EXPLICIT COPRODUCT FORMULA FOR QUANTUM GROUPS OF INFINITE SERIES

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ABSTRACT. We find an explicit form for the coproduct formula for PBW generators of quantum groups of infinite series $U_q(\mathfrak{sp}_{2n})$ and $U_q(\mathfrak{so}_{2n})$. Similar formulas for $U_q(\mathfrak{sl}_{n+1})$ and $U_q(\mathfrak{so}_{2n+1})$ are already known.

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1. INTRODUCTION

In the present paper, we prove an explicit coproduct formula for quantum groups $U_q(\mathfrak{g})$, where $\mathfrak{g} = \mathfrak{sp}_{2n}$ or $\mathfrak{g} = \mathfrak{so}_{2n}$ are simple Lie algebras of type C, D respectively. Consider a Weyl basis of the Lie algebra $\mathfrak{g}$,

$$u[k,m] = [\ldots[[x_k, x_{k+1}], x_{k+2}], \ldots, x_m],$$

see [21, Chapter VI, §4] or [6, Chapter IV, §3, XVII]. Here, $x_i = x_{2n-i}$ and in case C_n, we have $k \leq m \leq 2n-k$, whereas in case D_n, the sequence $x_1, x_2, \ldots, x_{2n-1}$ has no term x_{n-1} and $k \leq m < 2n-k$. If we replace the Lie operation by skew brackets, then the above basis becomes a set of PBW generators for the related quantum group $U_q(\mathfrak{g})$. We then find the coproduct of those PBW generators:

$$\Delta(u[k,m]) = u[k,m] \otimes 1 + g_{km} \otimes u[k,m] + \sum_{i=k}^{m-1} \tau_i (1-q^{-1}) g_{ki}\, u[i+1,m] \otimes u[k,i], \tag{1.1}$$

where g_{ki} is a group-like element that corresponds to $u[k,i]$, and almost all τ equal 1. More precisely, in case C_n, there is one exception: $\tau_{n-1} = 1 + q^{-1}$ if $m = n$. In case D_n, the exception is: $\tau_{n-1} = 0$ if $m = n$; and $\tau_{n-1} = p_{n\,n-1}$ otherwise.

Recall that the same formula is valid for $U_q(\mathfrak{sl}_{n+1})$ and $U_q(\mathfrak{so}_{2n+1})$. In case A_n there are no exceptions [13, Lemma 3.5]. In case B_n, the main parameter q becomes q^2, and we have an exception $\tau_n = q$, whereas in the sequence $x_1, x_2, \ldots, x_{2n}$, the variable x_n appears twice: $x_i = x_{2n-i+1}$, see [12, Theorem 4.3]. In case B_2, an explicit formula was established by M. Beattie, S. Dăscălescu, Ş. Raianu, [3].

In the formula, if $i \geq 2n-m$, then $u[i+1,m]$ does not appear in the list of the above PBW generators because $m > 2n-(i+1)$. The elements $u[k,m]$ with $m > 2n-k$ are defined in a similar manner,

$$u[k,m] = [x_k, [x_{k+1}, \ldots, [x_{m-1}, x_m]\ldots]].$$

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The formula remains valid for those elements as well, in which case all of the τ equal 1, except for $\tau_n = 1 + q^{-1}$ if $k = n$ in case C_n, and $\tau_n = 0$ if $k = n$ in case D_n (by definition, in case D_n, the sequence that defines $u[n, m]$ has the form $x_n, x_{n+2}, x_{n+3}, \ldots, x_m$). In other words, whereas the PBW generators do not span a subcoalgebra, the formula remains valid for a basis of the subcoalgebra generated by them. Furthermore, the formula demonstrates that the PBW generators span a left coideal.

We are reminded that M. Rosso [18] and H. Yamane [25] separately constructed PBW generators for $U_q(\mathfrak{sl}_{n+1})$. Then, G. Lusztig [17] found PBW bases for arbitrary $U_q(\mathfrak{g})$ in terms of his famous automorphisms defining the action of braid groups. A coproduct formula for PBW generators E_β in Lusztig form appeared in the paper by S.Z. Levendorski and Ya. S. Soibelman [15, Theorem 2.4.2]:

$$\Delta(E_\beta^n) - (E_\beta \otimes 1 + q^{H\beta} \otimes E_\beta)^n \in U_h(\mathfrak{n}_+)_\beta \otimes U_h(\mathfrak{B}_+). \tag{1.2}$$

Recently I. Heckenberger and H.-J. Schneider [5, Theorem 6.14] proved a similar formula within a more general context:

$$\Delta_{\mathfrak{B}(N)}(x) - x \otimes 1 \in \mathbf{k}\langle N_{\beta_{l-1}}\rangle \mathbf{k}\langle N_{\beta_{l-2}}\rangle \cdots \mathbf{k}\langle N_{\beta_1}\rangle \otimes \mathfrak{B}(N), \quad x \in N_{\beta_l}. \tag{1.3}$$

Although these formulas have no explicit form, they are convenient for inductive considerations, particularly in the study of one-sided coideal subalgebras.

We develop the coproduct formula by the same method as that in [12] for the case B_n. Firstly, we demonstrate that the values of the elements $u[k, m]$ in $U_q(\mathfrak{g})$ are almost independent of the arrangement of brackets (Lemmas 3.6, 3.7, 6.4, 6.5). Then, using this fact, we demonstrate that these values form a set of PBW generators (Propositions 4.1, 7.1). Next, we find the explicit shuffle representation of those elements (Propositions 4.2, 8.1). In case C_n (as well as in cases A_n and B_n) these PBW generators are proportional to shuffle comonomials. This proportionality makes it easy to find the coproduct of those elements inside the shuffle coalgebra. Because there is a clear connection (2.13) between the coproduct in $U_q(\mathfrak{g})$ and the coproduct in the shuffle coalgebra, we can set up the coproduct formula (Theorem 5.1). In case D_n, each PBW generator is either proportional to a comonomial or a linear combination of two comonomials. These two options allows one to find the coproduct inside the shuffle coalgebra and deduce the coproduct formula (Theorem 9.1).

The set of PBW generators for $U_q(\mathfrak{g})$ is the union of those sets for positive and negative quantum Borel subalgebras. Thus, we focus only on the positive quantum Borel subalgebra $U_q^+(\mathfrak{g})$.

2. Preliminaries

2.1. **Skew brackets.** Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of quantum variables; that is, associated with each x_i there are an element g_i of a fixed Abelian group G and a character $\chi^i : G \to \mathbf{k}^*$. For every word w in X, let g_w or $\mathrm{gr}(w)$ denote an element of G that appears from w by replacing each x_i with g_i. In the same manner, χ^w denotes a character that appears from w by replacing each x_i with χ^i.

Let $G\langle X\rangle$ denote the skew group algebra generated by G and $\mathbf{k}\langle X\rangle$ with the commutation rules $x_i g = \chi^i(g) g x_i$, or equivalently $wg = \chi^w(g) g w$, where w is an arbitrary word in X. If u, v are homogeneous in each x_i, $1 \leq i \leq n$ polynomials, then the skew brackets are defined by the formula

$$[u,v]=uv-\chi^u(g_v)vu. \tag{2.1}$$

We use the notation $\chi^u(g_v)=p_{uv}=p(u,v)$. The form $p(\text{-},\text{-})$ is bimultiplicative:

$$p(u,vt)=p(u,v)p(u,t),\quad p(ut,v)=p(u,v)p(t,v). \tag{2.2}$$

In particular $p(\text{-},\text{-})$ is completely defined by n^2 parameters $p_{ij}=\chi^i(g_j)$.

The brackets satisfy an analog of the Jacobi identity:

$$[[u,v],w]=[u,[v,w]]+p_{wv}^{-1}[[u,w],v]+(p_{vw}-p_{wv}^{-1})[u,w]\cdot v. \tag{2.3}$$

The antisymmetry identity transforms as follows:

$$[u,v]=-p_{uv}[v,u]+(1-p_{uv}p_{vu})u\cdot v \tag{2.4}$$

The Jacobi identity (2.3) implies a conditional identity:

$$[[u,v],w]=[u,[v,w]],\ \text{provided that } [u,w]=0. \tag{2.5}$$

By the evident induction on length, this result allows for the following generalization:

Lemma 2.1. [14, Lemma 2.2]. *If y_1, y_2, …, y_m are homogeneous linear combinations of words such that $[y_i,y_j]=0$, $1\leq i<j-1<m$, then the bracketed polynomial $[y_1y_2\dots y_m]$ is independent of the precise arrangement of brackets:*

$$[y_1y_2\dots y_m]=[[y_1y_2\dots y_s],[y_{s+1}y_{s+2}\dots y_m]],\ 1\leq s<m. \tag{2.6}$$

Another conditional identity is: if $[u,v]=0$ (that is, $uv=p_{uv}vu$), then

$$[u,[v,w]]=-p_{vw}[[u,w],v]+p_{uv}(1-p_{vw}p_{wv})v\cdot[u,w]. \tag{2.7}$$

The brackets are related to the product by ad-identities:

$$[u\cdot v,w]=p_{vw}[u,w]\cdot v+u\cdot[v,w], \tag{2.8}$$

$$[u,v\cdot w]=[u,v]\cdot w+p_{uv}v\cdot[u,w]. \tag{2.9}$$

It is easy to verify all of the identities developing the brackets by (2.1).

2.2. **Quantum Borel algebra.** The group G acts on the free algebra $\mathbf{k}\langle X\rangle$ by $g^{-1}ug=\chi^u(g)u$, where u is an arbitrary monomial in X. The skew group algebra $G\langle X\rangle$ has a natural Hopf algebra structure

$$\Delta(x_i)=x_i\otimes 1+g_i\otimes x_i,\quad i\in I,\quad \Delta(g)=g\otimes g,\ g\in G.$$

Let $C=||a_{ij}||$ be a symmetrizable Cartan matrix and let $D=\text{diag}(d_1,\dots,d_n)$ be such that $d_ia_{ij}=d_ja_{ji}$. We denote a Kac-Moody algebra defined by C, see [7], as $\mathfrak{g}$. Suppose that parameters p_{ij} are related by

$$p_{ii}=q^{d_i},\quad p_{ij}p_{ji}=q^{d_ia_{ij}},\quad 1\leq i,j\leq n. \tag{2.10}$$

In this case the multiparameter quantization $U_q^+(\mathfrak{g})$ is a homomorphic image of $G\langle X\rangle$ defined by Serre relations with the skew brackets in place of the Lie operation:

$$[\dots[[x_i,\underbrace{x_j],x_j],\dots,x_j]}_{1-a_{ji}\text{ times}}=0,\quad 1\leq i\neq j\leq n. \tag{2.11}$$

By [8, Theorem 6.1], the left-hand sides of these relations are skew-primitive elements in $G\langle X\rangle$. Therefore the ideal generated by these elements is a Hopf ideal, hence $U_q^+(\mathfrak{g})$ has the natural structure of a Hopf algebra.

2.3. **PBW basis.** Recall that a linearly ordered set V is said to be a *set of PBW generators* (of infinite heights) if the set of all products

$$g\cdot v_1^{n_1}\cdot v_2^{n_2}\cdot\ \cdots\ \cdot v_k^{n_k},\quad g\in G,\quad v_1<v_2<\ldots<v_k\in V \tag{2.12}$$

is a basis of $U_q^+(\mathfrak{g})$.

We fix the order $x_1>x_2>\ldots>x_n$ on the set X. On the set of all words in X, we fix the lexicographical order with the priority from left to right, where a proper beginning of a word is considered to be greater than the word itself.

A non-empty word u is called a *standard Lyndon-Shirshov* word if $vw>wv$ for each decomposition $u=vw$ with non-empty v,w. The *standard arrangement* of brackets $[u]$ on a standard word u is defined by induction: $[u]=[[v][w]]$, where v,w are the standard words such that $u=vw$ and v has the minimal length [22], [23], see also [16].

In [9], it was proven that the values of bracketed standard words corresponding to positive roots with the lexicographical order form a set of PBW generators (of infinite heights) for $U_q^+(\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra of infinite series A,B,C,D.

2.4. **Shuffle representation.** The $\mathbf{k}$-algebra A generated by values of x_i, $1\leq i\leq n$ in $U_q^+(\mathfrak{g})$ is not a Hopf subalgebra because it has no nontrivial group-like elements. Nevertheless, A is a Hopf algebra in the category of Yetter-Drinfeld modules over $\mathbf{k}[G]$. In particular, A has a structure of a braided Hopf algebra with a braiding $\tau(u\otimes v)=p(v,u)^{-1}v\otimes u$. The braided coproduct $\Delta^b: A\rightarrow A\underline{\otimes}A$ is connected with the coproduct on $U_q^+(\mathfrak{g})$ as follows

$$\Delta^b(u)=\sum_{(u)}u^{(1)}\mathrm{gr}(u^{(2)})^{-1}\underline{\otimes}u^{(2)},\ \text{where}\ \ \Delta(u)=\sum_{(u)}u^{(1)}\otimes u^{(2)}. \tag{2.13}$$

The tensor space $T(W)$, $W=\sum x_i\mathbf{k}$ also has the structure of a braided Hopf algebra, which is the *braided shuffle algebra* $Sh_\tau(W)$ with the coproduct

$$\Delta^b(u)=\sum_{i=0}^{m}(z_1\ldots z_i)\underline{\otimes}(z_{i+1}\ldots z_m), \tag{2.14}$$

where $z_i\in X$, and $u=(z_1z_2\ldots z_{m-1}z_m)$ is the tensor $z_1\otimes z_2\otimes\ldots\otimes z_{m-1}\otimes z_m$, called a *comonomial*, considered as an element of $Sh_\tau(W)$. The braided shuffle product satisfies

$$(w)(x_i)=\sum_{uv=w}p(x_i,v)^{-1}(ux_iv),\quad (x_i)(w)=\sum_{uv=w}p(u,x_i)^{-1}(ux_iv). \tag{2.15}$$

The map $x_i\rightarrow(x_i)$ defines a homomorphism of the braided Hopf algebra A into the braided Hopf algebra $Sh_\tau(W)$. This is extremely useful for calculating the coproducts due to formulae (2.13) and (2.14). If q is not a root of 1, then this representation is faithful. Otherwise, its kernel is the largest Hopf ideal in $A^{(2)}$, where $A^{(2)}$ is the ideal of A generated by values of x_ix_j, $1\leq i,j\leq n$. See details in P. Schauenberg [20], M. Rosso [19], M. Takeuchi [24], D. Flores de Chela and J.A. Green [4], N. Andruskiewitsch, H.-J. Schneider [1], V. K. Kharchenko [10].

3. Relations in $U_q^+(\mathfrak{sp}_{2n})$.

Throughout the following three sections, we fix a parameter q such that $q^3 \neq 1$, $q \neq -1$. If C is a Cartan matrix of type C_n, then relations (2.10) take the form

$$p_{ii} = q,\ 1 \leq i < n;\quad p_{i\,i-1}p_{i-1\,i} = q^{-1},\ 1 < i < n;\ p_{ij}p_{ji} = 1,\ j > i+1; \tag{3.1}$$

$$p_{nn} = q^2,\quad p_{n-1\,n}p_{n\,n-1} = q^{-2}. \tag{3.2}$$

In this case, the quantum Borel algebra $U_q^+(\mathfrak{sp}_{2n})$ is a homomorphic image of $G\langle X\rangle$ subject to the following relations

$$[x_i, [x_i, x_{i+1}]] = [[x_i, x_{i+1}], x_{i+1}] = 0,\ 1 \leq i < n-1;\quad [x_i, x_j] = 0,\quad j > i+1; \tag{3.3}$$

$$[[x_{n-1}, x_n], x_n] = [x_{n-1}, [x_{n-1}, [x_{n-1}, x_n]]] = 0. \tag{3.4}$$

Lemma 3.1. *If u is a standard word independent of x_n, then either $u = x_k x_{k+1} \ldots x_m$, $k \leq m < n$, or $[u] = 0$ in $U_q^+(\mathfrak{sp}_{2n})$. Here $[u]$ is a nonassociative word with the standard arrangement of brackets.*

Proof. The Hopf subalgebra of $U_q^+(\mathfrak{sp}_{2n})$ generated by x_i, $1 \leq i < n$ is the Hopf algebra $U_q^+(\mathfrak{sl}_n)$ defined by the Cartan matrix of type A_{n-1}. By this reason the third statement of [9, Theorem A_n] applies. □

Definition 3.2. In what follows, x_i, $n < i < 2n$ denotes the generator x_{2n-i}. Respectively, $v(k,m)$, $1 \leq k \leq m < 2n$ is the word $x_k x_{k+1} \cdots x_{m-1} x_m$, whereas $v(m,k)$ is the opposite word $x_m x_{m-1} \cdots x_{k+1} x_k$. If $1 \leq i < 2n$, then $\phi(i)$ denotes the number $2n - i$, so that $x_i = x_{\phi(i)}$.

Definition 3.3. If $k \leq i < m < 2n$, then we set

$$\sigma_k^m \stackrel{df}{=} p(v(k,m), v(k,m)), \tag{3.5}$$

$$\mu_k^{m,i} \stackrel{df}{=} p(v(k,i), v(i+1,m)) \cdot p(v(i+1,m), v(k,i)). \tag{3.6}$$

Lemma 3.4. *For each i, $k \leq i < m$ we have*

$$\mu_k^{m,i} = \sigma_k^m (\sigma_k^i \sigma_{i+1}^m)^{-1}. \tag{3.7}$$

Proof. Because $p(\text{-},\text{-})$ is a bimultiplicative map, there is a decomposition

$$p(ab, ab) = p(a,a)p(b,b) \cdot p(a,b)p(b,a). \tag{3.8}$$

Applying this equality to $a = v(k,i)$, $b = v(i+1,m)$, we get the required relation. □

Lemma 3.5. *If $1 \leq k \leq m < 2n$, then*

$$\sigma_k^m = \begin{cases} q^2, & \text{if } m = \phi(k); \\ q, & \text{otherwise.} \end{cases} \tag{3.9}$$

Proof. The bimultiplicativity of $p(\text{-},\text{-})$ implies that $\sigma_k^m = \prod_{k\leq s,t\leq m} p_{st}$ is the product of all coefficients of the $(m-k+1)\times(m-k+1)$-matrix $||p_{st}||$. By (3.1) all coefficients on the main diagonal equal q except $p_{nn}=q^2$.

If $m<n$ or $k>n$, then for non-diagonal coefficients, we have $p_{st}p_{ts}=1$ unless $|s-t|=1$, whereas $p_{s\,s+1}p_{s+1\,s}=q^{-1}$. Hence, $\sigma_k^m = q^{m-k+1}\cdot q^{-(m-k)}=q$.

If $m=n$ or $k=n$ but not both, then we have $p_{nn}=q^2$, $p_{n\,n-1}p_{n-1\,n}=q^{-2}$. By the above reasoning, we get $\sigma_k^m = q^{(m-k)+2}\cdot q^{-(m-k-1)-2}=q$. Of course, if $k=n=m$ then $\sigma_k^m=p_{nn}=q^2$.

In the remaining case, $k<n<m$, we use induction on $m-k$.

By (3.8) we have

$$\sigma_k^{m+1} = \sigma_k^m \cdot q\cdot p(v(k,m),x_{m+1})\cdot p(x_{m+1},v(k,m)). \tag{3.10}$$

We shall prove that if $k<n<m$, then

$$p(v(k,m),x_{m+1})\cdot p(x_{m+1},v(k,m)) = \begin{cases} 1, & \text{if } k=\phi(m)-1;\\ q^{-2}, & \text{if } k=\phi(m);\\ q^{-1}, & \text{otherwise.}\end{cases} \tag{3.11}$$

The left hand side of the above equality is $\prod_{k\leq t\leq m} p_{t\,m+1}p_{m+1\,t}$. If $m=n$, then by 3.1 and 3.2, the factor $p_{t\,n+1}p_{n+1\,t}$ differs from 1 only if $t\in\{n-2,n-1,n\}$ and related values are respectively q^{-1},q^2,q^{-2}. Hence, if $k<n-1=\phi(m)-1$, then the total product is q^{-1}; if $k=n-1=\psi(m)-1$, then this is 1; if $k=n=\phi(m)$, then this is q^{-2}.

If $m>n$, then the factor $p_{t\,m+1}p_{m+1\,t}$ differs from 1 only if

$$t\in\{\phi(m)-2,\phi(m)-1,\phi(m),m\}$$

and related values are respectively q^{-1},q^2,q^{-1},q^{-1}. Therefore if $k<\phi(m)-1$, then the whole product is q^{-1}; if $k=\phi(m)-1$, then this is 1; if $k=\phi(m)$, then this is q^{-2}; if $k>\phi(m)$, then this is again q^{-1}. This completes the proof of (3.11).

To complete the inductive step, we use (3.11) and inductive hypothesis: if $k=\phi(m)-1$, then $\sigma_k^{m+1}=q\cdot q\cdot 1=q^2$; if $k=\phi(m)$, then $\sigma_k^{m+1}=q^2\cdot q\cdot q^{-2}=q$; otherwise $\sigma_k^{m+1}=q\cdot q\cdot q^{-1}=q$. □

We define the bracketing of $v(k,m)$, $k\leq m<2n$ as follows.

$$v[k,m] = \begin{cases} [[[\ldots[x_k,x_{k+1}],\ldots],x_{m-1}],x_m], & \text{if } m<\phi(k);\\ [x_k,[x_{k+1},[\ldots,[x_{m-1},x_m]\ldots]]], & \text{if } m>\phi(k);\\ [\![v[k,m-1],x_m]\!], & \text{if } m=\phi(k),\end{cases} \tag{3.12}$$

where in the latter term, $[\![u,v]\!] \stackrel{df}{=} uv-q^{-1}p(u,v)vu$.

Conditional identity (2.6) demonstrates that the value of $v[k,m]$ in $U_q^+(\mathfrak{sp}_{2n})$ is independent of the precise arrangement of brackets, provided that $m\leq n$ or $k\geq n$. Now we are going to analyze what happens with the arrangement of brackets if $k<n<m\neq\phi(k)$.

Lemma 3.6. *If $k\leq n\leq m<\phi(k)$, then the value in $U_q^+(\mathfrak{sp}_{2n})$ of the bracketed word $[y_kx_nx_{n+1}\cdots x_m]$, where $y_k=v[k,n-1]$, is independent of the precise arrangement of brackets.*

Proof. To apply (2.6), it suffices to check $[y_k,x_t]=0$, where $n<t\leq m$ or, equivalently, $\phi(m)\leq t<n$. By (2.5) we have

$$[y_k,x_t]=\big[[v[k,t-2],v[t-1,n-1]],x_t\big]=\big[v[k,t-2],[v[t-1,n-1],x_t]\big].$$

By Lemma 3.1 the element $[v[t-1,n-1],x_t]$ equals zero in $U_q^+(\mathfrak{so}_{2n})$ because the word $u(t-1,n)x_t$ is independent of x_n, it is standard, and the standard bracketing is precisely $[v[t-1,n],x_t]$. □

Lemma 3.7. *If $k\leq n$, $\phi(k)<m$, then the value in $U_q^+(\mathfrak{sp}_{2n})$ of the bracketed word $[x_kx_{k+1}\cdots x_ny_m]$, where $y_m=v[n{+}1,m]$, is independent of the precise arrangement of brackets.*

Proof. To apply (2.6), we need the equalities $[x_t,y_m]=0$, $k\leq t<n$. The polynomial $[x_t,y_m]$ is independent of x_n. Moreover, $[x_t,y_m]$ is proportional to $[y_m,x_t]$ due to antisymmetry identity (2.4) because

$$p(x_t,y_m)p(y_m,x_t)=p_{t\,t+1}p_{tt}p_{t\,t-1}\cdot p_{t+1\,t}p_{tt}p_{t-1\,t}=1.$$

The equality $[y_m,x_t]=0$ turns to the proved above equality $[v[k,n-1],x_t]=0$ if one renames the variables $x_{n+1}\leftarrow x_k$, $x_{n+2}\leftarrow x_{k+1},\ldots.$ □

4. PBW generators of $U_q^+(\mathfrak{sp}_{2n})$

Proposition 4.1. *If $q^3\neq 1$, $q\neq -1$, then values of the elements $v[k,m]$, $k\leq m\leq \phi(k)$ form a set of PBW generators with infinite heights for the algebra $U_q^+(\mathfrak{sp}_{2n})$ over $\mathbf{k}[G]$.*

Proof. A word $v(k,m)$ is a standard Lyndon-Shirshov word provided that $k\leq m<\phi(k)$. By [9, Theorem C_n, p. 218] these words with the standard bracketing, say $[v(k,m)]$, become a set of PBW generators if we add to them the elements $[v_k]\stackrel{df}{=}[v[k,n-1],v[k,n]]$ $1\leq k<n$. We shall use induction on $m-k$ in order to demonstrate that the value in $U_q^+(\mathfrak{sp}_{2n})$ of $[v(k,m)]$, $k\leq m<\phi(k)$ is the same as the value of $v[k,m]$ with the bracketing given in (3.12).

If $m\leq n$, then the value of $v[k,m]$ is independent of the arrangement of brackets, see Lemma 2.1.

If $k<n<m$, then according to [9, Lemma 7.18], the brackets in $[v(k,m)]$ are set by the following recurrence formulae (we note that $[v(k,m)]=[v_{k\,\phi(m)}]$ in the notations of [9]):

$$\begin{aligned}&[v(k,m)]=[x_k[v(k+1,m)]], &&\text{if } m<\phi(k)-1;\\&[v(k,m)]=[[v(k,m-1)]x_m], &&\text{if } m=\phi(k)-1.\end{aligned}\tag{4.1}$$

In the latter case, the induction applies directly. In the former case, using induction and Lemma 3.6, we have

$$[v(k+1,m)]=v[k+1,m]=[v[k+1,n-1],v[n,m]].$$

At the same time $[x_k,x_t]=0$, $n\leq t\leq m$ because $x_t=x_{\phi(t)}$ and $\phi(t)\geq\phi(m)>k+1$. This implies $[x_k,v[n,m]]=0$. Applying conditional identity (2.5), we get

$$[v(k,m)]=[x_k[v[k+1,n-1],v[n,m]]]=\big[[x_kv[k+1,n-1]],v[n,m]\big]=v[k,m].$$

It remains to analyze the case $m=\phi(k)$. We have to demonstrate that if in the set V of PBW generators of Lyndon-Shirshov standard words one replaces the elements $[v_k]$ with $v[k,\phi(k)]$, $1\leq k<n$ then the obtained set is still a set of PBW generators. To do this, due to [11, Lemma 2.5] with $T\leftarrow\{v[k,\phi(k)],1\leq k<n\}$, $S\leftarrow U_q(\mathfrak{sp}_{2n})$, it suffices to see that the leading term of the PBW decomposition of $v[k,\phi(k)]$ in the generators V is proportional to $[v_k]$.

By definition (3.12) with $m = \phi(k)$, we have

$$v[k,m] = v[k,m-1]x_m - q^{-1}\pi x_m v[k,m-1]$$
$$= -q^{-1}\pi[x_m, v[k,m-1]] + (1-q^{-1}\pi\pi')v[k,m-1]\cdot x_m,$$

where $\pi = p(v(k,m-1),x_m)$, $\pi' = p(x_m, v(k,m-1))$. The second term of the latter sum is a basis element (2.12) in the PBW generators V. This element starts with $v[k,m-1]$ which is lesser than $[v_k]$. Hence it remains to analyze the bracket $[x_k, v[k,m-1]]$.

If $k = n-1$, then $[x_k, v[k,m-1]] = [x_{n-1}, [x_{n-1}, x_n]] = [v_k]$.

If $k < n-1$, then by Lemma 3.6 we have

$$[x_k, v[k,m-1]] = [x_k, [v[k,n], v[n+1,m-1]]]. \tag{4.2}$$

The basic relations (3.3) imply $[x_k,[x_k,x_{k+1}]] = 0$, $[x_k, v[k+2,n]] = 0$. By Lemma 2.1 value of $v[k,n]$ is independent of the arrangement of brackets,

$$v[k,n] = [[x_k, x_{k+1}], v[k+2,n]],$$

hence $[x_k, v[k,n]] = 0$.

By Eq. (2.7) with $u \leftarrow x_k$, $v \leftarrow v[k,n]$, $w \leftarrow v[n+1,m-1]$, the right hand side of (4.2) is a linear combination of the following two elements:

$$[x_k, v[n+1,m-1]], v[k,n]], \qquad v[k,n]\cdot[x_k, v[n+1,m-1]]. \tag{4.3}$$

The latter element starts with a factor $v[k,n]$ which is lesser than $[v_k]$. Hence it remains to prove that the leading term of the former element is proportional $[v_k]$.

By downward induction on k we shall prove the following decomposition

$$v[n+1,m-1] = \alpha\, v[k+1,n-1] + \sum_{s=k+2}^{n-1} \gamma_s\, v[s,n-1]\cdot U_s, \quad \alpha \neq 0. \tag{4.4}$$

If $k = n-2$, then this decomposition reduces to $x_{n+1} = x_{n-1}$. Let us apply $[\text{-}, x_m]$ to the both sides of the above equality. Using (2.4), we see that $[v[k+1,n-1], x_m]$ is proportional to $v[k,n-1] + \gamma_{k+1}\, v[k+1,n-1]\cdot x_m$, whereas (2.8) implies $[v[s,n-1]\cdot U_s, x_m] = v[s,n-1]\cdot[U_s, x_{k-1}]$ for $s \geq k+2$. This completes the inductive step.

Let us apply $[x_k, \text{-}]$ to both sides of the already proved Eq. (4.4). By (2.9), we get

$$[x_k, v[n+1,m-1]] = \alpha\, v[k,n-1] + \sum_{s=k+2}^{n-1} \gamma_s'\, v[s,n-1]\cdot[x_k, U_s]. \tag{4.5}$$

Finally, let us apply $[\text{-}, v[k,n]]$ to both sides of (4.5). In this way we find a decomposition of the first element of (4.3):

$$[[x_k, v[n+1,m-1]], v[k,n]] = \alpha\,[v_k] + \sum_{s=k+2}^{n-1} \gamma_s'\, v[s,n-1]\cdot[x_k,U_s]\cdot v[k,n] \tag{4.6}$$
$$- \sum_{s=k+2}^{n-1} \gamma_s''\, v[k,n]\cdot v[s,n-1]\cdot[x_k, U_s].$$

All summands, except the first one, start with $v[k,n]$, $v[s,n-1]$ that are lesser than $[v_k]$. Hence, the leading term, indeed, is proportional to $[v_k]$. □

Proposition 4.2. *Let $k \leq m < 2n$. In the shuffle representation, we have*

$$v[k,m] = \alpha_k^m \cdot (v(m,k)), \quad \alpha_k^m \stackrel{df}{=} \varepsilon_k^m (q-1)^{m-k} \cdot \prod_{k\leq i<j\leq m} p_{ij}, \tag{4.7}$$

where

$$\varepsilon_k^m = \begin{cases} 1+q, & \text{if } k \leq n \leq m, m \neq \phi(k); \\ 1+q^{-1}, & \text{if } m = \phi(k) \neq n; \\ 1, & \text{otherwise.} \end{cases} \tag{4.8}$$

Proof. We use induction on $m-k$. If $m=k$, then the equality reduces to $x_k = (x_k)$.

a). Consider first the case $m < \phi(k)$. By the inductive supposition, we have $v[k, m-1] = \alpha_k^{m-1} \cdot (w)$, $w = v(m-1,k)$. Using (2.15), we may write

$$v[k,m] = \alpha_k^{m-1}\{(w)(x_m) - p(w,x_m)\cdot (x_m)(w)\}$$

$$= \alpha_k^{m-1} \sum_{uv=w} \{p(x_m,v)^{-1} - p(v,x_m)\}(ux_mv), \tag{4.9}$$

where $p(v,x_m) = p(w,x_m)p(u,x_m)^{-1}$ because $w = uv$.

If $m \leq n$, then relations (3.2) imply $p(v,x_m)p(x_m,v) = 1$ with only one exception being $v = w$. Hence, sum (4.9) has just one term. The coefficient of $(x_m w) = (v(m,k))$ equals

$$\alpha_k^{m-1} p(w,x_m)(p(w,x_m)^{-1}p(x_m,w)^{-1} - 1) = \alpha_k^{m-1} \prod_{i=k}^{m-1} p_{im} \cdot (p_{m-1\,m}^{-1} p_{m\,m-1}^{-1} - 1).$$

If $m < n$, then the latter factor equals $q-1$, whereas if $m = n$, then it is $q^2 - 1 = \varepsilon_k^n (q-1)$.

Suppose that $m > n$ and still $m < \phi(k)$. In decomposition (4.9), we have $v = v(s,k)$, $k \leq s < m$ and hence $p(x_m,v)p(v,x_m) = \prod_{t=k}^{s} p_{mt}p_{tm}$. The product $p_{mt}p_{tm}$ differs from 1 only if $t \in \{\phi(m)-1, \phi(m), \phi(m)+1, m-1\}$; related values are q^{-1}, q^2, q^{-1}, q^{-1} if $m > n+1$, and they are q^{-1}, q^2, q^{-2} if $m = n+1$, $\phi(m)+1 = m-1$. This implies

$$p(x_m,v)p(v,x_m) = \begin{cases} q^{-1}, & \text{if } s = \phi(m)-1, \text{ or } s = m-1; \\ q, & \text{if } s = \phi(m); \\ 1, & \text{otherwise.} \end{cases} \tag{4.10}$$

Hence, in (4.9), only three terms remain with $s = \phi(m)-1$, $s = \phi(m)$, and $s = m-1$. If $s = \phi(m)-1$ or $s = \phi(m)$, then (ux_mv) equals

$$ux_mv = v(m-1, \phi(m)+1)x_m^2 v(\phi(m)-1,k),$$

whereas the coefficient of the comonomial (ux_mv) in sum (4.9) is

$$p(x_m,v_0)^{-1} - p(v_0,x_m) + p(x_m,x_mv_0)^{-1} - p(x_mv_0,x_m),$$

where $v_0 = v(\psi(m)-1,k)$. Taking into account (4.10), we find the above sum:

$$p(v_0,x_m)(q-1+q\cdot q^{-1} - q) = 0.$$

Thus, in (4.9) only one term remains, with $v = v(m-1,k)$, $u = \emptyset$. This term has the required coefficient:

$$\alpha_k^m = \alpha_k^{m-1}(p(x_m,w)^{-1} - p(w,x_m)) = \alpha_k^{m-1}p(w,x_m)(q-1).$$

b). Consider the case $m > \phi(k)$. By the inductive supposition, we have

$$v[k+1,m] = \alpha_{k+1}^m \cdot (w), \quad w = v(m,k+1).$$

Using (2.15), we get

$$v[k,m] = \alpha_{k+1}^m \{(x_k)(w) - p(x_k,w)\cdot(w)(x_k)\}$$

$$= \alpha_{k+1}^m \sum_{uv=w} \{p(u,x_k)^{-1} - p(x_k,w)p(x_k,v)^{-1}\}(ux_kv).$$

$$= \alpha_{k+1}^m \sum_{uv=v(m,k+1)} p(x_k,u\{p(u,x_k)^{-1}p(x_k,u)^{-1} - 1\}(ux_kv). \tag{4.11}$$

If $k \geq n$, then $p(u,x_k)p(x_k,u) = 1$, unless $u = w$. Hence, (4.11) has only one term, and the coefficient equals

$$\alpha_{k+1}^m p(x_k,w)(p(w,x_k)^{-1}p(x_k,w)^{-1} - 1) = \alpha_{k+1}^m p(x_k,w)(p_{k+1\,k}^{-1}p_{k\,k+1}^{-1} - 1).$$

If $k > n$, then the latter factor equals $q-1$, whereas if $k = n$, then it is $q^2 - 1$ $= (q-1)\varepsilon_n^m$ as claimed.

Suppose that $k < n$. In this case, $x_k = x_t$ with $m > t \overset{df}{=} \phi(k) > \phi(n) = n$. Let $u = v(m,s)$.

If $s > t+1$, then u depends only on x_i, $i < k-1$, and relations (3.1), (3.2) imply $p(x_k,u)p(u,x_k) = 1$.

If $s < t$, $s \neq k+1$, then $k+1 < n$ (otherwise $s = n = k+1$), and we have $p(x_k,u)p(u,x_k) = p_{k-1\,k}p_{kk}p_{k+1\,k}\cdot p_{k\,k-1}p_{kk}p_{k+1\,k} = 1$ because $x_t = x_k$.

Hence, three terms remain in (4.11) with $s = t$, $s = t+1$, and $s = k+1$. If $u = v(m,t)$ or $u = v(m,t+1)$, then $ux_kv = v(m,t+1)x_k^2v(t-1,k)$, whereas the coefficient of the corresponding tensor is

$$p(v(m,t+1),x_k)^{-1} - p(x_k,v(m,t+1)) + p(v(m,t),x_k)^{-1} - p(x_k,v(m,t))$$

$$= p(x_k,v(m,t+1))\{p_{k-1\,k}^{-1}p_{k\,k-1}^{-1} - 1 + p_{kk}^{-1}p_{k-1\,k}^{-1}p_{k\,k-1}^{-1} - p_{kk}\} = 0$$

because $p_{kk} = q$, $p_{k-1\,k}p_{k\,k-1} = q^{-1}$, and $p_{kr}p_{rk} = 1$ if $r > t+1$.

Thus, only one term remains in (4.9), and

$$\alpha_k^m = \alpha_{k+1}^m(p(w,x_k)^{-1} - p(x_k,w)) = \alpha_{k+1}^m p(x_k,w)(q-1).$$

c). Let $m = \phi(k) \neq n$. In this case, $x_m = x_k$, $p_{kk} = q$. By definition (3.12) we have

$$v[k,m] = v[k,m-1]\cdot x_k - q^{-1}p(v(k,m-1),x_m)x_k\cdot v[k,m-1]. \tag{4.12}$$

Case a) allows us to find the shuffle representation

$$v[k,m-1] = \alpha_k^{m-1}(w), \quad w = v(m-1,k).$$

Hence the right-hand side of (4.12) in the shuffle form is

$$\alpha_k^{m-1} \sum_{uv=w} \left(p(x_k,v)^{-1} - q^{-1}p(v(k,m-1),x_m)\cdot p(u,x_k)^{-1}\right)\cdot(ux_kv)$$

$$= \alpha_k^{m-1} \sum_{uv=v(m-1,k)} p(v,x_m)\left(p(x_k,v)^{-1}p(v,x_k)^{-1} - q^{-1}\right)\cdot(ux_kv). \tag{4.13}$$

The coefficient of $(v(k,m))$ related to $u=\emptyset$, $v=v(m-1,k)$ equals

$$\alpha_k^{m-1}p(v,x_m)\cdot(p(x_k,v)^{-1}p(v,x_k)^{-1}-q^{-1})=\alpha_k^{m-1}(1-q^{-1})\cdot\prod_{i=k}^{m-1}p_{im}. \tag{4.14}$$

Here we have used $x_k=x_m$ and Eq. (3.11) with $m\leftarrow m-1$, $k\leftarrow k$. It remains to show that all other terms in (4.13) are canceled. In this case we would have

$$\varepsilon_k^m=\varepsilon_k^{m-1}(1-q^{-1})(q-1)^{-1}=(1+q)(1-q^{-1})(q-1)^{-1}=1+q^{-1}$$

as required.

If $u=v(m-1,k)$, $v=\emptyset$ or $u=v(m-1,k+1)$, $v=x_k$, then

$$ux_kv=v(m-1,k+1)x_k^2,$$

whereas the total coefficient of the related comonomial is proportional to

$$1-q^{-1}\cdot p(u,x_m)p(u,x_k)^{-1}+p_{kk}^{-1}-q^{-1}\cdot p_{kk}=0.$$

Let $u=v(m-1,s)$, $v=v(s-1,k)$, $k+1<s<m$. The whole coefficient of the comonomial (ux_kv) takes the form

$$\alpha_k^{m-1}p(v(s-1,k),x_m)\cdot\left(p(x_k,v(s-1,k))^{-1}p(v(s-1,k),x_k)^{-1}-q^{-1})\right).$$

The latter factor equals $\prod_{t=k}^{s-1}p_{kt}^{-1}p_{tk}^{-1}-q^{-1}$. The product $p_{kt}p_{tk}$ differs from 1 only if $t\in\{k,k+1\}$ and related values are q^2 and q^{-1}. This implies that the coefficient of (ux_kv) has a factor $q^{-2}\cdot q-q^{-1}=0$. □

5. Coproduct formula for $U_q^+(\mathfrak{sp}_{2n})$

Theorem 5.1. *In $U_q^+(\mathfrak{sp}_{2n})$ the coproduct on the elements $v[k,m]$, $k\leq m<2n$ has the following explicit form*

$$\Delta(v[k,m])=v[k,m]\otimes 1+g_{km}\otimes v[k,m] + \sum_{i=k}^{m-1}\tau_i(1-q^{-1})g_{ki}\,v[i+1,m]\otimes v[k,i], \tag{5.1}$$

where $\tau_i=1$ with two exceptions, being $\tau_{n-1}=1+q^{-1}$ if $m=n$, and $\tau_n=1+q^{-1}$ if $k=n$. Here $g_{ki}=\mathrm{gr}(v(k,i))=g_kg_{k+1}\cdots g_i$.

Proof. By Proposition 4.2 we have the shuffle representation

$$v[k,m]=\alpha_k^m\cdot(v(m,k)). \tag{5.2}$$

Using (2.14), it is easy to find the braided coproduct of the comonomial shuffle:

$$\Delta_0^b((v(m,k))=\sum_{i=k}^{m-1}(v(m,i+1))\underline{\otimes}(v(i,k)),$$

where for short we put $\Delta_0^b(U)=\Delta^b(U)-U\underline{\otimes}1-1\underline{\otimes}U$. Taking into account (5.2), we have

$$\Delta_0^b(v[k,m])=\alpha_k^m\cdot\sum_{i=k}^{m-1}(\alpha_{i+1}^m)^{-1}v[i+1,m]\underline{\otimes}(\alpha_i^k)^{-1}v[k,i]. \tag{5.3}$$

Formula (2.13) demonstrates that the tensors $u^{(1)}\otimes u^{(1)}$ of the (unbraided) coproduct and tensors $u_b^{(1)}\underline{\otimes}u_b^{(1)}$ of the braided one are related by $u_b^{(1)}=u^{(1)}\mathrm{gr}(u^{(2)})^{-1}$,

$u_b^{(2)} = u^{(2)}$. The equality (5.3) provides the values of $u_b^{(1)}$ and $u_b^{(2)}$. Hence we may find $u^{(1)} = \alpha_k^m(\alpha_k^i\alpha_{i+1}^m)^{-1} \cdot v[i+1,m]g_{ki}$ and $u^{(2)} = v[k,i]$, where $g_{ki} = \mathrm{gr}(v[k,i])$. The commutation rules imply

$$v[i+1,m]g_{ki} = p(v(i+1,m), v(k,i))g_{ki}v[i+1,m].$$

Thus, the coproduct has the form (5.1), where

$$\tau_i(1-q^{-1}) = \alpha_k^m(\alpha_k^i\alpha_{i+1}^m)^{-1}p(v(i+1,m), v(k,i)).$$

The definition of α_k^m given in (4.7) shows that

$$\alpha_k^m(\alpha_k^i\alpha_{i+1}^m)^{-1} = \varepsilon_k^m(\varepsilon_k^i\varepsilon_{i+1}^m)^{-1} \cdot p(v(k,i), v(i+1,m))$$

because

$$\left(\prod_{k\le a<b\le i} p_{ab} \prod_{i+1\le a<b\le m} p_{ab}\right)^{-1} \prod_{k\le a<b\le m} p_{ab} = p(v(k,i), v(i+1,m)).$$

The definition of $\mu_k^{m,i}$ given in (3.6) implies

$$\tau_i(1-q^{-1}) = \varepsilon_k^m(\varepsilon_k^i\varepsilon_{i+1}^m)^{-1}(q-1)\mu_k^{m,i};$$

that is, $\tau_i = \varepsilon_k^m(\varepsilon_k^i\varepsilon_{i+1}^m)^{-1}q\mu_k^{m,i}$. By (3.7), we have $\mu_k^{m,i} = \sigma_k^m(\sigma_k^i\sigma_{i+1}^m)^{-1}$. Using (3.9) and (4.8), we see that

$$\varepsilon_k^m\sigma_k^m = \begin{cases} q^2+q, & \text{if } k \le n \le m, k \ne m; \\ q^2, & \text{if } k = n = m; \\ q, & \text{otherwise.} \end{cases} \tag{5.4}$$

Now, it is easy to check that the τ's have the following elegant form

$$\begin{aligned} \tau_i &= \varepsilon_k^m\sigma_k^m(\varepsilon_k^i\sigma_k^i)^{-1}(\varepsilon_{i+1}^m\sigma_{i+1}^m)^{-1}q \qquad (5.5) \\ &= \begin{cases} 1+q^{-1}, & \text{if } i = n-1, m = n; \text{ or } k = i = n; \\ 1, & \text{otherwise.} \end{cases} \end{aligned}$$

□

Remark 1. If q is a root of 1, say $q^t = 1$, $t > 2$, then the shuffle representation is not faithful. Therefore in this case, the formula (5.1) is proved only for the Frobenius-Lusztig kernel $u_q(\mathfrak{sp}_{2n})$. Nevertheless, all tensors in (5.1) have degree at most 2 in each variable. At the same time, general results on combinatorial representation of Nichols algebras [2, Section 5.5] demonstrate that in case C_n, the kernel of the natural projection $U_q(\mathfrak{sp}_{2n}) \to u_q(\mathfrak{sp}_{2n})$ is generated by polynomials of degree grater then 2 in (or independent of) each given variable. Hence (5.1) remains valid in this case as well.

6. Relations in $U_q^+(\mathfrak{so}_{2n})$

In what follows, we fix a parameter q such that $q \ne -1$. If C is a Cartan matrix of type D_n, then relations (2.10) take the form

$$p_{ii} = q, \ 1 \le i \le n; \ \ p_{i\,i-1}p_{i-1\,i} = p_{n-2\,n}p_{n\,n-2} = q^{-1}, \ 1 < i < n; \tag{6.1}$$

$$p_{ij}p_{ji} = p_{n-1\,n}p_{n\,n-1} = 1, \text{ if } j > i+1 \,\&\, (i,j) \ne (n, n-2). \tag{6.2}$$

The quantum Borel algebra $U_q^+(\mathfrak{so}_{2n})$ can be defined by the condition that the Hopf subalgebras U_{n-1} and U_n generated, respectively, by $x_1, x_2, \ldots, x_{n-1}$ and

$x_1, x_2, \ldots, x_{n-2}, x_n$ are Hopf algebras $U_q(\mathfrak{sl}_n)$ of type A_{n-1}, and by one additional relation

$$[x_{n-1}, x_n] = 0. \tag{6.3}$$

Recall that x_i, $n < i < 2n$ denotes the generator x_{2n-i}, whereas if $1 \leq i < 2n$, then $\phi(i)$ equals $2n - i$, so that $x_i = x_{\phi(i)}$, see Definition 3.2.

Definition 6.1. We define words $e(k,m)$, $1 \leq k \leq m < 2n$ in the following way:

$$e(k,m) = \begin{cases} x_k x_{k+1} \cdots x_{m-1} x_m, & \text{if } m < n \text{ or } k > n; \\ x_k x_{k+1} \cdots x_{n-2} x_n x_{n+1} \cdots x_m, & \text{if } k < n-1 < m; \\ x_n x_{n+1} \cdots x_m, & \text{if } k = n-1 < m; \\ x_n x_{n+2} x_{n+3} \cdots x_m, & \text{if } k = n. \end{cases} \tag{6.4}$$

Respectively, $e(m,k)$ is the word opposite to $e(k,m)$. Further, we define a word $e'(k,m)$ as a word that appears from $e(k,m)$ by replacing the subword $x_n x_{n+1}$, if any, with $x_{n-1}x_n$. Respectively, $e'(m,k)$ is the word opposite to $e'(k,m)$.

We see that $e(k,m)$ coincides with $v(k,m)$ if $m < n$ or $k > n$. If $k < n-1 < m$, then $e(k,m)$ appears from $v(k,m)$ by deleting the letter x_{n-1} (but not of x_{n+1}!). Similarly, if $k = n$, then $e(n,m)$ appears from $v(n,m)$ by deleting the letter x_{n+1}, whereas if $k = n-1$, then we have $e(n-1,m) = v(n,m)$. We have to stress that according to this definition $e(n-1,n) = e(n,n) = e(n,n+1) = x_n$.

Lemma 6.2. *If* $1 \leq k \leq m < 2n$, *then*

$$p(e(k,m), e(k,m)) = \sigma_k^m = \begin{cases} q^2, & \textit{if } m = \phi(k); \\ q, & \textit{otherwise.} \end{cases} \tag{6.5}$$

Proof. If the word $e(k,m)$ does not contain a subword $x_n x_{n+1}$, then it belongs to either U_n or U_{n-1} that are isomorphic to $U_q^+(\mathfrak{sl}_n)$. Hence we have $p(e(k,m), e(k,m)) = q$.

Let $k \leq n-1 < m$. In this case $e(k,m+1) = e(k,m)x_{m+1}$ which allows one to use induction on $m - n + 1$. If $m = n$, then $e(k,n)$ does not contain a sub-word $x_n x_{n+1}$. Because $p(\text{-},\text{-})$ is a bimultiplicative map, we may decompose

$$p(e(k,m+1), e(k,m+1)) = \sigma_k^m \cdot q \cdot p(e(k,m), x_{m+1}) \cdot p(x_{m+1}, e(k,m)). \tag{6.6}$$

Using relations 6.1 and 6.2 we shall prove

$$p(e(k,m), x_{m+1}) \cdot p(x_{m+1}, e(k,m)) = \begin{cases} 1, & \text{if } k = \phi(m) - 1; \\ q^{-2}, & \text{if } k = \phi(m); \\ q^{-1}, & \text{otherwise.} \end{cases} \tag{6.7}$$

The left hand side of the above equality is $\prod_{k \leq t \leq m,\, t \neq n-1} p_{t\, m+1} p_{m+1\, t}$.

If $m > n+1$, then by 6.1 and 6.2 the factor $p_{t\, m+1} p_{m+1\, t}$ differs from 1 only if $t \in \{\phi(m)-2, \phi(m)-1, \phi(m), m\}$ and related values are respectively $q^{-1}, q^2, q^{-1}, q^{-1}$ whereas the product of all those values is precisely q^{-1}. Hence, if $k < \phi(m)-1$, then the whole product is q^{-1}; if $k = \phi(m)-1$, then this is 1; if $k = \phi(m)$, then this is q^{-2}; if $k > \phi(m)$, then this is again q^{-1}. If $m = n+1$, then nontrivial factors are related to $t \in \{n-3, n-2, n, n+1\}$ with values $q^{-1}, q^2, q^{-1}, q^{-1}$, respectively. Hence, we arrive to the same conclusion with $k < n-2 = \phi(m) - 1$; $k = n-2 = \phi(m) - 1$; and $k = n - 1 = \phi(m)$.

Finally, if $m=n$, then there is just one nontrivial factor which relates to $t=n-2$ with value q^{-1}, so that if $k\leq n-2=\phi(m)-2$, then the total product is q^{-1}; if $k=n-1=\psi(m)-1$, then this is 1. This completes the proof of (6.7).

To complete the inductive step we use (6.7) and inductive hypothesis: if $k=\phi(m)-1$, then $\sigma_k^{m+1}=q\cdot q\cdot 1=q^2$; if $k=\phi(m)$, then $\sigma_k^{m+1}=q^2\cdot q\cdot q^{-2}=q$; otherwise $\sigma_k^{m+1}=q\cdot q\cdot q^{-1}=q$. □

Lemma 6.3. *If the word $e(k,m)$ contains the subword x_nx_{n+1}; that is $k<n<m$, then for each i, $k\leq i<m$ we have*

$$p(e(k,i),e(i+1,m))\cdot p(e(i+1,m),e(k,i))=\sigma_k^m(\sigma_k^i\sigma_{i+1}^m)^{-1}=\mu_k^{m,i}. \tag{6.8}$$

Proof. If $k<n<m$, then for $i\neq n-1$ there is a decomposition $e(k,m)=e(k,i)e(i+1,m)$ which implies (6.8) because the form $p(\text{-},\text{-})$ is bimultiplicative. For $i=n-1$ there is another equality $e'(k,m)=e(k,i)e(i+1,m)$. Certainly $p(e'(k,m),e'(k,m))=p(e(k,m),e(k,m))=\sigma_k^m$. Hence (6.8) is still valid. □

We define the bracketing of $e(k,m)$, $k\leq m<2n$ as follows.

$$e[k,m]=\begin{cases}[[[\ldots[x_k,x_{k+1}],\ldots],x_{m-1}],x_m], & \text{if } m<\phi(k);\\ [x_k,[x_{k+1},[\ldots,[x_{m-1},x_m]\ldots]]], & \text{if } m>\phi(k);\\ [\![e[k,m-1],x_m]\!], & \text{if } m=\phi(k),\end{cases} \tag{6.9}$$

where as above $[\![u,v]\!]=uv-q^{-1}p(u,v)vu$.

Conditional identity (2.6) demonstrates that the value of $e[k,m]$ in $U_q^+(\mathfrak{so}_{2n})$ is independent of the precise arrangement of brackets, provided that $m\leq n$ or $k\geq n$.

Lemma 6.4. *If $k<n<m<\phi(k)$, then the value in $U_q^+(\mathfrak{so}_{2n})$ of the bracketed word $[y_kx_{n+1}x_{n+2}\cdots x_m]$, where $y_k=e[k,n]$, is independent of the precise arrangement of brackets.*

Proof. To apply (2.6), it suffices to check $[y_k,x_t]=0$, where $n+1<t\leq m$ or, equivalently, $\phi(m)\leq t<n-1$. We have

$$[y_k,x_t]=\big[[e[k,t-2],e[t-1,n]],x_t\big]=\big[e[k,t-2],[e[t-1,n],x_t]\big].$$

The polynomial $[e[t-1,n],x_t]$ is independent of x_{n-1}, so that it belongs to the Hopf subalgebra $U_n=U_q^+(\mathfrak{sl}_n)$. By [9, Theorem A_n], the element $[e[t-1,n],x_t]$ equals zero in $U_q^+(\mathfrak{sl}_n)$ because the word $e(t-1,n)x_t$ is standard, and the standard bracketing is $[e[t-1,n],x_t]$. □

Lemma 6.5. *If $k<n$, $\phi(k)<m$, then the value in $U_q^+(\mathfrak{so}_{2n})$ of the bracketed word $[x_kx_{k+1}\cdots x_{n-2}x_ny_m]$, where $y_m=e[n+1,m]$, is independent of the precise arrangement of brackets.*

Proof. To apply (2.6), we need the equalities $[x_t,y_m]=0$, $k\leq t<n-1$. The polynomial $[x_t,y_m]$ belongs to the subalgebra U_{n-1}. Moreover, $[x_t,y_m]$ is proportional to $[y_m,x_t]$ due to antisymmetry identity (2.4) because $p(x_t,y_m)p(y_m,x_t)$ $=p_{t\,t+1}p_{tt}p_{t\,t-1}\cdot p_{t+1\,t}p_{tt}p_{t-1\,t}=1$. The equality $[y_m,x_t]=0$ turns to the proved above equality $[e[k,n],x_t]=0$ if one renames the variables $x_{n+1}\leftarrow x_k$, $x_{n+2}\leftarrow x_{k+1},\ldots$. □

7. PBW generators of $U_q^+(\mathfrak{so}_{2n})$

Proposition 7.1. *If $q \neq -1$, then values of the elements $e[k,m]$, $k \leq m < \phi(k)$ form a set of PBW generators with infinite heights for the algebra $U_q^+(\mathfrak{so}_{2n})$ over $\mathbf{k}[G]$.*

Proof. All words $e(k,m)$, $k \leq m < \phi(k)$ are standard Lyndon-Shirshov words, and by [9, Theorem D_n, p. 225] under the standard bracketing, say $[e(k,m)]$, they form a set of PBW generators with infinite heights.

By induction on $m-k$ we prove that the values in $U_q^+(\mathfrak{so}_{2n})$ of $[e(k,m)]$ equal the values of $e[k,m]$ with bracketing given in (6.9).

If $m \leq n$, then by Lemma 2.1 we have nothing to prove.

If $k < n < m$, then according to [9, Lemma 7.25], the brackets in $[e(k,m)]$ are set by the following recurrence formulae (we note that $[e(k,m)] = [e_{k\,\phi(m)}]$ in the notations of [9]):

$$[e(k,m)] = \begin{cases} [x_k[e(k+1,m)]], & \text{if } m < \phi(k)-1; \\ [[e(k,m-1)]x_m], & \text{if } m = \phi(k)-1. \end{cases} \tag{7.1}$$

In the latter case the induction applies directly. In the former case using induction and Lemma 6.4 we have $[e(k+1,m)] = e[k+1,m] = [e[k+1,n], e[n+1,m]]$. At the same time $[x_k, x_t] = 0$, $n < t \leq m$ because $x_t = x_{\phi(t)}$ and $\phi(t) \geq \phi(m) > k+1$ & $(k,\phi(t)) \neq (n-2,n)$. This implies $[x_k, e[n+1,m]] = 0$. Applying the conditional identity (2.5), we get

$$[e(k,m)] = [x_k[e[k+1,n], e[n+1,m]]] = \big[[x_k e[k+1,n]], e[n+1,m]\big] = e[k,m].$$

□

8. Shuffle representation for $U_q^+(\mathfrak{so}_{2n})$

In this section, we are going to find the shuffle representation of elements $e[k,m]$, $1 \leq k \leq m < 2n$. If $e(k,m)$ has not $x_n x_{n+1}$ as a subword, then $e[k,m]$ belongs to a Hopf subalgebra of type A_n: this is either $U_{n-1} = U_q^+(\mathfrak{sl}_n)$ or $U_n = U_q^+(\mathfrak{sl}_n)$. At the same time in the considered above case C_n, the elements $x_1, x_2, \ldots, x_{n-1}$ generate precisely a Hopf subalgebra $U_q(\mathfrak{sl}_n)$. Hence we may apply Proposition 4.2:

$$e[k,m] = \alpha_k^m \cdot (e(m,k)), \tag{8.1}$$

where

$$\alpha_k^m = \begin{cases} (q-1)^{m-k} \cdot \prod\limits_{k \leq i < j \leq m} p_{ij}, & \text{if } m < n \text{ or } k > n; \\ (q-1)^{m-n-1} \cdot \prod\limits_{n \leq i < j \leq m,\, i,j \neq n+1} p_{ij}, & \text{if } k = n; \\ (q-1)^{n-k-1} \cdot \prod\limits_{k \leq i < j \leq m,\, i,j \neq n-1} p_{ij}, & \text{if } m = n. \end{cases} \tag{8.2}$$

Proposition 8.1. *Let $1 \leq k < n < m < 2n$. In the shuffle representation, we have*

$$e[k,m] = \alpha_k^m \cdot \{(e(m,k)) + p_{n-1,n}(e'(m,k))\}, \tag{8.3}$$

where

$$\alpha_k^m = \epsilon_k^m (q-1)^{m-k-1} \cdot \prod_{k \leq i < j \leq m,\, i,j \neq n-1} p_{ij} \tag{8.4}$$

with

$$\epsilon_k^m = \begin{cases} q^{-1}, & \textit{if } m = \phi(k); \\ 1, & \textit{otherwise.} \end{cases} \tag{8.5}$$

Proof. a). Consider first the case $m < \phi(k)$. We use induction on $m-n$. Let $m-n=1$. Condition $n+1=m<\phi(k)$ implies $k<n-1$. Hence by Lemma 6.4 we have $e[k,n+1]=[e[k,n],x_{n+1}]$, whereas (8.1) implies $e[k,n]=\alpha_k^n(e(n,k))$. Using (2.15), we may write

$$e[k,n+1] = \alpha_k^n\{(e(n,k))(x_{n+1}) - p(e(n,k),x_{n+1})\cdot(x_{n+1})(e(n,k))\}$$

$$= \alpha_k^n \sum_{uv=e(n,k)} \{p(x_{n+1},v)^{-1} - p(v,x_{n+1})\}(ux_{n+1}v), \tag{8.6}$$

where $p(v,x_{n+1}) = p(e(n,k),x_{n+1})p(u,x_{n+1})^{-1}$ because $e(n,k)=uv$. We have

$$p(x_{n+1},v)^{-1} - p(v,x_{n+1}) = p(v,x_{n+1})\cdot\{p(x_{n+1},v)^{-1}p(v,x_{n+1})^{-1}-1\}.$$

At the same time equality $x_{n+1}=x_{n-1}$ and relations (6.1), (6.2) imply

$$p(x_{n+1},v)p(v,x_{n+1}) = \begin{cases} q^{-1}, & \text{if } v=e(n,k) \text{ or } v=e(n-2,k); \\ 1, & \text{otherwise.} \end{cases}$$

Hence in the decomposition (8.6) two terms remain

$$\alpha_k^n p(e(n,k),x_{n+1})(q-1)(x_{n+1}x_nx_{n-2}\cdots x_k) = \alpha_k^{n+1}(e(n+1,k))$$

and

$$\alpha_k^n p(e(n-2,k),x_{n+1})(q-1)(x_nx_{n-1}x_{n-2}\cdots x_k) = \alpha_k^{n+1}p_{n-1,n}(e'(n+1,k)),$$

for $p_{n,n+1}^{-1} = p_{n,n-1}^{-1} = p_{n-1,n}$ due to (6.2). This completes the first step of induction.

Suppose that equalities (8.3) and (8.4) are valid and still $m+1<\phi(k)$. Then Lemma 6.4 implies $e[k,m+1]=[e[k,m],x_{m+1}]$. By (2.15) we have

$$[(e(m,k)),(x_{m+1})] = \sum_{uv=e(m,k)} p(v,x_{m+1})\cdot\{p(x_{m+1},v)^{-1}p(v,x_{m+1})^{-1}-1\}(ux_{m+1}v).$$

Relations (6.1), (6.2) imply that

$$p(x_{m+1},v)p(v,x_{m+1}) = \begin{cases} q, & \text{if } v=e(\phi(m)-1,k); \\ q^{-1}, & \text{if } v=e(m,k) \text{ or } v=e(\phi(m)-2,k); \\ 1, & \text{otherwise.} \end{cases}$$

Thus in the decomposition just three terms remain. Two of them, corresponding to $v=e(\phi(m)-1,k)$ and $v=e(\phi(m)-2,k)$, are canceled:

$$p(x_{\phi(m)-1},x_{m+1})(q^{-1}-1)+(q-1) = q(q^{-1}-1)+(q-1)=0.$$

Thus

$$[(e(m,k)),(x_{m+1})] = \{(q-1)\cdot \prod_{k\le i\le m,\ i\ne n-1} p_{i\,m+1}\}\,(e(m+1,k)).$$

In perfect analogy, we have

$$[(e'(m,k)),(x_{m+1})] = \{(q-1)\cdot \prod_{k\le i\le m,\ i\ne n-1} p_{i\,m+1}\}\,(e'(m+1,k)).$$

The inductive supposition yields $e[m,k]=\alpha_k^m\cdot\{(e(m,k))+p_{n-1,n}(e'(k,m))\}$. Hence to complete the induction, it suffices to note that

$$\alpha_k^{m+1}=\alpha_k^m\cdot(q-1)\cdot\prod_{k\leq i\leq m,\ i\neq n-1}p_{i\,m+1}.$$

b). Similarly consider the case $m>\phi(k)$ using downward induction on $n-k$. Let $k=n-1$. Condition $m>\phi(k)$ implies $m\geq n+2$. Hence by Lemma 6.5 we have $e[n-1,m]=[x_n,e[n+1,m]]$, whereas (8.1) and (8.2) imply $e[n+1,m]$ $=\alpha_{n+1}^m(e(m,n+1))$. Using (2.15), we may write

$$e[n-1,m]=\alpha_{n+1}^m\{(x_n)(e(m,n+1))-p(x_n,e(m,n+1))\cdot(e(m,n+1))(x_n)\}$$

$$=\alpha_{n+1}^m\sum_{uv=e(m,n+1)}\{p(u,x_n)^{-1}-p(x_n,u)\}(ux_nv), \tag{8.7}$$

where $p(x_n,u)=p(x_n,e(m,n+1))p(x_n,v)^{-1}$ because $e(m,n+1)=uv$. We have

$$p(u,x_n)^{-1}-p(x_n,u)=p(x_n,u)\cdot\{p(u,x_n)^{-1}p(x_n,u)^{-1}-1\}.$$

Equality $x_{n+1}=x_{n-1}$ and relations (6.1), (6.2) imply that $p(u,x_n)p(x_n,u)=1$ unless $u=e(m,n+1)$ or $u=e(m,n+2)$. In these two exceptional cases, the product equals $p_{n+2\,n}p_{n\,n+2}=q^{-1}$ because $p_{n\,n+1}p_{n+1\,n}=1$. Hence in the decomposition (8.7) two terms remain

$$\alpha_{n+1}^m p(x_n,e(m,n+1))(q-1)(x_m\cdots x_{n+1}x_n)=\alpha_{n-1}^m(e(m,n-1))$$

and

$$\alpha_{n+1}^m p(x_n,e(m,n+2))(q-1)(x_m\cdots x_{n+2}x_nx_{n+1})=\alpha_{n-1}^m\cdot p_{n\,n+1}^{-1}(e'(m,n-1)).$$

This completes the first step of induction because $p_{n\,n+1}^{-1}=p_{n-1\,n}$.

Suppose that equalities (8.3) and (8.3) are valid and still $m>\phi(k-1)=\phi(k)+1$. Lemma 6.5 implies $e[k-1,m]=[x_{k-1},e[k,m]]$. We have

$$[(x_{k-1}),(e(m,k))]=\sum_{uv=e(m,k)}p(x_{k-1},u)\cdot\{p(u,x_{k-1})^{-1}p(x_{k-1},u)^{-1}-1\}(ux_{k-1}v).$$

Relations (6.1), (6.2) imply that

$$p(u,x_{k-1})p(x_{k-1},u)=\begin{cases}q & \text{if } u=e(m,\phi(k)+1);\\ q^{-1} & \text{if } u=e(m,k) \text{ or } u=e(\phi(k)+2,k);\\ 1 & \text{otherwise.}\end{cases}$$

Hence in the decomposition (8.6) just three terms remain. Two of them, corresponding to $u=e(m,\phi(k)+1)$ and $u=e(m,\phi(k))$, are canceled:

$$p(x_{k-1},x_{\phi(k)+1})(q^{-1}-1)+(q-1)=q(q^{-1}-1)+(q-1)=0.$$

Thus

$$[(x_{k-1}),(e(m,k)]=\{(q-1)\cdot\prod_{k\leq j\leq m,\ j\neq n-1}p_{k-1\,j}\}\,(v(m,k-1)).$$

In perfect analogy, we have

$$[(x_{k-1}),(e'(m,k)]=\{(q-1)\cdot\prod_{k\leq j\leq m,\ j\neq n-1}p_{k-1\,j}\}\,(v'(m,k-1)).$$

The inductive supposition states $e[m,k] = \alpha_k^m \cdot \{(e(m,k)) + p_{n-1,n}(e'(k,m))\}$. Hence it remains to note that

$$\alpha_{k-1}^m = \alpha_k^m \cdot (q-1) \cdot \prod_{k \leq j \leq m,\ j \neq n-1} p_{k-1\,j}.$$

c). Let $m = \phi(k) \neq n$. In this case, $x_m = x_k$, $\epsilon_k^m = q^{-1}$. If $k = n-1$, $m = n+1$, then $e(n-1,n) = x_n$ and by Definition 6.9 we have

$$e[n-1,n+1] = x_n x_{n+1} - q^{-1} p_{n\,n+1} x_{n+1} x_n = (1-q^{-1}) x_n x_{n+1}$$

since due to (6.3) we have $x_{n+1}x_n = p_{n-1\,n} x_n x_{n+1}$ with $x_{n+1} = x_{n-1}$ and $p_{n\,n+1} p_{n-1\,n} = 1$. In the shuffle form, we get

$$(x_n)(x_{n+1}) = (x_n x_{n+1}) + p_{n+1\,n}^{-1}(x_{n+1}x_n) = p_{n\,n+1} \cdot \{(x_{n+1}x_n) + p_{n-1\,n}(x_n x_{n+1})\}.$$

It remains to note that $e(n-1,n+1) = x_n x_{n+1}$, $e(n+1,n-1) = x_{n+1}x_n$, $e'(n-1,n+1) = x_{n-1}x_n$, $e'(n+1,n-1) = x_n x_{n-1} = x_n x_{n+1}$.

Let $k < n-1$. By definition (6.9) we have

$$e[k,m] = e[k,m-1] \cdot x_m - q^{-1} p(e(k,m-1), x_m) x_k \cdot e[k,m-1]. \tag{8.8}$$

Already done case a) allows us to find the shuffle representation

$$e[k,m-1] = \alpha_k^{m-1} \cdot \{(e(m-1,k)) + p_{n-1\,n}(e'(m-1,k))\}.$$

We have

$$[\![(e(m-1,k)),(x_m)]\!] = \sum_{uv = e(m-1,k)} p(v,x_m) \cdot \{p(x_m,v)^{-1} p(v,x_m)^{-1} - q^{-1}\}(u x_m v).$$

Relations (6.1), (6.2) imply that

$$p(x_m,v)p(v,x_m) = \begin{cases} 1, & \text{if } v = \emptyset \text{ or } v = e(m-1,k); \\ q^2, & \text{if } v = x_k; \\ q, & \text{otherwise.} \end{cases}$$

Therefore in the decomposition just three terms remain. Two of them, corresponding to $v = \emptyset$ and $v = x_k$, are canceled:

$$p(x_k,x_m)(q^{-2} - q^{-1}) + (1 - q^{-1}) = q(q^{-2} - q^{-1}) + (1 - q^{-1}) = 0.$$

Thus

$$[\![(e(m-1,k)),(x_m)]\!] = \{(1-q^{-1}) \cdot \prod_{k \leq i < m,\ i \neq n-1} p_{i\,m}\}\,(e(m,k)).$$

In perfect analogy, we have

$$[\![(e'(m-1,k)),(x_m)]\!] = \{(1-q^{-1}) \cdot \prod_{k \leq i < m,\ i \neq n-1} p_{i\,m}\}\,(e'(m,k)).$$

It suffices to note that $1 - q^{-1} = \epsilon_k^m (q-1)$, and by definition

$$\alpha_k^m = \alpha_k^{m-1} \cdot \epsilon_k^m (q-1) \cdot \prod_{k \leq i < m,\ i \neq n-1} p_{i\,m}.$$

The proposition is completely proved. □

9. Coproduct formula for $U_q(\mathfrak{so}_{2n})$

Theorem 9.1. *In $U_q^+(\mathfrak{so}_{2n})$ the coproduct on the elements $e[k,m]$, $k\leq m<2n$ has the following explicit form*

$$\Delta(e[k,m])=e[k,m]\otimes 1+g_{km}\otimes e[k,m] \tag{9.1}$$
$$+\sum_{i=k}^{m-1}\tau_i(1-q^{-1})g_{ki}\,e[i+1,m]\otimes e[k,i],$$

where $\tau_i=1$, with two exceptions, being $\tau_n=0$ if $k=n$, and $\tau_{n-1}=0$ if $m=n$; and $\tau_{n-1}=p_{n\,n-1}$ otherwise. Here $g_{ki}=\mathrm{gr}(e(k,i))$ is a group-like element that appears from the word $e(k,i)$ under the substitutions $x_\lambda\leftarrow g_\lambda$.

Proof. If the word $e(k,m)$ does not contain the subword x_nx_{n+1}, then $e[k,m]$ belongs to either U_{n-1} or U_n. Both of these Hopf algebras are isomorphic to $U_q^+(\mathfrak{sl}_n)$. Hence if $m\leq n$ or $k\geq n$, then we have nothing to prove.

Suppose that $k<n<m$. In this case by Proposition 8.1 we have the shuffle representation

$$e[k,m]=\alpha_k^m\cdot\{(e(m,k))+p_{n-1\,p}(e'(m,k))\}, \tag{9.2}$$

where $(e(m,k))$ is a comonomial shuffle

$$(e(m,k))=\begin{cases}(x_mx_{m-1}\cdots x_{n+2}x_{n+1}x_nx_{n-2}\cdots x_k), & \text{if } k<n-1;\\ (x_mx_{m-1}\cdots x_{n+2}x_{n+1}x_n), & \text{if } k=n-1,\end{cases}$$

whereas $(e'(m,k))$ is a related one:

$$(e'(m,k))=\begin{cases}(x_mx_{m-1}\cdots x_{n+2}x_nx_{n-1}x_{n-2}\cdots x_k), & \text{if } k<n-1;\\ (x_mx_{m-1}\cdots x_{n+2}x_nx_{n-1}), & \text{if } k=n-1.\end{cases}$$

Using (2.14) it is easy to find the braided coproduct of the comonomial shuffles:

$$\Delta_0^b((e(m,k)))=\sum_{i=k}^{n-2}(e(m,i+1))\underline{\otimes}(e(i,k))+\sum_{i=n}^{m-1}(e(m,i+1))\underline{\otimes}(e(i,k)),$$
$$\Delta_0^b((e'(m,k)))=\sum_{i=k}^{n-1}(e'(m,i+1))\underline{\otimes}(e(i,k))+\sum_{i=n+1}^{m-1}(e(m,i+1))\underline{\otimes}(e'(i,k)),$$

where for short we define $\Delta_0^b(U)=\Delta^b(U)-U\underline{\otimes}1-1\underline{\otimes}U$. Taking into account (9.2), we have

$$(\alpha_k^m)^{-1}\Delta_0^b(e[k,m])=\left(\sum_{i=k}^{n-2}(\alpha_{i+1}^m)^{-1}e[i+1,m]\underline{\otimes}(e(i,k))\right)+p_{n-1\,n}(e(m,n))\underline{\otimes}(e(n-1,k))$$
$$+(e(m,n+1))\underline{\otimes}(e(n,k))+\sum_{i=n+1}^{m-1}(e(m,i+1))\underline{\otimes}(\alpha_k^i)^{-1}e[k,i].$$

Relation (8.1) applied to $e[k,i]$, $i\leq n$ and $e[i+1,m]$, $i\geq n$ allows one to rewrite the right hand side of the above equality in terms of $e[i,j]$:

$$=\left(\sum_{i=k}^{n-2}(\alpha_k^i)^{-1}(\alpha_{i+1}^m)^{-1}e[i+1,m]\underline{\otimes}e[k,i]\right)+p_{n-1\,n}(\alpha_n^m)^{-1}(\alpha_k^{n-1})^{-1}e[n,m]\underline{\otimes}e[k,n-1]$$
$$+(\alpha_{n+1}^m)^{-1}(\alpha_k^n)^{-1}e[n+1,m]\underline{\otimes}e[k,n]+\sum_{i=n+1}^{m-1}(\alpha_{i+1}^m)^{-1}(\alpha_k^i)^{-1}e[i+1,m]\underline{\otimes}e[k,i].$$

Thus, we have

$$\Delta_0^b(e[k,m]) = \sum_{i=k}^{m-1} \gamma_i \, e[i+1,m]\underline{\otimes} e[k,i], \tag{9.3}$$

where

$$\gamma_i = p_{n-1\,n}^{\delta_{n-1}^i} \cdot \alpha_k^m (\alpha_k^i \alpha_{i+1}^m)^{-1}, \tag{9.4}$$

whereas δ_{n-1}^i is the Kronecker delta.

Our next step is to see that for all i, $k \leq i < m$ we have

$$\gamma_i = p_{n\,n-1}^{\delta_{n-1}^i} \cdot (q-1)\epsilon_k^m (\epsilon_k^i \epsilon_{i+1}^m)^{-1} p(e(k,i), e(i+1,m)). \tag{9.5}$$

All factors except the ϵ's in (9.4) have the form $(q-1)^s \prod_A p_{ab}$, where A is a suitable set of pairs (a,b) and s is an integer exponent. Due to bimultiplicativity of the form $p(\text{-},\text{-})$, the same is true for the right hand side of (9.5). Hence it suffices to demonstrate that the sum of exponents of the factors in (9.4) equals 1, and the resulting product domains in (9.4) and (9.5) are the same, or at least they define the same product.

If $i < n-1$, then using (9.4), (8.1), and Proposition 8.1, we have the required equality for the exponents,

$$(m-k-1) - (i-k) - (m-i-2) = 1,$$

and for the product domains:

$$\{k \leq a < b \leq m,\ a,b \neq n-1\} \setminus (\{k \leq a < b \leq i\} \cup \{i+1 \leq a < b \leq m,\ a,b \neq n-1\})$$
$$= \{k \leq a \leq i < b \leq m,\ a,b \neq n-1\}.$$

If $i \geq n$, then similarly we have the required equality for the exponents,

$$(m-k-1) - (i-k-1) - (m-i-1) = 1,$$

and for the product domains:

$$\{k \leq a < b \leq m,\ a,b \neq n-1\} \setminus (\{k \leq a < b \leq i,\ a,b \neq n-1\} \cup \{i+1 \leq a < b \leq m,\})$$
$$= \{k \leq a \leq i < b \leq m,\ a,b \neq n-1\}.$$

In the remaining case, $i = n-1$, we have $e(k,i) = x_k \cdots x_{k-2} x_{k-1}$, $e(i+1,m)$ $= x_n x_{n+2} \cdots x_m$. Due to (8.2), the exponent is

$$(m-k-1) - (n-1-k) - (m-n-1) = 1,$$

whereas the product domain of $\alpha_k^m (\alpha_k^{n-1} \alpha_n^m)^{-1}$ reduces to

$$\{k \leq a < b \leq m,\ a,b \neq n-1\} \setminus (\{k \leq a < b \leq n-1\} \cup \{n \leq a < b \leq m,\ a,b \neq n+1\})$$

$$= \{k \leq a < n-1 < b \leq m\} \cup \{n+1 = a < b \leq m\} \cup \{(n,n+1)\}. \tag{9.6}$$

However, in this case the product domain for α_k^{n-1} is not a subset of the product domain for α_k^m. Therefore additionally to the product defined by (9.6), there appears a factor $\prod\limits_{k \leq a < b = n-1} p_{ab}^{-1}$ and a factor $p_{n-1\,n}$ that comes from (9.4) due to $\delta_{n-1}^i = 1$. The latter factor cancels with the factor defined by the subset $\{(n,n+1)\}$ since $p_{n-1\,n} p_{n\,n+1} = 1$, whereas the product domain of the former factor must be added to the product domain of $p(e(k,i), e(i+1,m))$:

$$\{k \leq a \leq n-1 < b \leq m,\ b \neq n+1\} \cup \{k \leq a < b = n-1\}. \tag{9.7}$$

It remains to compare the products defined by (9.6) without the last pair and that defined by (9.7).

The set $\{k \le a < n-1 < b \le m,\, b \ne n+1\}$ is a subset of the first sets in (9.6) and (9.7). After cancelling the pairs from that set, (9.6) and (9.7) transform to, respectively,

$$\{k \le a < n-1 < b = n+1\} \cup \{n+1 = a < b \le m\} \tag{9.8}$$

and

$$\{a = n-1 < b \le m,\, b \ne n+1\} \cup \{k \le a < b = n-1\}. \tag{9.9}$$

The first set of (9.8) and the second set of (9.9) define the same product because $x_{n+1} = x_{n-1}$ and $p_{a\,n+1} = p_{a\,n-1}$. By the same reason $p_{n-1\,b} = p_{n+1\,b}$, hence the first set of (9.9) defines the same product as the second set of (9.8) up to one additional factor, $p_{n-1\,n}$ that corresponds to the pair $(n-1, n)$. This factor is canceled by the first factor $p_{n,n-1}$ that appears in (9.5) due to $\delta^i_{n-1} = 1$. The equality (9.5) is completely proved.

Now we are ready to consider the (unbraided) coproduct. Formula (2.13) demonstrates that the tensors $u^{(1)} \otimes u^{(1)}$ of the coproduct and tensors $u_b^{(1)} \underline{\otimes} u_b^{(1)}$ of the braided coproduct are related by $u_b^{(1)} = u^{(1)}\mathrm{gr}(u^{(2)})^{-1}$, $u_b^{(2)} = u^{(2)}$. The equality (9.3) provides the values of $u_b^{(1)}$ and $u_b^{(2)}$. Hence we may find $u^{(1)} = \gamma_i e[i+1, m] g_{ki}$ and $u^{(2)} = e[k, i]$, where $g_{ki} = \mathrm{gr}(e[k, i])$. The commutation rules imply

$$e[i+1, m] g_{ki} = p(e(i+1, m), e(k, i)) g_{ki} e[i+1, m].$$

Therefore the coproduct has the form (9.1), where

$$\tau_i(1 - q^{-1}) = \gamma_i\, p(e(i+1, m), e(k, i)).$$

Applying (9.5) and Lemma 6.3 we get

$$\tau_i(1 - q^{-1}) = p_{n\,n-1}^{\delta^i_{n-1}} \cdot (q-1)\epsilon_k^m (\epsilon_k^i \epsilon_{i+1}^m)^{-1} \cdot \sigma_k^m (\sigma_k^i \sigma_{i+1}^m)^{-1}.$$

Lemma 6.2 and Eq. (8.5) imply that $\epsilon_k^m \sigma_k^m$ equals q for all k, m without exceptions. Hence

$$\epsilon_k^m (\epsilon_k^i \epsilon_{i+1}^m)^{-1} \cdot \sigma_k^m (\sigma_k^i \sigma_{i+1}^m)^{-1} = q^{-1},$$

and we have

$$\tau_i = p_{n\,n-1}^{\delta^i_{n-1}} = \begin{cases} p_{n\,n-1}, & \text{if } i = n-1; \\ 1, & \text{otherwise.} \end{cases} \tag{9.10}$$

The theorem is completely proved. □

Remark 2. If $q^t = 1$, $t > 2$, then (9.1) remains valid due to precisely the same arguments that were given in Remark 1, see page 12.

Remark 3. In fact, the exceptions $\tau_n = 0$ if $k = n$, and $\tau_{n-1} = 0$ if $m = n$ can be omitted in the statement of the above theorem. Indeed, the related tensors are, respectively, $e[n+1, m] \otimes e[n, n]$ and $e[n, n] \otimes e[k, n-1]$, whereas by definition $e[n, n] = [\![x_n, x_n]\!] = x_n \cdot x_n - q^{-1} p(x_n, x_n) x_n \cdot x_n = 0$. So that, we may assume $\tau_n = 1$, $\tau_{n-1} = p_{n\,n-1}$ as well.

References

[1] N. Andruskiewitsch, H.-J. Schneider, Pointed Hopf algebras, in: S. Montgomery, H.-J. Schneider (Eds.) New Directions in Hopf Algebras, MSRI Publications, 43(2002), 1–68.

[2] I. Angiono, Nichols algebras with standard braiding, Alg. and Number Theory 3 Vol.1 (2009), 35–106.

[3] M. Beattie, S. Dăscălescu, Ş. Raianu, Lifting of Nichols algebras of type B_2, Israel Journal of Mathematics, 132(2002), 1-28.

[4] D. Flores de Chela, J.A. Green, On quantum symmetric algebra, Algebra and Representation Theory, 4, N1(2001), 55–76.

[5] I. Heckenberger and H.-J. Schneider, Right coideal subalgebras of Nichols algebras and the Duflo order on the Weyl groupoid, Israel Journal of Mathematics, 197, N1(2013), 139–187.

[6] N. Jacobson, Lie Algebras, Interscience Publishers, New York — London, 1962.

[7] V. Kac, Infinite dimensional Lie algebras, Cambridge University Press, 1990.

[8] V.K. Kharchenko, An algebra of skew primitive elements, Algebra i Logica, 37, N2(1998),181–224, English translation: Algebra and Logic, 37, N2(1998),101–127, arXiv: math. QA/0006077.

[9] V.K. Kharchenko, A combinatorial approach to the quantifications of Lie algebras, Pacific Journal of Mathematics, 203, N1(2002), 191–233.

[10] V.K. Kharchenko, Quantum Lie algebras and related problems, pp. 67–114, in Proceedings of the Third International Algebra Conference, Eds. Y. Fong; L.-S. Shiao, E. Zelmanov, Kluwer Academic Publishers, 2003.

[11] V.K. Kharchenko, PBW bases of coideal subalgebras and a freeness theorem, Transactions of the American Mathematical Society, v. 360, N10(2008), 5121–5143.

[12] V.K. Kharchenko, Right coideal subalgebras of $U_q^+(\mathfrak{so}_{2n+1})$, Journal of European Mathematical Society, 13(2011), 1677–1735.

[13] V.K. Kharchenko, A. A. Alvarez, On the combinatorial rank of Hopf algebras, Contemporary Mathematics, v. 376(2005), 299–308.

[14] V.K. Kharchenko, A.V. Lara Sagahon, Right coideal subalgebras in $U_q(\mathfrak{sl}_{n+1})$, Journal of Algebra, 319(2008), 2571–2625.

[15] S.Z. Levendorskii and Ya.S. Soibelman, Some applications of the quantum weyl groups, J. Geom. Phys. 7 (1990), 241–254.

[16] M. Lothaire, Algebraic Combinatorics on Words, Cambridge Univ. Press, 2002.

[17] G. Lusztig, Introduction to quantum groups, Birkhäuser, Boston, MA, 1993.

[18] M. Rosso, An analogue of the Poincaré-Birkhoff-Witt theorem and the universal R-matrix of $U_q(sl(N+1))$, Comm. Math. Phys., v.124(1989), 307–318.

[19] M. Rosso, Quantum groups and quantum Shuffles, Invent. Math. v.113, N2(1998), 399–416.

[20] P. Schauenberg, A characterization of the Borel-like subalgebras of quantum enveloping algebras, Comm. Algebra, v. 24, N9(1996), 2811–2823.

[21] J.-P. Serre, Algêbres de Lie semi-simples complexes, New York—Amsterdam, Benjamin, 1966.

[22] A.I. Shirshov, On free Lie rings, Mat. Sb. 45, 87(2)(1958), 113–122.

[23] A.I. Shirshov, Some algorithmic problems for Lie algebras, Sibirskii Math. J., 3(2)(1962), 292–296.

[24] M. Takeuchi, Survey of braided Hopf algebras, in: New Trends in Hopf Algebra Theory, Contemp. Math., vol. 267, AMS, Providence RI, 2000, 301–324.

[25] H. Yamane, A Poincaré-Birkhoff-Witt theorem for quantized universal enveloping algebras of type A_N, Publ. RIMS. Kyoto Univ., 25(1989), 503–520.

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