# The $q$-analog of higher order Hochschild homology and the Lie derivative


Abhishek Banerjee

*Dept. of Mathematics, Indian Institute of Science, Bengaluru - 560012, Karnataka, India.*
*Email: abhishekbanerjee1313@gmail.com*





**Abstract**

Let $A$ be a commutative algebra over $\mathbb{C}$. Given a pointed simplicial finite set $Y$ and $q \in \mathbb{C}$ a primitive $N$-th root of unity, we define the $q$-Hochschild homology groups $\{_qHH_n^Y(A)\}_{n\geq 0}$ of $A$ of order $Y$. When $D$ is a derivation on $A$, we construct the corresponding Lie derivative on the groups $\{_qHH_n^Y(A)\}_{n\geq 0}$. We also define the Lie derivative on $\{_qHH_n^Y(A)\}_{n\geq 0}$ for a higher derivation $\{D_n\}_{n\geq 0}$ on $A$. Finally, we describe the morphisms induced on the bivariant $q$-Hochschild cohomology groups $\{_qHH^n_Y(A,A)\}_{n\in\mathbb{Z}}$ of order $Y$ by a derivation $D$ on $A$.



**MSC(2010) Subj Classification:** 16W25
Keywords: Hochschild homology, higher derivations, Lie derivative


## 1 Introduction

Let $A$ be a commutative algebra over $\mathbb{C}$. Then, it is well known (see, for instance, [7, § 4.1]) that a derivation $D : A \longrightarrow A$ induces morphisms

$$L_D^n : HH_n(A) \longrightarrow HH_n(A) \qquad \forall\ n \geq 0 \tag{1.1}$$

on the Hochschild homology groups of the algebra $A$. The morphisms $L_D^n$, $n \geq 0$ play the role of the Lie derivative in noncommutative geometry. For more on these morphisms and for general properties of Hochschild homology, we refer the reader to [7]. Further, for any pointed simplicial finite set $Y$, Pirashvili [10] has introduced the Hochschild homology groups $\{HH_n^Y(A)\}_{n\geq 0}$ of $A$ of order $Y$ (see also Loday [6]). In particular, when $Y = S^1$ is the simplicial circle, the groups $\{HH_n^{S^1}(A)\}_{n\geq 0}$ reduce to the usual Hochschild homology groups of the algebra $A$. Let $q \in \mathbb{C}$ be a primitive $N$-th root of unity. The purpose of this paper is to introduce the $q$-analogues $\{_qHH_n^Y(A)\}_{n\geq 0}$ of these higher order Hochschild homology groups and study the morphisms induced on them by derivations on $A$.

More precisely, let $\Gamma$ denote the category whose objects are the finite sets $[n] = \{0, 1, 2, .., n\}$, $n \geq 0$ with basepoint $0 \in [n]$. Then, given the algebra $A$, we can define a functor $\mathcal{L}(A)$ from $\Gamma$ to the category $Vect$ of complex vector spaces that takes $[n]$ to $A \otimes A^{\otimes n}$ (see Section 2 for details). Then, we can prolong $\mathcal{L}(A)$ by means of colimits to a functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$ from the category $Fin_*$ of all finite sets with basepoint. Given a pointed simplicial finite set $Y$, i.e., a functor $Y : \Delta^{op} \longrightarrow Fin_*$ ($\Delta^{op}$ being the simplex category), we now have a simplicial vector space

$$\mathcal{L}^Y(A) : \Delta^{op} \xrightarrow{\ Y\ } Fin_* \xrightarrow{\ \mathcal{L}(A)\ } Vect \tag{1.2}$$

Let $d_i^j : \mathcal{L}^Y(A)_i \longrightarrow \mathcal{L}^Y(A)_{i-1}$, $0 \leq j \leq i$, $i \geq 0$ be the face maps of the simplicial vector space $\mathcal{L}^Y(A)$. We then construct the "$q$-Hochschild differentials":

$${}_qb_i : \mathcal{L}^Y(A)_i \longrightarrow \mathcal{L}^Y(A)_{i-1} \qquad {}_qb_i := \sum_{j=0}^{i} q^j d_i^j \tag{1.3}$$

Since $q \in \mathbb{C}$ is a primitive $N$-th root of unity, it follows that ${}_qb^N = 0$, i.e., $(\mathcal{L}^Y(A), {}_qb)$ is an $N$-complex in the sense of Kapranov [5]. We now define the $q$-Hochschild homology groups $\{{}_qHH_n^Y(A)\}_{n\geq 0}$ of $A$ of order $Y$ to be the homology objects of the $N$-complex $(\mathcal{L}^Y(A), {}_qb)$ (see Definitions 2.1 and 2.2). When $q = -1$ (and hence $N = 2$), ${}_qb$ reduces to the usual differential on the chain complex associated to the simplicial vector space $\mathcal{L}^Y(A)$ and we have ${}_{(-1)}HH_n^Y(A) = HH_n^Y(A)$, $\forall\, n \geq 0$. Then, the main result of Section 2 is as follows.

**Theorem 1.1.** *Let $D : A \longrightarrow A$ be a derivation on $A$. Then, for each $n \geq 0$, the derivation $D$ induces a morphism $L_D^{Y,n} : {}_qHH_n^Y(A) \longrightarrow {}_qHH_n^Y(A)$ of $q$-Hochschild homology groups of order $Y$. Additionally, if $\mathcal{H} = \mathcal{U}(Der(A))$ is the universal enveloping algebra of the Lie algebra $Der(A)$ of derivations on $A$, each ${}_qHH_n^Y(A)$ is a left $\mathcal{H}$-module, i.e., for any element $h \in \mathcal{H}$, there exist morphisms $L_h^{Y,n} : {}_qHH_n^Y(A) \longrightarrow {}_qHH_n^Y(A)$ of $q$-Hochschild homology groups of order $Y$.*

Thereafter, we consider a higher derivation $D = \{D_n\}_{n\geq 0}$ on $A$. We recall that a higher (or Hasse-Schmidt) derivation $D = \{D_n\}_{n\geq 0}$ on $A$ is a sequence of linear maps $D_n : A \longrightarrow A$ satisfying the following relation (see, for example, [8]):

$$D_n(a \cdot a') = \sum_{i=0}^{n} D_i(a) \cdot D_{n-i}(a') \qquad \forall\, a, a' \in A, n \geq 0 \tag{1.4}$$

In this paper, we restrict ourselves to normalized higher derivations, i.e., higher derivations $D = \{D_n\}_{n\geq 0}$ such that $D_0 = 1$. Then, in Section 3, we construct the Lie derivative on the $q$-Hochschild homology groups of order $Y$ corresponding to a higher derivation $D = \{D_n\}_{n\geq 0}$.

**Theorem 1.2.** *Let $D = \{D_n\}_{n\geq 0}$ be a normalized higher derivation on $A$. Then, for each $k \geq 0$, we have an induced morphism $L_D^{Y,k} : {}_qHH_*^Y(A) = \bigoplus_{n=0}^{\infty} {}_qHH_n^Y(A) \longrightarrow {}_qHH_*^Y(A) = \bigoplus_{n=0}^{\infty} {}_qHH_n^Y(A)$ on the $q$-Hochschild homology groups of $A$ of order $Y$.*

Further, in [9], Mirzavaziri has provided a characterization of normalized higher derivations on algebras over $\mathbb{C}$ from which it follows that if $D = \{D_k\}_{k\geq 0}$ is a higher derivation on $A$, each $D_k$ is an element of the Hopf algebra $\mathcal{H} = \mathcal{U}(Der(A))$ (see (3.7) for details). It follows therefore from Theorem 1.1 that for each $k$, the element $D_k \in \mathcal{H}$ induces a morphism $L_{D_k}^Y : {}_qHH_*^Y(A) \longrightarrow {}_qHH_*^Y(A)$. Then, in Section 3, we prove the following result.

**Theorem 1.3.** *Let $D = \{D_k\}_{k\geq 0}$ be a normalized higher derivation on $A$. Then, for each $k \geq 1$, we have $L^Y_{D_k} = L^{Y,k}_D$ as an endomorphism of ${}_qHH^Y_*(A) = \bigoplus_{n=0}^{\infty} {}_qHH^Y_n(A)$.*

In Section 4, we start by defining bivariant $q$-Hochschild cohomology groups $\{{}_qHH^n_Y(A,A)\}_{n\in\mathbb{Z}}$ of order $Y$. For this we consider the modules $\underline{Hom}(\mathcal{L}^Y(A),\mathcal{L}^Y(A))_n$, $n \in \mathbb{Z}$ where an element $f \in \underline{Hom}(\mathcal{L}^Y(A),\mathcal{L}^Y(A))_n$ is given by a family of morphisms $f = \{f_i : \mathcal{L}^Y(A)_i \longrightarrow \mathcal{L}^Y(A)_{i+n}\}_{i\in\mathbb{Z}}$. Further, we define a differential ${}_q\partial_n : \underline{Hom}(\mathcal{L}^Y(A),\mathcal{L}^Y(A))_n \longrightarrow \underline{Hom}(\mathcal{L}^Y(A),\mathcal{L}^Y(A))_{n-1}$ (see Definition 4.1). Then, $(\underline{Hom}(\mathcal{L}^Y(A),\mathcal{L}^Y(A)), {}_q\partial)$ is an $N$-complex and we let ${}_qHH^n_Y(A,A)$ be the $(-n)$-th homology object of $(\underline{Hom}(\mathcal{L}^Y(A),\mathcal{L}^Y(A)), {}_q\partial)$. We end with the following result.

**Theorem 1.4.** *Let $D : A \longrightarrow A$ be a derivation on $A$. Then, for each $n \in \mathbb{Z}$, the derivation $D$ induces a morphism $\underline{L}^{Y,n}_D : {}_qHH^n_Y(A,A) \longrightarrow {}_qHH^n_Y(A,A)$ on the bivariant $q$-Hochschild cohomology groups of order $Y$.*

## 2 Lie Derivative on higher order $q$-Hochschild homology

Let $Vect$ denote the category of vector spaces over $\mathbb{C}$. Let $A$ be a commutative $\mathbb{C}$-algebra. We recall here the definition of higher order Hochschild homology groups of a commutative algebra $A$ as introduced by Pirashvili [10] (see also Loday [6]). Let $\Gamma$ denote the category whose objects are the pointed sets $[n] = \{0,1,2,...,n\}$ with $0 \in [n]$ as base point for each $n \geq 0$. Then, a morphism $\phi : [m] \longrightarrow [n]$ in $\Gamma$ is a map $\phi : \{0,1,2,...,m\} \longrightarrow \{0,1,2,...,n\}$ of sets such that $\phi(0) = 0$. We now define a functor:

$$\mathcal{L}(A) : \Gamma \longrightarrow Vect \qquad [n] \mapsto A \otimes A^{\otimes n} \tag{2.1}$$

Given a morphism $\phi : [m] \longrightarrow [n]$ in $\Gamma$, we have an induced map in $Vect$:

$$\begin{gathered}\mathcal{L}(A)(\phi) : A \otimes A^{\otimes m} \longrightarrow A \otimes A^{\otimes n} \\ \mathcal{L}(A)(\phi)(a_0 \otimes a_1 \otimes .... \otimes a_m) = (b_0 \otimes b_1 \otimes ... \otimes b_n) \qquad b_j := \prod_{\phi(i)=j} a_i\end{gathered} \tag{2.2}$$

We now consider the category $Fin_*$ of finite pointed sets. There is a natural inclusion $\Gamma \hookrightarrow Fin_*$ of categories. Then, $\mathcal{L}(A) : \Gamma \longrightarrow Vect$ can be extended to a functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$ by setting:

$$\mathcal{L}(A) : Fin_* \longrightarrow Vect \qquad T \mapsto \underset{\Gamma\ni T' \longrightarrow T}{colim} \mathcal{L}(A)(T') \tag{2.3}$$

where the colimit in (2.3) is taken over all morphisms $T' \longrightarrow T$ in $Fin_*$ such that $T' \in \Gamma$. Let $\Delta$ be the simplex category, i.e., the category whose objects are sets $[n] = \{0,1,2,...,n\}$, $n \geq 0$ and whose morphisms are order preserving maps. Then, given a pointed simplicial finite set $Y$ corresponding to a functor $Y : \Delta^{op} \longrightarrow Fin_*$, we have a simplicial vector space $\mathcal{L}^Y(A)$ determined by the composition of functors:

$$\mathcal{L}^Y(A) : \Delta^{op} \xrightarrow{Y} Fin_* \xrightarrow{\mathcal{L}(A)} Vect \tag{2.4}$$

For any $n \geq 0$, let $HH_n^Y(A)$ denote the $n$-th homology group of the chain complex associated to the simplicial vector space $\mathcal{L}^Y(A)$. Following Pirashvili [10], when $Y = S^p$ ($S^p$ being the sphere of dimension $p \geq 1$), we say that the homology groups $\{HH_n^{S^p}(A)\}_{n\geq 0}$ are the Hochschild homology groups of $A$ of order $p$. When $p = 1$, i.e., $Y = S^1$ is the simplicial circle, the Hochschild homology groups $\{HH_n^{S^1}(A)\}_{n\geq 0}$ are identical to the usual Hochschild homology groups of $A$.

Our objective is to introduce a $q$-analog of the groups $HH_*^Y(A)$, where $q \in \mathbb{C}$ is a primitive $N$-th root of unity. For this, we consider the face maps $d_n^i : \mathcal{L}^Y(A)_n \longrightarrow \mathcal{L}^Y(A)_{n-1}$, $0 \leq i \leq n$, $n \geq 0$, of the simplicial vector space $\mathcal{L}^Y(A)$ defined in (2.4). We set:

$$ {}_qb_n : \mathcal{L}^Y(A)_n \longrightarrow \mathcal{L}^Y(A)_{n-1} \qquad {}_qb_n := \sum_{i=0}^{n} q^i d_n^i \tag{2.5}$$

For the sake of convenience, we will often write ${}_qb_n$ simply as ${}_qb$. Then, it is well known that the morphism ${}_qb$ satisfies ${}_qb^N = 0$ (this is true in general for any simplicial vector space; see, for instance, Kapranov [5, Proposition 0.2]). In particular, if $q = -1$, i.e., $N = 2$, we have ${}_{(-1)}b^2 = 0$ and ${}_{(-1)}b$ is the standard differential on the chain complex corresponding to the simplicial vector space $\mathcal{L}^Y(A)$. In general, the pair $(\mathcal{L}^Y(A), {}_qb)$, i.e., the simplicial vector space $\mathcal{L}^Y(A)$ equipped with the morphism ${}_qb$ is an "$N$-complex" in the sense defined below.

**Definition 2.1.** *(see [3, § 2] and [5, Definition 0.1]) Let $\mathcal{A}$ be an abelian category and $N \geq 2$ a positive integer. An $N$-complex in $\mathcal{A}$ is a sequence of objects and morphisms of $\mathcal{A}$*

$$C_* = \{\dots \longrightarrow C_1 \xrightarrow{b_1} C_0 \xrightarrow{b_0} C_{-1} \longrightarrow \dots\} \tag{2.6}$$

*such that the composition of any $N$ consecutive morphisms in (2.6) is 0. For any $n \in \mathbb{Z}$, the homology object $H_{\{n\}}(C_*, b)$ of the $N$-complex $(C_*, b)$ is defined as:*

$$H_{\{n\}}(C_*, b) := \bigoplus_{i=1}^{N-1} H_{\{i,n\}}(C_*, b) \qquad H_{\{i,n\}}(C_*, b) := \frac{Ker(b^i : C_n \longrightarrow C_{n-i})}{Im(b^{N-i} : C_{N-i+n} \longrightarrow C_n)} \tag{2.7}$$

**Definition 2.2.** *Let $A$ be a commutative algebra over $\mathbb{C}$ and let $Y$ be a pointed simplicial finite set. Let $q \in \mathbb{C}$ be a primitive $N$-th root of unity. Then, the q-Hochschild homology groups ${}_qHH_n^Y(A)$, $n \geq 0$ of $A$ of order $Y$ are defined to be the homology objects of the $N$-complex $(\mathcal{L}^Y(A), {}_qb)$ associated to the simplicial vector space $\mathcal{L}^Y(A)$; in other words, we define:*

$${}_qHH_n^Y(A) := H_{\{n\}}(\mathcal{L}^Y(A), {}_qb) \tag{2.8}$$

As with the ordinary Hochschild homology of an algebra (see, for instance, [7, § 4.1]), given a derivation $D : A \longrightarrow A$, we want to construct the Lie derivative $L_D^Y : HH_*^Y(A) \longrightarrow HH_*^Y(A)$ on the Hochschild homology of order $Y$. For this, we start with the following lemma.

**Lemma 2.3.** *Let $A$ be a commutative $\mathbb{C}$-algebra and let $D : A \longrightarrow A$ be a derivation on $A$. Then, the derivation $D$ induces an endomorphism $L_D : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$.*

*Proof.* We first consider the functor $\mathcal{L}(A)$ restricted to the subcategory $\Gamma$ of $Fin_*$, defined as in (2.1) and (2.2):

$$\mathcal{L}(A) : \Gamma \longrightarrow Vect \qquad [n] \mapsto A \otimes A^{\otimes n} \tag{2.9}$$

Given the derivation $D$ on $A$, we define morphisms (for all $n \geq 0$):

$$L_D([n]) : \mathcal{L}(A)([n]) \longrightarrow \mathcal{L}(A)([n]) \qquad (a_0 \otimes a_1 \otimes ... \otimes a_n) \mapsto \sum_{i=0}^{n} (a_0 \otimes a_1 \otimes ... \otimes D(a_i) \otimes ... \otimes a_n) \tag{2.10}$$

Further, for any morphism $\phi : [m] \longrightarrow [n]$ in $\Gamma$, we have, for any $(a_0 \otimes a_1 \otimes ... \otimes a_m) \in A \otimes A^{\otimes m}$:

$$\begin{aligned}
L_D([n]) \circ \mathcal{L}(A)(\phi)(a_0 \otimes a_1 \otimes ... \otimes a_m) &= L_D([n]) \left( \bigotimes_{j=0}^{n} \prod_{\phi(i)=j} a_i \right) \\
&= \sum_{k=0}^{n} \left( \bigotimes_{j=0}^{k-1} \prod_{\phi(i)=j} a_i \right) \otimes D \left( \prod_{\phi(i)=k} a_i \right) \otimes \left( \bigotimes_{j=k+1}^{n} \prod_{\phi(i)=j} a_i \right) \\
&= \sum_{k=0}^{n} \sum_{i \in \phi^{-1}(k)} \mathcal{L}(A)(\phi)(a_0 \otimes ... \otimes D(a_i) \otimes ... \otimes a_m) \\
&= \sum_{i=0}^{m} \mathcal{L}(A)(\phi)(a_0 \otimes ... \otimes D(a_i) \otimes ... \otimes a_m) \\
&= \mathcal{L}(A)(\phi) \circ L_D([m])(a_0 \otimes a_1 \otimes ... \otimes a_m)
\end{aligned}$$

It follows that the derivation $D$ induces an endomorphism $L_D$ of the functor $\mathcal{L}(A) : \Gamma \longrightarrow Vect$. More generally, for any object $T \in Fin_*$ and a morphism $T' \longrightarrow T$ in $Fin_*$ such that $T' \in \Gamma$, we have a morphism $L_D(T') : \mathcal{L}(A)(T') \longrightarrow \mathcal{L}(A)(T')$ as defined in (2.10). By definition, we know that $\mathcal{L}(A)(T) = \underset{\Gamma \ni T' \longrightarrow T}{colim} \mathcal{L}(A)(T')$ and hence we have an induced morphism

$$L_D(T) : \mathcal{L}(A)(T) = \underset{\Gamma \ni T' \longrightarrow T}{colim} \mathcal{L}(A)(T') \longrightarrow \mathcal{L}(A)(T) = \underset{\Gamma \ni T' \longrightarrow T}{colim} \mathcal{L}(A)(T') \tag{2.11}$$

From (2.11) it follows that the derivation $D$ induces an endomorphism $L_D : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$. This proves the claim.

□

**Proposition 2.4.** *Let $A$ be a commutative $\mathbb{C}$-algebra and let $D : A \longrightarrow A$ be a derivation on $A$. Let $Y$ be a pointed simplicial finite set. Then, for each $n \geq 0$, the derivation $D$ induces a morphism $L_D^{Y,n} : {}_qHH_n^Y(A) \longrightarrow {}_qHH_n^Y(A)$ of q-Hochschild homology groups of order $Y$, where $q \in \mathbb{C}$ is a primitive $N$-th root of unity.*

*Proof.* From Lemma 2.3, we know that the derivation $D$ induces an endomorphism $L_D : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$. Given the pointed simplicial finite set $Y$, the endomorphism $L_D : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of functors induces an endomorphism of the functor

$$\mathcal{L}^Y(A) : \Delta^{op} \xrightarrow{Y} Fin_* \xrightarrow{\mathcal{L}(A)} Vect \tag{2.12}$$

From (2.12), it follows that we have an endomorphism $L_D^Y : \mathcal{L}^Y(A) \longrightarrow \mathcal{L}^Y(A)$ of the simplicial vector space $\mathcal{L}^Y(A)$. Hence, we have induced morphisms $L_D^{Y,n} : {}_qHH_n^Y(A) \longrightarrow {}_qHH_n^Y(A)$ on the homology objects of the $N$-complex $(\mathcal{L}^Y(A), {}_qb)$ associated to the simplicial vector space $\mathcal{L}^Y(A)$ as in (2.5). $\square$

We now let $Der(A)$ denote the vector space of all derivations on the commutative $\mathbb{C}$-algebra $A$. Then, $Der(A)$ is a Lie algebra, endowed with the Lie bracket $[D, D'] := D \circ D' - D' \circ D$, $\forall$ $D$, $D' \in Der(A)$. Let $\mathcal{H} := \mathcal{U}(Der(A))$ denote the universal enveloping algebra of $Der(A)$. We will now show that for any pointed simplicial finite set $Y$, the operators $L_D^{Y,n}$, $D \in Der(A)$ on the $q$-Hochschild homology group of $A$ of order $Y$ make ${}_qHH_n^Y(A)$ into a module over the Hopf algebra $\mathcal{H} = \mathcal{U}(Der(A))$.

**Lemma 2.5.** *Let $q \in \mathbb{C}$ be a primitive $N$-th root of unity. Let $A$ be a commutative $\mathbb{C}$-algebra and let $D$, $D' \in Der(A)$ be derivations on $A$. Let $Y$ be a pointed simplicial finite set. Then, for each $n \geq 0$, the operators $L_D^{Y,n}$, $L_{D'}^{Y,n} : {}_qHH_n^Y(A) \longrightarrow {}_qHH_n^Y(A)$ satisfy $[L_D^{Y,n}, L_{D'}^{Y,n}] = L_D^{Y,n} \circ L_{D'}^{Y,n} - L_{D'}^{Y,n} \circ L_D^{Y,n} = L_{[D,D']}^{Y,n}$.*

*Proof.* For $D$, $D' \in Der(A)$, we consider the respective endomorphisms $L_D$, $L_{D'}$ of the functor $\mathcal{L}(A) : \Gamma \longrightarrow Vect$. By definition, for any object $[n] \in \Gamma$, we have morphisms:

$$\begin{array}{ll} L_D([n]) : \mathcal{L}(A)([n]) \longrightarrow \mathcal{L}(A)([n]) & (a_0 \otimes ... \otimes a_n) \mapsto \sum_{i=0}^{n} (a_0 \otimes ... \otimes D(a_i) \otimes ... \otimes a_n) \\ L_{D'}([n]) : \mathcal{L}(A)([n]) \longrightarrow \mathcal{L}(A)([n]) & (a_0 \otimes ... \otimes a_n) \mapsto \sum_{i=0}^{n} (a_0 \otimes ... \otimes D'(a_i) \otimes ... \otimes a_n) \end{array} \tag{2.13}$$

From (2.13), it may be verified easily that we have

$$(L_D \circ L_{D'} - L_{D'} \circ L_D)([n]) = L_{[D,D']}([n]) : \mathcal{L}(A)([n]) \longrightarrow \mathcal{L}(A)([n]) \qquad \forall\ n \geq 0 \tag{2.14}$$

and it follows that $L_D \circ L_{D'} - L_{D'} \circ L_D = L_{[D,D']}$ as endomorphisms of the functor $\mathcal{L}(A) : \Gamma \longrightarrow Vect$. More generally, for any object $T \in Fin_*$, we have $\mathcal{L}(A)(T) = \underset{\Gamma \ni T' \longrightarrow T}{colim} \mathcal{L}(A)(T')$ and hence $L_D \circ L_{D'} - L_{D'} \circ L_D = L_{[D,D']}$ as endomorphisms of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$. Finally, considering the composition of $\mathcal{L}(A) : Fin_* \longrightarrow Vect$ with the functor $Y : \Delta^{op} \longrightarrow Fin_*$ corresponding to the pointed simplicial finite set $Y$, it follows that $L_D^Y \circ L_{D'}^Y - L_{D'}^Y \circ L_D^Y = L_{[D,D']}^Y$ as endomorphisms of the functor $\mathcal{L}^Y(A) : \Delta^{op} \longrightarrow Vect$. Hence, we have $[L_D^{Y,n}, L_{D'}^{Y,n}] = L_D^{Y,n} \circ L_{D'}^{Y,n} - L_{D'}^{Y,n} \circ L_D^{Y,n} = L_{[D,D']}^{Y,n}$ on the homology objects ${}_qHH_n^Y(A)$, $n \geq 0$ of the $N$-complex $(\mathcal{L}^Y(A), {}_qb)$ associated to the simplicial vector space $\mathcal{L}^Y(A) : \Delta^{op} \longrightarrow Vect$ as in (2.5). $\square$

**Proposition 2.6.** *Let $q \in \mathbb{C}$ be a primitive $N$-th root of unity. Let $A$ be a commutative algebra over $\mathbb{C}$ and let $Der(A)$ denote the Lie algebra of derivations on $A$. Let $\mathcal{H} = \mathcal{U}(Der(A))$ denote the universal enveloping algebra of $Der(A)$. Then, for any pointed simplicial finite set $Y$ and any $n \geq 0$, the $q$-Hochschild homology group ${}_qHH_n^Y(A)$ of order $Y$ is a left module over the Hopf algebra $\mathcal{H}$.*

*Proof.* From Lemma 2.5, it follows that $Der(A)$ has a Lie algebra action on each ${}_qHH_n^Y(A)$, i.e., $[L_D^{Y,n}, L_{D'}^{Y,n}] = L_D^{Y,n} \circ L_{D'}^{Y,n} - L_{D'}^{Y,n} \circ L_D^{Y,n} = L_{[D,D']}^{Y,n}$ for any $D$, $D' \in Der(A)$. Since $\mathcal{H}$ is the universal enveloping algebra of $Der(A)$, it follows that this Lie algebra action of $Der(A)$ on ${}_qHH_n^Y(A)$ makes ${}_qHH_n^Y(A)$ into a left $\mathcal{H}$-module.

$\square$

## 3 Higher derivations and the Lie derivative

As before, we work with a commutative algebra $A$ over $\mathbb{C}$, a pointed simplicial finite set $Y$ and $q \in \mathbb{C}$ a primitive $N$-th root of unity. In this section, we will describe the Lie derivative on the $q$-Hochschild homology groups ${}_qHH_*^Y(A)$ corresponding to a higher derivation $D$ on $A$. Given an ordinary derivation $d$ on $A$, it is easy to verify that the sequence $\{D_n := d^n/n!\}_{n\geq 0}$ satisfies the following identity:

$$D_n(a \cdot a') = \sum_{i=0}^{n} D_i(a) \cdot D_{n-i}(a') \qquad \forall\ n \geq 0,\ a, a' \in A \tag{3.1}$$

More generally, we have the notion of a higher (or Hasse-Schmidt) derivation on $A$.

**Definition 3.1.** *(see, for instance, [8]) Let $A$ be a commutative algebra over $\mathbb{C}$. A sequence $D = \{D_n\}_{n\geq 0}$ of $\mathbb{C}$-linear maps on $A$ is said to be a higher (or Hasse-Schmidt) derivation on $A$ if it satisfies:*

$$D_n(a \cdot a') = \sum_{i=0}^{n} D_i(a) \cdot D_{n-i}(a') \qquad \forall\ n \geq 0,\ a, a' \in A \tag{3.2}$$

In this paper, we will only work with higher derivations $D = \{D_n\}_{n\geq 0}$ that are normalized, i.e., those higher derivations $D = \{D_n\}_{n\geq 0}$ which satisfy $D_0 = 1$. For a normalized higher derivation $D = \{D_n\}_{n\geq 0}$ it is easy to verify from relation (3.2) that $D_n(1) = 0$ for all $n > 0$. For more on the structure of higher derivations on an algebra, we refer the reader to [9], [11] and [12]. For a higher derivation on $A$, we have already described in [2] the corresponding Lie derivative on the ordinary Hochschild homology; we are now ready to introduce the action of a higher derivation on the $q$-Hochschild homology groups of order $Y$ of the algebra $A$.

**Lemma 3.2.** *Let $A$ be a commutative algebra over $\mathbb{C}$ and let $D = \{D_n\}_{n\geq 0}$ be a (normalized) higher derivation on $A$. Then, for any given $k \geq 0$, the higher derivation $D$ induces an endomorphism $L_D^k : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$.*

*Proof.* It suffices to prove that for each $k \geq 0$, we have an endomorphism $L_D^k : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$ restricted to the subcategory $\Gamma$ of $Fin_*$. Given the higher derivation $D = \{D_n\}_{n\geq 0}$ and the integer $k \geq 0$, we define morphisms ($\forall\ n \geq 0$)

$$\begin{array}{c} L_D^k([n]) : \mathcal{L}(A)([n]) \longrightarrow \mathcal{L}(A)([n]) \\ (a_0 \otimes a_1 \otimes ... \otimes a_n) \mapsto \sum\limits_{\substack{(p_0, p_1, ..., p_n) \\ p_0 + p_1 + ... + p_n = k}} (D_{p_0}(a_0) \otimes D_{p_1}(a_1) \otimes .... \otimes D_{p_n}(a_n)) \end{array} \tag{3.3}$$

For the sake of convenience, we will often denote a sum as in (3.3) taken over all ordered tuples $(p_0, p_1, ..., p_n)$ of non-negative integers such that $p_0 + p_1 + ... + p_n = k$ simply as

$$(a_0 \otimes a_1 \otimes ... \otimes a_n) \mapsto \sum_{p_0+p_1+...+p_n=k} (D_{p_0}(a_0) \otimes D_{p_1}(a_1) \otimes .... \otimes D_{p_n}(a_n)) \tag{3.4}$$

Let $\phi : [m] \longrightarrow [n]$ be a morphism in $\Gamma$. We let $N(j)$ denote the cardinality of the set $\phi^{-1}(j) \subseteq [m]$ for any $0 \leq j \leq n$. Then, we have, for any $(a_0 \otimes a_1 \otimes ... \otimes a_m) \in A \otimes A^{\otimes m}$:

$$\begin{aligned}
L_D^k([n]) \circ \mathcal{L}(A)(\phi)(a_0 \otimes a_1 \otimes ... \otimes a_m) &= L_D^k([n]) \left( \bigotimes_{j=0}^{n} \prod_{\phi(i)=j} a_i \right) \\
&= \sum_{p_0+p_1+...+p_n=k} \left( \bigotimes_{j=0}^{n} D_{p_j} \left( \prod_{\phi(i)=j} a_i \right) \right) \\
&= \sum_{p_0+p_1+...+p_n=k} \left( \bigotimes_{j=0}^{n} \sum_{q_1+...+q_{N(j)}=p_j} \prod_{\phi(i)=j} D_{q_i}(a_i) \right) \\
&= \sum_{r_0+r_1+...+r_m=k} \left( \bigotimes_{j=0}^{n} \prod_{\phi(i)=j} D_{r_i}(a_i) \right) \\
&= \sum_{r_0+r_1+...+r_m=k} \mathcal{L}(A)(\phi) \left( \bigotimes_{i=0}^{m} D_{r_i}(a_i) \right) \\
&= \mathcal{L}(A)(\phi) \circ L_D^k([m])(a_0 \otimes a_1 \otimes ... \otimes a_m)
\end{aligned} \tag{3.5}$$

From (3.5), it follows that for each $k \geq 0$, $L_D^k : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ is an endomorphism of the functor $\mathcal{L}(A)$ restricted to $\Gamma$ and hence, taking colimits as in the proof of Lemma 2.3, $L_D^k$ induces an endomorphism of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$.

□

**Proposition 3.3.** *Let $q \in \mathbb{C}$ be a primitive $N$-th root of unity. Let $A$ be a commutative algebra over $\mathbb{C}$ and let $Y$ be a pointed simplicial finite set. Then, given a higher derivation $D = \{D_n\}_{n\geq 0}$ on $A$, for each $k \geq 0$, we have an induced morphism:*

$$L_D^{Y,k} : {}_qHH_*^Y(A) = \bigoplus_{n=0}^{\infty} {}_qHH_n^Y(A) \longrightarrow {}_qHH_*^Y(A) = \bigoplus_{n=0}^{\infty} {}_qHH_n^Y(A) \tag{3.6}$$

*on the $q$-Hochschild homology groups of $A$ of order $Y$.*

*Proof.* From Lemma 3.2, we know that for any $k \geq 0$, we have an endomorphism $L_D^k : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$. Composing with the functor $Y : \Delta^{op} \longrightarrow Fin_*$ corresponding to the pointed simplicial finite set $Y$, we have an induced endomorphism $L_D^{Y,k} : \mathcal{L}^Y(A) \longrightarrow \mathcal{L}^Y(A)$ of the functor $\mathcal{L}^Y(A) = \mathcal{L}(A) \circ Y : \Delta^{op} \xrightarrow{Y} Fin_* \xrightarrow{\mathcal{L}(A)} Vect$. Accordingly, $L_D^{Y,k}$ induces an endomorphism on the homology objects of the $N$-complex $(\mathcal{L}^Y(A), {}_qb)$ associated to the simplicial vector space $\mathcal{L}^Y(A)$ as in (2.5). Hence, we have induced morphisms $L_D^{Y,k} : {}_qHH_*^Y(A) \longrightarrow {}_qHH_*^Y(A)$ on the $q$-Hochschild homology groups of order $Y$.

□

We have already shown in the last section that ${}_qHH_*^Y(A)$ is a left module over the universal enveloping algebra $\mathcal{H} = \mathcal{U}(Der(A))$ of the Lie algebra of derivations on $A$. Given a higher derivation $D = \{D_k\}_{k\geq 0}$

on a $\mathbb{C}$-algebra $A$, Mirzavaziri [9] has shown that the higher derivation $D$ may be expressed as follows: there exists a sequence of ordinary derivations $\{d_n\}_{n\geq 0}$, $d_n \in Der(A)$ such that:

$$D_k = \sum_{i=1}^{k} \left( \sum_{\sum_{j=1}^{i} r_j = k} \left( \prod_{j=1}^{i} \frac{1}{r_j + ... + r_i} \right) d_{r_1} .... d_{r_i} \right) \tag{3.7}$$

From (3.7), it is clear that given a higher derivation $D = \{D_k\}_{k\geq 0}$ on $A$, each $D_k$ is an element of the Hopf algebra $\mathcal{H} = \mathcal{U}(Der(A))$. Hence, it follows from Proposition 2.6 that each operator $D_k \in \mathcal{H}$ induces a morphism $L^Y_{D_k} : {}_qHH^Y_*(A) \longrightarrow {}_qHH^Y_*(A)$ on the $q$-Hochschild homology groups of order $Y$. We will now show that the morphisms $L^Y_{D_k}$, $k \geq 1$ are identical to the morphisms $L^{Y,k}_D : {}_qHH^Y_*(A) \longrightarrow {}_qHH^Y_*(A)$ described in Proposition 3.3.

**Proposition 3.4.** *Let $q \in \mathbb{C}$ be a primitive $N$-th root of unity. Let $A$ be a commutative algebra over $\mathbb{C}$ and let $Y$ be a pointed simplicial finite set. Let $D = \{D_k\}_{k\geq 0}$ denote a higher derivation on $A$. For any $k \geq 1$, let $L^Y_{D_k} : {}_qHH^Y_*(A) \longrightarrow {}_qHH^Y_*(A)$ be the morphism induced by $D_k \in \mathcal{H}$ as in Proposition 2.6 and let $L^{Y,k}_D : {}_qHH^Y_*(A) \longrightarrow {}_qHH^Y_*(A)$ be the morphism induced by $D$ as in Proposition 3.3. Then, we have $L^Y_{D_k} = L^{Y,k}_D : {}_qHH^Y_*(A) \longrightarrow {}_qHH^Y_*(A)$.*

*Proof.* From the proofs of Lemma 2.3 and Lemma 2.5, it follows that the element $D_k \in \mathcal{H} = \mathcal{U}(Der(A))$ of the universal enveloping algebra $\mathcal{H}$ defines an endomorphism $L_{D_k} : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$. From the proofs of Proposition 2.4 and Proposition 2.6, it is clear that the morphism $L^Y_{D_k} : {}_qHH^Y_*(A) \longrightarrow {}_qHH^Y_*(A)$ is obtained from the endomorphism $L^Y_{D_k} : \mathcal{L}^Y(A) \longrightarrow \mathcal{L}^Y(A)$ of the functor $\mathcal{L}^Y(A) = \mathcal{L}(A) \circ Y$ induced by $L_{D_k} : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$.

Similarly, from Lemma 3.2, it follows that the higher derivation $D$ induces an endomorphism $L^k_D : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$. From the proof of Proposition 3.3, it follows that the morphism $L^{Y,k}_D : {}_qHH^Y_*(A) \longrightarrow {}_qHH^Y_*(A)$ is obtained from the endomorphism $L^{Y,k}_D : \mathcal{L}^Y(A) \longrightarrow \mathcal{L}^Y(A)$ of the functor $\mathcal{L}^Y(A) = \mathcal{L}(A) \circ Y$ induced by $L^k_D : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$. Hence, in order to prove the result, we need to show that $L^k_D = L_{D_k}$ as endomorphisms of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$. As before, it suffices to show that $L^k_D = L_{D_k}$ as endomorphisms of the functor $\mathcal{L}(A)$ restricted to the subcategory $\Gamma$ of $Fin_*$.

Let $\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ denote the coproduct on $\mathcal{H}$. For any $h \in \mathcal{H}$ and any $n \geq 0$, we write $\Delta^n(h) = \sum h_{(1)} \otimes h_{(2)} \otimes ... \otimes h_{(n+1)}$. Then, we have an induced endomorphism $L_h : \mathcal{L}(A) \longrightarrow \mathcal{L}(A)$ of the functor $\mathcal{L}(A) : Fin_* \longrightarrow Vect$. Further, we note that the equation

$$L_h([n])(a_0 \otimes a_1 \otimes ... \otimes a_n) = \sum (h_{(1)}(a_0) \otimes h_{(2)}(a_1) \otimes ... h_{(n+1)}(a_n)) \quad \forall (a_0 \otimes ... \otimes a_n) \in \mathcal{L}(A)([n]) \tag{3.8}$$

holds for all $h \in Der(A) \subseteq \mathcal{H}$ and hence for all $h \in \mathcal{H} = \mathcal{U}(Der(A))$. From the definition of $L^k_D$ in Lemma 3.2, we now see that in order to show that $L^k_D = L_{D_k}$, it suffices to show that

$$\Delta^n(D_k) = \sum_{\sum_{i=0}^{n} p_i = k} D_{p_0} \otimes D_{p_1} \otimes ... \otimes D_{p_n} \qquad \forall\, n \geq 0 \tag{3.9}$$

We will prove (3.9) by induction on $k$. For any given $n \geq 0$, it is clear that the equation (3.9) holds for $k=0$ and $k=1$. We now suppose that its holds for any $0 \leq k \leq K$. From [9, Proposition 2.1], we know that

$$D_{M+1} = \frac{1}{M+1} \sum_{m=0}^{M} d_{m+1} D_{M-m} \qquad \forall\ M \geq 0 \tag{3.10}$$

where the $d_{m+1}$ are the derivations corresponding to the higher derivation $D = \{D_n\}_{n\geq 0}$ as described in (3.7). From (3.10), it follows that $\Delta^n(D_{K+1}) = \frac{1}{K+1}\sum_{m=0}^{K} \Delta^n(d_{m+1})\Delta^n(D_{K-m})$ and hence

$$\Delta^n(D_{K+1}) = \frac{1}{K+1} \sum_{m=0}^{K} \left( \sum_{j=0}^{n} d_{m+1}^{j} \right) \left( \sum_{\sum_{i=0}^{n} p_i = K-m} D_{p_0} \otimes D_{p_1} \otimes ... \otimes D_{p_n} \right) \tag{3.11}$$

where $d_{m+1}^{j}$ denotes the term $1 \otimes 1 \otimes ... \otimes d_{m+1} \otimes ... \otimes 1$ (i.e., $d_{m+1}$ at the $j$-th position) appearing in the expression for $\Delta^n(d_{m+1})$. We now consider ordered tuples $(p'_0, p'_1, ..., p'_n)$ of non-negative integers such that $p'_0 + p'_1 + .... + p'_n = K+1$. Then, we can write:

$$\begin{aligned}
&\sum_{m=0}^{K} \left( \sum_{j=0}^{n} d_{m+1}^{j} \right) \left( \sum_{\sum_{i=0}^{n} p_i = K-m} D_{p_0} \otimes D_{p_1} \otimes ... \otimes D_{p_n} \right) \\
&= \sum_{\sum_{i=0}^{n} p'_i = K+1} \ \sum_{j=0, p'_j \geq 1}^{n} \ \sum_{m=0}^{p'_j - 1} d_{m+1}^{j} \cdot (D_{p'_0} \otimes ... \otimes D_{p'_j - m - 1} \otimes ... \otimes D_{p'_n}) \\
&= \sum_{\sum_{i=0}^{n} p'_i = K+1} \ \sum_{j=0, p'_j \geq 1}^{n} \ \sum_{m=0}^{p'_j - 1} (D_{p'_0} \otimes ... \otimes d_{m+1} D_{p'_j - m - 1} \otimes ... \otimes D_{p'_n})
\end{aligned} \tag{3.12}$$

From (3.10), it follows that $\sum_{m=0}^{p'_j - 1} d_{m+1} D_{p'_j - m - 1} = p'_j \cdot D_{p'_j}$ and hence:

$$\sum_{m=0}^{p'_j - 1} (D_{p'_0} \otimes ... \otimes d_{m+1} D_{p'_j - m - 1} \otimes ... \otimes D_{p'_n}) = p'_j \cdot (D_{p'_0} \otimes ... \otimes D_{p'_j} \otimes ... \otimes D_{p'_n}) \tag{3.13}$$

Combining (3.11), (3.12) and (3.13), it follows that:

$$\begin{aligned}
\Delta^n(D_{K+1}) &= \tfrac{1}{K+1} \left( \sum_{\sum_{i=0}^{n} p'_i = K+1} \ \sum_{j=0, p'_j \geq 1}^{n} p'_j \cdot (D_{p'_0} \otimes ... \otimes D_{p'_j} \otimes ... \otimes D_{p'_n}) \right) \\
&= \tfrac{1}{K+1} \left( \sum_{\sum_{i=0}^{n} p'_i = K+1} (K+1) \cdot (D_{p'_0} \otimes ... \otimes D_{p'_j} \otimes ... \otimes D_{p'_n}) \right) \\
&= \sum_{\sum_{i=0}^{n} p'_i = K+1} (D_{p'_0} \otimes ... \otimes D_{p'_j} \otimes ... \otimes D_{p'_n})
\end{aligned} \tag{3.14}$$

This proves the result of (3.9) for $K+1$.

$\square$

# 4 Action on bivariant $q$-Hochschild cohomology groups

Let $A$ be a commutative algebra over $\mathbb{C}$ and let $q \in \mathbb{C}$ be a primitive $N$-th root of unity. Let $Y$ be a pointed simplicial finite set. In this section, we will define the bivariant $q$-Hochschild cohomology groups $\{HH^n_Y(A,A)\}_{n\in\mathbb{Z}}$ of $A$ of order $Y$ and show that a derivation $D$ on $A$ induces a morphism $\underline{L}_D^{Y,n}(A,A) : {}_qHH^n_Y(A,A) \longrightarrow {}_qHH^n_Y(A,A)$. For the ordinary bivariant Hochschild cohomology groups $\{HH^n(A,A)\}_{n\in\mathbb{Z}}$, we have already studied this morphism in [1]. For the definition and properties of ordinary bivariant Hochschild cohomology, we refer the reader to [7, § 5.1] (see also the original paper of Jones and Kassel [4]). We start by defining the bivariant $q$-Hochschild cohomology groups of order $Y$.

**Definition 4.1.** *Let $(\mathcal{L}^Y(A), {}_qb)$ be the $N$-complex corresponding to the simplicial vector space $\mathcal{L}^Y(A)$ as defined in (2.5). We consider the q-Hom complex $\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))$ of these $N$-complexes which is defined as follows:*

$$\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n := \prod_{i\in\mathbb{Z}} Hom_{Vect}(\mathcal{L}^Y(A)_i, \mathcal{L}^Y(A)_{i+n}) \tag{4.1}$$

*Further, if the family $f = \{f_i : \mathcal{L}^Y(A)_i \longrightarrow \mathcal{L}^Y(A)_{i+n}\}_{i\in\mathbb{Z}}$ is an element of $\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n$, then the differential ${}_q\partial_n : \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n \longrightarrow \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_{n-1}$ is defined by setting:*

$$\begin{aligned} {}_q\partial_n(f) &:= \{{}_q\partial_n(f)_i : \mathcal{L}^Y(A)_i \longrightarrow \mathcal{L}^Y(A)_{i+n-1}\}_{i\in\mathbb{Z}} \\ {}_q\partial_n(f)_i &= {}_qb_{i+n} \circ f_i - q^n f_{i-1} \circ {}_qb_i \end{aligned} \tag{4.2}$$

*For any given $n \in \mathbb{Z}$, we define the bivariant q-Hochschild cohomology group ${}_qHH^n_Y(A,A)$ of $A$ of order $Y$ to be the homology object*

$${}_qHH^n_Y(A,A) := H_{\{-n\}}(\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), {}_q\partial) \tag{4.3}$$

*of the $N$-complex $(\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), {}_q\partial)$.*

We mention that it follows from [5, Proposition 1.8] that the $q$-Hom complex $(\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), {}_q\partial)$ as defined in (4.1) and (4.2) is also an $N$-complex. We now make the convention that if $M = \oplus_{i\in\mathbb{Z}} M_i$ is a graded vector space and $f = \{f_i : M_i \longrightarrow M_{i+m}\}_{i\in\mathbb{Z}}$ and $g = \{g_i : M_i \longrightarrow M_{i+n}\}_{i\in\mathbb{Z}}$ are two morphisms of homogenous degree $m$ and $n$ respectively, we will write $[f,g] := f\circ g - q^{mn} g \circ f$ for their graded $q$-commutator.

**Lemma 4.2.** *Let $L^m = \{L^m_i\}_{i\in\mathbb{Z}}$ denote a collection of maps $L^m_i : \mathcal{L}^Y(A)_i \longrightarrow \mathcal{L}^Y(A)_{i+m}$. Given an element $f = \{f_i\}_{i\in\mathbb{Z}}$ in $\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n$, we define $\underline{L}^m(f) \in \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_{m+n}$ by setting:*

$$\underline{L}^m(f)_i : \mathcal{L}^Y(A)_i \longrightarrow \mathcal{L}^Y(A)_{i+m+n} \qquad \underline{L}^m(f)_i := L^m_{i+n} \circ f_i - q^{mn} f_{i+m} \circ L^m_i \tag{4.4}$$

*Then, if $q^{2m} = 1$, the endomorphism $\underline{L}^m : \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)) \longrightarrow \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))$ of homogeneous degree $m$ satisfies the following relation:*

$$[{}_q\partial, \underline{L}^m](f) = [{}_qb, L^m]f + q^{mn+m+n} f[L^m, {}_qb] \qquad \forall\ f \in \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n, n \in \mathbb{Z} \tag{4.5}$$

*Proof.* We consider:

$$
\begin{aligned}
((_q\partial \circ \underline{L}^m)(f))_i \; &= {}_qb_{i+m+n} \circ \underline{L}^m(f)_i - q^{m+n}\underline{L}^m(f)_{i-1} \circ {}_qb_i \\
&= {}_qb_{i+m+n} \circ L^m_{i+n} \circ f_i - q^{mn}{}_qb_{i+m+n} \circ f_{i+m} \circ L^m_i \\
&\quad - q^{m+n}L^m_{i+n-1} \circ f_{i-1} \circ {}_qb_i + q^{mn+m+n}f_{i+m-1} \circ L^m_{i-1} \circ {}_qb_i \\
((\underline{L}^m \circ {}_q\partial)(f))_i \; &= L^m_{i+n-1} \circ {}_q\partial(f)_i - q^{m(n-1)}{}_q\partial(f)_{i+m} \circ L^m_i \\
&= L^m_{i+n-1} \circ {}_qb_{i+n} \circ f_i - q^nL^m_{i+n-1} \circ f_{i-1} \circ {}_qb_i \\
&\quad - q^{m(n-1)}{}_qb_{i+m+n} \circ f_{i+m} \circ L^m_i + q^{mn-m+n}f_{i+m-1} \circ {}_qb_{i+m} \circ L^m_i
\end{aligned}
\tag{4.6}
$$

From (4.6), it follows that:

$$
\begin{aligned}
&([_q\partial, \underline{L}^m](f))_i = ((_q\partial \circ \underline{L}^m)(f))_i - q^{-m}((\underline{L}^m \circ {}_q\partial)(f))_i \\
&= (_qb_{i+m+n} \circ L^m_{i+n} - q^{-m}L^m_{i+n-1} \circ {}_qb_{i+n}) \circ f_i + f_{i+m-1} \circ q^{mn+m+n}(L^m_{i-1} \circ {}_qb_i - q^{-2m}(q^{-m}{}_qb_{i+m} \circ L^m_i)) \\
&-q^{mn}(1-q^{-2m})_qb_{i+m+n} \circ f_{i+m} \circ L^m_i - q^{m+n}(1-q^{-2m})L^m_{i+n-1} \circ f_{i-1} \circ {}_qb_i
\end{aligned}
$$

Combining with the fact that $q^{2m} = 1$, it follows from the above expression that:

$$
[_q\partial, \underline{L}^m](f) = [_qb, L^m]f + q^{mn+m+n}f[L^m, {}_qb] \tag{4.7}
$$

□

**Proposition 4.3.** *Let $q \in \mathbb{C}$ be a primitive $N$-th root of unity. Let $A$ be a commutative algebra over $\mathbb{C}$ and let $D : A \longrightarrow A$ be a derivation on $A$. Let $Y$ be a pointed simplicial finite set. Then, for each $n \in \mathbb{Z}$, the derivation $D$ on $A$ induces a morphism*

$$
\underline{L}^{Y,n}_D : {}_qHH^n_Y(A,A) \longrightarrow {}_qHH^n_Y(A,A) \tag{4.8}
$$

*on the bivariant $q$-Hochschild cohomology groups of order $Y$.*

*Proof.* From the proof of Proposition 2.6, we know that the derivation $D$ induces an endomorphism $L^Y_D : \mathcal{L}^Y(A) \longrightarrow \mathcal{L}^Y(A)$ of the simplicial vector space $\mathcal{L}^Y(A)$. Accordingly, we have a collection of maps $L^Y_D = \{L^Y_{D,i} : \mathcal{L}^Y(A)_i \longrightarrow \mathcal{L}^Y(A)_i\}_{i\in\mathbb{Z}}$ determined by the endomorphism $L^Y_D$. Applying Lemma 4.2 with $m = 0$ (and hence $q^{2m} = 1$), it follows that $L^Y_D$ determines a morphism

$$
\underline{L}^Y_D : \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)) \longrightarrow \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)) \tag{4.9}
$$

of homogeneous degree $m = 0$ satisfying:

$$
[_q\partial, \underline{L}^Y_D](f) = [_qb, L^Y_D]f + q^nf[L^Y_D, {}_qb] \qquad \forall\ f \in \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))_n, n \in \mathbb{Z} \tag{4.10}
$$

Again, since $L^Y_D : \mathcal{L}^Y(A) \longrightarrow \mathcal{L}^Y(A)$ is a morphism of simplicial vector spaces, the morphisms $\{L^Y_{D,i} : \mathcal{L}^Y(A)_i \longrightarrow \mathcal{L}^Y(A)_i\}_{i\in\mathbb{Z}}$ commute with the face maps $d^j_i : \mathcal{L}^Y(A)_i \longrightarrow \mathcal{L}^Y(A)_{i-1}$, $0 \leq j \leq i$, $i \geq 0$ of the simplicial vector space $\mathcal{L}^Y(A)$. By definition, ${}_qb_i := \sum_{j=0}^i q^jd^j_i$ and hence we have:

$$
[_qb, L^Y_D] = [L^Y_D, {}_qb] = 0 \tag{4.11}
$$

Applying this to (4.10), it follows that:

$$[{}_q\partial, \underline{L}_D^Y] = {}_q\partial \circ \underline{L}_D^Y - q^{-m}\underline{L}_D^Y \circ {}_q\partial = {}_q\partial \circ \underline{L}_D^Y - \underline{L}_D^Y \circ {}_q\partial = 0 \tag{4.12}$$

From (4.12), it follows that the endomorphism $\underline{L}_D^Y : \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)) \longrightarrow \underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))$ of degree zero commutes with the differential ${}_q\partial$ on the $N$-complex $\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A))$. This induces morphisms ($\forall\ n \in \mathbb{Z}$):

$$\begin{array}{c} {}_qHH_Y^n(A,A) = H_{\{-n\}}(\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), {}_q\partial) \\ \underline{L}_D^{Y,n} \Big\downarrow \\ {}_qHH_Y^n(A,A) = H_{\{-n\}}(\underline{Hom}(\mathcal{L}^Y(A), \mathcal{L}^Y(A)), {}_q\partial) \end{array} \tag{4.13}$$

on the bivariant $q$-Hochschild cohomology groups of order $Y$. □

## References


[1] A. Banerjee, Cartan homotopy formulae and the bivariant Hochschild complex, *C. R. Math. Acad. Sci. Paris. Ser I*, Vol **347**, (2009), 997–1000.

[2] A. Banerjee, Higher derivations and Hochschild homology, *C. R. Math. Rep. Acad. Sci. Canada* Vol. **33** (2) (2011), pp. 33–43.

[3] M. Dubois-Violette, $d^N = 0$: generalized homology, *K-Theory* **14** (1998), no. 4, 371–404.

[4] J. D. S. Jones, C. Kassel, Bivariant cyclic theory, *K-Theory* **3** (4) (1989) 339–365.

[5] M. M. Kapranov, On the $q$-analog of homological algebra, Preprint (1996), arXiv: q-alg/9611005.

[6] J.-L. Loday, Opérations sur l'homologie cyclique des algèbres commutatives. *Invent. Math.* Vol. **96** (1989), no. 1, 205–230.

[7] J.-L. Loday, Cyclic homology. Appendix E by Maria O. Ronco. Second edition. Chapter 13 by the author in collaboration with Teimuraz Pirashvili. *Grundlehren der Mathematischen Wissenschaften*, **301**. Springer-Verlag, Berlin, 1998.

[8] H. Matsumura, Commutative Ring Theory, Cambridge Studies in Advanced Mathematics, Vol. 8, Cambridge University Press, Cambridge, 1986.

[9] M. Mirzavaziri, Characterization of higher derivations on algebras. *Comm. Algebra*, Vol. **38** (2010), 981–987.

[10] T. Pirashvili, Hodge decomposition for higher order Hochschild homology. *Ann. Sci. École Norm. Sup.* (4) **33** (2000), no. 2, 151–179.

[11] A. Roy, R. Sridharan, Higher derivations and central simple algebras, *Nagoya Math. J.*, Vol. **32** (1968), 21–30.

[12] R. M. Skjelnes, Computing Hasse-Schmidt derivations and Weil restrictions over jets, *Journal of Algebra*, Vol. **411** (2014), 114–128.