in 2\mathbf{Z}, \\ (-1)k_1 + 2k_2 + (-1)k_3 & \in 2\mathbf{Z}, \\ (-1)k_2 + 2k_3 + (-1)k_4 & \in 2\mathbf{Z}, \\ \vdots & \\ (-1)k_{r-2} + 2k_{r-1} + (-1)k_r & \in 2\mathbf{Z}, \\ (-1)k_{r-1} + 2k_r & \in 2\mathbf{Z}. \end{cases} \tag{3.29}$$

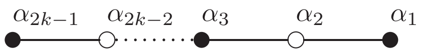
It follows that k_2 is even and so are the successive $k_4, ..., k_r$ (we have assumed r to be even). Similarly, k_{r-1} is even, and therefore all $k_{r-3}, ..., k_1$ are also even. That is $Y_{Q,2} = J$.

Note that we could simply apply the lemma to get $Y_{Q,2} = J$, which corresponds to the fact there is only the empty set for Ω satisfying properties (i) and (ii). There is nothing to check in this case, and the character $\overline{\chi}$ with $\overline{\chi}\big(s_J(t)\big) = 1, t \in T_J = J \otimes F^\times$ will be a distinguished character.

Case 2: $r = 2k-1$ is odd. The consideration as in (3.29) works. However, for convenience we will apply the lemma to get $[Y_{Q,2}, J] = 2$ with a basis of $Y_{Q,2}$ given by

$$\big\{\alpha_{r,[2]}^\vee, \alpha_{r-1,[2]}^\vee, ..., \alpha_{2,[2]}^\vee, e_\Omega = \sum_{m=1}^{k} \alpha_{2m-1}^\vee\big\}.$$

The nontrivial coset corresponds to the set Ω indicated by alternating bold circles below



A basis for $J = 2Y^{sc}$ is given by

$$\left\{\alpha^\vee_{r,[2]}, \alpha^\vee_{r-1,[2]}, ..., \alpha^\vee_{2,[2]}, 2e_\Omega\right\}.$$

Now the construction of the distinguished $\overline{\chi}_\psi$ in previous section is given by

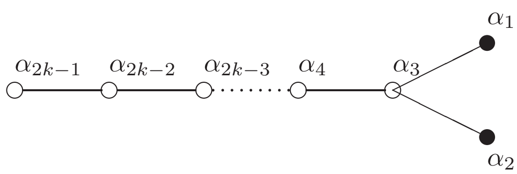
$$\begin{cases} \overline{\chi}_\psi(1, \alpha^\vee_{i,[2]} \otimes a) = 1 \\ \overline{\chi}_\psi(1, e_\Omega \otimes a) = \gamma_\psi(a)^{(2-1)Q(e_\Omega)} = \gamma_\psi(a)^{|\Omega|}. \end{cases} \tag{3.30}$$

This agrees with the formula in Savin when we substitute $a = \varpi$ in $\gamma_\psi(a)$ for ψ of conductor $\mathcal{O}_F$. See [Sav04, pg. 118].

The D_r case.

We also have two cases.

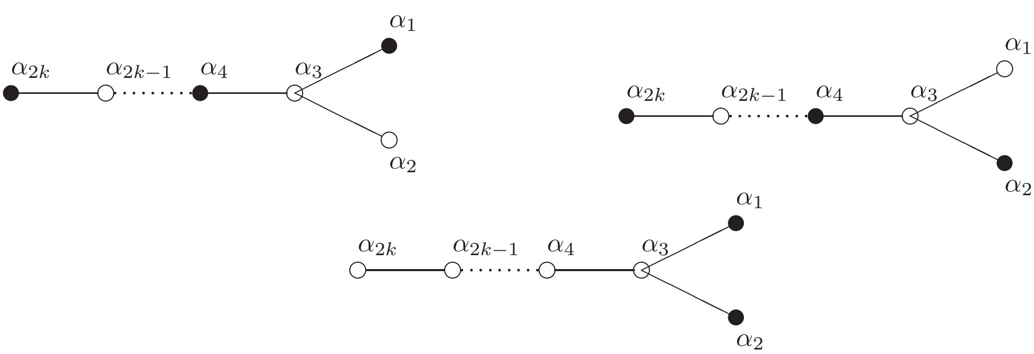
Case 1: $r = 2k - 1$ is odd with $k \geq 2$. Then $[Y_{Q,2}, J] = 2$ with the nontrivial $\Omega = \left\{\alpha_1, \alpha_2\right\}$:



Consider the basis $\left\{\alpha^\vee_{i,[2]} : 2 \leq i \leq r\right\} \cup \left\{e_\Omega\right\}$ of $Y_{Q,2}$, then the construction of distinguished $\overline{\chi}_\psi$ in previous section is determined by

$$\overline{\chi}_\psi(1, e_\Omega \otimes a) = \gamma_\psi(a)^{(2-1)Q(e_\Omega)} = \gamma_\psi(a)^{|\Omega|}.$$

Case 2: $r = 2k$ is even. Then $[Y_{Q,2} : J] = 4$. There are three nontrivial sets Ω_i for $i = 1, 2, 3$ as indicated by the bold circles below.



That is, $\Omega_1 = \left\{\alpha_1\right\} \cup \left\{\alpha_{2m} : 2 \leq m \leq k\right\}$, $\Omega_2 = \left\{\alpha_2\right\} \cup \left\{\alpha_{2m} : 2 \leq m \leq k\right\}$ and $\Omega_3 = \left\{\alpha_1, \alpha_2\right\}$. Note $|\Omega_1| = |\Omega_2| = k$ and $|\Omega_2| = 2$.

A basis of $Y_{Q,2}$ is given by

$$\left\{\alpha_{i,[2]}^{\vee} : 3 \leq i \leq 2k-1\right\} \cup \left\{e_{\Omega_1}, e_{\Omega_2}, e_{\Omega_3}\right\}.$$

However, the construction of distinguished characters in previous section utilized the elementary divisor theorem. Thus we have to provide bases for $Y_{Q,2}$ and J aligned in a proper way. To achieve this, consider the alternative basis of $Y_{Q,2}$ given by

$$\left\{\alpha_{i,[2]}^{\vee} : 3 \leq i \leq 2k-1\right\} \cup \left\{e_{\Omega_1} + e_{\Omega_2} + e_{\Omega_3}, e_{\Omega_2} + e_{\Omega_3}, e_{\Omega_3}\right\}.$$

Then it is easy to check that the set

$$\left\{\alpha_{i,[2]}^{\vee} : 3 \leq i \leq 2k-1\right\} \cup \left\{e_{\Omega_1} + e_{\Omega_2} + e_{\Omega_3}, 2(e_{\Omega_2} + e_{\Omega_3}), 2e_{\Omega_3}\right\}$$

is a basis for J. Note

$$Q(e_{\Omega_2} + e_{\Omega_3}) = |\Omega_2| + Q(2\alpha^{\vee}) = |\Omega_2| + 4.$$

Thus a distinguished character could be determined by

$$\begin{cases} \overline{\chi}_{\psi}(1, (e_{\Omega_2} + e_{\Omega_3}) \otimes a) = \gamma_{\psi}(a)^{Q(e_{\Omega_2}+e_{\Omega_3})} = \gamma_{\psi}(a)^{|\Omega_2|}, \\ \overline{\chi}_{\psi}(1, e_{\Omega_3} \otimes a) = \gamma_{\psi}(a)^{|\Omega_3|}. \end{cases}$$

However, since we have assumed that D takes the special form given by (3.28), we have

$$\begin{aligned} D(e_{\Omega_1}, e_{\Omega_2} + e_{\Omega_3}) &= |\Omega_1| \\ D(e_{\Omega_2}, e_{\Omega_3}) &= Q(\alpha^{\vee}) = 1. \end{aligned}$$

Thus

$$\begin{aligned} \overline{\chi}_{\psi}(1, e_{\Omega_1} \otimes a) \cdot \overline{\chi}_{\psi}(1, (e_{\Omega_2} + e_{\Omega_3}) \otimes a) &= (a,a)_2^{|\Omega_1|} \cdot \overline{\chi}_{\psi}(1, (e_{\Omega_1} + e_{\Omega_2} + e_{\Omega_3}) \otimes a) \\ \overline{\chi}_{\psi}(1, e_{\Omega_2} \otimes a) \cdot \overline{\chi}_{\psi}(1, e_{\Omega_3} \otimes a) &= (a,a)_2 \cdot \overline{\chi}_{\psi}(1, (e_{\Omega_2} + e_{\Omega_3}) \otimes a) \end{aligned}$$

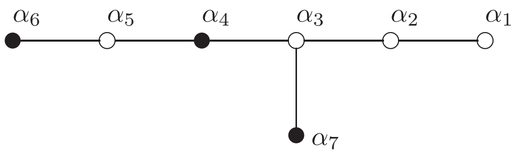
Recall $\gamma_{\psi}(a)^2 = (a,a)_2$. This combined with the above results gives

$$\begin{cases} \overline{\chi}_{\psi}(1, e_{\Omega_1} \otimes a) = \gamma_{\psi}(a)^{|\Omega_1|}, \\ \overline{\chi}_{\psi}(1, e_{\Omega_2} \otimes a) = \gamma_{\psi}(a)^{|\Omega_2|}, \\ \overline{\chi}_{\psi}(1, e_{\Omega_3} \otimes a) = \gamma_{\psi}(a)^{|\Omega_3|}. \end{cases}$$

It agrees with the character given by Savin.

The E_6, E_7, E_8 case.

For E_6 and E_8, $Y_{Q,2} = J$ and so the situation is trivial. Consider E_7, then $[Y_{Q,n} : J] = 2$. The nontrivial Ω is given by $\Omega = \{\alpha_4, \alpha_6, \alpha_7\}$.



The set $\{\alpha^\vee_{i,[2]} : 1 \le i \le 6\} \cup \{e_\Omega\}$ is a basis of $Y_{Q,2}$, while $\{\alpha^\vee_{i,[2]} : 1 \le i \le 6\} \cup \{2e_\Omega\}$ a basis for J.

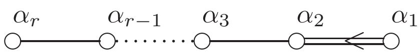
Our distinguished character is determined by

$$\overline{\chi}_\psi(1, e_\Omega \otimes a) = \gamma_\psi(a)^{|\Omega|}.$$

This agrees with Savin also.

3.4.2 The case C_r and compatibility with the classical metaplectic double cover $\overline{\mathrm{Sp}}_{2r}(F)$

Let Sp_{2r} be the simply-connected simple group with Dynkin diagram:



Let $\{\alpha_1^\vee, \alpha_2^\vee, ..., \alpha_r^\vee\}$ be the set of simple coroots with $\alpha_1^\vee$ the short one. Let $n = 2$. As mentioned, $\overline{\mathrm{Sp}}_{2r}(F) = \mathrm{Hs}_{\mathrm{Sp}_{2r}}(\overline{\mathrm{Sp}_{2r}})$ is the two-fold cover of $\mathrm{Sp}_{2r}(F)$. Here $\overline{\mathrm{Sp}}_{2r}$ is determined by the unique Weyl-invariant quadratic form Q on Y with $Q(\alpha_1^\vee) = 1$.

It follows $n_{\alpha_1} = 2$. Also $Q(\alpha_i^\vee) = 2$ and $n_{\alpha_i} = 1$ for $2 \le i \le r$. Moreover,

$$\begin{aligned} Y_{Q,2} = Y^{sc} &= \langle \alpha_1^\vee, \alpha_2^\vee, ..., \alpha_r^\vee \rangle_{\mathbf{Z}}, \\ Y_{Q,2}^{sc} &= \langle 2\alpha_1^\vee, \alpha_2^\vee, ..., \alpha_r^\vee \rangle_{\mathbf{Z}}. \end{aligned}$$

Since $J = Y^{sc}_{Q,2}$, by the construction of distinguished character $\overline{\chi}_\psi$, it is determined by

$$\begin{cases} \overline{\chi}_\psi(1, \alpha_i^\vee \otimes a) = 1, \text{ if } i = 2, 3, ..., r; \\ \overline{\chi}_\psi(1, \alpha_1^\vee \otimes a) = \gamma_\psi(a)^{(2-1)Q(\alpha_1^\vee)} = \gamma_\psi(a). \end{cases} \tag{3.31}$$

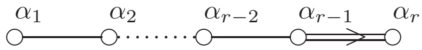
This uniquely determined a genuine character of $\overline{T}$ which is abelian. It can be checked that this agrees with the classical one (cf. [Rao93] or [Kud96] for example).

3.4.3 The B_r, F_4 and G_2 case

For completeness, we also give the explicit form of the distinguished character constructed in previous section for the double cover $\overline{G}$ of the simply connected group G of type B_r, F_4 and G_2. Recall that when $n = 2$ we have $J = Y^{sc}_{Q,2}$.

The B_r case

Consider the Dynkin diagram of B_r:



Let Q be the unique Weyl-invariant quadratic form with $Q(\alpha_i^\vee)=1$ for $1\le i\le r-1$. It gives $Q(\alpha_r^\vee)=2$. We have also assumed that the double cover $\overline{G}$ is incarnated by a fair bisector D. The discussion now will be split into two cases according to the parity of r.

<u>CASE 1:</u> r is odd. Direct computation gives $Y_{Q,2}^{sc}=Y_{Q,n}$ and therefore this case is trivial.

<u>CASE 2:</u> r is even. It is not difficult to compute the index $[Y_{Q,2}:Y_{Q,2}^{sc}]=2$. In fact, a basis of $Y_{Q,2}$ is given by

$$\left\{\alpha_1^\vee+\alpha_3^\vee+...+\alpha_{r-1}^\vee\right\}\cup\left\{2\alpha_i^\vee:2\le i\le r-1\right\}\cup\left\{\alpha_r^\vee\right\}.$$

This gives a basis of $J=Y_{Q,2}^{sc}$:

$$\left\{2(\alpha_1^\vee+\alpha_3^\vee+...+\alpha_{r-1}^\vee)\right\}\cup\left\{2\alpha_i^\vee:2\le i\le r-1\right\}\cup\left\{\alpha_r^\vee\right\}.$$

We have

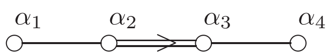
$$Q(\alpha_1^\vee+\alpha_3^\vee+...+\alpha_{r-1}^\vee)=r/2.$$

By the construction of distinguished character $\overline{\chi}_\psi$, it is determined by

$$\begin{cases}\overline{\chi}_\psi\big((1,(\alpha_1^\vee+\alpha_3^\vee+...+\alpha_{r-1}^\vee)\otimes a)\big)=\gamma_\psi(a)^{r/2};\\ \overline{\chi}_\psi\big((1,(2\alpha_i^\vee)\otimes a)\big)=1,\ \text{for } 2\le i\le r-1;\\ \overline{\chi}_\psi\big((1,\alpha_r^\vee\otimes a)\big)=1.\end{cases}\tag{3.32}$$

The F_4 case

Consider the Dynkim diagram of F_4:



Let Q be such that $Q(\alpha_i^\vee)=1$ for $i=1,2$. It implies $Q(\alpha_i^\vee)=2$ for $i=3,4$. Clearly $n_{\alpha_i}=2$ for $i=1,2$ and $n_{\alpha_i}=1$ for $i=3,4$. We can compute

$$B_Q(\alpha_1^\vee,\alpha_2^\vee)=-1,\quad B_Q(\alpha_2^\vee,\alpha_3^\vee)=-2,\quad B_Q(\alpha_3^\vee,\alpha_4^\vee)=-2.$$

Also $B_Q(\alpha_i^\vee,\alpha_j^\vee)=0$ if α_i and α_j are not adjacent in the Dynkin diagram.

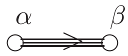
Moreover, any $\sum_i k_i\alpha_i^\vee\in Y^{sc}$ with certain $k_i\in\mathbf{Z}$ belongs to $Y_{Q,2}$ if and only if $2|B_Q(\sum_i k_i\alpha_i^\vee,\alpha_j^\vee)$ for all $1\le j\le 4$, i.e., in explicit terms:

$$\begin{cases}2k_1+(-1)k_2\in 2\mathbf{Z},\\ (-1)k_1+2k_2+(-2)k_3\in 2\mathbf{Z},\\ (-2)k_2+4k_3+(-2)k_4\in 2\mathbf{Z},\\ (-2)k_3+4k_4\in 2\mathbf{Z}.\end{cases}$$

Equivalently, $k_1, k_2 \in 2\mathbf{Z}$. This shows $Y_{Q,2} = Y_{Q,2}^{sc}$, and thus the situation is trivial. Note that this agrees with the fact that the dual group $\overline{G}^\vee$ in this case has to be of type F_4, and there is a unique one. This gives a priori the equality $Y_{Q,2} = Y_{Q,2}^{sc}$.

The G_2 case

Consider the Dynkin diagram of G_2:



Let Q be such that $Q(\alpha^\vee) = 1$. This determines $Q(\beta^\vee) = 3$. Note $B_Q(\alpha^\vee, \beta^\vee) = -Q(\alpha^\vee) = -3$.

Since the computation is straightforward, we may assume $n \in \mathbf{N}_{\geq 1}$ is general instead of 2. It follows $n_\alpha = n$ and $n_\beta = n/\gcd(n, 3)$. Then $k_1\alpha^\vee + k_2\beta^\vee$ lies in $Y_{Q,n}$ if and only if

$$\begin{cases} 2k_1 - 3k_2 \in n\mathbf{Z}, \\ -3k_1 + 6k_2 \in n\mathbf{Z}. \end{cases}$$

Equivalently, $k_1 \in n\mathbf{Z}$ and k_2 divisible by $n/\gcd(n, 3)$. This exactly shows $Y_{Q,n} = Y_{Q,n}^{sc}$ for arbitrary n. This also agrees with the a priori fact that the dual group $\overline{G}^\vee$ of $\overline{G}$ must be of type G_2 and there is a unique one, which enforces the equality $Y_{Q,n} = Y_{Q,n}^{sc}$ to hold.

Also in this case, it is trivial to define the distinguished character for the fair D.

Remark 3.4.2. Fix $n = 2$ and Q such that $Q(\alpha^\vee) = 1$ for short coroots $\alpha^\vee$. In retrospect we see that for type B_r, F_4, G_2 one can have ad hoc descriptions of the cosets representative of $Y_{Q,n}/J$ in a similar way as Lemma 3.4.1. For example, for B_r and F_4 we modify the Dynkin diagram of these two by removing the short roots, then there is a correspondence between coset representative of $Y_{Q,n}/J$ and subsets of nodes in the modified Dynkin diagram satisfying (i) and (ii) as in Lemma 3.4.1. For G_2, it is easy to have similar description as well.

3.5 An equivalent construction of ${}^L\overline{T}$ and LLC by Deligne

Consider the case when $\overline{G} = \overline{T}$ over a local field F, Deligne gives a canonical construction ${}^{\mathcal{D}}\overline{T}$ with $\overline{T}^\vee \hookrightarrow {}^{\mathcal{D}}\overline{T} \twoheadrightarrow \mathrm{W}_F$, which is isomorphic to ${}^L\overline{T}$ (cf. [We14] also). In fact ${}^{\mathcal{D}}\overline{T}$ is defined to be ${}^{\mathcal{D}}\overline{T} = \mathrm{Rec}^*({}^{\mathcal{D}}E_{\overline{T}})$ for a certain ${}^{\mathcal{D}}E_{\overline{T}}$ in

$$Z(\overline{G}^\vee) \hookrightarrow {}^{\mathcal{D}}E_{\overline{T}} \twoheadrightarrow F^\times .$$

The construction of ${}^{\mathcal{D}}E_{\overline{T}}$ is canonical and along the way one obtains an analogous version of the local Langlands correspondence (${}^{\mathcal{D}}$LLC). After recalling the construction of ${}^{\mathcal{D}}E_{\overline{T}}$ and the ${}^{\mathcal{D}}$LLC, we will show that there is a natural isomorphism $E_{\overline{T}} \simeq {}^{\mathcal{D}}E_{\overline{T}}$, which gives the compatibility of LLC and ${}^{\mathcal{D}}$LLC.

The construction of ${}^{\mathcal{D}}E_{\overline{T}}$ and ${}^{\mathcal{D}}$LLC

Recall for $\overline{T} \in \mathsf{CExt}(T, \mu_n)$ of BD type, the center $Z(\overline{T})$ sits in the exact sequence

$$\begin{tikzcd} \mu_n \arrow[r, hook] & Z(\overline{T}) \arrow[r, two heads] & T^\dagger . \end{tikzcd}$$

Moreover, we have the map $i_{Q,n} : T_{Q,n} \twoheadrightarrow T^\dagger$ and the pull-back $\overline{T}_{Q,n}$:

$$\begin{tikzcd} \mu_n \arrow[r, hook] & Z(\overline{T}) \arrow[r, two heads] & T^\dagger \\ \mu_n \arrow[u, equal] \arrow[r, hook] & \overline{T}_{Q,n} \arrow[u, two heads] \arrow[r, two heads] & T_{Q,n}. \arrow[u, two heads, "i_{Q,n}"'] \end{tikzcd} \tag{3.33}$$

Use the fixed embedding $\epsilon : \mu_n \longrightarrow \mathbf{C}^\times$ to obtain the push-out $\epsilon_*(Z(\overline{T}))$. At the same time, any $\overline{\chi} \in \mathrm{Hom}_\epsilon(Z(\overline{T}), \mathbf{C}^\times)$ gives rise to a splitting of $\epsilon_*(Z(\overline{T}))$ into $\mathbf{C}^\times$.

$$\begin{tikzcd} \mu_n \arrow[r, hook] \arrow[d, hook, "\epsilon"'] & Z(\overline{T}) \arrow[r, two heads] \arrow[d] \arrow[dl, "\overline{\chi}"] & T^\dagger \arrow[d, equal] \\ \mathbf{C}^\times \arrow[r, hook] & \epsilon_*(Z(\overline{T})) \arrow[r, two heads] \arrow[l, dotted, bend left, "s_{\overline{\chi}}"] & T^\dagger. \end{tikzcd} \tag{3.34}$$

Here the splitting $s_{\overline{\chi}} : \epsilon_*(Z(\overline{T})) \longrightarrow \mathbf{C}^\times$ is given by

$$s_{\overline{\chi}} : \quad [(z, \overline{t})] \longmapsto z \cdot \overline{\chi}^{-1}(\overline{t}),$$

where $[(z, \overline{t})]$ denote the class of $(z, \overline{t}) \in \mathbf{C}^\times \times Z(\overline{T})$ in $\epsilon_*(Z(\overline{T}))$. Clearly $s_{\overline{\chi}}$ gives a splitting $\rho^{\mathcal{D}}_{\overline{\chi}}$ of $\epsilon_*(Z(\overline{T}))$ over $T^\dagger$ given by

$$\rho^{\mathcal{D}}_{\overline{\chi}} : \quad t \longmapsto [(\overline{\chi}(\overline{t}), \overline{t})],$$

where $\overline{t} \in Z(\overline{T})$ is any preimage of t.

Note by definition $\overline{T}^\vee = X_{Q,n} \otimes \mathbf{C}^\times$ and also $T_{Q,n} = Y_{Q,n} \otimes F^\times$, where $X_{Q,n} := \mathrm{Hom}(Y_{Q,n}, \mathbf{Z})$ is the lattice dual to $Y_{Q,n}$. By abuse of notation, we still use $i_{Q,n}$ to denote the naturally induced map $i_{Q,n} : X_{Q,n} \otimes T_{Q,n} \longrightarrow X_{Q,n} \otimes T^\dagger$.

Let m be the map

$$m : F^\times \longrightarrow X_{Q,n} \otimes T_{Q,n}$$

given by

$$\big(m(a)\big)(y) = y \otimes a, \ y \in Y_{Q,n},$$

where we have identified $X_{Q,n} \otimes T_{Q,n} \simeq \mathrm{Hom}(Y_{Q,n}, T_{Q,n})$. Then ${}^{\mathcal{D}}E_{\overline{T}}$ is defined to be the pull-back of $X_{Q,n} \otimes \epsilon_*(Z(\overline{T}))$ via $i_{Q,n} \circ m$:

$$\begin{tikzcd} \overline{T}^{\vee} \arrow[r, hook] \arrow[d, equal] & X_{Q,n} \otimes \epsilon_*(Z(\overline{T})) \arrow[r, two heads] & X_{Q,n} \otimes T^{\dagger} \arrow[l, bend right, "\rho_{\overline{\chi}}^{\mathcal{D}}"'] \\ \overline{T}^{\vee} \arrow[r, hook] & {}^{\mathcal{D}}E_{\overline{T}} \arrow[r, two heads] \arrow[u] & F^{\times}. \arrow[u, "i_{Q,n}\circ m"'] \arrow[l, bend left, "\rho_{\overline{\chi}}^{\mathcal{D}}"] \end{tikzcd} \tag{3.35}$$

Meanwhile, there is the inherited splitting of ${}^{\mathcal{D}}E_{\overline{T}}$ over $F^{\times}$, still denoted by $\rho_{\overline{\chi}}^{\mathcal{D}}$.

Definition 3.5.1. We define ${}^{\mathcal{D}}\overline{T}$ to be $\mathrm{Rec}^*({}^{\mathcal{D}}E_{\overline{T}})$. We will call ${}^{\mathcal{D}}$LLC the canonical map $\mathrm{Hom}_{\epsilon}(Z(\overline{T}), \mathbf{C}^{\times}) \longrightarrow \mathfrak{S}({}^{\mathcal{D}}E_{\overline{T}}, F^{\times})$ given by

$$\overline{\chi} \longmapsto \rho_{\overline{\chi}}^{\mathcal{D}}$$

from above discussion. Since we have the canonical isomorphism $\mathfrak{S}({}^{\mathcal{D}}E_{\overline{T}}, F^{\times}) \simeq \mathfrak{S}({}^{\mathcal{D}}\overline{T}, \mathrm{W}_F)$, $\rho_{\overline{\chi}}^{\mathcal{D}}$ could be viewed as a splitting of ${}^{\mathcal{D}}\overline{T}$ over W_F.

Note that we could work over $\overline{T}_{Q,n}$ and obtain similarly $\epsilon_*(\overline{T}_{Q,n})$ and also $X_{Q,n} \otimes \epsilon_*(\overline{T}_{Q,n})$ in the extension

$$X_{Q,n} \otimes \mathbf{C}^{\times} \hookrightarrow X_{Q,n} \otimes \epsilon_*(\overline{T}_{Q,n}) \twoheadrightarrow X_{Q,n} \otimes T_{Q,n}\ .$$

The the pull-back $m^*\big(X_{Q,n} \otimes \epsilon_*(\overline{T}_{Q,n})\big)$, by tracing through the compatible construction from (3.33), is canonically isomorphic to ${}^{\mathcal{D}}E_{\overline{T}}$:

$$\begin{tikzcd} \overline{T}^{\vee} \arrow[r, hook] \arrow[d, equal] & {}^{\mathcal{D}}E_{\overline{T}} \arrow[r, two heads] & F^{\times} \arrow[d, equal] \\ \overline{T}^{\vee} \arrow[r, hook] & m^*\big(X_{Q,n} \otimes \epsilon_*(\overline{T}_{Q,n})\big) \arrow[r, two heads] \arrow[u, "\simeq"] & F^{\times}\ . \end{tikzcd}$$

Since any $\overline{\chi} \in \mathrm{Hom}_{\epsilon}(Z(\overline{T}), \mathbf{C}^{\times})$ could be viewed as a character of $\overline{T}_{Q,n}$, we will have a splitting of $m^*\big(X_{Q,n} \otimes \epsilon_*(\overline{T}_{Q,n})\big)$ over $F^{\times}$. It is canonically identified with $\rho_{\overline{\chi}}^{\mathcal{D}}$ via the isomorphism in above commutative diagram.

To summarize, we see that for any genuine character $\overline{\chi}$ of $Z(\overline{T})$, there is a canonically defined splitting of ${}^{\mathcal{D}}E_{\overline{T}}$ over $F^{\times}$, and therefore also a canonical splitting of ${}^{\mathcal{D}}\overline{T}$ over W_F. In fact, we have

Proposition 3.5.2 ([We14])**.** *Above construction gives an isomorphism $Hom_{\epsilon}(\overline{T}_{Q,n}, \mathbf{C}^{\times}) \simeq \mathfrak{S}({}^{\mathcal{D}}E_{\overline{T}}, F^{\times})$. Therefore, the ${}^{\mathcal{D}}$LLC could be viewed as a composition of the following canonical maps:*

$$Hom_{\epsilon}(Z(\overline{T}), \mathbf{C}^{\times}) \hookrightarrow Hom_{\epsilon}(\overline{T}_{Q,n}, \mathbf{C}^{\times}) \xrightarrow{\simeq} \mathfrak{S}({}^{\mathcal{D}}E_{\overline{T}}, F^{\times}) \xrightarrow{\simeq} \mathfrak{S}({}^{\mathcal{D}}\overline{T}, W_F).$$

Compatibility of LLC and ${}^{\mathcal{D}}$LLC

We now describe a natural isomorphism $\Theta : E_{\overline{T}} \longrightarrow {}^{\mathcal{D}}E_{\overline{T}}$, and check that ${}^{\mathcal{D}}$LLC and LLC are compatible with respect to Θ.

We view the tensor functor $X_{Q,n}\otimes -$ as $\mathrm{Hom}(Y_{Q,n}, -)$. By definition in (3.35), elements of ${}^{\mathcal{D}}E_{\overline{T}}$ are of the form $({}^{\mathcal{D}}[R_a], a) \in \mathrm{Hom}(Y_{Q,n}, \epsilon_*(Z(\overline{T}))) \times F^\times$, given by

$$ {}^{\mathcal{D}}[R_a] : \quad y \longmapsto \big[\big(R_a(y), (1, y\otimes a)\big)\big] \in \epsilon_*(\overline{T}_{Q,n}), $$

where $R_a : Y_{Q,n} \longrightarrow \mathbf{C}^\times$ is a certain function. Since ${}^{\mathcal{D}}[R_a]$ is a homomorphism, this gives

$$ R_a(y_1 + y_2) = R_a(y_1) \cdot R_a(y_2) \cdot (a, a)_n^{D(y_1, y_2)}, $$

which is a necessary and sufficient condition for ${}^{\mathcal{D}}[R_a]$ to be a homomorphism. We mention in passing that the inverse of $({}^{\mathcal{D}}[R_a], a)$ is $({}^{\mathcal{D}}[R_a^{-1} \cdot (a, a)_n^{Q(-)}], a^{-1})$.

Recall the definition of $E_{\overline{T}}$ as the Baer sum $E_{2,\overline{T}} \oplus_B E_{1,\overline{T}}$. The group $E_{2,\overline{T}}$ consists of $([P_a], a) \in \mathrm{Hom}(\mathcal{E}_{Q,n}, \mathbf{C}^\times) \times F^\times$ such that $[P_a]|_{F^\times/n} = h_a$, where $h_a(b) = (b, a)_n$. Meanwhile we have written

$$ [P_a] : \quad (b, y) \longmapsto (b, a)_n \cdot P_a(y), $$

where $P_a : Y_{Q,n} \longrightarrow \mathbf{C}^\times$ is some function. That fact of $[P_a]$ being a homomorphism is equivalent to the following property of the map P_a:

$$ P_a(y_1 + y_2) = P_a(y_1) \cdot P_a(y_2) \cdot (a, a)_n^{D(y_1, y_2)}. $$

It is now clear that we can define a map $\Theta : E_{\overline{T}} \longrightarrow {}^{\mathcal{D}}E_{\overline{T}}$ by

$$ \Theta : \quad ([P_a], a) \oplus_B (1, a)_{E_1} \longmapsto ({}^{\mathcal{D}}[P_a], a). $$

It is easy to check that Θ is an isomorphism.

Let $\overline{\chi}$ be a genuine character of $Z(\overline{T})$. The ${}^{\mathcal{D}}$LLC gives the splitting $\rho^{\mathcal{D}}_{\overline{\chi}}$. Tracing through the construction of $\rho^{\mathcal{D}}_{\overline{\chi}}$ we see that it takes the explicit form: for $a \in F^\times$,

$$ \rho^{\mathcal{D}}_{\overline{\chi}}(a) = \big({}^{\mathcal{D}}[\overline{\chi}(1, -\otimes a)], a\big) \in {}^{\mathcal{D}}E_{\overline{T}}. $$

That is, $R_a(y) = \overline{\chi}\big((1, y\otimes a)\big)$ for all $y \in Y_{Q,n}$.

Corollary 3.5.3. *We have the equality*

$$ \rho^{\mathcal{D}}_{\overline{\chi}} = \Theta \circ \rho_{\overline{\chi}}, $$

where $\rho_{\overline{\chi}}$ is the splitting given by LLC as in Proposition (3.2.13). That is, the LLC and ${}^{\mathcal{D}}$LLC are compatible with respect to Θ.

Equivalently, if by abuse of notation we use Θ *to denote the induced isomorphism* ${}^L\overline{T} \longrightarrow {}^{\mathcal{D}}\overline{T}$, $\rho_{\overline{\chi}}$ *and* $\rho^{\mathcal{D}}_{\overline{\chi}}$ *for the induced splittings in* $\mathfrak{S}({}^L\overline{T}, W_F)$, $\mathfrak{S}({}^{\mathcal{D}}\overline{T}, W_F)$ *respectively, then the following diagram commutes:*

$$\begin{tikzcd} {}^L\overline{T} \arrow[rr, "\Theta"] \arrow[dr] & & {}^{\mathcal{D}}\overline{T} \arrow[dl] \\ & W_F \arrow[ul, dashed, bend left, "\rho_{\overline{\chi}}"] \arrow[ur, dashed, bend right, "\rho^{\mathcal{D}}_{\overline{\chi}}"'] & \end{tikzcd}$$

3.6 Discussion on the global situation

Although in previous sections the discussion has been for local L-groups, some of the results can be extended easily to the global situation.

Let F be a number field with $\mu_n \subseteq F^\times$. Let $\overline{\mathbb{G}}$ be a BD-type $\mathbb{K}_2$-torsor over F incarnated by (D, η). It gives rise to $\overline{\mathbb{G}}(\mathbb{A}_F)$ whose L-group ${}^L\overline{G}$ is defined in section 2.4.3.

Following the Definition 3.2.1 we may similarly define admissible splittings of ${}^L\overline{G}$ over W_F to be those which gives an isomorphism ${}^L\overline{G} \simeq \overline{G}^\vee \times \mathrm{W}_F$. Since ${}^L\overline{G}$ is derived from the fundamental extension $E_{\mathbb{A}_F}$ by pull-back, admissible splittings of the former are just splittings of

$$Z(\overline{G}^\vee) \hookrightarrow E_{\mathbb{A}_F} \twoheadrightarrow F^\times \backslash \mathbb{A}_F^\times .$$

Consider $(D, \mathbb{1})$ fair, i.e. with D fair. As in the local case, to split $E_{\mathbb{A}_F}$ it suffices to find a genuine automorphic character

$$\overline{\chi} : \quad \mathbb{T}_{Q,n}(F) \backslash \overline{\mathbb{T}}_{Q,n}(\mathbb{A}_F) \longrightarrow \mathbf{C}^\times,$$

which is trivial on the image of $\mathbb{T}_J(\mathbb{A}_F)$ in $\overline{\mathbb{T}}_{Q,n}(\mathbb{A}_F)$ by the map locally given by (3.22). Here as before $J = nY + Y^{sc}_{Q,n}$. We use s_ϕ to denote the map given by

$$s_\phi : \mathbb{T}_J(\mathbb{A}_F) \longrightarrow \overline{\mathbb{T}}_{Q,n}(\mathbb{A}_F), \quad \textstyle\prod_v y \otimes a_v \longmapsto \prod_v (1, y \otimes a_v). \tag{3.36}$$

A character $\overline{\chi}$ satisfying $\overline{\chi} \circ s_\phi = \mathbb{1}$ will descend to an automorphic character of the center $Z(\overline{\mathbb{T}}(\mathbb{A}_F))$.

The construction in section 3.3 generalizes. More precisely, we fix an additive character $\psi = \bigotimes_v \psi_v$ of $\mathbb{A}_F$. As in section 3.3, we obtain a local distinguished character $\overline{\chi}_{\psi_v}$ of $\overline{\mathbb{T}}_{Q,n}(F_v)$. Then

$$\overline{\chi}_\psi := \bigotimes_v \overline{\chi}_{\psi_v}$$

will be a well-defined automorphic character of $\overline{\mathbb{T}}_{Q,n}(\mathbb{A}_F)$ such that $\overline{\chi}_\psi \circ s_\phi = \mathbb{1}$. Such a global character $\overline{\chi}_\psi$ will be Weyl-invariant.

Chapter 4

The Gindikin-Karpelevich formula and the local Langlands-Shahidi L-functions

4.1 Satake isomorphism and unramified representations

We recall basic facts on the Satake isomorphism as in [McN12], [Li12], [We14-2] or [We14], and refer to the papers for detailed discussions and proofs.

Let $\overline{G} \in \mathsf{CExt}(\mathbb{G}(F), \mu_n)$ be an object incarnated by some $(D, \eta) \in \mathsf{Bis}_{\mathbb{G}}^{Q}$ over a local field F. We are interested in ϵ-genuine representation of $\overline{G}$, i.e. with $\mu_n \subseteq \overline{G}$ acting by the fixed faithful character $\epsilon : \mu_n \hookrightarrow \mathbf{C}^\times$.

Recall from Definition 2.3.1 that the group $\overline{G}$ is called unramified if and only if

(i) $\gcd(n, p) = 1$ and,

(ii) there exists a splitting $s_K : K_G \hookrightarrow \overline{G}$ of the maximal compact subgroup K_G.

To simplify notation, we may omit s_K whenever necessary, and no confusion will arise. From now, unless otherwise stated, we assume $\overline{G}$ is unramified with respect to a fixed s_K.

Definition 4.1.1. An irreducible genuine representation $\sigma \in \mathrm{Irr}_\epsilon(\overline{G})$ is called unramified if and only if

$$\sigma^{K_G} \neq 0.$$

That is, the space of K_G-fixed vectors is nonzero.

The key to understanding unramified representation is the Satake isomorphism as in the linear algebraic group case. Let $\mathcal{H}_\epsilon(\overline{G}, K_G)$ be the $\mathbf{C}$-algebra consists of locally constant and compactly supported functions $f : \overline{G} \longrightarrow \mathbf{C}$ satisfying

$$f(\xi k_1 \overline{g} k_2) = \epsilon(\xi) f(\overline{g}), \text{ for all } \xi \in \mu_n, k_1, k_2 \in K_G \text{ and } \overline{g} \in \overline{G}.$$

Similarly, we can define the Hecke algebra for $\mathcal{H}_\epsilon(\overline{T}, K_T)$ for the covering torus $\overline{T}$ with respect to $K_T := T \cap K$.

The Satake transform

$$\mathcal{S}: \quad \mathcal{H}_\epsilon(\overline{G}, K_G) \longrightarrow \mathcal{H}_\epsilon(\overline{T}, K_T)$$

is given by

$$\mathcal{S}(f)(\overline{t}) := \delta(\overline{t})^{1/2} \int_U f(\overline{t}u)du \text{ for all } f \in \mathcal{H}_\epsilon(\overline{G}, K_G).$$

Any function $f \in \mathcal{H}_\epsilon(\overline{T}, K_T)$ has support in the centralizer $C_{\overline{T}}(K_T)$ of K_T in $\overline{T}$. This follows from the chain of equalities

$$f(\overline{t}) = f(\overline{t}k) = [\overline{t}, k] \cdot f(k\overline{t}) = [\overline{t}, k] \cdot f(\overline{t}),$$

where $\overline{t} \in \overline{T}$ and $k \in K_T$ are arbitrary, and $[-,-]$ is the commutator on $\overline{T}$.

Clearly, $Z(\overline{T})K_T \subseteq C_{\overline{T}}(K_T)$. We refer to [McN12, Lm. 1], [We14-2, §3.2] and [We14] for the following.

Lemma 4.1.2. *The centralizer $C_{\overline{T}}(K_T)$ of K_T in $\overline{T}$ is equal to $Z(\overline{T})K_T$. Moreover, $C_{\overline{T}}(K_T) = Z(\overline{T})K_T$ is a maximal abelian subgroup of $\overline{T}$ containing $Z(\overline{T})$.*

We may identify the Weyl group W with $(N(T) \cap K_G)/K_T$. Then W acts on $Z(\overline{T})$ and also K_T and therefore on $Z(\overline{T})K_T$, which induces a well-defined action on $\mathcal{H}_\epsilon(\overline{T}, K_T)$. Moreover, the group K_T is invariant under W.

As in the linear algebraic group case, we have:

Theorem 4.1.3. *The Satake transform $\mathcal{S}$ gives an isomorphism of algebras:*

$$\mathcal{S}: \quad \mathcal{H}_\epsilon(\overline{G}, K_G) \longrightarrow \mathcal{H}_\epsilon(\overline{T}, K_T)^W.$$

The proof of above theorem could be found in [McN12, §13.10] where F is assumed to contain $2n$-th root of unity. See also [Li12, §3.2] [We14-2] and [We14] for a discussion and proof in general.

From above one has by restriction of functions the isomorphism

$$\mathcal{V}: \quad \mathcal{H}_\epsilon(\overline{T}, K_T) \xrightarrow{\simeq} \mathcal{H}_\epsilon(Z(\overline{T})K_T, K_T),$$

where the right hand side consists of ϵ-genuine compactly supported functions on $Z(\overline{T})K_T$ invariant under K_T. Clearly, it is isomorphic to $\mathcal{H}_\epsilon(Z(\overline{T}), T^\dagger \cap K_T)$ since $Z(\overline{T})K_T/K_T \simeq Z(\overline{T})/(T^\dagger \cap K_T)$.

For this purpose we define $\overline{Y}_{Q,n} := \Big(Z(\overline{T})/(T^\dagger \cap K_T) \simeq Z(\overline{T})K_T/K_T\Big)$ as in

$$\begin{tikzcd}
 & & T^\dagger \cap K_T \arrow[dl, hook', "s_K"'] \arrow[d, hook] \\
\mu_n \arrow[r, hook] \arrow[d, equal] & Z(\overline{T}) \arrow[r, two heads] \arrow[d] & T^\dagger \arrow[d] \\
\mu_n \arrow[r, hook] & \overline{Y}_{Q,n} \arrow[r, two heads] & Y_{Q,n}.
\end{tikzcd}$$

The abelian group $\overline{Y}_{Q,n}$ inherits a W-action from that of W on $Z(\overline{T})K_T$. Then the algebra $\mathcal{H}_\epsilon(Z(\overline{T}), T^\dagger \cap K_T)$ is isomorphic to

$$\mathbf{C}_\epsilon[\overline{Y}_{Q,n}] := \frac{\mathbf{C}[\overline{Y}_{Q,n}]}{\langle \zeta - \epsilon(\zeta)^{-1} : \zeta \in \mu_n \rangle},$$

which in turn gives a W-equivariant isomorphism

$$\mathcal{H}_\epsilon(\overline{T}, K_T) \xrightarrow{\simeq} \mathbf{C}_\epsilon[\overline{Y}_{Q,n}].$$

Corollary 4.1.4 ([We14]). *There is a natural isomorphism of algebras*

$$\mathcal{H}_\epsilon(\overline{G}, K_G) \xrightarrow{\mathcal{S}} \mathcal{H}_\epsilon(\overline{T}, K_T)^W \xrightarrow{\mathcal{V}} \mathbf{C}_\epsilon[\overline{Y}_{Q,n}]^W.$$

The Satake isomorphism thus gives the natural bijections between the following isomorphism classes:

$$\begin{array}{c} \{\textit{irred. unramified genuine representations of } \overline{G}\} \\ \Big\updownarrow \mathcal{S}^* \\ \{W\textit{-orbits of irred. unramified genuine representations of } \overline{T}\} \\ \Big\updownarrow \mathcal{V}^* \\ \{W\textit{-orbits of unramified genuine characters of } Z(\overline{T})\}. \end{array}$$

Here a character $\overline{\chi}$ of $Z(\overline{T})$ is called unramified if it is trivial on $T^\dagger \cap K_T$, or equivalently, it is the pull-back of a certain character on $\overline{Y}_{Q,n}$. Also the W-action on unramified representations of $\overline{T}$ is via the correspondence between unramified representations of $\overline{T}$ and unramified characters of $Z(\overline{T})$ induced by the isomorphism $\mathcal{V}$.

We also have

Corollary 4.1.5. *For any unramfied representation $\sigma \in Irr_\epsilon(\overline{G})$,*

$$dim_{\mathbf{C}}\ \sigma^{K_G} = 1.$$

Given $\overline{\chi} \in \mathrm{Hom}_\epsilon(Z(\overline{T}), \mathbf{C}^\times)$ unramified, we will give an elaborate discussion on the correspondences $\mathcal{S}^*$ and $\mathcal{V}^*$ and the normalized unramified vector in the resulting unramified representation of $\overline{G}$.

4.1.1 Unramified representations of $\overline{T}$

Let $\overline{T}$ be a degree n cover of T of BD type. Then $\overline{T}$ is an example of a Heisenberg group, i.e. a group whose commutator subgroup lies in the center. We first state the following general Stone-von Neumann theorem which works for general $\overline{T}$, unramified or not.

Proposition 4.1.6 ([We09, Thm. 3.1]). *Let $Z(\overline{T})$ be the center of $\overline{T}$, and let A be any maximal abelain subgroup of $\overline{T}$ which contains $Z(\overline{T})$. Let $\overline{\chi} : Z(\overline{T}) \longrightarrow \mathbf{C}^\times$ be a genuine character of $Z(\overline{T})$, and let $\overline{\chi}' : A \longrightarrow \mathbf{C}^\times$ be any extension to A. Write $Ind_A^{\overline{T}}(\overline{\chi}')$ for the induced representation on $\overline{T}$.*

The construction $\overline{\chi} \mapsto Ind_A^{\overline{T}}(\overline{\chi}')$ gives, up to isomorphism, a bijection between

$$Hom_\epsilon\big(Z(\overline{T}), \mathbf{C}^\times\big) \rightsquigarrow Irr_\epsilon(\overline{T}).$$

In particular, the isomorphism class of $Ind_A^{\overline{T}}(\overline{\chi}')$ does not depend on the choice of A and the extension $\overline{\chi}'$, and therefore we may write $i(\overline{\chi})$ for the induced representation instead.

For $\overline{T}$ unramified, there is a natural choice of A which we take to be $C_{\overline{T}}(K_T) = Z(\overline{T})K_T$ by Lemma 4.1.2. It is a maximal abelian subgroup of $\overline{T}$ and contains $Z(\overline{T})$ by Lemma 4.1.2.

As mentioned, to give an unramified genuine character $\overline{\chi}$ on $Z(\overline{T})$ is equivalent to giving a genuine character on $\overline{Y}_{Q,n}$. Therefore, there is a natural extension $\overline{\chi}'$ to $C_{\overline{T}}(K_T)$ which is trivial on K_T by inflation, since we also have $\overline{Y}_{Q,n} \simeq C_{\overline{T}}(K_T)/K_T$.

Consider $i(\overline{\chi}) := \mathrm{Ind}_A^{\overline{T}}(\overline{\chi}')$. It is clear that $Z(\overline{T})$ acts on $i(\overline{\chi})$ by $\overline{\chi}$. Moreover, $i(\overline{\chi})$ is unramified and $\dim_{\mathbf{C}} i(\overline{\chi})^{K_T} = 1$.

Conversely, let π be an unramified representation of $\overline{T}$. Then, by the Stone von-Neumann theorem π is isomorphic to $i(\overline{\chi})$ for some unramified character $\overline{\chi}$ of $Z(\overline{T})$. Therefore, we see that the correspondence $\mathcal{V}^*$ in Corollary 4.1.4 is basically realized as

$$\overline{\chi} \rightsquigarrow i(\overline{\chi}) := \mathrm{Ind}_A^{\overline{T}}(\overline{\chi}').$$

4.1.2 Unramified principal series representations of $\overline{G}$

Given any $\overline{G}$, let $\overline{B}$ be the Borel $B \subseteq G$ with the decomposition $\overline{B} = \overline{T}N$. Let $i(\overline{\chi})$ be a genuine irreducible unramified representation of $\overline{T}$ obtained above. We now define the principal series representations of $\overline{G}$.

Definition 4.1.7. Consider any $i(\overline{\chi}) \in \mathrm{Irr}_\epsilon(\overline{T})$. We define the normalized induced representation $I_{\overline{B}}^{\overline{G}}(i(\overline{\chi}))$ as follows.

First let

$$L(i(\overline{\chi})) = \big\{ f : \overline{G} \to i(\overline{\chi}) |\ f(\overline{b}\overline{g}) = \delta_{\overline{B}}^{1/2}(\overline{b}) \cdot i(\overline{\chi})(\overline{b}) f(\overline{g}) \big\}.$$

Then $I_{\overline{B}}^{\overline{G}}(i(\overline{\chi}))$ are the smooth vectors of $L(i(\overline{\chi}))$. Here $\delta_{\overline{B}}$ is the modular character of $\overline{B}$ and we view $i(\overline{\chi})$ as a representation of $\overline{B}$ by the inflation $\overline{B} \twoheadrightarrow \overline{T}$.

For simplicity we may write $I(\overline{\chi})$ for $I_{\overline{B}}^{\overline{G}}(i(\overline{\chi}))$ and call it a principal series representation of $\overline{G}$. Since the modular character $\delta_{\overline{B}}$ factors through that of B, i.e. $\delta_{\overline{B}}(\overline{b}) = \delta_B(b)$ where $b \in B$ is the image of $\overline{b}$, we may use interchangeably both notations $\delta_{\overline{B}}$ and δ_B.

Lemma 4.1.8. *Suppose $\overline{\chi}$ is unramified. Then the principal series $I(\overline{\chi})$ has a one-dimensional space of K_G-fixed vectors.*

Proof. The proof can be taken verbatim from [McN12, Lm. 2]. As it is important to have an explicit description of the unramified vectors of $I(\overline{\chi})$, we give a sketch here.

The key is that we have the isomorphism of vector spaces

$$I(\overline{\chi})^{K_G} \longrightarrow i(\overline{\chi})^{K_T}, \text{ given by } f \longmapsto f(1_{\overline{G}}).$$

Write $\mathbb{f} = f(1_{\overline{G}}) \in i(\overline{\chi})^{K_T}$. Then $\mathbb{f}(\overline{a}\overline{t}) = \overline{\chi}'(\overline{a})\mathbb{f}(\overline{t})$ for all $\overline{t} \in \overline{T}$ and $\overline{a} \in A = C_{\overline{T}}(K_T) = Z(\overline{T})K_T$, where $\overline{\chi}'$ is the extension of $\overline{\chi}$ to A which is trivial on K_T.

Moreover, for all $\overline{t}$ and $k \in K_T$,

$$\mathbb{f}(\overline{t}) = \big(i(\overline{\chi})(k)\mathbb{f}\big)(\overline{t}) = \mathbb{f}(\overline{t}k) = [\overline{t}, k]\overline{\chi}'(k)\mathbb{f}(\overline{t}) = [\overline{t}, k]\overline{\chi}'(k)\mathbb{f}(\overline{t}) = [\overline{t}, k]\mathbb{f}(\overline{t}).$$

Thus the support of $\mathbb{f}$ is A, and since $\mathbb{f}|_A$ transform under $\overline{\chi}'$, $\mathbb{f}$ is uniquely determined by $\mathbb{f}(1_{\overline{T}})$. That is, $\dim_{\mathbf{C}} I(\overline{\chi})^{K_G} = \dim_{\mathbf{C}} i(\overline{\chi})^{K_T} = 1$. $\square$

Thus, we can argue as in the linear case that the correspondence $\mathcal{S}^*$ in Corollary 4.1.4 is basically given by

$$i(\overline{\chi}) \leadsto \text{ the unramified component of } I(\overline{\chi}).$$

4.2 Intertwining operators

4.2.1 Notations and basic set-up

As before, we use the boldface $\mathbf{w}$ to denote an element of W. We have defined in section 2.1.4 the following elements of $\mathbb{SL}_2$:

$$\mathbb{e}_+(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad \mathbb{e}_-(a) = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix},$$

$$\mathbb{w}_o(a) = \mathbb{e}_+(a)\mathbb{e}_-(a^{-1})\mathbb{e}_+(a) = \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}, \quad \mathbb{h}_o(a) = \mathbb{w}_o(a)\mathbb{w}_o(-1) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

From section 2.1.4 there is the associated morphism $\varphi_\alpha : \mathbb{SL}_2 \longrightarrow \mathbb{G}^{sc}$ for any coroot $\alpha^\vee \in \Psi^\vee$. We also have the elements in $\mathbb{G}^{sc}$: $\mathbb{e}_\alpha(a) = \varphi_\alpha(\mathbb{e}_+(a))$, $\mathbb{e}_{-\alpha}(t) = \varphi_\alpha(\mathbb{e}_-(a))$. Similarly, $\mathbb{w}_\alpha(a) := \varphi_\alpha(\mathbb{w}_o(a))$, $\mathbb{h}_\alpha(a) := \varphi_\alpha(\mathbb{h}_o(a))$. Clearly $\mathbb{w}_\alpha(a) \in N(\mathbb{T})(F)$.

More importantly, for $\overline{\mathbb{G}}$ and its pull-back to $\overline{\mathbb{G}}^{sc}$ via $\mathbb{G}^{sc} \longrightarrow \mathbb{G}$, there are the natural Brylinski-Deligne liftings of above elements in $\overline{\mathbb{G}}^{sc}$ (cf. section 2.1.4), which we have denoted by

$$\overline{\mathbb{e}}_\alpha, \quad \overline{\mathbb{w}}_\alpha, \quad \overline{\mathbb{h}}_\alpha^{[b]}.$$

We are interested in the image of the F-rational points of these elements in $\overline{G} = \overline{\mathbb{G}}(F)$. Recall we have used $\Phi_{D,\eta}$ to denote the natural map as in the following diagram:

$$\begin{tikzcd} \mu_n \arrow[r, hook] \arrow[d, equal] & \overline{G}^{sc} \arrow[r, two heads] \arrow[d, "\Phi_{D,\eta}"] & G^{sc} \arrow[d, "\Phi"] \\ \mu_n \arrow[r, hook] & \overline{G} \arrow[r, two heads] & G, \end{tikzcd}$$

which is just the pull-back of $\overline{G}$ to G^{sc}.

From now on, to simplify and fix our notations we will write for any root $\alpha \in \Psi$

$$e_\alpha(a) := \Phi(\mathbb{e}_\alpha(a)), \quad w_\alpha(a) := \Phi(\mathbb{w}_\alpha(a)), \quad h_\alpha(a) := \Phi(\mathbb{h}_\alpha(a)) \in G,$$

which all lie in G.

For any $a, b \in F^\times$, we have the elements $\overline{\mathbb{e}}_\alpha(a)$, $\overline{\mathbb{w}}_\alpha(a)$ and $\overline{\mathbb{h}}_\alpha^{[b]}(a) \in \overline{G}^{sc}$. We thus introduce the following notation

$$\overline{e}_\alpha(a) := \Phi_{D,\eta}(\overline{\mathbb{e}}_\alpha(a)), \quad \overline{w}_\alpha(a) := \Phi_{D,\eta}(\overline{\mathbb{w}}_\alpha(a)), \quad \overline{h}_\alpha^{[b]}(a) := \Phi_{D,\eta}(\overline{\mathbb{h}}_\alpha^{[b]}(a)) \in \overline{G}. \tag{4.1}$$

They all lie in $\overline{G}$ for any root $\alpha \in \Psi$. As in Proposition 3.2.3, we will write $\overline{\mathbb{h}}_\alpha(a)$ and $\overline{h}_\alpha(a)$ for $\overline{\mathbb{h}}_\alpha^{[b]}(a) \in \overline{G}^{sc}$ and $\overline{h}_\alpha^{[b]}(a) \in \overline{G}$ respectively, whenever the latters are independent of the choice $b \in F^\times$. For example, we have used the notation $\overline{\mathbb{h}}_\alpha(a^{n_\alpha})$ in Proposition 3.2.3.

Furthermore, for convenience the following notations may also be used

$$e_\alpha := e_\alpha(1), \quad w_\alpha := w_\alpha(1) \ \in G.$$

Similarly,

$$\overline{e}_\alpha := \overline{e}_\alpha(1), \quad \overline{w}_\alpha := \overline{w}_\alpha(1), \quad \overline{h}_\alpha^{[b]} := \overline{h}_\alpha^{[b]}(1) \ \in \overline{G}.$$

In particular, $\overline{h}_\alpha^{[1]} = 1_{\overline{G}}$.

We will follow [Sav04] and consider the group $W^{K_G} \subseteq K_G$ generated by $w_\alpha(-1)$ for $\alpha \in \Psi$, which lies in the exact sequence

$$T^{sgn} \hookrightarrow W^{K_G} \twoheadrightarrow W, \qquad w_\alpha(-1) \mapsto \mathbf{w}_\alpha,$$

where $T^{sgn} \subseteq T$ is the finite group generated by $h_\alpha(-1)$ for $\alpha \in \Psi$. For application of global purpose, there is no loss of generalities to choose representatives for the Weyl group W which lie in K_G.

There is a preferred section of W^{K_G} over W. Let $\mathbf{w} \in W$, write $\mathbf{w} = \mathbf{w}_{\alpha_k} ... \mathbf{w}_{\alpha_2} ... \mathbf{w}_{\alpha_1} \in W$ in the form of a minimum decomposition. We would choose the following as a representative of $\mathbf{w}$:

$$s_W(\mathbf{w}) := w_{\alpha_k} \cdot w_{\alpha_{k-1}} ... \cdot w_{\alpha_2} \cdot w_{\alpha_1} \in W^{K_G}.$$

One property of s_W is that the representative $s_W(\mathbf{w})$ is independent of the minimum decomposition of $\mathbf{w}$. Moreover, it is multiplicative (cf. [Hum75, §29.4]). That is, if $l(\mathbf{w}\mathbf{w}') = l(\mathbf{w}) + l(\mathbf{w}')$, then

$$s_W(\mathbf{w}\mathbf{w}') = s_W(\mathbf{w}) \cdot s_W(\mathbf{w}').$$

For simplicity, we may also write w for $s_W(\mathbf{w})$, i.e. by definition

$$w := s_W(\mathbf{w}).$$

The finite subgroup T^{sgn} does not lie in $T^\dagger$, and this follows from considering the commutator $[h_\alpha(-1), y \otimes \varpi] = (-1, \varpi)_n^{B_Q(\alpha^\vee, y)}$ for any $y \in Y$. The obstruction is given by $(-1, \varpi)_n$. Thus the conjugation action of W^{K_G} on $\overline{T}$ does not descend to W. However, we have seen that the action of W on the isomorphism class of representations of $\overline{T}$ is well-defined.

We may consider W^{K_G} as a subgroup of $\overline{G}$ via the splitting s_K of K_G; then its finite subgroup T^{sgn} viewed as a subgroup of $\overline{G}$ does not lie in the center $Z(\overline{T})$ from above consideration. Since the splitting s_K agrees with the unipotent splitting i_u as in Corollary 2.3.3, we have

$$\begin{aligned} s_K(w_{\alpha_i}) =& s_K(e_{\alpha_i} \cdot e_{-\alpha_i} \cdot e_{\alpha_i}) \\ =& s_K(e_{\alpha_i}) \cdot s_K(e_{-\alpha_i}) \cdot s_K(e_{\alpha_i}) \\ =& i_u(e_{\alpha_i}) \cdot i_u(e_{-\alpha_i}) \cdot i_u(e_{\alpha_i}) \\ =& \overline{e}_{\alpha_i} \cdot \overline{e}_{-\alpha_i} \cdot \overline{e}_{\alpha_i} \\ =& \overline{w}_{\alpha_i} \end{aligned}$$

It follows in general

$$\begin{aligned} s_K(w) &= s_K(w_{\alpha_k}) \cdot s_K(w_{\alpha_{k-1}}) \cdot \ldots \cdot s_K(w_{\alpha_2}) \cdot s_K(w_{\alpha_1}) \\ &= \overline{w}_{\alpha_k} \cdot \overline{w}_{\alpha_{k-1}} \cdot \ldots \cdot \overline{w}_{\alpha_2} \cdot \overline{w}_{\alpha_1}. \end{aligned}$$

The expression is independent of the minimum decomposition of $\mathbf{w}$. Therefore, we may also write $\overline{w}$ for $s_K(w) = s_K \circ s_W(\mathbf{w})$ without any ambiguity.

To summarize for the notations, for every $\mathbf{w} \in W$, we have a well-defined representative $w \in W^{K_G} \subseteq K_G$ and $\overline{w} \in s_K(W^{K_G}) \subseteq \overline{W^{K_G}}$ (the preimage of W^{K_G} in $\overline{G}$):

$$\begin{tikzcd} & \mu_n \arrow[d, hook] & \\ & \overline{W^{K_G}} \arrow[d] & \\ T^{sgn} \arrow[r, hook] & W^{K_G} \arrow[u, bend left, "s_K"] \arrow[r, two heads] & W. \arrow[l, bend right, "s_W"'] \end{tikzcd}$$

The canonical isomorphism ${}^w i(\overline{\chi}) \simeq i({}^{\mathbf{w}}\overline{\chi})$

Let $i(\overline{\chi})$ be an irreducible representation of $\overline{T}$. Then we define an action of $w \in W^{K_G}$ (or equivalently that of $\overline{w}$) on $i(\overline{\chi})$ by

$${}^w i(\overline{\chi})(\overline{t}) = i(\overline{\chi})(w^{-1}\overline{t}w).$$

We have defined ${}^{\mathbf{w}}\overline{\chi}$ (see section 3.2.3) by also considering the conjugation action of w on the genuine character $\overline{\chi}$ of $Z(\overline{T})$. Since it actually only depends on $\mathbf{w} \in W$, we have written ${}^{\mathbf{w}}\overline{\chi}$ previously with no ambiguity.

Now we can compare ${}^{w}i(\overline{\chi})$ with $i({}^{\mathbf{w}}\overline{\chi})$.

Lemma 4.2.1. *The two representations are isomorphic:*

$$ {}^{w}i(\overline{\chi}) \simeq i({}^{\mathbf{w}}\overline{\chi}). $$

If $i(\overline{\chi})$ is unramified with unramified character $\overline{\chi}$, then both ${}^{w}i(\overline{\chi})$and $i({}^{\mathbf{w}}\overline{\chi})$ are unramified. Since any isomorphism between the two spaces preserves unramified vectors, there is a canonical isomorphism which identifies the normalized unramified vectors of the two sides.

Proof. Note that both spaces are irreducible with $Z(\overline{T})$ acting by the same character ${}^{\mathbf{w}}\overline{\chi}$. Therefore, by the Stone von-Neumann theorem in Proposition 4.1.6, they are isomorphic.

Now assume $i(\overline{\chi})$ is unramified with $\overline{\chi}$ a unramified character of $Z(\overline{T})$. Then ${}^{\mathbf{w}}\overline{\chi}$ is unramified as well since $\mathbf{w}$ preserves $T^{\dagger} \cap K_T$. Therefore $i({}^{\mathbf{w}}\overline{\chi})$ as well as ${}^{w}i(\overline{\chi})$ are both unramified since $Z(\overline{T})$ in both representations acts by the unramified character ${}^{\mathbf{w}}\overline{\chi}$.

It is easy to see that any isomorphism between ${}^{w}i(\overline{\chi})$and $i({}^{\mathbf{w}}\overline{\chi})$ preserves unramified vectors, and thus the normalization in the lemma could be achieved. $\square$

We could give some analysis on the unramified vectors on both sides of above lemma.

Identify $i(\overline{\chi})$ with $\mathrm{Ind}_A^{\overline{T}}(\overline{\chi}')$, where $A = C_{\overline{T}}(K_T)$ and $\overline{\chi}'$ the natural unramified extension to A introduced before. Let $\mathfrak{f}_{i(\overline{\chi})} : \overline{T} \longrightarrow \mathbf{C}$ be the normalized unramified vector of $i(\overline{\chi})$. Then the support of $\mathfrak{f}_{i(\overline{\chi})}$ is A and it takes value 1 at $1_{\overline{T}}$. We claim $\mathfrak{f}_{i(\overline{\chi})} \in {}^{w}i(\overline{\chi})$ is unramified under the ${}^{w}i(\overline{\chi})$ action as well.

Let $k \in K_T$. Then $w^{-1}kw \in K_T$ since $w \in K_G$, and this gives

$$ {}^{w}i(\overline{\chi})(k)\mathfrak{f}_{i(\overline{\chi})}(\overline{t}) = \mathfrak{f}_{i(\overline{\chi})}(\overline{t} \cdot w^{-1}kw) = \mathfrak{f}_{i(\overline{\chi})}(\overline{t}). $$

This shows that indeed $\mathfrak{f}_{i(\overline{\chi})}$ is ${}^{w}i(\overline{\chi})$-unramified.

Moreover, we see that $\mathfrak{f}_{i(\overline{\chi})}$ transforms under ${}^{\mathbf{w}}(\overline{\chi}')$ when restricted to A. This can be seen as follows. Let $a \in A$, then $w^{-1}aw \in A$ and thus

$$ {}^{w}i(\overline{\chi})(a)\mathfrak{f}_{i(\overline{\chi})}(\overline{t}) = \mathfrak{f}_{i(\overline{\chi})}(\overline{t} \cdot w^{-1}aw) = \begin{cases} 0 = {}^{\mathbf{w}}(\overline{\chi}')(a) \cdot \mathfrak{f}_{i(\overline{\chi})}(\overline{t}) & \text{if } \overline{t} \notin A, \\ \mathfrak{f}_{i(\overline{\chi})}(w^{-1}aw \cdot \overline{t}) = {}^{\mathbf{w}}(\overline{\chi}')(a) \cdot \mathfrak{f}_{i(\overline{\chi})}(\overline{t}) & \text{if } \overline{t} \in A, \end{cases} $$

where the first case holds since the support of f is in A.

On the other hand, consider $i({}^{\mathbf{w}}\overline{\chi})$ and identify it with $\mathrm{Ind}_A^{\overline{T}}\big(({}^{\mathbf{w}}\overline{\chi})'\big)$, where $({}^{\mathbf{w}}\overline{\chi})'$ is the natural unramified extension of ${}^{\mathbf{w}}\overline{\chi}$ to A. The normilized unramified vector of $i({}^{\mathbf{w}}\overline{\chi})$ is a function $\mathfrak{f}_{i({}^{\mathbf{w}}\overline{\chi})} : \overline{T} \longrightarrow \mathbf{C}^{\times}$ with support in A that transforms under $({}^{\mathbf{w}}\overline{\chi})'$ when restricted to A. Also we require $\mathfrak{f}_{i({}^{\mathbf{w}}\overline{\chi})}(1_{\overline{T}}) = 1$.

Note however, since $\overline{\chi}$ is unramified, we have ${}^{\mathbf{w}}(\overline{\chi}') = ({}^{\mathbf{w}}\overline{\chi})'$. Therefore, the rigidified isomorphism in above lemma is requiring the normalized vector $\mathfrak{f}_{i(\overline{\chi})}$ sent to the normalized vector $\mathfrak{f}_{i({}^{\mathbf{w}}\overline{\chi})}$.

In fact, it is not hard to check that the canonical isomorphism is given by

$$r_w : {}^{w}i(\overline{\chi}) \xrightarrow{\simeq} i({}^{\mathbf{w}}\overline{\chi}), \quad \mathfrak{f} \longmapsto r_w(\mathfrak{f})(\overline{t}) := \mathfrak{f}(w^{-1}\overline{t}w). \tag{4.2}$$

Meanwhile, if we take another representative $w' \in W^{K_G}$ of $\mathbf{w}$, then we have the following commutative diagram

$$\begin{tikzcd} & {}^{w}i(\overline{\chi}) \arrow[dl, "r_{w,w'}"'] \arrow[d, "r_w"] \\ {}^{w'}i(\overline{\chi}) \arrow[r, "r_{w'}"] & i({}^{\mathbf{w}}\overline{\chi}), \end{tikzcd} \tag{4.3}$$

where $r_{w,w'}$ is given by $r_{w,w'}(\mathfrak{f})(\overline{t}) = \mathfrak{f}(w^{-1}w' \cdot \overline{t} \cdot w'^{-1}w)$. Note that the unramified vectors $\mathfrak{f}_{{}^{w}i(\overline{\chi})}$ and $\mathfrak{f}_{{}^{w'}i(\overline{\chi})}$ are the same functions and equal to the unramified vector $\mathfrak{f}_{i(\overline{\chi})} \in i(\overline{\chi})$. Moreover, the map $r_{w,w'}$ restricts to the identity map from $\mathbf{C} \cdot \mathfrak{f}_{{}^{w}i(\overline{\chi})}$ to $\mathbf{C} \cdot \mathfrak{f}_{{}^{w'}i(\overline{\chi})}$ (with both identified with $\mathbf{C} \cdot \mathfrak{f}_{i(\overline{\chi})}$).

As a consequence of the above discussion, it follows that for all $h_\alpha(-1) \in T^{sgn}$,

$${}^{h_\alpha(-1)}i(\overline{\chi}) \simeq i(\overline{\chi}).$$

From now, we will fix the isomorphism ${}^{w}i(\overline{\chi}) \simeq i({}^{\mathbf{w}}\overline{\chi})$ by requiring that the normalized unramified functions on two sides correspond to each other as in the proof of the above lemma.

4.2.2 Intertwining operators and cocycle relations

For linear case, we have the cocycle relations for intertwining operators for induced representations. As before let $w = w_{\alpha_k} w_{\alpha_{k-1}} ... w_{\alpha_2} w_{\alpha_1}$ be the element of W^{K_G} representing $\mathbf{w} = \mathbf{w}_{\alpha_k} \mathbf{w}_{\alpha_{k-1}} ... \mathbf{w}_{\alpha_2} \mathbf{w}_{\alpha_1}$ in a minimum expansion. We have also defined the element $\overline{w} = s_K(w) \in \overline{G}$ which is independent of the factorization of $\mathbf{w}$ as well.

We are interested in the map:

$$T\big(w, i(\overline{\chi})\big) : \quad I(i(\overline{\chi})) \to I({}^{w}i(\overline{\chi})), \quad f \mapsto \int_{N^w} f(\overline{w}^{-1}\overline{u}g)du. \tag{4.4}$$

and its factorization properties. Here $N^w = N \cap wN^- w^{-1}$, N the unipotent radical of the Borel $\overline{B} = \overline{T}N$ and N^- the unipotent radical opposite to N.

First of all,

Lemma 4.2.2. *The map* $T\big(w, i(\overline{\chi})\big) : I(i(\overline{\chi})) \longrightarrow I({}^{w}i(\overline{\chi}))$ *is well-defined. That is,* $T\big(w, i(\overline{\chi})\big)(f) \in I({}^{w}i(\overline{\chi}))$. *It intertwines the two unramified representations and sends unramified vector to unramified.*

Proof. The proof is routine and follows the same argument as in the linear case, see [Sha10, §4.1].

For example, to check $T\big(w, i(\overline{\chi})\big)(f) \in I({}^w i(\overline{\chi}))$, let $u_o\overline{t} \in \overline{B} = N\overline{T}$, we compute

$$\begin{aligned}
T\big(w, i(\overline{\chi})\big)(f)(u_o\overline{t}\cdot\overline{g}) &= \int_{N^w} f(\overline{w}^{-1}\overline{u}\cdot u_o\overline{t}\cdot\overline{g})du \\
&= \int_{N^w} f(\overline{w}^{-1}\overline{t}\cdot\overline{t}^{-1}u\overline{t}\cdot\overline{g})du \\
&= \prod_{\substack{\alpha\in\Psi^+ \\ \mathbf{w}^{-1}\alpha<0}} |\alpha(t)|_F \cdot \int_{N^w} f(\overline{w}^{-1}\overline{t}\cdot u\cdot\overline{g})du \\
&= \prod_{\substack{\alpha\in\Psi^+ \\ \mathbf{w}^{-1}\alpha<0}} |\alpha(t)|_F \cdot \int_{N^w} f(\overline{w}^{-1}\overline{t}\overline{w}\cdot\overline{w}^{-1}u\cdot\overline{g})du \\
&= \prod_{\substack{\alpha\in\Psi^+ \\ \mathbf{w}^{-1}\alpha<0}} |\alpha(t)|_F \cdot \delta_{\overline{B}}(w^{-1}\overline{t}w)^{1/2}\cdot {}^w i(\overline{\chi})(\overline{t}) \int_{N^w} f(\overline{w}^{-1}u\cdot\overline{g})du. \\
&= \prod_{\substack{\alpha\in\Psi^+ \\ \mathbf{w}^{-1}\alpha<0}} |\alpha(t)|_F \cdot \delta_B(w^{-1}tw)^{1/2}\cdot {}^w i(\overline{\chi})(\overline{t}) T\big(w, i(\overline{\chi})\big)(f)(\overline{g}).
\end{aligned}$$

Here any root $\alpha \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_{\mathrm{mul}})$ is viewed as a character of $\mathrm{Hom}(T, F^\times)$. Also we have the following equality (cf. [Sha10, pg. 83] for example),

$$\delta_{\overline{B}}(\overline{t})^{1/2} = \prod_{\substack{\alpha\in\Psi^+ \\ \mathbf{w}^{-1}\alpha<0}} |\alpha(t)|_F \cdot \delta_B(w^{-1}tw)^{1/2}.$$

That is, the intertwining operator $T\big(w, i(\overline{\chi})\big)$ is well-defined. It is easy to see that it sends an unramified vector to an unramified one. □

Let $f_{i(\overline{\chi})}$ and $f_{{}^w i(\overline{\chi})}$ be the normalized unramified vectors in $I(i(\overline{\chi}))$ and $I({}^w i(\overline{\chi}))$ respectively. Write $c(w, \overline{\chi}) \in \mathbf{C}$ for the coefficient such that

$$T\big(w, i(\overline{\chi})\big) f_{i(\overline{\chi})} = c(w, \overline{\chi}) f_{{}^w i(\overline{\chi})}.$$

The coefficient $c(w, \overline{\chi})$, which depends a priori on the preferred representative w of $\mathbf{w}$, does not in the following sense. Take any other representative $w' \in K_G$ of $\mathbf{w}$, we have the intertwining operator $T\big(w', i(\overline{\chi})\big) : I(i(\overline{\chi})) \longrightarrow I({}^{w'} i(\overline{\chi}))$ and

$$T\big(w', i(\overline{\chi})\big) f_{i(\overline{\chi})} = c(w', \overline{\chi}) f_{{}^{w'} i(\overline{\chi})}.$$

We have the following commutative diagram

$$\begin{tikzcd}
I(i(\overline{\chi})) \arrow[r, "T(w, i(\overline{\chi}))"] \arrow[d, "T(w', i(\overline{\chi}))"'] & I({}^w i(\overline{\chi})) \arrow[d, "r_w^*"] \arrow[dl, "r_{w,w'}^*"] \\
I({}^{w'} i(\overline{\chi})) \arrow[r, "r_{w'}^*"'] & I({}^{\mathbf{w}}\overline{\chi}),
\end{tikzcd} \tag{4.5}$$

where the maps from the right lower triangle are induced from those in (4.3). It is easy to see $r^*_{w,w'}(f_{^w i(\overline{\chi})}) = f_{^{w'} i(\overline{\chi})}$, and it follows

$$c(w, \overline{\chi}) = c(w', \overline{\chi}).$$

Therefore, we may write $c(\mathbf{w}, \overline{\chi})$ instead of $c(w, \overline{\chi})$. To take another view of $c(\mathbf{w}, \overline{\chi})$, we may consider the compositions from above diagram:

$$T(\mathbf{w}, \overline{\chi}) = r^*_w \circ T(w, i(\overline{\chi})) : \quad I(i(\overline{\chi})) \longrightarrow I(^w i(\overline{\chi})) \longrightarrow I(i(^{\mathbf{w}}\overline{\chi})). \tag{4.6}$$

It is justified from above diagram that this map is independent of the choice of representative in W^{K_G} for fixed $\mathbf{w} \in W$, hence the notation. In brief, we may write

$$T(\mathbf{w}, \overline{\chi}) : \quad I(\overline{\chi}) \longrightarrow I(^{\mathbf{w}}\overline{\chi}).$$

Thus $T(\mathbf{w}, \overline{\chi})$ is the intertwining map between unramified spaces uniquely determined by

$$T(\mathbf{w}, \overline{\chi}) f_{i(\overline{\chi})} = c(\mathbf{w}, \overline{\chi}) f_{i(^{\mathbf{w}}\overline{\chi})}.$$

Remark 4.2.3. Following a suggestion of Wee Teck Gan, we could consider representatives of W in a larger group. From the Bruhat-Tits theory, we have obtained a model $\mathbf{G}$ of $\mathbb{G}$ over $\mathcal{O}_F$, from which $K_G := \mathbf{G}(\mathcal{O}_F)$. This gives models $\mathbf{T}$ and $N(\mathbf{T})$ for $\mathbb{T}$ and $N(\mathbb{T})$ respectively, and we have an exact sequence

$$\mathbf{T}(\mathcal{O}_F) \hookrightarrow N(\mathbf{T})(\mathcal{O}_F) \twoheadrightarrow W\ .$$

The discussion in this (and the previous) subsection applies with representatives of W chosen from $N(\mathbf{T})(\mathcal{O}_F)$. For example, given any $n \in N(\mathbf{T})(\mathcal{O}_F)$ mapping to $\mathbf{w} \in W$, as in (4.2) the morphism

$$r_n : {}^n i(\overline{\chi}) \xrightarrow{\simeq} i(^{\mathbf{w}}\overline{\chi}), \quad f \longmapsto r_n(f)(\overline{t}) := f(n^{-1}\overline{t}n)$$

gives the desired canonical isomorphism. We can define $T(n, \overline{\chi})$ as in (4.4), and also the composition $r^*_n \circ T(n, \overline{\chi}) : I(\overline{\chi}) \longrightarrow I(^{\mathbf{w}}\overline{\chi})$ as in (4.6). The latter does not depend on the choice of n and thus can be denoted by $T(\mathbf{w}, \overline{\chi})$ as well.

This is certainly a more natural and less restrictive approach, as the more restrictive group W^{K_G} is essentially $N(\mathbf{T})(\mathbf{Z})$. The consideration of $N(\mathbf{T})(\mathcal{O}_F)$ would allow for a treatment for general quasi-split $\mathbb{G}$, which may not have a canonical integral model over $\mathbf{Z}$.

Keep notations as above, and write $\Psi_{\mathbf{w}} := \{\alpha \in \Psi^+ :\ \mathbf{w}(\alpha) \in \Psi^-\}$.

Proposition 4.2.4. *Let $\mathbf{w} = \mathbf{w}_{\alpha_k} ... \mathbf{w}_{\alpha_2}\mathbf{w}_{\alpha_1} \in W$ be an expansion of minimum length into simple reflections, and let $w = w_{\alpha_k} ... w_{\alpha_2} w_{\alpha_1}$ be defined as above. Then the intertwining operator factorizes as*

$$T\big(w, i(\overline{\chi})\big) = T\big(w_{\alpha_k}, {}^{w_{\alpha_{k-1}} w_{\alpha_{k-2}} ... w_{\alpha_1}} i(\overline{\chi}))\big) \circ ... \circ T\big(w_{\alpha_2}, {}^{w_{\alpha_1}} i(\overline{\chi})\big) \circ T\big(w_{\alpha_1}, i(\overline{\chi})\big).$$

Proof. The proof is as in the linear case, see [CKM04, pg. 139-140]. We will apply induction. For convenience, we write w_o for $w_{\alpha_{k-1}} w_{\alpha_{k-2}} ... w_{\alpha_1}$ and also $\overline{w}_o$ for $s_K(w_o)$, i.e. the element $\overline{w}_{\alpha_{k-1}} \overline{w}_{\alpha_{k-2}} ... \overline{w}_{\alpha_1}$. Since $w = w_{\alpha_k} w_o$, by induction it suffices to show

$$T\big(w, i(\overline{\chi})\big) = T\big(w_{\alpha_k}, {}^{w_o} i(\overline{\chi})\big) \circ T\big(w_o, i(\overline{\chi})\big).$$

Note $U^w = \prod_{\alpha \in \Psi_{\mathbf{w}}} e_\alpha(u_\alpha), u_\alpha \in F$. In fact,

$$\begin{aligned} T\big(w, i(\overline{\chi})\big) f(\overline{g}) &= \int_{U^w} f(\overline{w}^{-1} \overline{u}\, \overline{g}) du \\ &= \int_{U^w} f(\overline{w}^{-1} \overline{u}\, \overline{w} \cdot \overline{w}^{-1} \overline{g}) du \\ &= \int_{U^{-,w}} f(\overline{u}\, \overline{w}^{-1} \overline{g}) du, \end{aligned}$$

where $U^{-,w} = \prod_{\alpha \in \Psi_{\mathbf{w}}} e_{-\alpha}(u_\alpha), u_\alpha \in F$. Similar equality holds for $T(w_o, i(\overline{\chi}))$ with replacing $U^{-,w}$ by U^{-,w_o}.

The set $\Psi_{\mathbf{w}}$ is equal to the union of two disjoint sets:

$$\Psi_{\mathbf{w}} = \big\{ \mathbf{w}_o^{-1}(\alpha_k) \big\} \cup \Psi_{\mathbf{w}_o}.$$

Thus $U^{-,w} = U^{-,w_o} \cdot U_{-\mathbf{w}_o^{-1}(\alpha_k)}$, which gives

$$\begin{aligned} T\big(w, i(\overline{\chi})\big) f(\overline{g}) &= \int_{U_{-\mathbf{w}_o^{-1}(\alpha_k)}} \int_{U^{-,w_o}} f(\overline{u}_o \overline{e}_{-\mathbf{w}_o^{-1}(\alpha_k)}(u) \cdot \overline{w}_o^{-1} \overline{w}_{\alpha_k}^{-1} \overline{g}) du_o du \\ &= \int_{U_{-\mathbf{w}_o^{-1}(\alpha_k)}} \int_{U^{-,w_o}} f(\overline{u}_o \overline{w}_o^{-1} \cdot \overline{w}_o \overline{e}_{-\mathbf{w}_o^{-1}(\alpha_k)}(u) \overline{w}_o^{-1} \cdot \overline{w}_{\alpha_k}^{-1} \overline{g}) du_o du \\ &= \int_{U_{-\alpha_k}} \int_{U^{-,w_o}} f(\overline{u}_o \overline{w}_o^{-1} \cdot \overline{e}_{-\alpha_k}(u) \cdot \overline{w}_{\alpha_k}^{-1} \overline{g}) du_o du \\ &= \int_{U_{-\alpha_k}} T\big(w_o, i(\overline{\chi})\big) f(\overline{e}_{-\alpha_k}(u) \cdot \overline{w}_{\alpha_k}^{-1} \overline{g}) du \\ &= T\big(w_{\alpha_k}, {}^{w_o} i(\overline{\chi})\big) \circ T\big(w_o, i(\overline{\chi})\big) f(\overline{g}). \end{aligned}$$

□

Immediately it follows:

Corollary 4.2.5. *With notations above, one has*

$$c(\mathbf{w}, \overline{\chi}) = \prod_{m=1}^{k} c(\mathbf{w}_{\alpha_m}, {}^{\mathbf{w}_{\alpha_{m-1}} ... \mathbf{w}_{\alpha_2} \mathbf{w}_{\alpha_1}} \overline{\chi}), \tag{4.7}$$

which will be referred to as the cocycle relation. Equivalently, we have

$$T(\mathbf{w}, \overline{\chi}) = T(\mathbf{w}_{\alpha_k}, {}^{\mathbf{w}_{\alpha_{k-1}} \mathbf{w}_{\alpha_{k-2}} ... \mathbf{w}_{\alpha_1}} \overline{\chi}) \circ ... \circ T(\mathbf{w}_{\alpha_2}, {}^{\mathbf{w}_{\alpha_1}} \overline{\chi}) \circ T(\mathbf{w}_{\alpha_1}, \overline{\chi}).$$

4.3 The crude Gindikin-Karpelevich formula

Let $\overline{G} \in \mathsf{CExt}(G, \mu_n)$ be an unramified central cover of BD type over a local field F. Choose a uniformizer ϖ of F. The Gindikin-Karpelevich formula is obtained in [McN11, Thm. 6.4] by using a crystal basis decomposition of the domain of integration. Recently, as a consequence of the Casselman-Shalika formula computed, the GK formula is obtained as in [McN14, Thm. 12.1]. However, we will compute directly below using a straightforward method as in the linear case and remove restrictions such as $\mu_{2n} \subseteq F^\times$. Moreover, the GK formula here is expressed in terms of naturally defined terms and could be considered as a refinement of [McN11] and above. This allows us to interpret it as local Langlands-Shahidi L-function later.

Before we proceed, we state the following simple but useful results.

Lemma 4.3.1. *Let du be an additive measure of F such that the measure of $\mathcal{O}_F$ is equal 1, or equivalently the measure of $\mathcal{O}_F^\times$ with respect to du is $1 - 1/q$. If $k \in \mathbf{Z}$, then*

$$\int_{\mathcal{O}_F^\times} (u, \varpi)_n^k \, du = \begin{cases} 1 - 1/q & \text{if } n|k, \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

Lemma 4.3.2. *For any root $\alpha \in \Psi$ and any $u \in F^\times$, the following relations hold:*

$$\overline{w}_\alpha^{-1}\overline{e}_\alpha(u) = \overline{h}_\alpha^{[1]}(u^{-1})\overline{e}_\alpha(-u)\overline{e}_{-\alpha}(-u^{-1}) \in \overline{G}.$$

Proof. It suffices to show the following

$$\overline{\mathbb{w}}_\alpha^{-1}\overline{\mathbb{e}}_\alpha(u) = \overline{\mathbb{h}}_\alpha^{[1]}(u^{-1})\overline{\mathbb{e}}_\alpha(-u)\overline{\mathbb{e}}_{-\alpha}(-u^{-1}) \in \overline{G}^{sc},$$

from which we could apply the morphism $\Phi_{D,\eta} : \overline{G}^{sc} \longrightarrow \overline{G}$ to get the desired result by the definition of the elements in above equality, see section 4.2.1. Note $\overline{\mathbb{w}}_\alpha = \overline{\mathbb{w}}_\alpha(1)$ by definition, and therefore $\overline{\mathbb{w}}_\alpha^{-1} = \overline{\mathbb{w}}_\alpha(-1)$.

We start with

$$\begin{aligned}
\text{RHS} &= \overline{\mathbb{h}}_\alpha^{[1]}(u^{-1}) \cdot \overline{\mathbb{e}}_\alpha(-u)\overline{\mathbb{e}}_{-\alpha}(-u^{-1})\overline{\mathbb{e}}_\alpha(-u) \cdot \overline{\mathbb{e}}_\alpha(u) \\
&= \overline{\mathbb{h}}_\alpha^{[1]}(u^{-1})\overline{\mathbb{w}}_\alpha(-u) \cdot \overline{\mathbb{e}}_\alpha(u) \\
&= \overline{\mathbb{h}}_\alpha^{[1]}(u^{-1}) \cdot \overline{\mathbb{w}}_\alpha(-u)\overline{\mathbb{w}}_\alpha(-1) \cdot \overline{\mathbb{w}}_\alpha(1)\overline{\mathbb{e}}_\alpha(u) \\
&= \overline{\mathbb{h}}_\alpha^{[1]}(u^{-1})\overline{\mathbb{h}}_\alpha^{[1]}(-u) \cdot \overline{\mathbb{w}}_\alpha(1)\overline{\mathbb{e}}_\alpha(u) \\
&= (u^{-1}, -u)_n^{Q(\alpha^\vee)} \cdot \overline{\mathbb{h}}_\alpha^{[1]}(-1)\overline{\mathbb{w}}_\alpha(1)\overline{\mathbb{e}}_\alpha(u) \\
&= \overline{\mathbb{w}}_\alpha(-1)\overline{\mathbb{w}}_\alpha(-1)\overline{\mathbb{w}}_\alpha(1)\overline{\mathbb{e}}_\alpha(u) \\
&= \overline{\mathbb{w}}_\alpha(-1)\overline{\mathbb{e}}_\alpha(u)
\end{aligned}$$

The proof is completed since $\overline{\mathbb{w}}_\alpha(-1) = \overline{\mathbb{w}}_\alpha^{-1}$. $\square$

Recall the convention on notations in section 4.2.1: we write $\overline{h}_\alpha(a^{n_\alpha})$ for $\overline{h}_\alpha^{[b]}(a^{n_\alpha})$ since the latter does not depend on $b \in F^\times$. Now we state the one-dimensional computation of $T\big(w_\alpha, i(\overline{\chi})\big)$.

Proposition 4.3.3. *Let $i(\overline{\chi})$ be an unramified representation of $\overline{T}$ for some unramified character $\overline{\chi}$ of $Z(\overline{T})$. Let $\alpha \in \Delta$, consider the intertwining operator*

$$T\big(w_\alpha, i(\overline{\chi})\big): \quad I(i(\overline{\chi})) \longrightarrow I({}^{w_\alpha} i(\overline{\chi}))$$

between unramified principal series respectively of $\overline{G}$. Let $f_{i(\overline{\chi})}$ and $f_{{}^{w_\alpha}i(\overline{\chi})}$ be the normalized unramified vectors of $I(i(\overline{\chi}))$ and $I({}^{w_\alpha}i(\overline{\chi}))$ respectively. Write $T\big(w_\alpha, i(\overline{\chi})\big) f_{i(\overline{\chi})} = c(\mathbf{w}_\alpha, \overline{\chi}) f_{{}^{w_\alpha}i(\overline{\chi})}$. Then

$$c(\mathbf{w}_\alpha, \overline{\chi}) = \frac{1 - q^{-1}\overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))}{1 - \overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))},$$

which is independent on the uniformizer ϖ chosen.

Proof. Write $\overline{\pi} := i(\overline{\chi})$. The unities $1_{\overline{G}}, 1_{\overline{T}}$ of all subgroups $\overline{G}, \overline{T}$ etc of $\overline{G}$ are all the same; but for convenience, we use the subindex to remind us the group on which the representation lives.

By the definition of $T\big(w_\alpha, i(\overline{\chi})\big)$, it suffices to compute

$$\begin{aligned}
&T\big(w_\alpha, i(\overline{\chi})\big)(f_{i(\overline{\chi})})(1_{\overline{G}})(1_{\overline{T}}) \\
=& \int_{U_\alpha} f\big(\overline{w}_\alpha^{-1}\overline{e}_\alpha(u)\big)(1_{\overline{T}}) du \\
=& \int_{0<|u|\le 1} f\big(\overline{w}_\alpha^{-1}\overline{e}_\alpha(u)\big)(1_{\overline{T}}) du + \int_{|u|>1} f\big(\overline{w}_\alpha^{-1}\overline{e}_\alpha(u)\big)(1_{\overline{T}}) du.
\end{aligned}$$

Here du is the additive Haar measure as in Lemma 4.3.1. For the first integral, $\overline{w}_\alpha^{-1}\overline{e}_\alpha(u) \in K_G \subseteq \overline{G}$ for all $0 < |u| \le 1$. Since f is K_G-invariant, the integrand is equal to $f(1_{\overline{G}})(1_{\overline{T}}) = 1$. Therefore, the first integral is equal to 1.

For the second integral, we apply the previous lemma to get it equal to

$$\begin{aligned}
&\int_{|u|>1} f\big(\overline{h}_\alpha^{[1]}(u^{-1})\overline{e}_\alpha(-u)\overline{e}_{-\alpha}(-u^{-1})\big)(1_{\overline{T}}) du \\
=& \int_{|u|>1} f\big(\overline{h}_\alpha^{[1]}(u^{-1})\overline{e}_\alpha(-u)\big)(1_{\overline{T}}) du \\
=& \int_{|u|>1} \delta_B^{1/2}(h_\alpha(u^{-1})) \cdot \Big(\overline{\pi}\big(\overline{h}_\alpha^{[1]}(u^{-1})\big) f(1_{\overline{G}})\Big)(1_{\overline{T}}) du
\end{aligned}$$

Note

$$\delta_B^{1/2}(h_\alpha(u)) = |u|_F^{\langle \rho_B, \alpha^\vee \rangle} = |u|_F.$$

Use the partition $\{u : |u| > 1\} = \bigcup_{k\geq 1} \mathcal{O}_F^\times \varpi^{-k}$, the integral is then equal to

$$\begin{aligned}
&\sum_{k\geq 1}\int_{u\in\varpi^{-k}\mathcal{O}^\times} \delta_B^{1/2}(h_\alpha(\varpi^k))\cdot(u,\varpi^k)_n^{Q(\alpha^\vee)}\cdot\Big(\overline{\pi}\big(\overline{h}_\alpha^{[1]}(\varpi^k)\overline{h}_\alpha(u)\big)f(1_{\overline{G}})\Big)(1_{\overline{T}})du.\\
=&\sum_{k\geq 1}\int_{u\in\mathcal{O}_F^\times} |\varpi|_F^k\cdot(u,\varpi)_n^{kQ(\alpha^\vee)}\Big(\overline{\pi}\big(\overline{h}_\alpha^{[1]}(\varpi^k)\big)f(1_{\overline{G}})\Big)(1_{\overline{T}})\cdot|\varpi^{-k}|_F du\\
=&\sum_{k\geq 1, n_\alpha|k}\int_{u\in\mathcal{O}_F^\times}\Big(\pi\big(\overline{h}_\alpha^{[1]}(\varpi^k)\big)f(1_{\overline{G}})\Big)(1_{\overline{T}})du \quad \text{by Lemma 4.3.1}\\
=&\sum_{k\geq 1, n_\alpha|k}\overline{\chi}\big(\overline{h}_\alpha(\varpi^k)\big)\cdot f(1_{\overline{G}})(1_{\overline{T}})\cdot(1-q^{-1})\\
=&(1-q^{-1})\frac{\overline{\chi}\big(\overline{h}_\alpha(\varpi^{n_\alpha})\big)}{1-\overline{\chi}\big(\overline{h}_\alpha(\varpi^{n_\alpha})\big)},
\end{aligned}$$

where the last equality is due to the fact that $\overline{\chi}\big(\overline{h}_\alpha(\varpi^{r\cdot n_\alpha})\big) = \overline{\chi}^r\big(\overline{h}_\alpha(\varpi^{n_\alpha})\big)$ for $r\in\mathbf{N}_{\geq 1}$. Now combine the first and second integral, we obtain the desired formula.

It remains to show the independence of the chosen uniformizer, we have for $u\in\mathcal{O}_F^\times$,

$$\begin{aligned}
\overline{h}_\alpha\big((\varpi u)^{n_\alpha}\big) =&\overline{h}_\alpha(\varpi^{n_\alpha})\cdot\overline{h}_\alpha(u^{n_\alpha})\cdot(\varpi,u)_n^{n_\alpha^2 Q(\alpha^\vee)}\\
=&\overline{h}_\alpha(\varpi^{n_\alpha})\cdot\overline{h}_\alpha(u^{n_\alpha}).
\end{aligned}$$

However, since $\overline{h}_\alpha(u^{n_\alpha})\in Z(\overline{T})\cap K_T\subseteq s_K(K_G)$, it follows that $\overline{\chi}$ takes the value 1 at it and the independence on uniformizer could be concluded. ☐

Now we come back to general $\mathbf{w}\in W$ and consider the intertwining operator

$$T(\mathbf{w},\overline{\chi}) : I(\overline{\chi})\longrightarrow I({}^{\mathbf{w}}\overline{\chi}).$$

We can use the cocycle relation to deduce:

Corollary 4.3.4. *Let $f_{i(\overline{\chi})}$ and $f_{i({}^{\mathbf{w}}\overline{\chi})}$ be the normalized unramified vectors in $I(\overline{\chi})$ and $I({}^{\mathbf{w}}\overline{\chi})$ respectively. Then one has $T(\mathbf{w},\overline{\chi})f_{i(\overline{\chi})} = c(\mathbf{w},\overline{\chi})f_{i({}^{\mathbf{w}}\overline{\chi})}$ with*

$$c(\mathbf{w},\overline{\chi}) = \prod_{\alpha\in\Psi_{\mathbf{w}}}\frac{1-q^{-1}\overline{\chi}\big(\overline{h}_\alpha(\varpi^{n_\alpha})\big)}{1-\overline{\chi}\big(\overline{h}_\alpha(\varpi^{n_\alpha})\big)},$$

where $\Psi_{\mathbf{w}} = \{\alpha\in\Psi^+ :\ \mathbf{w}\alpha\in\Psi^-\}$.

Proof. Let $\mathbf{w}_k\mathbf{w}_{k-1}...\mathbf{w}_2\mathbf{w}_1$ be a minimum decomposition of $\mathbf{w}$, where $\mathbf{w}_i$ represents a simple reflection $\mathbf{w}_{\alpha_i}$ with $\alpha_i\in\Delta$.

By the cocycle condition in Corollary 4.7, it suffices to compute the general coefficient $c(\mathbf{w}_m, {}^{\mathbf{w}_{m-1}...\mathbf{w}_1}\overline{\chi})$ for $m=1,2,...,k$. First note, since Q is Weyl-invariant, it gives

$$n_{\mathbf{w}_1...\mathbf{w}_{m-1}\alpha_m} = n_{\alpha_m}.$$

Meanwhile, we have

$$\begin{aligned}
&{}^{\mathbf{w}_{m-1}\ldots\mathbf{w}_1}\overline{\chi}\big(\overline{h}_{\alpha_m}(\varpi^{n_{\alpha_m}})\big)\\
=&\overline{\chi}\big((w_{m-1}\ldots w_1)^{-1}\cdot \overline{h}_{\alpha_m}(\varpi^{n_{\alpha_m}})\cdot w_{m-1}\ldots w_1)\big)\\
=&\overline{\chi}\big((w_1^{-1}\ldots w_{m-1}^{-1}\cdot \overline{h}_{\alpha_m}(\varpi^{n_{\alpha_m}})\cdot w_{m-1}\ldots w_1)\big).
\end{aligned}$$

To proceed, consider in general the element $w_\beta^{-1}\overline{h}_\alpha^{[b]}(a)w_\beta$ for $\alpha,\beta\in\Psi$ and $a,b\in F^\times$. We have

$$\begin{aligned}
&w_\beta^{-1}\overline{h}_\alpha^{[b]}(a)w_\beta\\
=&w_\beta^{-1}\cdot\overline{w}_\alpha(ab)\overline{w}_\alpha(-b)\cdot w_\beta\\
=&w_\beta^{-1}\overline{w}_\alpha(ab)w_\beta\cdot w_\beta^{-1}\overline{w}_\alpha(-b)w_\beta.
\end{aligned}$$

At the same time, for general $c\in F^\times$, the following equalities hold

$$\begin{aligned}
&w_\beta^{-1}\overline{w}_\alpha(c)w_\beta\\
=&w_\beta^{-1}\cdot\overline{e}_\alpha(c)\overline{e}_{-\alpha}(c^{-1})\overline{e}_\alpha(c)\cdot w_\beta\\
=&\overline{e}_{\mathbf{w}_\beta(\alpha)}(\epsilon c)\cdot\overline{e}_{-\mathbf{w}_\beta(\alpha)}(\epsilon c^{-1})\cdot\overline{e}_{\mathbf{w}_\beta(\alpha)}(\epsilon c),\\
=&\overline{w}_{\mathbf{w}_\beta(\alpha)}(\epsilon c),
\end{aligned}$$

where the second last equality follows from the fact that the unipotent splitting is G-equivariant as from Proposition 2.3.2, and $\epsilon\in\{\pm1\}$ is a certain sign (depending on α,β) associated with the Chevalley system of épinglage, see [BrTi84, §3.2.2].

Now it follows

$$w_\beta^{-1}\overline{h}_\alpha^{[b]}(a)w_\beta=\overline{w}_{\mathbf{w}_\beta(\alpha)}(\epsilon ab)\cdot\overline{w}_{\mathbf{w}_\beta(\alpha)}(-\epsilon b)=\overline{h}^{[\epsilon b]}_{\mathbf{w}_\beta(\alpha)}(a).$$

In the case of $a=\varpi^{n_\alpha}$, the element $\overline{h}^{[\epsilon b]}_{\mathbf{w}_\beta(\alpha)}(a)$ is independent of ϵ and b. Compute inductively we obtain

$${}^{\mathbf{w}_{m-1}\ldots\mathbf{w}_1}\overline{\chi}\big(\overline{h}_{\alpha_m}(\varpi^{n_{\alpha_m}})\big)=\overline{\chi}\big(\overline{h}_{\mathbf{w}_1\ldots\mathbf{w}_{m-1}\alpha_m}(\varpi^{n_{\mathbf{w}_1\cdots\mathbf{w}_{m-1}\alpha_m}})\big).$$

Lastly, we have the equality $\Psi_{\mathbf{w}}=\{\mathbf{w}_1\ldots\mathbf{w}_{m-1}\alpha_m: m=1,2,\ldots,k\}$, from which the result follows from combining all $c(\mathbf{w}_m,{}^{\mathbf{w}_{m-1}\ldots\mathbf{w}_1}\overline{\chi})$'s. $\square$

Remark 4.3.5. The usage of the element $\overline{h}(\varpi^{n_\alpha})$ (or in general $\overline{h}^{[1]}(a)$) from the Brylinski-Deligne section enables us to remove the assumption $\mu_{2n}\subseteq F^\times$ as in [McN11] and [McN12] for example. In fact, the computation of the metaplectic Casselman-Shalika formula in [McN14] could be carried over using such naturally defined elements. It can be checked that in the case of double cover of $\mathbb{S}\mathbb{p}_{2r}(F)$, McNamara's formula [McN14, Thm. 13.1] recovers that of Szpruch in [Szp11, Thm. 8.1] which does not reply on the assumption that F contains $2n$-th root of unity, provided we make use of these naturally defined elements as in Lemma 4.3.2: $\overline{w}_\alpha,\overline{e}_\alpha(u),\overline{h}^{[1]}(u)$ etc.

Remark 4.3.6. Recall that for any root $\alpha \in \Psi$ the morphism $\varphi_\alpha : \mathbb{SL}_2 \longrightarrow \mathbb{G}$ induces a covering $\overline{SL}_2^\alpha \in \mathsf{CExt}(\mathrm{SL}_2, \mu_n)$ from any given $\overline{G} \in \mathsf{CExt}(\overline{G}, \mu_n)$ of BD type. Let T_o and T be the tori of SL_2 and G, and let $\overline{T}$ and $\overline{T}^\alpha$ be the covering tori of $\overline{G}$ and $\overline{SL}_2^\alpha$ respectively. Let $Z(\overline{T})$ and $Z(\overline{T}^\alpha)$ be the centers of the two covering tori respectively. Then we have the following commutative diagram

$$\begin{tikzcd}
& \overline{T} \arrow[r, two heads] & T \\
Z(\overline{T}) \arrow[ur, hook] & \overline{T}^\alpha \arrow[u, "\overline{\varphi}_\alpha"] \arrow[r, two heads] & T_o \arrow[u, "\varphi_\alpha"] \\
\overline{\varphi}_\alpha^*(Z(\overline{T})) \arrow[u, "\overline{\varphi}_\alpha"] \arrow[ur, hook] \arrow[r, hook] & Z(\overline{T}^\alpha) \arrow[u, hook] &
\end{tikzcd} ,$$

where $\overline{\varphi}_\alpha^*(Z(\overline{T}))$ is the pull-back. By definition, $Z(\overline{T}^\alpha)$ and $\overline{\varphi}_\alpha^*(Z(\overline{T}))$ are closely related to $\alpha_{[n]}^\vee/\gcd(2, n_\alpha)$ and $\alpha_{[n]}^\vee$. More precisely, define

$$\mathbf{n} = \begin{cases} 1 & \text{if } \alpha_{[n]}^\vee/\gcd(2, n_\alpha) \in Y_{Q,n} \\ 2 & \text{otherwise.} \end{cases}$$

Then it is not hard to see

$$Z(\overline{T}^\alpha)/\overline{\varphi}_\alpha^*(Z(\overline{T})) \simeq F^\times/\mathbf{n}.$$

That is, in general $\overline{\varphi}_\alpha^*(Z(\overline{T}))$ is not equal to the whole group $Z(\overline{T}^\alpha)$ and $Z(\overline{T}^\alpha)/\overline{\varphi}_\alpha^*(Z(\overline{T}))$ is a torsion 2 group. In fact, since we have assumed $\gcd(n, p) = 1$, $F^\times/\mathbf{n} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

This has the following implication. For any genuine character $\overline{\chi}$ on $Z(\overline{T})$ we may write $\overline{\chi}_\alpha := \overline{\chi} \circ \overline{\varphi}_\alpha$, which is a genuine character on $\overline{\varphi}_\alpha^*(Z(\overline{T}))$. In the unramified case, the rank one intertwining operator $T(\mathbf{w}_\alpha, \overline{\chi})$, or equivalently the scalar $c(\mathbf{w}_\alpha, \overline{\chi})$ such that $T(\mathbf{w}_\alpha, \overline{\chi}) f_{i(\overline{\chi})} = c(\mathbf{w}_\alpha, \overline{\chi}) f_{i({}^{\mathbf{w}_\alpha}\overline{\chi})}$, can be determined from computing the following intertwining operator on $\overline{SL}_2^\alpha$:

$$T(\mathbf{w}_\alpha, \overline{\chi}) : \quad I\Big(\mathrm{Ind}_{\overline{\varphi}_\alpha^*(Z(\overline{T}))}^{Z(\overline{T}^\alpha)}(\overline{\chi}_\alpha)\Big) \longrightarrow I\Big({}^{\mathbf{w}_\alpha}\big(\mathrm{Ind}_{\overline{\varphi}_\alpha^*(Z(\overline{T}))}^{Z(\overline{T}^\alpha)}(\overline{\chi}_\alpha)\big)\Big).$$

Note however, if in general we have $Z(\overline{T}^\alpha)/\overline{\varphi}_\alpha^*(Z(\overline{T})) \simeq (\mathbf{Z}/2\mathbf{Z})^2$, then $\mathrm{Ind}_{\overline{\varphi}_\alpha^*(Z(\overline{T}))}^{Z(\overline{T}^\alpha)}(\overline{\chi}_\alpha) = \bigoplus_{i=1}^4 \overline{\chi}_{\alpha,i}$ is a 4-dimensional representation of $Z(\overline{T}^\alpha)$.

Thus $T(\mathbf{w}_\alpha, \overline{\chi})$ is given by

$$T(\mathbf{w}_\alpha, \overline{\chi}) = \bigoplus_{i=1}^4 T(\mathbf{w}_\alpha, \overline{\chi}_{\alpha,i}) : \quad \bigoplus_{i=1}^4 I(\overline{\chi}_{\alpha,i}) \longrightarrow \bigoplus_{i=1}^4 I({}^{\mathbf{w}_\alpha}\overline{\chi}_{\alpha,i}).$$

Write $T(\mathbf{w}_\alpha, \overline{\chi}_{\alpha,i})(f_{i(\overline{\chi}_{\alpha,i})}) = c(\mathbf{w}_\alpha, \overline{\chi}_{\alpha,i}) f_{i({}^{\mathbf{w}_\alpha}\overline{\chi}_{\alpha,i})}$. Then it follows from the computation in this section that

$$c(\mathbf{w}_\alpha, \overline{\chi}_{\alpha,i}) = c(\mathbf{w}_\alpha, \overline{\chi}_{\alpha,j}) \text{ for any } i \text{ and } j,$$

since it is shown that $c(\mathbf{w}_\alpha, \overline{\chi}_{\alpha,i})$ depends only on $\overline{\chi}_{\alpha,i}$ restricted to $\overline{\varphi}_\alpha^*(Z(\overline{T}))$ and these characters are all equal there.

Example 4.3.7. Consider $\overline{\mathbb{GL}}_2$ with $(Q, \mathcal{E}, \phi)$, where Q is given by

$$Q: \quad Y \longrightarrow \mathbf{Z}, \quad (y_1, y_2) \mapsto -y_1 y_2.$$

Let $n = 2$ and one obtains $\overline{GL}_2 \in \mathsf{CExt}(GL_2, \mu_2)$.

Here we identity $Y = \mathbf{Z}^2$, with respect to which the element $(1, -1) \in \mathbf{Z}^2$ gives the coroot $\alpha^\vee$ of $\mathbb{SL}_2 \subseteq \mathbb{GL}_2$. So $Q(\alpha^\vee) = 1$ and $n_\alpha = 2$.

One thus has the degree two covers $\overline{SL}_2$ and $\overline{GL}_2$ of SL_2 and GL_2 respectively. It is easy to check $\alpha^\vee_{[n]}/\gcd(2, n_\alpha) = \alpha^\vee \notin Y_{Q,n}$, and therefore $Z(\overline{T}^\alpha)$ is not equal to $\overline{\varphi}^*_\alpha(Z(\overline{T}))$.

4.4 The GK formula as local Langlands-Shahidi L-functions

4.4.1 Adjoint action and the GK formula for principal series

Locally for $\overline{G} \in \mathsf{CExt}(G, \mu_n)$ of BD type, the L-group ${}^L\overline{G}$ sits in the exact sequence

$$\overline{G}^\vee \hookrightarrow {}^L\overline{G} \twoheadrightarrow \mathrm{W}_F .$$

One can define the adjoint action of ${}^L\overline{G}$ on its Lie algebra which is simply the Lie algebra $\overline{\mathfrak{g}}^\vee$ of $\overline{G}^\vee$. Thus we obtain the adjoint representation

$$Ad: \quad {}^L\overline{G} \longrightarrow GL(\overline{\mathfrak{g}}^\vee).$$

By definition ${}^L\overline{G} = j_*^{\overline{G}^\vee} \circ \mathrm{Rec}^*(E_{\overline{G}})$, where $E_{\overline{G}}$ is the fundamental extension over $F^\times$ by $Z(\overline{G}^\vee)$. This implies that the adjoint action Ad depends only on the first coordinate in $\overline{G}^\vee$, where by definition we are viewing ${}^L\overline{G}$ as

$${}^L\overline{G} = \frac{\overline{G}^\vee \times \mathrm{Rec}^*(E_{\overline{G}})}{\nabla Z(\overline{G}^\vee)}.$$

More precisely, define $\overline{G}^\vee_{ad} := \overline{G}^\vee / Z(\overline{G}^\vee)$ and consider the extension $\overline{G}^\vee_{ad} \times W_F$ over W_F. Then there is a natural map $q*$ such that the following commutes:

$$\begin{tikzcd}
\overline{G}^\vee \arrow[r, hook] \arrow[d, "q"] & {}^L\overline{G} \arrow[r, two heads] \arrow[d, "q^*"] & \mathrm{W}_F \arrow[d, equal] \\
\overline{G}^\vee_{ad} \arrow[r, hook] & \overline{G}^\vee_{ad} \times W_F \arrow[r, two heads] \arrow[l, bend left, "s^{\mathrm{Tr}}"] & \mathrm{W}_F,
\end{tikzcd}$$

which is equipped with a canonical splitting s^{Tr}.

The adjoint representation Ad factors through the usual complex adjoint representation $Ad^{\mathbf{C}}$ of $\overline{G}_{ad}^{\vee}$ on $\overline{\mathfrak{g}}^{\vee}$ with respect to the map $s^{\mathrm{Tr}} \circ q^*$:

$$\begin{tikzcd} {}^L\overline{G} \arrow[r, "Ad"] \arrow[d, "s^{\mathrm{Tr}}\circ q^*"'] & GL(\overline{\mathfrak{g}}^{\vee}) \\ \overline{G}_{ad}^{\vee}. \arrow[ur, dashed, "Ad^{\mathbf{C}}"'] \end{tikzcd} \tag{4.9}$$

Now for any parabolic $\mathbb{P} = \mathbb{M}\mathbb{U}$ of $\mathbb{G}$ we obtain $\overline{M}$ and therefore ${}^L\overline{M}$ as well. Recall that there exists a canonical map ${}^L\varphi$ such that the following diagram commutes:

$$\begin{tikzcd} \overline{M}^{\vee} \arrow[r, hook] \arrow[d, hook, "\varphi^{\vee}"'] & {}^L\overline{M} \arrow[r, two heads] \arrow[d, hook, "{}^L\varphi"'] & \mathrm{W}_F \arrow[d, equal] \\ \overline{G}^{\vee} \arrow[r, hook] & {}^L\overline{G} \arrow[r, two heads] & \mathrm{W}_F. \end{tikzcd}$$

By restriction, this gives rise to a representation of ${}^L\overline{M}$ which factors through the complex representation $Ad^{\mathbf{C}}$ of $\overline{M}^{\vee}/Z(\overline{G}^{\vee})$:

$$Ad^{\mathbf{C}}: \quad \overline{M}^{\vee}/Z(\overline{G}^{\vee}) \longrightarrow GL(\overline{\mathfrak{g}}^{\vee}).$$

Since $\overline{M}^{\vee}$ is a Levi subgroup of $\overline{G}^{\vee}$, we can define a complex unipotent $\overline{U}^{\vee}$ such that it is the unipotent radical of the parabolic subgroup $\overline{M}^{\vee}\overline{U}^{\vee}$ of $\overline{G}^{\vee}$.

The group $\overline{M}^{\vee}/Z(\overline{G}^{\vee})$ and therefore ${}^L\overline{M}$ act on the Lie algebra $\overline{\mathfrak{u}}^{\vee}$ of $\overline{U}^{\vee}$, which is a invariant space under the adjoint action of $\overline{M}^{\vee}$. That is, we have as in (4.9)

$$\begin{tikzcd} {}^L\overline{M} \arrow[r, "Ad"] \arrow[d] & GL(\overline{\mathfrak{u}}^{\vee}) \\ \overline{M}^{\vee}/Z(\overline{G}^{\vee}), \arrow[ur, dashed, "Ad^{\mathbf{C}}"'] \end{tikzcd} \tag{4.10}$$

where the vertical map is the composition ${}^L\varphi \circ s^{\mathrm{Tr}} \circ q^*$.

Now we specialize to the case where $\mathbb{P} = \mathbb{B}$ is the Borel subgroup of $\mathbb{G}$. To be consistent with previous notations, we write $\mathbb{B} = \mathbb{T}\mathbb{N}$. Then $\overline{\mathfrak{n}}^{\vee}$ is generated by eigenvectors of the form $E_{\alpha_{[n]}^{\vee}}$ for all $\alpha \in \Psi^+$. For any $\mathbf{w} \in W$, recall we have defined $\Psi_{\mathbf{w}} = \left\{\alpha \in \Psi^+ : \ \mathbf{w}\alpha \in \Psi^-\right\}$. We are interested in the space

$$\overline{\mathfrak{n}}_{\mathbf{w}}^{\vee} = \bigoplus_{\alpha \in \Psi_{\mathbf{w}}} \mathbf{C} \cdot E_{\alpha_{[n]}^{\vee}} \subseteq \overline{\mathfrak{n}}^{\vee}.$$

The space $\overline{\mathfrak{n}}_{\mathbf{w}}^{\vee}$ is invariant under the adjoint action $Ad^{\mathbf{C}}$ of $\overline{T}^{\vee}/Z(\overline{G}^{\vee})$ and this allows us to write $Ad_{\mathbf{w}} := Ad|_{\overline{\mathfrak{n}}_{\mathbf{w}}^{\vee}}$. Clearly $Ad_{\mathbf{w}}$ has the decomposition

$$Ad_{\mathbf{w}} = \bigoplus\nolimits_{\alpha \in \Psi_{\mathbf{w}}} Ad_{\alpha} : \quad {}^L\overline{T} \longrightarrow GL(\overline{\mathfrak{n}}_{\mathbf{w}}^{\vee}),$$

where each Ad_{α} is the one-dimensional representation on $\overline{\mathfrak{n}}_{\alpha}^{\vee} := \mathbf{C} \cdot E_{\alpha_{[n]}^{\vee}}$.

Let $I(\overline{\chi}) = I(i(\overline{\chi}))$ be an unramified principal series of $\overline{G}$. Let $\rho_{\overline{\chi}} : \mathrm{W}_F \longrightarrow {}^L\overline{T}$ be the splitting of ${}^L\overline{T}$ over W_F associated with $\overline{\chi}$ by the local Langlands correspondence. We could identify $\rho_{\overline{\chi}}$ with the splitting $\rho_{\overline{\chi}} : F^\times \longrightarrow E_{\overline{T}}$, which arises from the canonical isomorphism $\mathfrak{S}(E_{\overline{T}}, F^\times) \simeq \mathfrak{S}({}^L\overline{T}, \mathrm{W}_F)$. Note we may view $F^\times$ as the abelianization W_F^{ab} via the Artin reciprocity map.

We have the following ad hoc definition of unramified $\rho_{\overline{\chi}}$.

Definition 4.4.1. The splitting (or L-parameter) $\rho_{\overline{\chi}}$ associated with $\overline{\chi}$ is called unramified if and only if $\overline{\chi}$ is unramified.

Note that for unramified character $\overline{\chi}$, the element $\rho_{\overline{\chi}}(\varpi) \in E_{\overline{T}}$ may still depend on the choice of the uniformizer ϖ as $\rho_{\overline{\chi}}$ is a splitting of $F^\times$ into $E_{\overline{T}}$. However, recall the canonical map $\mathcal{C}$ from $E_{\overline{T}}$ to $\overline{T}^\vee/Z(\overline{G}^\vee)$ as in Corollary 2.4.9. Then the element $\mathcal{C} \circ \rho_{\overline{\chi}}(\varpi) \in \mathcal{C}(E_{\overline{T}})$ is independent of the uniformizer.

Proposition 4.4.2. *Let $\rho_{\overline{\chi}} \in \mathfrak{S}(E_{\overline{T}}, F^\times)$ be an unramified splitting. Identify $\overline{T}^\vee/Z(\overline{G}^\vee)$ with $Hom(Y_{Q,n}^{sc}, \mathbf{C})$. Then for all $\alpha \in \Psi$, we have that*

$$\mathcal{C} \circ \rho_{\overline{\chi}}(\varpi)(\alpha_{[n]}^\vee) = \overline{\chi}\big(\overline{h}(\varpi^{n_\alpha})\big) \in \mathbf{C}$$

is independent of the uniformizer ϖ chosen.

Proof. The identify is from Corollary 3.2.14, while the independence of the uniformizer follows from by Proposition 4.3.3. □

Write $\overline{\pi} := i(\overline{\chi})$. Then the local Langlands L-function $L(s, \overline{\pi}, Ad_\alpha)$ is given by the Artin L-function associated with

$$Ad_\alpha \circ \rho_{\overline{\chi}} : \quad \mathrm{W}_F \longrightarrow {}^L\overline{T} \longrightarrow GL(\overline{\mathfrak{n}}_\alpha^\vee), .$$

That is,

$$L(s, \overline{\pi}, Ad_\alpha) := \frac{1}{\det\big(1 - q^{-s} \cdot Ad_\alpha \circ \rho_{\overline{\chi}}(\mathrm{Frob})|_{\overline{\mathfrak{n}}_\alpha^{\vee I}}\big)}.$$

For unramified $\rho_{\overline{\chi}}$ which we view as an element in $\mathfrak{S}(E_{\overline{T}}, F^\times)$, is given by

$$L(s, \overline{\pi}, Ad_\alpha) = \frac{1}{\det\big(1 - q^{-s} \cdot Ad_\alpha \circ \rho_{\overline{\chi}}(\varpi)|_{\overline{\mathfrak{n}}_\alpha^{\vee I}}\big)}.$$

Now fix $\mathbf{w} \in W$.

Theorem 4.4.3. *The eigenvalue of $\rho_{\overline{\chi}}(\varpi)$ of the representation Ad_α on the one-dimensional invariant space $\overline{\mathfrak{n}}_\alpha^\vee$ is given by*

$$Ad_\alpha(\rho_{\overline{\chi}}(\varpi))(E_{\alpha_{[n]}^\vee}) = \overline{\chi}\big(\overline{h}_\alpha(\varpi^{n_\alpha})\big) \cdot E_{\alpha_{[n]}^\vee}, \tag{4.11}$$

which is independent of the uniformizer chosen.

It follows that the GK formula for the intertwining operator $T(\mathbf{w}, \overline{\chi})$ acting on the unramified representation $I(\overline{\chi})$ can be rewritten as $T(\mathbf{w}, \overline{\chi}) f_{i(\overline{\chi})} = c(\mathbf{w}, \overline{\chi})(f_{i({}^{\mathbf{w}}\overline{\chi})})$ with

$$c(\mathbf{w}, \overline{\chi}) = \prod_{\alpha \in \Psi_{\mathbf{w}}} \frac{L(0, \overline{\pi}, Ad_\alpha)}{L(1, \overline{\pi}, Ad_\alpha)}.$$

Proof. It suffices to show the equality in (4.11) regarding the eigenvalue.

Identify the splitting $\rho_{\overline{\chi}}$ as that of $E_{\overline{T}}$ over $F^\times$. Then the adjoint action of ${}^L\overline{T}$ naturally factor through that of $E_{\overline{T}}$:

$$\begin{tikzcd} {}^L\overline{T} \arrow[r, "Ad"] \arrow[d] & GL(\overline{\mathfrak{n}}_\alpha^\vee) \\ E_{\overline{T}}. \arrow[ur, dashed, "Ad"'] & \end{tikzcd}$$

Further more, the adjoint representation of $E_{\overline{T}}$ on $\overline{\mathfrak{n}}_\alpha^\vee$ also factors through $\overline{T}^\vee / Z(\overline{G}^\vee)$:

$$\begin{tikzcd} E_{\overline{T}} \arrow[r, "Ad_\alpha"] \arrow[d, "\mathcal{C}"'] & GL(\overline{\mathfrak{n}}_\alpha^\vee) \\ \overline{T}^\vee / Z(\overline{G}^\vee). \arrow[ur, "Ad_\alpha^{\mathbf{C}}"'] & \end{tikzcd}$$

Therefore, with $\rho_{\overline{\chi}}$ viewed as a splitting of $E_{\overline{T}}$ over $F^\times$, we have the composition

$$Ad_\alpha^{\mathbf{C}} \circ \mathcal{C} \circ \rho_{\overline{\chi}} : \quad F^\times \longrightarrow E_{\overline{T}} \longrightarrow \overline{T}^\vee / Z(\overline{G}^\vee) \longrightarrow GL(\overline{\mathfrak{n}}_\alpha^\vee).$$

Now it suffices to compute the eigenvalue of $Ad_\alpha^{\mathbf{C}} \circ \mathcal{C} \circ \rho_{\overline{\chi}}(\varpi)$ on $E_{\alpha_{[n]}^\vee}$, which is equal to $\mathcal{C} \circ \rho_{\overline{\chi}}(\varpi)(\alpha_{[n]}^\vee)$ since $E_{\alpha_{[n]}^\vee}$ is the unipotent vector corresponding to the root $\alpha_{[n]}^\vee$ of $\overline{G}^\vee$. Here we view $\mathcal{C} \circ \rho_{\overline{\chi}}(\varpi) \in \overline{T}^\vee / Z(\overline{G}^\vee) \simeq \mathrm{Hom}(Y_{Q,n}^{sc}, \mathbf{C}^\times)$.

The proof is thus completed in view of the equality $\mathcal{C} \circ \rho_{\overline{\chi}}(\varpi)(\alpha_{[n]}^\vee) = \overline{\chi}\big(\overline{h}(\varpi^{n_\alpha})\big)$ and the independence statement from the previous proposition. □

Remark 4.4.4. When $\mathbb{G}$ is not necessarily split over F but over an unramified extension E of F, the computation in [McN14, Thm. 12.1] gives the rank one GK coefficient as

$$\frac{\big(1 + (-1)^{n_\alpha} \cdot q^{-1} \overline{h}_\alpha(\varpi^{n_\alpha})\big)\big(1 - (-1)^{n_\alpha} \cdot q^{-2} \overline{h}_\alpha(\varpi^{n_\alpha})\big)}{1 - \overline{h}_\alpha(\varpi^{n_\alpha})^2},$$

where $\overline{h}_\alpha(\varpi^{n_\alpha})^2 = \overline{h}_\alpha(\varpi^{2n_\alpha})$. We believe that a proper understanding of the L-group construction for BD covers of nonsplit $\mathbb{G}$ in [We14] will enable us to express the GK coefficient obtained as Langlands-Shahidi type L-function as well. We leave the investigation of this to a future work.

4.4.2 The GK formula for induction from maximal parabolic

In this subsection, we consider parabolic induction from a maximal parabolic of $\overline{G}$. Let $\mathbb{G}$ be a reductive group split over F with root datum $(X, \Psi, Y, \Psi^\vee)$. As before, we have fixed a set of simple roots $\Delta \subseteq \Psi^+$ with $\Psi = \Psi^+ \sqcup \Psi^-$. Consider any simple root $\beta \in \Delta$, and let $\mathbb{P} = \mathbb{M}\mathbb{U}$ be a maximal parabolic of $\mathbb{G}$ associated with $\Delta \backslash \{\beta\}$.

Let $2\rho_P$ be the sum of positive roots in $\mathbb{U}$, define

$$\beta_P = \frac{\rho_P}{\langle \rho_P, \beta^\vee \rangle},$$

where $\langle -, - \rangle$ is the pairing between X and Y. Then $\beta_P \in X \otimes \mathbf{Q}$ is the fundamental weight associated with β.

From the μ_n-extension $\overline{G}$, we obtain a μ_n extension $\overline{M}$ of $M = \mathbb{M}(F)$. Since the construction of μ_n extension $\overline{G}$ is functorial, the extension $\overline{M}$ over the Levi M

$$\mu_n \hookrightarrow \overline{M} \twoheadrightarrow M$$

is obtained from the data inherited from those associated with $\overline{\mathbb{G}}$.

Let $\overline{\pi}$ be an irreducible unramified genuine representation of $\overline{M}$. Consider the normalized induced representation $I_{\overline{P}}^{\overline{G}}(\overline{\pi} \otimes \mathbb{1}_U)$. For simplicity we may just write $I(\overline{\pi})$ for it.

Since $\overline{\pi}$ is unramified, by the Satake isomorphism in Corollary 4.1.4 there is a ϵ-genuine character $\overline{\chi}$ of $Z(\overline{T})$ such that

$$\overline{\pi} \hookrightarrow I_{\overline{B}_M}^{\overline{M}}(\overline{\chi}) ,$$

where $\overline{B}_M = \overline{T} N_M$ is the Borel of $\overline{M}$ whose Levi factor is just $\overline{T}$. Then the representation $\mathrm{Ind}_{\overline{P}}^{\overline{G}}(I_{\overline{B}_M}^{\overline{M}}(\overline{\chi}) \otimes \mathbb{1}_U$, by induction in stages, is just the unramified principle series $I(\overline{\chi}) = I(i(\overline{\chi}))$ of $\overline{G}$ introduced before. Moreover, we have

$$I(\overline{\pi}) \hookrightarrow I(\overline{\chi}).$$

It is known (cf. [CKM04, pg. 122]) that there exists a unique $\mathbf{w} \in W$ such that

$$\mathbf{w}(\Delta \backslash \{\beta\}) \subseteq \Delta \text{ and } \mathbf{w}(\beta) \in \Psi^-.$$

In fact, $\mathbf{w}(\beta_P) = -\beta_P$. From now on, we fix this $\mathbf{w}$ whenever we consider intertwining operators for induction from maximal parabolic.

Let $w = s_W(\mathbf{w}) \in W^{K_G}$ and $\overline{w} = s_K(w)$ be the representatives of $\mathbf{w}$ defined in section 4.2.1. We are interested in the intertwining operator

$$T(w, \overline{\pi}) : \quad I(\overline{\pi}) \longrightarrow I({}^w\overline{\pi})$$

given by

$$f \mapsto T(w, \overline{\pi}) f(\overline{g}) = \int_{U^w} f(\overline{w}^{-1} \overline{u}\, \overline{g}) du,$$

where $U^w = N \cap wU^- w^{-1}$ with U^- the unipotent radical opposed to U.

As in the linear algebraic case, the following diagram commutes:

$$\begin{tikzcd}
I(i(\overline{\chi})) \arrow[r, "T(w, i(\overline{\chi}))"] & I({}^w i(\overline{\chi})) \\
I(\overline{\pi}) \arrow[u, hook] \arrow[r, "T(w, \overline{\pi})"] & I({}^w \overline{\pi}) \arrow[u, hook]
\end{tikzcd} .$$

Let $f_{\overline{\pi}}, f_{{}^w\overline{\pi}}$ be the normalized unramified vectors of $I(\overline{\pi}), I({}^w\overline{\pi})$ respectively. We view them as vectors in the unramified principal series $I(i(\overline{\chi}))$ and $I({}^w i(\overline{\chi}))$, normalized such that $f_{\overline{\pi}}(1_{\overline{G}})(1_{\overline{T}}) = f_{{}^w\overline{\pi}}(1_{\overline{G}})(1_{\overline{T}}) = 1$. We want to compute the constant $c(\mathbf{w}, \overline{\pi})$ that appears in $T(w, \overline{\pi}) f_{\overline{\pi}} = c(\mathbf{w}, \overline{\pi}) f_{{}^w\overline{\pi}}$.

Coupled with the computation of the GK formula in Theorem 4.4.3, we see that

$$c(\mathbf{w}, \overline{\pi}) = \prod_{\alpha \in \Psi_{\mathbf{w}}} \frac{L(0, Ad_\alpha \circ \rho_{\overline{\chi}})}{L(1, Ad_\alpha \circ \rho_{\overline{\chi}})},$$

where each $Ad_\alpha \circ \rho_{\overline{\chi}} : \mathrm{W}_F \longrightarrow \mathbf{C}^\times$ is a character.

Consider the adjoint action

$$Ad_{\overline{\mathfrak{u}}^\vee} : \quad {}^L\overline{M} \longrightarrow GL(\overline{\mathfrak{u}}^\vee),$$

where $\overline{\mathfrak{u}}^\vee$ is the Lie algebra of $\overline{U}^\vee$ such that $\overline{M}^\vee \overline{U}^\vee$ is a parabolic subgroup of $\overline{G}^\vee$. It factors through $Ad^{\mathbf{C}}_{\overline{\mathfrak{u}}^\vee}$:

$$Ad_{\overline{\mathfrak{u}}^\vee} : \quad \begin{tikzcd} {}^L\overline{M} \arrow[r] \arrow[d] & GL(\overline{\mathfrak{u}}^\vee) \\ \overline{M}^\vee / Z(\overline{G}^\vee). \arrow[ur, "Ad^{\mathbf{C}}_{\overline{\mathfrak{u}}^\vee}"'] & \end{tikzcd}$$

Therefore, irreducible subspaces of $\overline{\mathfrak{u}}^\vee$ for $Ad_{\overline{\mathfrak{u}}^\vee}$ are in correspondence with irreducible subspaces with respect to $Ad^{\mathbf{C}}_{\overline{\mathfrak{u}}^\vee}$, which are familiar (cf. [Lan71]). More precisely, we consider the decomposition of $Ad_{\overline{\mathfrak{u}}^\vee}$ into irreducibles

$$Ad_{\overline{\mathfrak{u}}^\vee} = \bigoplus_{i=1}^m Ad_i.$$

Let $V_i \subseteq \overline{\mathfrak{u}}^\vee$ be the irreducible space for Ad_i. Then as observed by Langlands, V_i is given by

$$V_i = \bigoplus_{\langle \beta_P / n_\beta, \alpha^\vee_{[n]} \rangle = i} \mathbf{C} \cdot E_{\alpha^\vee_{[n]}}.$$

Moreover the following equality holds:

$$\Psi_{\mathbf{w}} = \bigsqcup_{i=1}^m \{ \alpha \in \Psi^+ : \langle \beta_P / n_\beta, \alpha^\vee_{[n]} \rangle = i \}.$$

Recall the local Artin L-function $L(s, \overline{\pi}, Ad_i)$ is by definition $L(s, Ad_i \circ \rho_{\overline{\chi}})$ associated with $Ad_i \circ \rho_{\overline{\chi}}$:

$$L(s, \overline{\pi}, Ad_i) = \frac{1}{\det\left(1 - q^{-s} Ad_i \circ \rho_{\overline{\chi}}(\mathsf{Frob})|_{V_i^I}\right)},$$

where we also write $\rho_{\overline{\chi}}$ for the composition $\mathrm{W}_F \longrightarrow {}^L\overline{T} \longrightarrow {}^L\overline{M}$. For unramified $\overline{\pi}$, if we identify $\rho_{\overline{\chi}}$ with an unramified splitting of $E_{\overline{T}}$ over $F^\times$, then the inertia group I acts trivially on V_i by Theorem 4.4.3. We thus have

$$L(s, \overline{\pi}, Ad_i) = \frac{1}{\det\left(1 - q^{-s} Ad_i \circ \rho_{\overline{\chi}}(\varpi)|_{V_i}\right)}.$$

In view of Theorem 4.4.3, we could summarize our discussion above as:

Theorem 4.4.5. *The G-K formula takes the form* $T(w,\overline{\pi})f_{\overline{\pi}} = c(\mathbf{w},\overline{\pi})f_{w\overline{\pi}}$ *with*

$$c(\mathbf{w},\overline{\pi}) = \prod_{i=1}^{m} \frac{L(0,\overline{\pi},Ad_i)}{L(1,\overline{\pi},Ad_i)},$$

where in this case we have the equality

$$L(s,\overline{\pi},Ad_i) = \prod_{\substack{\alpha\in\Psi^+ \\ \langle \beta_P/n_\beta, \alpha^\vee_{[n]}\rangle = i}} L(s, Ad_\alpha \circ \rho_{\overline{\chi}}).$$

Chapter 5

Automorphic L-function, constant term of Eisenstein series and residual spectrum

The aim of this chapter is to compute the constant term of Eisenstein series, and to write the coefficient of global intertwining operators in terms of certain Langlands-Shahidi type L-functions. The main tool is the theory of Eisenstein series and the computation of its constant terms in terms of intertwining operators as given in [MW95]. It is important to bridge from the global situation to the local ones and thus to utilize the local results which we have obtained in the previous chapter.

Since the global theory is developed systematically in the book by Moeglin-Waldspurger ([MW95]), we will just give a brief review below and refer to the book for properties of Eisenstein series, e.g. meromorphic continuation and functional equations etc. We also refer to [MW95] for detailed introduction on automorphic forms on $\overline{\mathbb{G}}(\mathbb{A}_F)$ and the spectral decomposition of $L^2(\mathbb{G}(F)\backslash\overline{\mathbb{G}}(\mathbb{A}_F))$.

In this chapter we fix F to be a number field, and ${}^L\overline{G}$ the global L-group.

5.1 Automorphic L-function

Let $\overline{\sigma} = \bigotimes_v \overline{\sigma}_v$ be a genuine automorphic representation of $\overline{\mathbb{G}}(\mathbb{A}_F)$. Then for almost all v, $\overline{\sigma}_v$ is unramified and $\overline{\sigma}_v \hookrightarrow I(\overline{\chi}_v)$. It gives rise to a splitting of the local L-group ${}^L\overline{G}_v$ as the composition

$$\rho_v : \quad \mathrm{W}_{F_v} \longrightarrow {}^L\overline{T}_v \hookrightarrow {}^L\overline{G}_v,$$

where the first map is given by $\rho_{\overline{\chi}_v}$ from the LLC in Proposition 3.2.13. Recall that for all $v \in |F|$, there is the canonical map ${}^L\overline{G}_v \longrightarrow {}^L\overline{G}$.

Definition 5.1.1. Let $R : {}^L\overline{G} \longrightarrow GL(V)$ be any finite dimensional representation. For any v, let $R_v : {}^L\overline{G}_v \longrightarrow {}^L\overline{G} \xrightarrow{R} GL(V)$ be the post composition with R. Then the global partial L-function of $\overline{\sigma}$ with respect to R is defined to be

$$L^S(s,\overline{\sigma},R) = \prod_{v\notin S} L(s,\overline{\sigma}_v,R),$$

where $L(s,\overline{\sigma}_v,R)$ is the local Artin L-function associated with the unramified representation $\rho_{v,R} : \mathrm{W}_{F_v} \xrightarrow{\rho_v} {}^L\overline{G}_v \xrightarrow{R_v} GL(V)$. More precisely,

$$L(s,\overline{\sigma}_v,R) := \frac{1}{\det\left(1 - q_v^{-s}\cdot\rho_{v,R}(\mathsf{Frob}_v)|_{V^{I_v}}\right)}.$$

Since ${}^L\overline{G}$ is a disconnected reductive complex Lie group, it is not easy to give its irreducible representations. However, there is a natural family of representations which are of interest to us, namely the adjoint type L-functions. As noted in section 4.4.1, we have the commutative diagram

$$\begin{tikzcd}
\overline{G}^\vee \arrow[r, hook] \arrow[d, "q"] & {}^L\overline{G} \arrow[r, two heads] \arrow[d, "q^*"] & W_F \arrow[d, equal] \\
\overline{G}^\vee_{ad} \arrow[r, hook] & \overline{G}^\vee_{ad}\times W_F \arrow[r, two heads] \arrow[l, bend left, "s^{\mathrm{Tr}}"] & W_F.
\end{tikzcd}$$

Then any representation of $\overline{G}^\vee_{ad}$ pulls back to a representation of ${}^L\overline{G}$. In particular, the adjoint representation Ad of $\overline{G}^\vee_{ad}$ gives the adjoint representations of ${}^L\overline{G}$ on the Lie algebra $\overline{\mathfrak{g}}^\vee$. More generally, for $\mathbb{P}=\mathbb{M}\mathbb{U}$ a parabolic subgroup of $\mathbb{G}$, we obtain the dual group $\overline{M}^\vee$ embedded in $\overline{G}^\vee$. Then the adjoint representation of $\overline{M}^\vee/Z(\overline{G}^\vee)$ on the Lie algebra $\overline{\mathfrak{u}}^\vee$ can be pulled back to give representation of ${}^L\overline{M}$.

Moreover, from ${}^L\overline{G}_v \longrightarrow {}^L\overline{G}$ the adjoint representation Ad of ${}^L\overline{G}$ can be pulled back to ${}^L\overline{G}_v$ to give the adjoint representation of the local L-group discussed in previous chapter. One thus obtains the Langlands-Shahidi type L-functions with respect to these adjoint representations, which are of the main interest in the following sections.

5.2 Eisenstein series and its constant terms

For simplicity, we concentrate on the maximal parabolic case, while the general case follows from similar treatment despite the complication in notations. Before we proceed, we recall two equivalent realizations of induced representations.

Follow notations in section 4.4.2, let $\mathbb{P}=\mathbb{M}\mathbb{U}$ be a maximal parabolic of $\mathbb{G}$ corresponding to $\Delta\backslash\{\beta\}$. Let $2\rho_P$ be the sum of positive roots in $\mathbb{U}$ and β_P be the fundamental weight corresponding to β.

Consider the character group $X^*(\mathbb{M})$ of $\mathbb{M}$, and also the real and complex vector space

$$X^*(\mathbb{M})_{\mathbf{R}} = X^*(\mathbb{M})\bigotimes_{\mathbf{Z}}\mathbf{R},\quad X^*(\mathbb{M})_{\mathbf{C}} = X^*(\mathbb{M})\bigotimes_{\mathbf{Z}}\mathbf{C}.$$

Any $\nu_o \in X^*(\mathbb{M})$ could be viewed as a character on $\mathbb{M}(\mathbb{A}_F)$ valued in $\mathbb{A}_F^\times$. Further composition with the valuation of $\mathbb{A}_F^\times$ gives us a character of $\mathbb{M}(\mathbb{A}_F)$ valued in $\mathbf{C}^\times$.

Similarly, for any $\nu = \nu_o \otimes s \in X^*(\mathbb{M})_{\mathbf{C}}$ with $\nu_o \in X^*(\mathbb{M})$ and $s \in \mathbf{C}$, we denote by $\boldsymbol{\delta}^\nu$ the following character of $\mathbb{M}(\mathbb{A}_F)$:

$$\boldsymbol{\delta}^\nu : \quad \mathbb{M}(\mathbb{A}_F) \longrightarrow \mathbf{C}, \qquad m \longmapsto |\nu_o(m)|_{\mathbb{A}_F}^s.$$

The relation between $\boldsymbol{\delta}$ and the modular character δ_P is that

$$\boldsymbol{\delta}^{\rho_P \otimes 1} = \delta_P^{1/2}.$$

In the case of maximal parabolic, $X^*(\mathbb{M}/Z(\mathbb{G})) \bigotimes \mathbf{C}$ is of dimension one over $\mathbf{C}$ with $\beta_P \otimes 1$ or $\rho_P \otimes 1$ as a basis vector. Henceforth, we will write

$$\boldsymbol{\delta}^s := \boldsymbol{\delta}^{\beta_P \otimes s}, \quad s \in \mathbf{C}.$$

For example for $\mathbb{SL}_2$ with positive root β, $\rho_P = \beta/2$ and $\beta_P = \rho_P$. Then $\boldsymbol{\delta}^s = \delta_P^{s/2}$, with δ_P the modular character of the Borel subgroup P.

Let $\overline{\pi}$ be a genuine cuspidal automorphic representation of $\overline{\mathbb{M}}(\mathbb{A}_F)$, i.e., $\overline{\pi}$ occurs as a direct summand $V_{\overline{\pi}}$ in the decomposition of $L^2_{\text{cusp}}(\mathbb{M}(F)\backslash\overline{\mathbb{M}}(\mathbb{A}_F))$. Here $\overline{\pi}$ is a unitary representation.

We take $\boldsymbol{\delta}^s$ to be a character of the covering $\overline{\mathbb{M}}(\mathbb{A}_F)$ by the inflation via the surjection $\overline{\mathbb{M}}(\mathbb{A}_F) \twoheadrightarrow \mathbb{M}(\mathbb{A}_F)$. Now we consider the induction $I(s,\overline{\pi}) := Ind_{\overline{P}}^{\overline{G}}(\boldsymbol{\delta}^s\overline{\pi}) \otimes \mathbb{1}$. We have the tensor product decomposition

$$I(s,\overline{\pi}) = \bigotimes_v I(s,\overline{\pi}_v),$$

where $I(s,\overline{\pi}_v)$ is unramified for almost all v.

For the purpose of defining Eisenstein series valued in $\mathbf{C}$, one would like to consider the following alternative description of $I(s,\overline{\pi})$ (cf. [MW95, §I.2.17]).

We have the Iwasawa decomposition

$$\overline{\mathbb{G}}(\mathbb{A}_F) = \mathbb{U}(\mathbb{A}_F)\overline{\mathbb{M}}(\mathbb{A}_F)\overline{K},$$

where $\overline{K} \subseteq \overline{\mathbb{G}}(\mathbb{A}_F)$ is the preimage of $K = \prod_v K_v$, which is a fixed maximal compact subgroup of $\mathbb{G}(\mathbb{A}_F)$ such that $K_v = \mathbb{G}(\mathcal{O}_v)$ for almost all v.

It follows

$$\mathbb{U}(\mathbb{A}_F)\mathbb{M}(F)\backslash\overline{\mathbb{G}}(\mathbb{A}_F) \simeq \big(\mathbb{M}(F)\backslash\overline{\mathbb{M}}(\mathbb{A}_F)\big) \cdot \overline{K}.$$

We thus define a space $V_{P,\overline{\pi}}$ of functions

$$\phi : \quad \mathbb{U}(\mathbb{A}_F)\mathbb{M}(F)\backslash\overline{\mathbb{G}}(\mathbb{A}_F) \longrightarrow \mathbf{C},$$

satisfying

(i) ϕ is right $\overline{K}$-finite. That is, the space spanned by ϕ_k where $\phi_k(\overline{g}) = \phi(\overline{g}k)$ is finite dimensional.

(ii) for each $k \in \overline{K}$, the function given by $\overline{m} \mapsto \phi(\overline{m}k), \overline{m} \in \overline{\mathbb{M}}(\mathbb{A}_F)$, lies in $V_{\overline{\pi}}$ which we recall is a realization of $\overline{\pi}$ in $L^2_{\text{cusp}}(\mathbb{M}(F)\backslash\overline{\mathbb{M}}(\mathbb{A}_F))$.

Consider the space

$$\mathbf{I}(s, \overline{\pi}) = \{\phi \cdot \boldsymbol{\delta}^{s+\rho_P} : \phi \in V_{P,\overline{\pi}}\},$$

where we have written $\boldsymbol{\delta}^{s+\rho_P} := \boldsymbol{\delta}^{\beta_P \otimes s + \rho_P \otimes 1}$ in abbreviation. Here $\boldsymbol{\delta}^s$ is also used to denote the map

$$\boldsymbol{\delta}^s : \mathbb{U}(\mathbb{A}_F)\mathbb{M}(F)\backslash\overline{\mathbb{G}}(\mathbb{A}_F) \longrightarrow \mathbf{C}^\times, \quad \overline{g} = u\overline{m}k \longmapsto \boldsymbol{\delta}^s(\overline{m}),$$

which is well-defined as $\boldsymbol{\delta}^s$ is trivial on $\overline{\mathbb{M}}(\mathbb{A}_F) \cap \overline{K}$. Roughly speaking, the space $\mathbf{I}(s, \overline{\pi})$ is endowed with an action of right translation, i.e. for all $\overline{g} \in \overline{\mathbb{G}}(\mathbb{A}_F)$,

$$\mathbf{I}(s, \overline{\pi})(\overline{g}) : \quad \phi(x) \cdot \boldsymbol{\delta}^{s+\rho_P}(x) \longmapsto \phi(x\overline{g}) \cdot \boldsymbol{\delta}^{s+\rho_P}(x\overline{g}), x \in \overline{\mathbb{G}}(\mathbb{A}_F).$$

Note that in a more rigorous way $\overline{\mathbb{G}}(\mathbb{A}_F)$ does not act directly on it. Rather, the representation space (or rather module) we are interested in is endowed with a structure of $\big(\text{Lie}(\overline{\mathbb{G}}(\mathbb{A}_\infty)) \otimes_{\mathbf{R}} \mathbf{C}, \overline{K}\big) \times \overline{\mathbb{G}}(\mathbb{A}_{\text{fin}})$-module. Here we just fix ideas and refer to [MW95] for a more careful and rigorous treatment.

Now we can define a map

$$I(s, \overline{\pi}) \longrightarrow \mathbf{I}(s, \overline{\pi}), \qquad f_s \longmapsto \text{ev}_{1_{\overline{P}}} \circ f_s,$$

where $f_s : \overline{\mathbb{G}}(\mathbb{A}_F) \longrightarrow \overline{\pi}$ is an element in $I(s, \overline{\pi})$, and $\text{ev}_{1_{\overline{P}}}$ is the evaluation map at the identity $1_{\overline{P}} = 1_{\overline{G}}$. Regarding the well-definedness one can check that the function

$$\overline{g} \quad \mapsto \quad \frac{\text{ev}_{1_{\overline{P}}} \circ f_s}{\boldsymbol{\delta}^{-(s+\rho_P)}}(\overline{g})$$

is independent of s and lies in $V_{P,\overline{\pi}}$. Equivalently, $\text{ev}_{1_{\overline{P}}} \circ f_s \in \mathbf{I}(s, \overline{\pi})$. In fact, above map is an isomorphism, with respect to which $I(s, \overline{\pi})$ and $\mathbf{I}(s, \overline{\pi})$ can be identified.

Let $\phi_s = \phi \cdot \boldsymbol{\delta}^{s+\rho_P} \in \mathbf{I}(s, \overline{\pi})$ be corresponding to a certain $f_s \in I(s, \overline{\pi})$. Define the Eisenstein series

$$E(s, \overline{\pi}, \phi_s, \overline{g}) = \sum_{\gamma \in \mathbb{P}(F)\backslash\mathbb{G}(F)} \phi_s(\gamma\overline{g}) = \sum_{\gamma \in \mathbb{P}(F)\backslash\mathbb{G}(F)} \phi(\gamma\overline{g}) \cdot \boldsymbol{\delta}^{s+\rho_P}(\gamma\overline{g}).$$

We may also write $E(s, \overline{\pi}, \phi, \overline{g})$ for $E(s, \overline{\pi}, \phi_s, \overline{g})$.

Recall we have the unique $\mathbf{w} \in W$ such that $\mathbf{w}(\Delta\backslash\{\beta\}) \subseteq \Delta$ and $\mathbf{w}(\beta) \in \Psi^-$.

Definition 5.2.1. The parabolic $\mathbb{P}$ is called self-associated if $\mathbf{w}(\Delta\backslash\{\beta\}) = \Delta\backslash\{\beta\}$.

Pick the representative $w \in \mathbb{G}(F)$ as in section 4.2.1, which implies $w \in \mathbb{G}(F_v)$ for all v. That is, the embedding $\mathbb{G}(F) \hookrightarrow \mathbb{G}(\mathbb{A}_F)$ gives $w \hookrightarrow (w_v)_v$ such that $w_v = w \in \mathbb{G}(F_v)$ for all v.

Moreover, the splitting $\mathbb{G}(F) \hookrightarrow \overline{\mathbb{G}}(\mathbb{A}_F)$ gives $w \hookrightarrow (\widetilde{w}_v)_v$ with $\widetilde{w}_v \in \overline{\mathbb{G}}(F_v)$ for all v. Since w is generated by unipotent element; for almost all v, the element $\widetilde{w}_v$ is in fact equal to the lifting of $w_v \in \mathbb{G}(F_v)$ by s_K. That is, for almost all v we have $w_v = w \in W^{K_{G_v}}$ and furthermore $\widetilde{w}_v = \overline{w}_v \in s_K(W^{K_{G_v}})$.

Consider the parabolic associated with $\mathbf{w}(\Delta \backslash \{\beta\}) \subseteq \Delta$ whose Levi subgroup is then given by $\mathbb{M}' := w\mathbb{M}w^{-1}$. As in [MW95, §II.1.6], denote by ${}^w\overline{\pi}$ the representation on $\overline{\mathbb{M}'}(\mathbb{A}_F)$. Then we have the global intertwining operator

$$T(w, s, \overline{\pi}) = \bigotimes_v T(\widetilde{w}_v, s, \overline{\pi}_v): \quad I(s, \overline{\pi}) \longrightarrow I(-s, {}^w\overline{\pi})$$

induces an intertwining operator from $\mathbf{I}(s, \overline{\pi})$ to $\mathbf{I}(-s, {}^w\overline{\pi})$, still denoted by $T(w, s, \overline{\pi})$.

The main properties of Eisenstein series could be summarized as

Theorem 5.2.2 ([MW95, §I.1.5, §IV.1.10-11])**.** *The Eisenstein series $E(s, \overline{\pi}, \phi_s, \overline{g})$ has the following properties:*

1. *$E(s, \overline{\pi}, \phi_s, \overline{g})$ is absolutely convergent for $Re(s) > \langle \rho_P, \beta^\vee \rangle$.*
2. *$E(s, \overline{\pi}, \phi_s, \overline{g})$ and $T(w, s, \overline{\pi})$ both have meromorphic continuation to $s \in \mathbf{C}$ and satisfy a functional equation*
$$E(s, \overline{\pi}, \phi_s, \overline{g}) = E(-s, {}^w\overline{\pi}, T(w, s, \overline{\pi})\phi_s, \overline{g}), \quad T(w, -s, {}^w\overline{\pi})T(w, s, \overline{\pi}) = id.$$
3. *$E(s, \overline{\pi}, \phi, \overline{g})$ and $T(w, s, \overline{\pi})$ are holomorphic on $Re(s) = 0$.*
4. *The singularities of $E(s, \overline{\pi}, \phi, \overline{g})$ and $T(w, s, \overline{\pi})$ are the same. In the region $Re(s) > 0$, there are only finitely many of them, all are simple and on the interval $(0, \langle \rho_P, \beta^\vee \rangle)$.*

The last part of the above theorem is due to the fact that the intertwining operator $T(w, s, \overline{\pi})$ figures itself in the computation of the constant term of Eisenstein series, which has the same singularities as the Eisenstein series $E(s, \overline{\pi}, \phi, \overline{g})$.

By definition, the constant term of $E(s, \overline{\pi}, \phi, \overline{g})$ along a parabolic subgroup $\mathbb{P}_1 = \mathbb{M}_1\mathbb{U}_1$ in general is defined by

$$E_{P_1}(s, \overline{\pi}, \phi, \overline{g}) = \int_{\mathbb{U}_1(F)\backslash \mathbb{U}_1(\mathbb{A}_F)} E(s, \overline{\pi}, \phi, u\overline{g}) du,$$

where for the global integration we use the Tamagawa measure normalized such that $F \backslash \mathbb{A}_F$ has measure 1.

Recall that the maximal parabolic $\mathbb{P}$ associated with $\Delta \backslash \{\beta\}$ is called self-associated if $\mathbf{w}(\Delta \backslash \{\beta\}) = \Delta \backslash \{\beta\}$ (cf. Definition 5.2.1).

Theorem 5.2.3 ([MW95, §II.1.7]). *We have the following*

(1) If $\mathbb{P}_1$ is neither $\mathbb{P}$ nor the parabolic $\mathbb{P}'$ associated with $\mathbf{w}(\Delta\backslash\{\beta\}) \subseteq \Delta$, then

$$E_{P_1}(s, \overline{\pi}, \phi, \overline{g}) = 0.$$

Therefore, the interesting cases are $E_P(s, \overline{\pi}, \phi, \overline{g})$ and $E_{P'}(s, \overline{\pi}, \phi, \overline{g})$. For this we have

(2) If $\mathbb{P}$ is self-associated, i.e. $\mathbb{P} = \mathbb{P}'$, then

$$E_P(s, \overline{\pi}, \phi, \overline{g}) = E_{P'}(s, \overline{\pi}, \phi, \overline{g}) = \phi_s(\overline{g}) + T(w, s, \overline{\pi})\phi_s(\overline{g}),$$

where $\phi_s(\overline{g}) = \phi(\overline{g}) \cdot \boldsymbol{\delta}^{s+\rho_P}$.

(2)' If $\mathbb{P}$ is not self-associated, i.e. $\mathbb{P} \neq \mathbb{P}'$, then the two cases of interest are given by

$$E_P(s, \overline{\pi}, \phi, \overline{g}) = \phi_s(\overline{g}),$$
$$E_{P'}(s, \overline{\pi}, \phi, \overline{g}) = \phi_s(\overline{g}) + T(w, s, \overline{\pi})\phi_s(\overline{g}).$$

The poles of $E(s, \overline{\pi}, \phi, \overline{g})$ agree with those of $T(w, s, \overline{\pi})$ by Theorem 5.2.2. The intertwining operator has a tensor product decomposition

$$T(w, s, \overline{\pi}) = \bigotimes_v T(\widetilde{w}_v, s, \overline{\pi}_v) : \quad \bigotimes_v I(s, \overline{\pi}_v) \longrightarrow \bigotimes_v I(s, {}^{\widetilde{w}_v}\overline{\pi}_v).$$

For finite set $S \subseteq |F|$ big enough and for all $v \notin S$, we have $\widetilde{w}_v = \overline{w}_v$ and thus the operator $T(\overline{w}_v, s, \overline{\pi}_v)$ intertwins between unramified representations. We computed in previous chapter the coefficient $c(\overline{w}_v, s, \overline{\pi}_v)$ such that $T(\overline{w}_v, s, \overline{\pi}_v) f_{\overline{\pi}_v} = c(\overline{w}_v, s, \overline{\pi}_v) f_{\overline{w}_v \overline{\pi}_v}$. By applying the results in Theorem 4.4.5 to $\boldsymbol{\delta}_v^s \otimes \overline{\pi}_v$ we get:

Theorem 5.2.4. *Let $f = \bigotimes_{v\notin S} f_{\overline{\pi}_v} \otimes \bigotimes_{v\in S} f_v \in \overline{\sigma}$. The global intertwining operator $T(w, s, \overline{\pi})f$ is then given by*

$$T(w, s, \overline{\pi})f = \prod_{i=1}^{m} \frac{L^S(n_\beta i \cdot s, \overline{\pi}, Ad_i)}{L^S(1 + n_\beta i \cdot s, \overline{\pi}, Ad_i)} \bigotimes_{v\notin S} f_{\overline{w}_v\overline{\pi}_v} \otimes \bigotimes_{v\in S} T(\widetilde{w}_v, s, \overline{\pi}_v) f_v.$$

Here $L^S(s, \overline{\pi}, Ad_i)$ is the automorphic partial L-function (cf. section 5.1) associated with the adjoint representations $Ad_{\overline{\mathfrak{u}}^\vee} = \bigoplus_{i=1}^m Ad_i$ of ${}^L\overline{M}$ on $\overline{\mathfrak{u}}^\vee$. More explicitly, it is given by

$$L^S(s, \overline{\pi}, Ad_i) = \prod_{v\notin S} L(s, \overline{\pi}_v, Ad_i),$$

where the local L-function $L(s, \overline{\pi}_v, Ad_i)$ is the one determined in Theorem 4.4.5.

Proof. Observe that the adjoint representation $Ad_{\overline{\mathfrak{u}}^\vee}$ of the global ${}^L\overline{M}$ on $\overline{\mathfrak{u}}^\vee$ restricts to a representation of the local ${}^L\overline{M}_v$, which is just the adjoint representation of ${}^L\overline{M}_v$ on $\overline{\mathfrak{u}}^\vee$. They both factor through the complex adjoint representation $Ad^{\mathbf{C}}$ of $\overline{M}^\vee$. We will use $Ad = \bigoplus_{i=1}^m Ad_i$ to represent both local and global situations, and no confusion will arise.

Thus, in view of Theorem 4.4.5 it suffices to show that for almost all v the equality $L(0, \boldsymbol{\delta}_v^s \otimes \overline{\pi}_v, Ad_i) = L(n_\beta i \cdot s, \overline{\pi}_v, Ad_i)$ holds. Write $\mathbb{T}^\dagger(\mathbb{A}_F)$ for the image of $Z(\overline{\mathbb{T}}(\mathbb{A}_F))$ in $\mathbb{T}(\mathbb{A}_F)$. Consider the character

$$\chi_{\boldsymbol{\delta}^s}: \quad \mathbb{T}^\dagger(\mathbb{A}_F) \hookrightarrow \mathbb{T}(\mathbb{A}_F) \hookrightarrow \mathbb{M}(\mathbb{A}_F) \xrightarrow{\boldsymbol{\delta}^s} \mathbf{C}^\times,$$

which can be decomposed as $\chi_{\boldsymbol{\delta}^s} = \bigotimes_v \chi_{\boldsymbol{\delta}_v^s}$. We then have $I(s, \overline{\chi}) \simeq I(\overline{\chi} \otimes \chi_{\boldsymbol{\delta}^s})$ and thus $I(s, \overline{\pi}) \hookrightarrow I(\overline{\chi} \otimes \chi_{\boldsymbol{\delta}^s})$. Therefore we have locally for $v \notin S$,

$$\begin{aligned} L(0, \boldsymbol{\delta}_v^s \otimes \overline{\pi}_v, Ad_i) &= \prod_{\substack{\alpha \in \Psi^+ \\ \langle \beta_P/n_\beta, \alpha_{[n]}^\vee \rangle = i}} L(0, Ad_\alpha \circ \rho_{\overline{\chi}_v \otimes \chi_{\boldsymbol{\delta}_v^s}}) \\ &= \prod_{\substack{\alpha \in \Psi^+ \\ \langle \beta_P/n_\beta, \alpha_{[n]}^\vee \rangle = i}} \frac{1}{1 - \overline{\chi}_v \otimes \chi_{\boldsymbol{\delta}_v^s}\left(\overline{h}_\alpha(\varpi_v^{n_\alpha})\right)} \\ &= \prod_{\substack{\alpha \in \Psi^+ \\ \langle \beta_P/n_\beta, \alpha_{[n]}^\vee \rangle = i}} \frac{1}{1 - \overline{\chi}_v\left(\overline{h}_\alpha(\varpi^{n_\alpha})\right) \cdot |\varpi_v^{\langle \beta_P, \alpha_{[n]}^\vee \rangle}|_F^s} \\ &= \prod_{\substack{\alpha \in \Psi^+ \\ \langle \beta_P/n_\beta, \alpha_{[n]}^\vee \rangle = i}} L(n_\beta i \cdot s, Ad_\alpha \circ \rho_{\overline{\chi}_v}), \end{aligned}$$

which is clearly equal to $L(n_\beta i \cdot s, \overline{\pi}_v, Ad_i)$. The proof is completed. □

In view of Theorem 5.2.2, we immediately have

Corollary 5.2.5. *The product of the partial L-functions*

$$\prod_{i=1}^m \frac{L^S(n_\beta i \cdot s, \overline{\pi}, Ad_i)}{L^S(1 + n_\beta i \cdot s, \overline{\pi}, Ad_i)}$$

has meromorphic continuation to the whole complex plane $\mathbf{C}$. *In particular, if* $m = 1$, *the L-function* $L^S(s, \overline{\pi}, Ad)$ *has meromorphic continuation to* $\mathbf{C}$.

Remark 5.2.6. If $m \geq 2$, one would like to show the meromorphic continuation of each $L^S(s, \overline{\pi}, Ad_i)$. However, in this case, the induction method for linear algebraic groups as in [Sha88, Lm. 4.2] or [Sha90, Prop. 4.1] can not be applied directly. The proof of the induction lemma mentioned in these references is largely due to a classification of the pair of reductive group and the Levi subgroups of its maximal parabolic subgroups. Such classification is lacking in the covering group setting. It would be interesting to see how this and other analytic properties of the individual L-functions above could be achieved, by either using an analogous method or alternatives.

5.3 The residual spectrum for $\overline{\mathbb{SL}}_2(\mathbb{A}_F)$

In this section, we determine completely the residual spectrum for $\overline{\mathbb{SL}}_2(\mathbb{A}_F)$ associated with arbitrary $n \in \mathbf{N}_{\geq 1}$ and quadratic form Q on $Y = Y^{sc}$ of $\mathbb{SL}_2$.

Therefore, we fix n such that $\mu_n \subseteq F^\times$. Let $\alpha^\vee$ be the positive coroot, then Q is uniquely determined by $Q(\alpha^\vee) \in \mathbf{Z}$. We can readily compute the complex dual group $\overline{SL}_2^\vee$ and the L-group ${}^L\overline{SL}_2$. There are two cases accordingly.

(1) $\gcd(n_\alpha, 2) = 1$, then $\overline{T}^\vee \simeq \mathbf{C}^\times$ and $\overline{SL}_2^\vee = \mathbb{PGL}_2(\mathbf{C})$. By construction there are canonical isomorphisms ${}^L\overline{T} \simeq \mathbf{C}^\times \times \mathrm{W}_F$ and ${}^L\overline{SL}_2 \simeq \mathbb{PGL}_2(\mathbf{C}) \times \mathrm{W}_F$.

(2) $\gcd(n_\alpha, 2) = 2$, then $\overline{SL}_2^\vee = \mathbb{SL}_2(\mathbf{C})$. In this case, the L-group ${}^L\overline{SL}_2$ is isomorphic to $\mathbb{SL}_2(\mathbf{C}) \times \mathrm{W}_F$, but not canonically so.

We obtain the global n-covering:

$$\mu_n \hookrightarrow \overline{\mathbb{SL}}_2(\mathbb{A}_F) \twoheadrightarrow \mathbb{SL}_2(\mathbb{A}_F)\,.$$

Let $\mathbb{P} = \mathbb{B} = \mathbb{T}\mathbb{N}$ be the parabolic (Borel) subgroup of $\mathbb{SL}_2$. Let $\overline{\pi}$ be a genuine irreducible representation of $\overline{\mathbb{T}}(\mathbb{A}_F)$. By [We14] or [We14-2, Thm. 4.15], the group $\mathbb{T}(F)Z(\overline{\mathbb{T}}(\mathbb{A}_F))$ is a maximal abelian subgroup of $\overline{\mathbb{T}}(\mathbb{A}_F)$, and it follows that

$$\overline{\pi} \simeq \mathrm{Ind}_{\mathbb{T}(F)Z(\overline{\mathbb{T}}(\mathbb{A}_F))}^{\overline{\mathbb{T}}(\mathbb{A}_F)} \overline{\chi}, \tag{5.1}$$

where $\overline{\chi}$ is a global genuine unitary character of $\mathbb{T}(F)Z(\overline{\mathbb{T}}(\mathbb{A}_F))$ that is trivial on the intersection $\mathbb{T}(F) \cap Z(\overline{\mathbb{T}}(\mathbb{A}_F))$. Since there is the surjection

$$\overline{\mathbb{T}}_{Q,n}(\mathbb{A}_F) \twoheadrightarrow Z(\overline{\mathbb{T}}(\mathbb{A}_F)),$$

the character $\overline{\chi}$ could be viewed as one of $\overline{\mathbb{T}}_{Q,n}(\mathbb{A}_F)$ which descends.

We have $\overline{\pi} = \bigotimes_v \overline{\pi}_v$, where $\overline{\pi}_v = \mathrm{Ind}_{Z(\overline{T}_v)}^{\overline{T}_v} \overline{\chi}_v$, and for almost all v it is unramified with unramified character $\overline{\chi}_v$. We could compute the Satake parameter $\rho_{\overline{\chi}_v}(\mathrm{Frob}_v) \in {}^L\overline{SL}_2$ and there are two cases as above.

(1) $\gcd(n_\alpha, 2) = 1$, then the Satake parameter is given by

$$\rho_{\overline{\chi}_v}(\mathrm{Frob}_v) = \left(\begin{bmatrix} \overline{\chi_v}\big(\overline{h}(\varpi_v^{n_\alpha})\big) & 0 \\ 0 & 1 \end{bmatrix}, \mathrm{Frob}_v \right) \in {}^L\overline{T}_v \subseteq \mathbb{PGL}_2(\mathbf{C}) \times \mathrm{W}_{F_v}.$$

(2) $\gcd(n_\alpha, 2) = 2$. Identify $\rho_{\overline{\chi}_v}$ with a splitting of $E_{\overline{T}_v}$ over F_v, and the Satake parameter is determined by $\rho_{\overline{\chi}_v}(\varpi_v) \in E_{\overline{T}_v}$. Meanwhile, we may identify $E_{\overline{T}_v}$ with $\overline{T}_v^\vee \times E_{\overline{G}_v} / \nabla Z(\overline{SL}_2^\vee)$ as in Proposition 2.4.8, with respect to which we write $\rho_{\overline{\chi}_v}(\varpi_v) = (s_v, e_v) \in \overline{T}_v^\vee \times E_{\overline{G}_v} / \nabla Z(\overline{SL}_2^\vee)$. By tracing through the discussion from Proposition 2.4.8 to Corollary 2.4.9, it is possible to obtain explicit form of s_v and e_v. However, for our purpose, it suffices to note that the element (not uniquely determined, only so in the quotient of $\mathbb{SL}_2(\mathbf{C})$ modulo its center)

$$s_v = \begin{bmatrix} z_v & 0 \\ 0 & z_v^{-1} \end{bmatrix} \in \mathbb{SL}_2(\mathbf{C})$$

always has the property that $z_v^2 = \overline{\chi_v}\big(\overline{h}(\varpi_v^{n_\alpha})\big)$.

The fundamental weight for $\mathbb{P}$ is $\beta_P = \alpha/2$. Considering the induced representation $\overline{\sigma} = I(s, \overline{\pi})$, we form the Eisenstein series $E(s, \overline{\pi}, \phi, \overline{g})$ as before.

Clearly in our case $\mathbb{P}$ is self-associated, and thus the pole of $E(s, \overline{\pi}, \phi, \overline{g})$ agrees with that of $E_P(s, \overline{\pi}, \phi, \overline{g})$, which is given by (cf. Theorem 5.2.3)

$$E_P(s, \overline{\pi}, \phi, \overline{g}) = \phi_s(\overline{g}) + T(w, s, \overline{\pi})\phi_s(\overline{g}).$$

Let $f = \bigotimes_{v \notin S} f_{\overline{\pi}_v} \otimes \bigotimes_{v \in S} f_v \in \overline{\sigma}$, by Theorem 5.2.4, we have

$$T(w, s, \overline{\pi})f = \frac{L^S(n_\alpha s, \overline{\pi}, Ad)}{L^S(1 + n_\alpha s, \overline{\pi}, Ad)} \bigotimes_{v \notin S} f_{\overline{w}_v \overline{\pi}_v} \otimes \bigotimes_{v \in S} T(\widetilde{w}_v, s, \overline{\pi}_v) f_v.$$

Also,

$$L^S(n_\alpha s, \overline{\pi}, Ad) = \prod_{v \notin S} L(n_\alpha s, \overline{\pi}_v, Ad) = \prod_{v \notin S} \frac{1}{1 - q_v^{-n_\alpha s} \overline{\chi}(\overline{h}_\alpha(\varpi_v^{n_\alpha}))}.$$

To state it in another way, consider the diagram below

$$\begin{tikzcd}
\mu_n \arrow[r, hook] & Z(\overline{\mathbb{T}}(\mathbb{A}_F)) \arrow[r, two heads] & \mathbb{T}^\dagger(\mathbb{A}_F) \\
& & \mathbb{T}^\ddagger(\mathbb{A}_F) \arrow[u, hook] \arrow[ul, dashed, "s_{\mathbb{A}_F}"] \\
& & \mathbb{T}^{sc}_{Q,n}(\mathbb{A}_F). \arrow[u, "(-)^{n_\alpha}"']
\end{tikzcd}$$

First we explain the groups and maps in the diagram. The group $\mathbb{T}^\dagger(\mathbb{A}_F)$ (respectively $\mathbb{T}^\ddagger(\mathbb{A}_F)$) is the image of $\mathbb{T}_{Q,n}(\mathbb{A}_F)$ (resp. $\mathbb{T}^{sc}_{Q,n}(\mathbb{A}_F)$) in $\mathbb{T}(\mathbb{A}_F)$ induced from the embedding $i_{Q,n} : Y_{Q,n} \hookrightarrow Y$ (resp. $i^{sc}_{Q,n} : Y^{sc}_{Q,n} \hookrightarrow Y$) of the rank-one lattices. If we identify $\mathbb{T}^{sc}_{Q,n}(\mathbb{A}_F)$ and $\mathbb{T}(\mathbb{A}_F)$ both with $\mathbb{A}_F^\times$, then the induced map $\mathbb{T}^{sc}_{Q,n}(\mathbb{A}_F) \longrightarrow \mathbb{T}(\mathbb{A}_F)$ is simply the n_α-power, whence the notation $(-)^{n_\alpha}$ in the diagram.

Moreover, the extension $Z(\overline{\mathbb{T}}(\mathbb{A}_F))$ of $\mathbb{T}^\dagger(\mathbb{A}_F)$ splits over $\mathbb{T}^\ddagger(\mathbb{A}_F) \subseteq \mathbb{T}^\dagger(\mathbb{A}_F)$, which is given by

$$\mathbf{s}_{\mathbb{A}_F} : \quad \big((a_v)_v\big)^{n_\alpha} \longmapsto \big(\overline{h}_\alpha(a_v^{n_\alpha})\big)_v$$

for any $(a_v)_v \in \mathbb{A}_F^\times \simeq \mathbb{T}^{sc}_{Q,n}(\mathbb{A}_F)$.

Using this splitting, let $\chi^{sc} = \bigotimes_v \chi_v^{sc}$ be the following composition

$$\chi^{sc} = \overline{\chi} \circ s_{\mathbb{A}_F} \circ (-)^{n_\alpha} : \quad \mathbb{A}_F^\times \twoheadrightarrow \mathbb{T}^\ddagger(\mathbb{A}_F) \hookrightarrow Z(\overline{\mathbb{T}}(\mathbb{A}_F)) \xrightarrow{\overline{\chi}} \mathbf{C}^\times.$$

Then χ^{sc} is a unitary Hecke character and it follows $L(n_\alpha s, \overline{\pi}_v, Ad) = L(n_\alpha s, \chi_v^{sc})$ for $v \notin S$. Fix an additive character $\psi = \bigotimes_v \psi_v : \mathbb{A}_F \longrightarrow \mathbf{C}$. Then we can normalize and rewrite above formula for $T(w, s, \overline{\pi})$ as

$$T(w, s, \overline{\pi})f = \frac{L(n_\alpha s, \chi^{sc})}{L(1 + n_\alpha s, \chi^{sc})} \bigotimes_{v \notin S} f_{\overline{w}_v \overline{\pi}_v} \otimes \bigotimes_{v \in S} N(\widetilde{w}_v, s, \overline{\pi}_v) f_v,$$

where for all $v \in |F|$,

$$N(\widetilde{w}_v, s, \overline{\pi}_v) f_v = \frac{L(1 + n_\alpha s, \chi_v^{sc})}{L(n_\alpha s, \chi_v^{sc}) \varepsilon(n_\alpha s, \chi_v^{sc}, \psi_v)} T(\widetilde{w}_v, s, \overline{\pi}_v) f_v.$$

To determine the residual spectrum, we require

Lemma 5.3.1. *For all $v \in S$, the normalized operator $N(\widetilde{w}_v, s, \overline{\pi}_v)$ is holomorphic and nonvanishing for $Re(s) > 0$.*

Proof. It is easy to check the nonvanishing of $T(\widetilde{w}_v, s, \overline{\pi}_v)$ and $L(s, \chi_v^{sc})$ for $\mathrm{Re}(s) > 0$. Moreover, since χ_v^{sc} is unitary, the local $L(s, \chi_v^{sc})$ contains no poles.

The gives the desired result. □

Also,

Lemma 5.3.2. *For all $v \in |F|$, the images of $N(\widetilde{w}_v, 1/n_\alpha, \overline{\pi}_v)$ and $T(\widetilde{w}_v, 1/n_\alpha, \overline{\pi}_v)$ are both irreducible and nonzero.*

Proof. The normalizing factor $L(1 + n_\alpha s, \chi_v^{sc}) L(n_\alpha s, \chi_v^{sc})^{-1} \varepsilon(n_\alpha s, \chi_v^{sc}, \psi_v)^{-1}$ has no pole or zero at $s = 1/n_\alpha$, therefore it suffices to prove the lemma for $T(\widetilde{w}_v, 1/n_\alpha, \overline{\pi}_v)$. However, since $s = 1/n_\alpha > 0$, it follows from the Langlands classification theorem (cf. [BaJa13, Thm. 4.1]) that the image of $T(\widetilde{w}_v, 1/n_\alpha, \overline{\pi}_v)$ is irreducible and equal to the Langlands quotient of $I(s, \overline{\pi})$. □

In fact, one can show that $s = 1/n_\alpha$ is a reducibility point for the local induced representation, though we do not need such fact here. Now denote by $\mathcal{J}(1/n_\alpha, \overline{\pi}_v)$ the irreducible images of $N(\widetilde{w}_v, 1/n_\alpha, \overline{\pi}_v)$. If $\overline{\chi}$ is such that $\chi^{sc} = \mathbb{1}$, then there is a simple pole of $E_P(s, \overline{\pi}, \phi, \overline{g})$ at $s = 1/n_\alpha$ which arises from the Hecke L-series $L(n_\alpha s, \chi^{sc})$.

Under this condition,

$$\mathrm{Res}_{s=1/n_\alpha} E_P(s, \overline{\pi}, \phi, \overline{g}) = \bigotimes_v \mathcal{J}(1/n_\alpha, \overline{\pi}_v).$$

Taking constant terms commutes with taking residues:

$$\begin{tikzcd}
I(s,\overline{\pi}) \arrow[r, "\phi_s \mapsto \mathrm{Res}_{s_0} E(\phi_s)"] \arrow[dr, "\phi_s \mapsto \mathrm{Res}_{s_0} E_P(\phi_s)"'] & \mathcal{A}^2(\mathbb{SL}_2(F) \backslash \overline{\mathbb{SL}}_2(\mathbb{A}_F)) \arrow[d, "\text{take const. term}"] \\
& \mathcal{A}(\mathbb{N}(\mathbb{A}_F)\mathbb{T}(F) \backslash \overline{\mathbb{SL}}_2(\mathbb{A}_F)).
\end{tikzcd}$$

Moreover, the right vertical map is injective on the image of the top horizontal map (cf. [MW95, pg. 45]). Thus, we may identify $\mathrm{Res}_{s=1/n_\alpha} E(s, \overline{\pi}, \phi, \overline{g})$ with $\mathrm{Res}_{s=1/n_\alpha} E_P(s, \overline{\pi}, \phi, \overline{g})$ as abstract representations of $\overline{\mathbb{SL}}_2(\mathbb{A}_F)$. Since $\mathbf{w}(s\beta_P) = \mathbf{w}(\alpha/2n_\alpha) = -\alpha/2n_\alpha$, it follows from the Langlands' criterion (cf. [MW95, §I.4]) that these residues are square integrable.

Let $\mathfrak{A}$ be the collection of unitary characters $\overline{\chi} : Z(\overline{\mathbb{T}}(\mathbb{A}_F)) \longrightarrow \mathbf{C}^\times$ such that

$$\begin{cases} (1) & \overline{\chi}(\overline{g}) = 1 \text{ for all } \overline{g} \in \mathbb{T}(F) \cap Z(\overline{\mathbb{T}}(\mathbb{A}_F)), \\ (2) & \chi^{sc} = \mathbb{1}. \end{cases}$$

Let $\overline{\pi} = i(\overline{\chi})$ be the globall induced representation of $\overline{\mathbb{T}}(\mathbb{A}_F)$ as in (5.1). Write $\mathcal{J}(1/n_\alpha, \overline{\pi}) = \bigotimes_v \mathcal{J}(1/n_\alpha, \overline{\pi}_v)$. Let $L^2_{res}(\mathbb{SL}_2(F)\backslash\overline{\mathbb{SL}}_2(\mathbb{A}_F))$ denote the residual spectrum. Then we have

Theorem 5.3.3. *The representation $\mathcal{J}(1/n_\alpha, \overline{\pi})$ occurs in the residual spectrum of $\overline{\mathbb{SL}}_2(\mathbb{A}_F)$ for such $\overline{\mathbb{SL}}_2$. In fact, we have the decomposition*

$$L^2_{res}(\mathbb{SL}_2(F)\backslash\overline{\mathbb{SL}}_2(\mathbb{A}_F)) = \bigoplus_{\substack{\overline{\pi}=i(\overline{\chi}) \\ \overline{\chi}\in\mathfrak{A}}} \mathcal{J}(1/n_\alpha, \overline{\pi}).$$

In view of the existence of global Weyl-group invariant characters as discussed in section 3.6, we could have an alternative description of the condition $\mathfrak{A}$ and thus also the residual spectrum.

For simplicity, we restrict ourselves to consider the case when $n|2Q(\alpha^\vee)$, under which assumption n_α could be equal to either 1 or 2. This covers the linear case when $n = 1$ and the classical metaplectic double cover of $\mathbb{SL}_2(\mathbb{A}_F)$ when $n = 2, Q(\alpha^\vee) = 1$.

Though the general case could be considered in a similar way, with above assumption we see that $Y_{Q,n} = Y$ and thus $\overline{\mathbb{T}}(\mathbb{A}_F)$ is abelian, i.e. $Z(\overline{\mathbb{T}}(\mathbb{A}_F)) = \overline{\mathbb{T}}(\mathbb{A}_F)$. Therefore the first condition on $\overline{\chi} \in \mathfrak{A}$ is equivalent to

$$(1)' \quad \overline{\chi} \text{ is a unitary Hecke character on } \mathbb{T}(F)\backslash\overline{\mathbb{T}}(\mathbb{A}_F).$$

For (2), we fix an additive character $\psi = \bigotimes_v \psi_v : \mathbb{A}_F \longrightarrow \mathbf{C}^\times$. Then we obtain a Weyl invariant genuine character $\overline{\chi}_\psi = \bigotimes \overline{\chi}_{\psi_v} : \overline{\mathbb{T}}(\mathbb{A}_F) \longrightarrow \mathbf{C}^\times$ from section 3.3. In our case, the local Weyl invariant genuine character $\overline{\chi}_{\psi_v}$ is determined by

$$\overline{\chi}_{\psi_v} : \quad \overline{T}_v \longrightarrow \mathbf{C}^\times, \qquad (1, \alpha^\vee \otimes a_v) \longmapsto \gamma_{\psi_v}(a_v)^{2(n_\alpha - 1)Q(\alpha^\vee)/n},$$

where $\overline{T}_v := \overline{\mathbb{T}}(F_v)$ and γ_{ψ_v} is the Weil index. In fact, $\overline{\chi}_\psi$ is an automorphic character, i.e. trivial on $\mathbb{T}(F)$. Then with respect to $\overline{\chi}_\psi$, any unitary genuine character $\overline{\chi}$ can be written as

$$\overline{\chi} = \overline{\chi}_\psi \cdot \chi \text{ for some unitary } \chi \in \mathrm{Hom}(\mathbb{T}(F)\backslash\mathbb{T}(\mathbb{A}_F), \mathbf{C}^\times).$$

If we identify $\mathbb{T}(\mathbb{A}_F)$ with $\mathbb{A}_F^\times$, then we could write $\chi : F^\times\backslash\mathbb{A}_F^\times \longrightarrow \mathbf{C}^\times$ which is a unitary Hecke-character.

Keep notations as before, in the local setting the splitting of $\overline{T}_v$ over $T_v^\ddagger$ is given by

$$T_v^\ddagger \longrightarrow \overline{T}_v, \qquad \alpha^\vee_{[n]} \otimes a_v \in T_v^\ddagger \longmapsto \overline{h}_\alpha(a_v^{n_\alpha}) \in \overline{T}_v,$$

where in fact $\overline{h}_\alpha(a_v^{n_\alpha}) = (1, \alpha_{[n]}^\vee \otimes a_v)$ in terms of the coordinates on $\overline{T}_v$.

Note that by the defining property of $\overline{\chi}_{\psi_v}$, it is trivial on $\overline{h}_\alpha(a_v^{n_\alpha})$. Therefore for all $a_v \in F_v^\times$,

$$\begin{aligned}\chi_v^{sc}(a_v) &= \overline{\chi}_v\big(\overline{h}_\alpha(a_v^{n_\alpha})\big)\\ &= \overline{\chi}_{\psi_v}\big(\overline{h}_\alpha(a_v^{n_\alpha})\big)\cdot\chi_v\big(h_\alpha(a_v^{n_\alpha})\big)\\ &= \chi_v\big(h_\alpha(a_v^{n_\alpha})\big)\\ &= \chi_v^{n_\alpha}(a_v)\end{aligned}$$

Thus globally $\chi^{sc} = \chi^{n_\alpha}$. The second condition for $\overline{\chi} \in \mathfrak{A}$ is then equivalent to

$$(2)' \quad \chi^{n_\alpha} = \mathbb{1}.$$

To summarize,

Theorem 5.3.4. *Suppose $n|2Q(\alpha^\vee)$. Keep notations as above, and denote by $\mathfrak{A}'$ characters χ of $F^\times\backslash\mathbb{A}_F^\times = \mathbb{T}(F)\backslash\mathbb{T}(\mathbb{A}_F)$ satisfying $\chi^{n_\alpha} = \mathbb{1}$. Then we have the decomposition of the residual spectrum*

$$L^2_{res}(\mathbb{SL}_2(F)\backslash\overline{\mathbb{SL}}_2(\mathbb{A}_F)) = \bigoplus_{\chi\in\mathfrak{A}'}\mathcal{J}(1/n_\alpha, \overline{\chi}_\psi\otimes\chi).$$

5.4 The residual spectrum of $\overline{\mathbb{GL}}_2(\mathbb{A}_F)$

Since the derived group of $\mathbb{GL}_2$ is $\mathbb{SL}_2$ which is simply-connected, any $(D,\eta) \in \mathsf{Bis}^Q_{\mathbb{GL}_2}$ is isomorphic to a fair $(D, \mathbb{1})$, cf. (2.6). Therefore, without loss of generality we work with a fair $(D, \mathbb{1})$ and consider $\overline{\mathbb{GL}}_2$ incarnated by such fair object.

Let e_1 and e_2 be two $\mathbf{Z}$-basis of the cocharacter group Y of $\mathbb{GL}_2$ such that the coroot is $\alpha^\vee = e_1 - e_2$. Any Weyl-invariant bilinear form B_Q is uniquely determined by the two numbers $B_Q(e_1,e_1) = 2Q(e_1) = 2Q(e_2) = B_Q(e_2,e_2)$ and $B_Q(e_1,e_2) = B_Q(e_2,e_1)$. Write $Q(e_1) = \mathbf{p} \in \mathbf{Z}$, $B_Q(e_1,e_2) = \mathbf{q} \in \mathbf{Z}$, then B_Q is determined by the matrix $B(e_i,e_j), i,j = 1,2$:

$$B_Q(e_i,e_j) = \begin{bmatrix} 2\mathbf{p} & \mathbf{q} \\ \mathbf{q} & 2\mathbf{p}\end{bmatrix}.$$

It follows $Q(\alpha^\vee) = 2\mathbf{p} - \mathbf{q}$. Fix a natural number $n \in \mathbf{N}_{\geq 1}$, and thus by definition

$$n_\alpha = \frac{n}{\gcd(n, 2\mathbf{p}-\mathbf{q})}.$$

Define $Y_{Q,n}$ and $Y_{Q,n}^{sc}$ as before with $Y_{Q,n}^{sc}$ generated by $\alpha_{[n]}^\vee$.

Note that in general the complex dual group $\overline{GL}_2^\vee$ may not be equal to $\mathbb{GL}_2(\mathbf{C})$. For instance, consider the case where $\mathbf{q} = 2\mathbf{p}$ and $n = 2\mathbf{q}$. Then $Q(\alpha^\vee) = 0$ and thus $n_\alpha = 1$. We see that

$$Y_{Q,n} = \big\{k_1e_1 + k_2e_2 : k_1 \equiv k_2 \mod 2\big\}.$$

The root data of $\overline{GL}_2^\vee$ is given by $\left(Y_{Q,n},\ \{\alpha_{[n]}^\vee\},\ \mathrm{Hom}(Y_{Q,n},\mathbf{Z}),\ \{\alpha/n_\alpha\}\right)$. However, for this example it is not difficult to see that $(2n_\alpha)^{-1}\alpha \in \mathrm{Hom}(Y_{Q,n},\mathbf{Z})$. In fact, the complex dual group in this case is $\overline{GL}_2^\vee = \mathbb{GL}_2(\mathbf{C})/\mu_2$.

To proceed with the general case, we start with the following lemma.

Lemma 5.4.1. *There exists an element $y_o \in Y_{Q,n}$ such that $\alpha_{[n]}^\vee$ and y_o form a $\mathbf{Z}$-basis of the lattice $Y_{Q,n}$.*

Proof. First, we show that if $k\alpha^\vee \in Y_{Q,n}$ for some $k \in \mathbf{Z}$, then $n_\alpha | k$. If $k\alpha^\vee \in Y_{Q,n}$, then

$$k \cdot B_Q(\alpha^\vee, e_i) \in n\mathbf{Z} \text{ for } i = 1, 2.$$

It follows that n divides $\pm k(2\mathbf{p} - \mathbf{q})$. Therefore $n_\alpha | k$.

Now let y_1, y_2 be a basis of $Y_{Q,n}$ and let $\alpha_{[n]}^\vee = a_1 y_1 + a_2 y_2$ for some $a_i \in \mathbf{Z}$. By above observation, $\gcd(a_1, a_2) = 1$. Write

$$a_1 b_1 + a_2 b_2 = 1, b_i \in \mathbf{Z}.$$

Let $y_o = b_2 y_1 + (-b_1) y_2$. Consider the set $\{\alpha_{[n]}^\vee, y_o\}$. We claim that it forms a basis for $Y_{Q,n}$. It suffices to show $\{\alpha_{[n]}^\vee, y_o\}$ can generate y_1, y_2, and this follows from the following equalities which could be verified easily

$$\begin{cases} y_1 = b_1 \alpha_{[n]}^\vee + a_2 y_o, \\ y_2 = b_2 \alpha_{[n]}^\vee + (-a_1) y_o. \end{cases}$$

The proof is completed. $\square$

With respect to the one-dimensional lattice $Y_{Q,n}^o$ spanned by y_o, we have

$$Y_{Q,n} = Y_{Q,n}^{sc} \oplus Y_{Q,n}^o.$$

Denote by $\mathbb{T}_{Q,n}^{sc}$ and $\mathbb{T}_{Q,n}^o$ the tori corresponding to $Y_{Q,n}^{sc}$ and $Y_{Q,n}^o$ respectively. Then $Z(\overline{\mathbb{T}}(\mathbb{A}_F))$ is equal to the image of $\overline{\mathbb{T}}_{Q,n}^{sc}(\mathbb{A}_F) \times \overline{\mathbb{T}}_{Q,n}^o(\mathbb{A}_F)/\nabla\mu_n$ in $\overline{\mathbb{T}}(\mathbb{A}_F)$. To give a genuine character of $Z(\overline{\mathbb{T}}(\mathbb{A}_F))$ is equivalent to give $\overline{\chi}^{sc} \otimes \overline{\chi}^o$, where $\overline{\chi}^{sc}$ and $\overline{\chi}^o$ are genuine characters of $\overline{\mathbb{T}}_{Q,n}^{sc}(\mathbb{A}_F)$ and $\overline{\mathbb{T}}_{Q,n}^o(\mathbb{A}_F)$ respectively, which both descend to $\overline{\mathbb{T}}(\mathbb{A}_F)$.

Clearly the fundamental weight α_P is equal to $\rho_P = \alpha/2$. Identify s with $\alpha_P \otimes s \in X^*(\mathbb{T})_{\mathbf{C}}$ as in section 4.4.2. Write $\overline{\pi} = i(\overline{\chi}^{sc} \otimes \overline{\chi}^o)$. Define the Eisenstein series $E(s, \overline{\pi}, \phi, \overline{g})$ for the representation $I(s, \overline{\pi})$.

We see that from Theorem 5.2.4 the residue is determined by

$$T(w, s, \overline{\pi}) f = \frac{L^S(n_\alpha s, i(\overline{\chi}^{sc} \otimes \overline{\chi}^o), Ad)}{L^S(1 + n_\alpha s, i(\overline{\chi}^{sc} \otimes \overline{\chi}^o), Ad)} \bigotimes_{v \notin S} f_{\overline{w}_v \overline{\pi}_v} \otimes \bigotimes_{v \in S} T(\widetilde{w}_v, s, \overline{\pi}_v) f_v,$$

where $f = \bigotimes_{v \notin S} f_{\overline{\pi}_v} \otimes \bigotimes_{v \in S} f_v$.

More explicitly,

$$L^S(s, i(\overline{\chi}^{sc} \otimes \overline{\chi}^o), Ad) = \prod_{v \notin S} \frac{1}{1 - q_v^{-s} \cdot \overline{\chi}_v^{sc}(\overline{h}_\alpha(\varpi_v^{n_\alpha}))}.$$

As in the $\mathbb{SL}_2$ case in previous section, let χ^{sc} be the linear character:

$$\chi^{sc} = \bigotimes_v \chi_v^{sc} : \quad \mathbb{A}_F^\times \twoheadrightarrow \mathbb{T}^\ddagger(\mathbb{A}_F) \xhookrightarrow{\mathsf{s}_{\mathbb{A}_F}} \overline{\mathbb{T}}(\mathbb{A}_F) \xrightarrow{\overline{\chi}^{sc} \otimes \overline{\chi}^o} \mathbf{C}^\times,$$

where as before we identify $\mathbb{T}_{Q,n}^{sc}(\mathbb{A}_F)$ with $\mathbb{A}_F^\times$ and $\mathbb{T}^\ddagger(\mathbb{A}_F)$ denotes its image in $\overline{\mathbb{T}}(\mathbb{A}_F)$.

It follows,

$$T(w, s, \overline{\pi}) f = \frac{L^S(n_\alpha s, \chi^{sc})}{L^S(1 + n_\alpha s, \chi^{sc})} \bigotimes_{v \notin S} f_{\overline{w}_v \overline{\pi}_v} \otimes \bigotimes_{v \in S} T(\widetilde{w}_v, s, \overline{\pi}_v) f_v.$$

To determine the residues of $E(s, i(\overline{\chi}^{sc} \otimes \overline{\chi}^o), \phi, \overline{g})$, we follow the proof of the previous section exactly, and details may be omitted here. In particular, we denote by $\mathcal{J}(1/n_\alpha, i(\overline{\chi}_v^{sc} \otimes \overline{\chi}_v^o))$ the irreducible and nonzero image of the certain normalized operator $N(\widetilde{w}_v, 1/n_\alpha, i(\overline{\chi}_v^{sc} \otimes \overline{\chi}_v^o))$. Write $\mathcal{J}(1/n_\alpha, i(\overline{\chi}^{sc} \otimes \overline{\chi}^o)) = \bigotimes_v \mathcal{J}(1/n_\alpha, i(\overline{\chi}_v^{sc} \otimes \overline{\chi}_v^o))$.

Finally, let $\mathfrak{B}$ be the collection of characters $\overline{\chi}^{sc} \otimes \overline{\chi}^o$ of $Z(\overline{\mathbb{T}}(\mathbb{A}_F))$ trivial on $\mathbb{T}(F) \cap Z(\overline{\mathbb{T}}(\mathbb{A}_F))$ such that χ^{sc} defined above is trivial. Then

Theorem 5.4.2. *The residual spectrum $L^2_{res}(\mathbb{GL}_2(F)\backslash\overline{\mathbb{GL}}_2(\mathbb{A}_F))$ has a decomposition of the form*

$$L^2_{res}(\mathbb{GL}_2(F)\backslash\overline{\mathbb{GL}}_2(\mathbb{A}_F)) = \bigoplus_{\overline{\chi}^{sc} \otimes \overline{\chi}^o \in \mathfrak{B}} \mathcal{J}(1/n_\alpha, i(\overline{\chi}^{sc} \otimes \overline{\chi}^o)).$$

5.5 The residual spectrum of $\overline{\mathbb{Sp}}_4(\mathbb{A}_F)$

Let $\Delta = \{\alpha_1, \alpha_2\}$ be two simple roots of $\mathbb{Sp}_4$ with α_1 the the long root. Let Q be the Weyl-invariant quadratic form on $Y = Y^{sc}$ uniquely determined by $Q(\alpha_1^\vee) = 1$. Let $n = 2$, then we obtain the classical metaplectic group

$$\mu_2 \hookrightarrow \overline{\mathbb{Sp}}_4(\mathbb{A}_F) \twoheadrightarrow \mathbb{Sp}_4(\mathbb{A}_F).$$

The residual spectrum $L^2_{res}(\mathbb{Sp}_4(F)\backslash\overline{\mathbb{Sp}}_4(\mathbb{A}_F))$ is completely determined in [Gao12], and therefore we will not give any elaborate discussion here.

However, as an example, we will show that the partial L-functions appearing in the constant terms of Eisenstein series induced from the two maximal parabolic subgroups as in [Gao12] agree with the ones given by Theorem 5.2.4.

Let $\mathbb{P}_j = \mathbb{M}_j\mathbb{U}_j$ be the maximal parabolic subgroups generated by α_j. We may call $\mathbb{P}_2$ and $\mathbb{P}_1$ the Siegel and non-Siegel parabolic subgroups respectively.

In this case, the complex dual group is $\overline{Sp}_4^\vee = \mathbb{Sp}_4(\mathbf{C})$. The complex dual group $\overline{M}_j^\vee$ is contained in some parabolic $\overline{P}_j^\vee = \overline{M}_j^\vee \overline{U}_j^\vee$ generated by the two simple roots $\alpha_{j,[2]}^\vee := 2\alpha_j^\vee/\gcd(2, Q(\alpha_j^\vee))$ of $\overline{Sp}_4^\vee$ respectively, with $\alpha_{1,[2]}^\vee$ being the long root of $\overline{Sp}_4^\vee$. That is, $\overline{P}_1^\vee$ is the non-Siegel parabolic subgroup of $\overline{Sp}_4^\vee$, while $\overline{P}_2^\vee$ the Siegel parabolic.

The non-Siegel parabolic $\mathbb{P}_1$ case

For $j=1$, i.e. the non-Siegel parabolic $\mathbb{P}_1$. Write $\mathbb{M}_1 = \mathbb{GL}_1 \times \mathbb{Sp}_2$, we have

$$\overline{\mathbb{M}}_1(\mathbb{A}_F) \simeq \overline{\mathbb{GL}}_1(\mathbb{A}_F) \times \overline{\mathbb{Sp}}_2(\mathbb{A}_F)/\nabla\mu_2.$$

Any genuine cuspidal representation of $\overline{\mathbb{M}}_1(\mathbb{A}_F)$, by using certain global Weyl-invariant character $\overline{\chi}_\psi$ defined as before, could be identified with $\overline{\chi} \boxtimes \overline{\pi}$, where $\overline{\chi} = \overline{\chi}_\psi \otimes \chi$ is a genuine character of $\overline{\mathbb{GL}}_1(\mathbb{A}_F)$ with χ being a unitary Hecke character and $\overline{\pi}$ a cuspidal representation of the degree two cover $\overline{\mathbb{Sp}}_2(\mathbb{A}_F)$.

Let $I(s, \overline{\chi} \boxtimes \overline{\pi})$ be the induced representation, where $s := (\alpha_1/2 + \alpha_2) \otimes s \in X^*(\mathbb{M}_1)_{\mathbf{C}}$. The Weyl group element of interest is $\mathbf{w} = \mathbf{w}_{\alpha_2}\mathbf{w}_{\alpha_1}\mathbf{w}_{\alpha_2}$.

We have $n_{\alpha_2} = 1$. By Theorem 5.2.4, the partial L-functions which appear in the constant term of Eisenstein series in this case is given by

$$\prod_{i=1}^{m=2} \frac{L^S(n_{\alpha_2} i \cdot s, \overline{\chi} \boxtimes \overline{\pi}, Ad_i)}{L^S(1 + n_{\alpha_2} i \cdot s, \overline{\chi} \boxtimes \overline{\pi}, Ad_i)} = \frac{L^S(s, \chi \times \overline{\pi})}{L^S(1 + s, \chi \times \overline{\pi}))} \cdot \frac{L^S(2s, \chi^2)}{L^S(1 + 2s, \chi^2)}.$$

Here the Rankin-Selberg product $L^S(s, \chi \times \overline{\pi})$, or more precisely its local counterpart, is given in [Szp11, §7]. It agrees with [Gao12, §4.2].

The Siegel parabolic $\mathbb{P}_2$ case

For $j=2$, the Siegel parabolic $\mathbb{P}_2$ has Levi $\mathbb{M}_2 \simeq \mathbb{GL}_2$. Therefore,

$$\overline{\mathbb{M}}_2(\mathbb{A}_F) \simeq \overline{\mathbb{GL}}_2(\mathbb{A}_F).$$

Using an additive character ψ of $\mathbb{A}_F$, there is a genuine character $\overline{\mathbb{GL}}_2(\mathbb{A}_F)$ which is also denoted by $\overline{\chi}_\psi$ by abuse of notation. Any cuspidal representation $\overline{\pi}$ of $\overline{\mathbb{GL}}_2(\mathbb{A}_F)$ could be written as $\overline{\pi} = \pi \otimes \overline{\chi}_\psi$, where π is a cuspidal representation of $\mathbb{GL}_2(\mathbb{A}_F)$.

Identify s with $(\alpha_1 + \alpha_2) \otimes s \in X^*(\mathbb{M}_2)_{\mathbf{C}}$ and Let $I(s, \overline{\pi})$ be the induced representation. We will consider the intertwining operator for $\mathbf{w} = \mathbf{w}_{\alpha_1}\mathbf{w}_{\alpha_2}\mathbf{w}_{\alpha_1}$.

Note $n_{\alpha_1} = 2$. By Theorem 5.2.4, the partial L-function that appears in the constant term of Eisenstein series in this case is given by

$$\prod_{i=1}^{m=1} \frac{L^S(n_{\alpha_1} i \cdot s, \overline{\pi}, Ad_i)}{L^S(1 + n_{\alpha_1} i \cdot s, \overline{\pi}, Ad_i)} = \frac{L^S(2s, \pi, \mathrm{Sym}^2)}{L^S(1 + 2s, \pi, \mathrm{Sym}^2)},$$

which also agrees with [Gao12, §3.2].

Chapter 6

Discussions and future work

There are questions which have been left with inconclusiveness in our discussion on the splitting of L-groups, the computation of the GK formula and the interpretation in terms of Langlands-Shahidi type L-functions. For instance, one may wonder about the characterization of conditions on the existence of distinguished characters. Also, as in Remark 5.2.6, it is not completely satisfactory to have only the meromorphic continuation of the whole product of partial L-functions in Theorem 5.2.4.

There are also problems and questions that could be imposed immediately based on the discussions in previous chapters. It is expected that some of these problems could be addressed without much difficulty by extending our argument, while others might stimulate for investigations which can be completed only as a long-time project. For the latter, one could readily impose problems by taking an analogy and comparing with the linear algebraic case, as in some sense every question that exists for linear algebraic groups could be asked for BD-type covering groups with proper modifications.

In the following, we will list some questions and problems, which are by no means exhaustive. We only give a glimpse of such problems and questions reflecting our current interest.

Note that we only dealt with the case where $\mathbb{G}$ is split, whereas the construction of L-group in [We14] works more generally for nonsplit groups as well. It is thus natural to consider (see Remark 4.4.4 also)

> Problem[1]. To compute the GK formula for coverings of quasi-split groups and to interpret it analogously in terms of adjoint L-functions.

One of the most ambitious goals for the theory of genuine automorphic representation would be to postulate and prove functoriality in the spirit of the Langlands' program. There are subtleties for this matter even for some simple covers. A more restrictive question is on how genuine automorphic forms on $\overline{\mathbb{G}}(\mathbb{A}_F)$ are related to automorphic forms on some linear algebraic $\mathbb{G}'(\mathbb{A}_F)$. A complete answer would have to be able to recover existing links and yield potentially new results. It is not only the answer that is important, any machinery and delicate analysis which could shed light on the question would have extensive applications as well. This could already be seen from the theory of theta correspondence between the degree two covering $\overline{\mathbb{Sp}}_{2r}(\mathbb{A}_F)$ and the orthogonal $\mathbb{SO}_{2k+1}(\mathbb{A}_F)$, for which there has been a rich literature.

For example, we may even restrict ourselves to a more specific question:

> Problem[2]. Determine whether the (completed in some way) L-functions which appear in the GK formula for the global intertwining operators (cf. Theorem 5.2.4) are equal to the Langlands-Shahidi L-functions for certain linear algebraic groups.

First of all, to make sense of the problem which addresses the global *completed* L-function, a theory of local L-functions or local γ-factors is to be developed. For generic representations of the double cover $\overline{\mathbb{Sp}}_{2r}(F_v)$, a Langlands-Shahidi method is developed in [Szp11], where a theory of local γ-factor is developed.

There is no need to emphasize the importance of local L-function. In particular, one way to attack Problem[2] is to apply the converse theorem for L-functions for admissible representations of global groups. It is clear that the partial L-functions in the GK formula in Theorem 5.2.4 could be associated with *admissible* representations of global linear groups; however, to prove that they are *automorphic* L-functions, there is the essential ingredient of local γ-factors in order to apply the converse theory. See [CKM04] for a very readable introduction to converse theorems in the linear case.

Thus, a problem closed related to Problem[2] is

> Probelm[3]. To develop a theory of local L-functions, based on which one could develop converse theorems and give answers to problems including but not restricted to the previous one.

As an example, one might like to look at the Kazhdan-Patterson coverings $\overline{\mathbb{GL}}_r(\mathbb{A}_F)$ of $\mathbb{GL}_r(\mathbb{A}_F)$, which arise from the Brylinski-Deligne framework, see [GaG14, §13.2].

> Problem[4]. To work out the Kazhdan-Patterson coverings from the BD perspective in details.

Beyond the theory of L-functions, one may also explore other facets of the BD type covering groups from different angles. For instance, it would be very rewarding to understand the harmonic analysis on BD covering groups which is relevant to representation theory. We especially refer to the analysis of orbital integrals, and more foundationally the (stable) conjugacy classes of $\overline{G}_v$. Therefore, one may consider as the first step the following problem, which we do not phrase in precise terms.

> Problem[5]. Work out the geometry of the local covering groups $\overline{G}_v$.

Again, for double cover $\overline{\mathbb{Sp}}_{2r}(F_v)$, a theory of endoscopy is developed in the thesis of W.-W. Li. In a sequel of works starting with [Li12], he also gives local analysis on more general central covering groups.

Beyond these, there is also the central question on the applications of automorphic forms on covering groups to arithmetic. Certainly this question is closely related to the consideration stated before Problem[2], which concerns how genuine automorphic forms on covering groups can be understood in terms of linear algebraic groups, in which case arithmetic applications of the former would be available via the latter by a detour consideration.

However, one may ask for direct applications which would in turn gives hint on the relations between BD covering and linear algebraic groups. For instance, in the work of Lieman [Lie94], it is shown that the L-function of a certain twisted elliptic curves is closely related to the Whittaker coefficient of certain Eisenstein series on degree three covers of $GL_3(\mathbb{A}_F)$.

More generally, along such direction there has been the deep study of general Whittaker coefficients of automorphic forms on covering groups by many mathematicians, notably B. Brubaker, D. Bump, S. Friedberg and J. Hoffstein etc. See for instance [BBFH07], [BBF11] and [BBF11-2]. They showed that certain Weyl group multiple Dirichlet series arising from those coefficients satisfy the nice expected properties such as meromorphic continuation and functional equation. Moreover, it is also shown that such series make constant appearance in the theory of crystal graph, quantum groups etc (cf. [BBCFG], [BBF11-2]).

For arithmetic application of L-functions of automorphic forms or more generally their coefficients, one will inevitably mention the study of L-functions arising from prehomogeneous vectors spaces with actions by a reductive group. We refer to the wok of T. Shintani, F. Sato, A. Yukie and M. Bhargava for expositions, especially for the applications in determining distribution of number fields etc. A program along such line has been undertaken, and this motivates us to ask what role covering groups might play.

We summarize the discussion from several paragraphs above with the following generic problem.

> Problem[6]. To explore any possible arithmetic applications of automorphic forms and representations of BD-type coverings, which preferably allow for a systematic treatment.

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