DISPERSIVE AND DIFFUSIVE LIMITS FOR OSTROVSKY-HUNTER TYPE EQUATIONS

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ABSTRACT. We consider the equation

$$\partial_x(\partial_t u + \partial_x f(u) - \beta\partial_{xxx}^3 u) = \gamma u,$$

that includes the short pulse, the Ostrovsky-Hunter, and the Korteweg-deVries ones. We consider here the asymptotic behavior as $\gamma \to 0$. The proof relies on deriving suitable a priori estimates together with an application of the compensated compactness method in the L^p setting.

1. INTRODUCTION

The nonlinear evolution equation

$$\partial_x(\partial_t u + \partial_x f(u) - \beta\partial_{xxx}^3 u) = 0, \tag{1.1}$$

with $\beta \in \mathbb{R}$ and $f(u) = \frac{u^2}{2}$, was derived by Korteweg-deVries to model internal solitary waves in the atmosphere and ocean. Here $u(t,x)$ is the amplitude of an appropriate linear long wave mode, with linear long wave speed C_0. However, when the effects of background rotation through the Coriolis parameter κ need to be taken into account, an extra term is needed, and (1.1) is replaced by

$$\partial_x(\partial_t u + \partial_x f(u) - \beta\partial_{xxx}^3 u) = \gamma u, \tag{1.2}$$

where $\gamma = \frac{\kappa^2}{2C_0}$ (see [9, 12]), which is known as the Ostrovsky equation (see [23]).

Mathematical properties of the Ostrovsky equation (1.2) were studied recently in many details, including the local and global well-posedness in energy space [10, 16, 19, 31], stability of solitary waves [14, 17, 20], and convergence of solutions in the limit of the Korteweg-deVries equation [15, 20]. We shall consider the limit of no high-frequency dispersion $\beta = 0$, therefore (1.2) reads

$$\partial_x(\partial_t u + \partial_x f(u)) = \gamma u, \quad f(u) = \frac{u^2}{2}. \tag{1.3}$$

(1.3) is deduced considering two asymptotic expansions of the shallow water equations, first with respect to the rotation frequency and then with respect to the amplitude of the waves (see [9, 12]). It is known under different names such as the reduced Ostrovsky equation [24, 29], the Ostrovsky-Hunter equation [1], the short-wave equation [11], and the Vakhnenko equation [21, 25].

Integrating (1.3) on x we gain the integro-differential formulation of (1.3) (see [18])

$$\partial_t u + \partial_x f(u) = \gamma \int^x u(t,y)dy, \tag{1.4}$$

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that is equivalent to

$$\partial_t u + \partial_x f(u) = \gamma P, \qquad \partial_x P = u. \tag{1.5}$$

The unique useful conserved quantities are

$$t \longmapsto \int u(t,x)dx = 0, \qquad t \longmapsto \int u^2(t,x)dx. \tag{1.6}$$

In the sense that if $u(t,\cdot)$ has zero mean at time $t = 0$, then it will have zero mean at any time $t > 0$. In addition, the L^2 norm of $u(t,\cdot)$ is constant with respect to t.

In [4, 7, 9], it is proved that (1.3) admits an unique entropy solutions in the sense of the following definition

Definition 1.1. *We say that* $u \in L^\infty((0,T)\times\mathbb{R})$, $T > 0$, *is an entropy solution of the initial value problem* (1.3), *if*

i) *u is a distributional solution of* (1.4) *or equivalently of* (1.5)*;*
ii) *for every convex function* $\eta \in C^2(\mathbb{R})$ *the entropy inequality*

$$\partial_t \eta(u) + \partial_x q(u) - \gamma\eta'(u)P \le 0, \qquad q(u) = \int^u f'(\xi)\eta'(\xi)\, d\xi, \tag{1.7}$$

holds in the sense of distributions in $(0,\infty)\times\mathbb{R}$*.*

In [2], it is proved the wellposedness of the entropy solutions of (1.4), or (1.5), for the non-homogeneous initial boundary problem, while in [5] it is proved the convergence of the solutions of (1.2) to the discontinuous solutions of (1.4), or (1.5).

If $f(u) = -\frac{1}{6}u^3$, (1.2) reads,

$$\partial_x \left(\partial_t u - \frac{1}{6}\partial_x(u^3) - \beta\partial_{xxx}^3 u\right) = \gamma u. \tag{1.8}$$

(1.8) is known as the regularized short pulse equation, and was derived by Costanzino, Manukian and Jones [8] in the context of the nonlinear Maxwell equations with high-frequency dispersion.

If we send $\beta \to 0$ in (1.8), we pass from (1.8) to the equation

$$\partial_x \left(\partial_t u - \frac{1}{6}\partial_x(u^3)\right) = \gamma u, \tag{1.9}$$

or equivalently (see [26]),

$$\partial_t u - \frac{1}{6}\partial_x(u^3) = \gamma P, \quad \partial_x P = u. \tag{1.10}$$

(1.9) is known as the short pulse equation, and was introduced recently by Schäfer and Wayne [27] as a model equation describing the propagation of ultra-short light pulses in silica optical fibers. It provides also an approximation of nonlinear wave packets in dispersive media in the limit of few cycles on the ultra-short pulse scale. In [3, 7, 9], it is proved the wellposedness of the entropy solution of (1.9) in sense of Definition (1.1), for the initial boundary problem and for the Cauchy problem, while, in [6], it is proved the convergence of the solutions of (1.8) to the discontinuous solutions of (1.9).

The deep difference between the two equations is in the flux. If we have a function that preserves the conserved quantities we can make sense of (1.3) using the distribution theory because the flux is quadratic and the L^2 norm is preserved. On the contrary the same argument does not apply to (1.9). Indeed, the flux is cubic and we do not have any information on the L^3 norm of the solution. In [3], we solved this problem proving that the solutions are bounded, and the argument is much more delicate than the one in [2].

In this paper, we study the dispersion-diffusion of (1.2) and of (1.5), when $\gamma \to 0$ (that is, when $\kappa \to 0$, or $C_0 \to \infty$). We prove that, if $\gamma \to 0$, the solution of (1.2) and of (1.5) converge to the to the discontinuous solutions of the following equation

$$\partial_t u + \partial_x (u^2) = 0, \tag{1.11}$$

which is known as Burgers' equation. Likewise, when $\gamma \to 0$, the solutions of (1.8) and of (1.9) converge to the discontinuous solutions of the following scalar conservation law

$$\partial_t u - \frac{1}{6}\partial_x (u^3) = 0. \tag{1.12}$$

The paper is organized in three sections. In Section 2 we prove the convergence of (1.5) and of (1.9) to (1.11) and (1.12), respectively. In Section 3, we prove the convergence of (1.2) to the (1.11), while in Section 4, we prove the convergence of (1.8) to (1.12).

2. Ostrovsky-Hunter equation and short pulse one: $\gamma \to 0$.

In this section, we consider the following Cauchy problem

$$\begin{cases} \partial_t u + \partial_x f(u) = \gamma P, & t > 0,\ x \in \mathbb{R}, \\ \partial_x P = u, & t > 0,\ x \in \mathbb{R}, \\ P(t, -\infty) = 0, & t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

or equivalently,

$$\begin{cases} \partial_t u + \partial_x f(u) = \gamma \int_{-\infty}^{x} u(t, y) dy, & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.2}$$

On the initial datum, we assume that

$$u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad \int_{\mathbb{R}} u_0(x) dx = 0, \tag{2.3}$$

while, on the function

$$P_0(x) = \int_{-\infty}^{x} u_0(y) dy, \quad x \in \mathbb{R}, \tag{2.4}$$

we assume that

$$\int_{\mathbb{R}} P_0(x) dx = \int_{\mathbb{R}} \left(\int_{-\infty}^{x} u_0(y) dy \right) dx = 0. \tag{2.5}$$

Moreover, the flux $f \in C^2(\mathbb{R})$ is assumed to be smooth.
If $\gamma = 0$, (2.1) reads

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.6}$$

which is a scalar conservation law.

Fix three small numbers $0 < \varepsilon, \delta, \gamma < 1$, and let $u_{\varepsilon,\delta,\gamma} = u_{\varepsilon,\delta,\gamma}(t, x)$ be the unique classical solution of the following mixed problem:

$$\begin{cases} \partial_t u_{\varepsilon,\delta,\gamma} + \partial_x f(u_{\varepsilon,\delta,\gamma}) = \gamma P_{\varepsilon,\delta,\gamma} + \varepsilon \partial_{xx}^2 u_{\varepsilon,\delta,\gamma}, & t > 0,\ x \in \mathbb{R}, \\ -\delta \partial_t P_{\varepsilon,\delta,\gamma} + \partial_x P_{\varepsilon,\delta,\gamma} = u_{\varepsilon,\delta,\gamma}, & t > 0, x \in \mathbb{R}, \\ P_{\varepsilon,\delta,\gamma}(t, -\infty) = 0, & t > 0, \\ u_{\varepsilon,\delta,\gamma}(0, x) = u_{\varepsilon,\delta,\gamma,0}(x), & x \in \mathbb{R}, \end{cases} \tag{2.7}$$

where $u_{\varepsilon,\delta,\gamma,0}$ is a C^{∞} approximation of u_0 such that

$$\begin{aligned}
&u_{\varepsilon,\delta,\gamma,0} \to u_0 \quad \text{in } L^p_{loc}(\mathbb{R}),\ 1 \le p < \infty, \text{ as } \varepsilon,\ \delta,\ \gamma \to 0,\\
&\|u_{\varepsilon,\delta,\gamma,0}\|_{L^{\infty}(\mathbb{R})} \le \|u_0\|_{L^{\infty}(\mathbb{R})}, \quad \varepsilon,\ \delta,\ \gamma > 0,\\
&\|u_{\varepsilon,\delta,\gamma,0}\|^2_{L^2(\mathbb{R})} + \delta\gamma \|P_{\varepsilon,\delta,\gamma,0}\|^2_{L^2(\mathbb{R})} + \delta \|\partial_x P_{\varepsilon,\delta,\gamma,0}\|^2_{L^2(\mathbb{R})} \le C_0, \quad \varepsilon,\ \delta,\ \gamma > 0,\\
&\int_{\mathbb{R}} u_{\varepsilon,\delta,\gamma,0}(x)dx = 0, \quad \int_{\mathbb{R}} P_{\varepsilon,\delta,\gamma,0}(x)dx = 0, \quad \varepsilon,\ \delta,\ \gamma > 0,
\end{aligned} \tag{2.8}$$

and C_0 is a constant independent on ε, δ and γ.

The main result of this section is the following theorem.

Theorem 2.1. *Fix $T > 0$. Assume* (2.3)*,* (2.4)*,* (2.5) *and* (2.8)*. If*

$$\gamma = \mathcal{O}(\varepsilon^{\frac{1}{3}}\delta). \tag{2.9}$$

There exists three sequences $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\{\beta_k\}_{k\in\mathbb{N}}$, $\{\gamma_k\}_{k\in\mathbb{N}}$, with $\varepsilon_k, \beta_k, \gamma_k \to 0$, such that

$$u_{\varepsilon_k,\beta_k,\gamma_k} \to u \quad \textit{strongly in } L^p_{loc}((0,T)\times\mathbb{R}), \textit{ for each } 1 \le p < \infty,$$

where u is the unique entropy solution of (2.6)*. Moreover, we have that*

$$\int_{\mathbb{R}} u(t,x)dx = 0, \quad t > 0. \tag{2.10}$$

Let us prove some a priori estimates on $u_{\varepsilon,\delta,\gamma}$ and $P_{\varepsilon,\delta,\gamma}$, denoting with C_0 the constants which depend on the initial datum, and $C(T)$ the constants which depend also on T.

Lemma 2.1. *For each $t > 0$,*

$$P_{\varepsilon,\delta,\gamma}(t,\infty) = \partial_x P_{\varepsilon,\delta,\gamma}(t,-\infty) = \partial_x P_{\varepsilon,\delta,\gamma}(t,\infty) = 0. \tag{2.11}$$

In particular, we have that

$$\int_{\mathbb{R}} u_{\varepsilon,\delta,\gamma}(t,x)dx = -\delta\frac{d}{dt}\int_{\mathbb{R}} P_{\varepsilon,\delta,\gamma}(t,x)dx, \quad t > 0. \tag{2.12}$$

Proof. Arguing as [2, Lemma 3.1], we have (2.11).

Let us show that (2.12) holds. Integrating the second equation in (2.7) on $(-\infty, x)$, we have

$$\begin{aligned}
P_{\varepsilon,\delta,\gamma}(t,x) &= \int_{-\infty}^{x} u_{\varepsilon,\delta,\gamma}(t,y)dy + \delta\int_{-\infty}^{x} \partial_t P_{\varepsilon,\delta,\gamma}(t,y)dy\\
&= \int_{-\infty}^{x} u_{\varepsilon,\delta,\gamma}(t,y)dy + \delta\frac{d}{dt}\int_{-\infty}^{x} P_{\varepsilon,\delta,\gamma}(t,y)dy.
\end{aligned} \tag{2.13}$$

Therefore, (2.12) follows from (2.11) and (2.13). □

Lemma 2.2. *For each $t > 0$,*

$$\int_{\mathbb{R}} P_{\varepsilon,\delta,\gamma}(t,x)dx = 0. \tag{2.14}$$

In particular, we have that

$$\int_{\mathbb{R}} u_{\varepsilon,\delta,\gamma}(t,x)dx = 0, \quad t > 0. \tag{2.15}$$

Proof. Let $t > 0$. Integrating the first equation in (2.7) on $\mathbb{R}$, we have

$$\int_{\mathbb{R}} \partial_t u_{\varepsilon,\delta,\gamma}(t,x)dx = \frac{d}{dt}\int_{\mathbb{R}} u_{\varepsilon,\delta,\gamma}(t,x)dx = \gamma \int_{\mathbb{R}} P_{\varepsilon,\delta,\gamma}(t,x)dx. \tag{2.16}$$

Differentiating (2.12) with respect to t, we get

$$\frac{d}{dt}\int_{\mathbb{R}} u_{\varepsilon,\delta,\gamma}(t,x)dx = -\delta \frac{d^2}{d^2t}\int_{\mathbb{R}} P_{\varepsilon,\delta,\gamma}(t,x)dx. \tag{2.17}$$

Therefore, (2.16) and (2.17) give

$$\frac{d^2}{d^2t}\int_{\mathbb{R}} P_{\varepsilon,\delta,\gamma}(t,x)dx + \frac{\gamma}{\delta}\int_{\mathbb{R}} P_{\varepsilon,\delta,\gamma}(t,x)dx = 0.$$

Then,

$$\int_{\mathbb{R}} P_{\varepsilon,\delta,\gamma}(t,x)dx = C_1 \cos\left(\sqrt{\frac{\gamma}{\delta}}t\right) + C_2 \sin\left(\sqrt{\frac{\gamma}{\delta}}t\right), \tag{2.18}$$

where C_1, C_2 are two constants.

It follows from (2.8) that

$$\frac{d}{dt}\int_{\mathbb{R}} P_{\varepsilon,\delta,\gamma,0}(x)dx = 0. \tag{2.19}$$

Thanks to (2.8) and (2.19), to compute C_1, C_2, we must solve the following system:

$$\begin{cases} C_1 \cos\left(\sqrt{\frac{\gamma}{\delta}}t\right) + C_2 \sin\left(\sqrt{\frac{\gamma}{\delta}}t\right) = 0, \\ -C_1\sqrt{\frac{\gamma}{\delta}} \sin\left(\sqrt{\frac{\gamma}{\delta}}t\right) + C_2\sqrt{\frac{\gamma}{\delta}} \cos\left(\sqrt{\frac{\gamma}{\delta}}t\right) = 0. \end{cases} \tag{2.20}$$

(2.20) says that

$$C_1 = C_2 = 0. \tag{2.21}$$

Then, (2.18) and (2.21) give (2.14).

Finally, let us show that (2.15) holds. Differentiating (2.14) with respect to t, we get

$$\frac{d}{dt}\int_{\mathbb{R}} P_{\varepsilon,\delta,\gamma}(t,x)dx = 0, \quad t > 0. \tag{2.22}$$

Therefore, (2.15) follows from (2.12) and (2.22). □

Lemma 2.3. *For each $t > 0$,*

$$\|u_{\varepsilon,\delta,\gamma}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \delta\gamma \|P_{\varepsilon,\delta,\gamma}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon \int_0^t \|\partial_x u_{\varepsilon,\delta,\gamma}(s,\cdot)\|^2_{L^2(\mathbb{R})} dx \le C_0. \tag{2.23}$$

Moreover, fixed $T > 0$, there exists $C(T) > 0$, independent on ε, δ and γ, such that

$$\|P_{\varepsilon,\delta,\gamma}\|_{L^2((0,T)\times\mathbb{R})} \le \frac{C(T)}{\delta^{\frac{1}{2}}\gamma^{\frac{1}{2}}}, \quad 0 < t < T. \tag{2.24}$$

Proof. Let $t > 0$. Multiplying by $P_{\varepsilon,\delta,\gamma}$ the second equation in (2.7), we have

$$-\delta P_{\varepsilon,\delta,\gamma}\partial_t P_{\varepsilon,\delta,\gamma} + P_{\varepsilon,\delta,\gamma}\partial_x P_{\varepsilon,\delta,\gamma} = u_{\varepsilon,\delta,\gamma}P_{\varepsilon,\delta,\gamma}. \tag{2.25}$$

Due to (2.11), an integration on $\mathbb{R}$ gives

$$\begin{aligned} -\frac{d}{dt}\left(\frac{\delta}{2}\|P_{\varepsilon,\delta,\gamma}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right) &= \int_{\mathbb{R}} u_{\varepsilon,\delta,\gamma}P_{\varepsilon,\delta,\gamma}dx - \frac{1}{2}\int_{\mathbb{R}} \partial_x\left(P^2_{\varepsilon,\delta,\gamma}\right)dx \\ &= \int_{\mathbb{R}} u_{\varepsilon,\delta,\gamma}P_{\varepsilon,\delta,\gamma}dx. \end{aligned} \tag{2.26}$$

Multiplying by $u_{\varepsilon,\delta,\gamma}$ the first equation in (2.7), an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|u_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=&\int_{\mathbb{R}}u_{\varepsilon,\delta,\gamma}\partial_t u_{\varepsilon,\delta,\gamma}dx\\ =&-\int_{\mathbb{R}}u_{\varepsilon,\delta,\gamma}f'(u_{\varepsilon,\delta,\gamma})\partial_x u_{\varepsilon,\delta,\gamma}dx+\gamma\int_{\mathbb{R}}u_{\varepsilon,\delta,\gamma}P_{\varepsilon,\delta,\gamma}dx\\ &+\varepsilon\int_{\mathbb{R}}u_{\varepsilon,\delta,\gamma}\partial^2_{xx}u_{\varepsilon,\delta,\gamma}dx\\ =&\gamma\int_{\mathbb{R}}u_{\varepsilon,\delta,\gamma}P_{\varepsilon,\delta,\gamma}dx-\varepsilon\left\|\partial_x u_{\varepsilon,\delta,\gamma}(t,x)\right\|^2_{L^2(\mathbb{R})},\end{aligned}$$

that is

$$\frac{d}{dt}\left\|u_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|\partial_x u_{\varepsilon,\delta,\gamma}(t,x)\right\|^2_{L^2(\mathbb{R})}=2\gamma\int_{\mathbb{R}}u_{\varepsilon,\delta,\gamma}P_{\varepsilon,\delta,\gamma}dx. \tag{2.27}$$

It follows from (2.26) and (2.27) that

$$\frac{d}{dt}\left(\left\|u_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\delta\gamma\left\|P_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)+2\varepsilon\left\|\partial_x u_{\varepsilon,\delta,\gamma}(t,x)\right\|^2_{L^2(\mathbb{R})}=0. \tag{2.28}$$

Integrating (2.28) on $(0,t)$, from (2.8), we have (2.23).

Finally, we prove (2.24). Let $T>0$. We begin by observing that, from (2.23), we have that

$$\delta\gamma\int_{\mathbb{R}}P^2_{\varepsilon,\delta,\gamma}dx\le C_0.$$

An integration on $(0,T)$ gives

$$\delta\gamma\int_0^T\int_{\mathbb{R}}P^2_{\varepsilon,\delta,\gamma}dtdx\le C_0T=C(T),$$

that is (2.24). □

Lemma 2.4. *Let $T>0$. There exists $C(T)>0$, independent on ε, δ and γ, such that*

$$\left\|\partial_x P_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|_{L^2(\mathbb{R})}\le\frac{C(T)}{\delta\sqrt{\varepsilon}}, \tag{2.29}$$

for every $0<t<T$. Moreover,

$$\left\|P_{\varepsilon,\delta,\gamma}\right\|_{L^\infty((0,T)\times\mathbb{R})}\le\frac{C(T)}{\delta^{\frac{3}{4}}\gamma^{\frac{1}{4}}\varepsilon^{\frac{1}{4}}}. \tag{2.30}$$

Proof. Let $0<t<T$. Differentiating the second equation in (2.7) with respect to x, we have

$$\delta\partial^2_{tx}P_{\varepsilon,\delta,\gamma}=-\partial_x u_{\varepsilon,\delta,\gamma}+\partial^2_{xx}P_{\varepsilon,\delta,\gamma}. \tag{2.31}$$

Multiplying (2.31) by $\partial_x P_{\varepsilon,\delta,\gamma}$, an integration on $\mathbb{R}$ and (2.11) give

$$\begin{aligned}\frac{d}{dt}\left(\delta\left\|\partial_x P_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)=&-2\int_{\mathbb{R}}\partial_x u_{\varepsilon,\delta,\gamma}\partial_x P_{\varepsilon,\delta,\gamma}dx+\int_{\mathbb{R}}\partial_x\left(\partial_x P_{\varepsilon,\delta,\gamma}\right)^2dx\\ =&-2\int_{\mathbb{R}}\partial_x u_{\varepsilon,\delta,\gamma}\partial_x P_{\varepsilon,\delta,\gamma}dx.\end{aligned} \tag{2.32}$$

Due to the Young inequality,

$$-2\int_{\mathbb{R}}\partial_x u_{\varepsilon,\delta,\gamma}\partial_x P_{\varepsilon,\delta,\gamma}dx\le 2\left|\int_{\mathbb{R}}\partial_x u_{\varepsilon,\delta,\gamma}\partial_x P_{\varepsilon,\delta,\gamma}dx\right|$$

$$
\begin{aligned}
&\leq 2\int_{\mathbb{R}}\left|\frac{\partial_x u_{\varepsilon,\delta,\gamma}}{\sqrt{\delta}}\right|\left|\sqrt{\delta}\partial_x P_{\varepsilon,\delta,\gamma}\right| dx\\
&\leq \frac{1}{\delta}\left\|\partial_x u_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\delta\left\|\partial_x P_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, we get

$$
\frac{d}{dt}\left(\delta\left\|\partial_x P_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\leq \frac{1}{\delta}\left\|\partial_x u_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\delta\left\|\partial_x P_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
$$

that is

$$
\frac{d}{dt}\left(\delta\left\|\partial_x P_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)-\delta\left\|\partial_x P_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\leq \frac{1}{\delta}\left\|\partial_x u_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
$$

The Gronwall Lemma and (2.8) give

$$
\begin{aligned}
\delta\left\|\partial_x P_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &\leq C_0e^t+\frac{e^t}{\delta}\int_0^t e^{-s}\left\|\partial_x u_{\varepsilon,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\leq C(T)+\frac{C(T)}{\delta}\int_0^t\left\|\partial_x u_{\varepsilon,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds.
\end{aligned}\tag{2.33}
$$

Due to (2.23),

$$
\frac{1}{\delta}\int_0^t\left\|\partial_x u_{\varepsilon,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds=\frac{\varepsilon}{\delta\varepsilon}\int_0^t\left\|\partial_x u_{\varepsilon,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\leq\frac{C_0}{\delta\varepsilon}.\tag{2.34}
$$

Since $0<\varepsilon,\ \delta<1$, it follows from (2.33) and (2.34) that

$$
\delta\left\|\partial_x P_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\leq C(T)\left(1+\frac{1}{\delta\varepsilon}\right)\leq C(T)\left(\frac{\delta\varepsilon+1}{\delta\varepsilon}\right)\leq\frac{C(T)}{\delta\varepsilon}.
$$

Hence,

$$
\left\|\partial_x P_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\leq\frac{C(T)}{\delta^2\varepsilon},
$$

which gives (2.29).
Let us show that (2.30) holds. We begin by observing that, thanks to the Hölder inequality,

$$
\begin{aligned}
P^2_{\varepsilon,\delta,\gamma}(t,x)&=2\int_{-\infty}^x P_{\varepsilon,\delta,\gamma}(t,y)\partial_x P_{\varepsilon,\delta,\gamma}(t,y)dy\\
&\leq 2\int_{\mathbb{R}}\left|P_{\varepsilon,\delta,\gamma}(t,y)\right|\left|\partial_x P_{\varepsilon,\delta,\gamma}(t,y)dy\right|dx\\
&\leq\left\|P_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x P_{\varepsilon,\delta,\gamma}(t,\cdot)\right\|_{L^2(\mathbb{R})}.
\end{aligned}\tag{2.35}
$$

It follows from (2.23) and (2.29) that

$$
\left\|P_{\varepsilon,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\leq\frac{C_0}{\sqrt{\delta\gamma}}\frac{C(T)}{\delta\sqrt{\varepsilon}}\leq\frac{C(T)}{\delta^{\frac{3}{2}}\gamma^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}},
$$

which gives (2.30). □

Lemma 2.5. *Let $T>0$. Assume (2.9). Then, there exists $C(T)>0$, independent on ε, δ and γ, such that*

$$
\left\|u_{\varepsilon,\delta,\gamma}\right\|_{L^\infty((0,T)\times\mathbb{R})}\leq\left\|u_0\right\|_{L^\infty(\mathbb{R})}+C(T).\tag{2.36}
$$

Proof. We begin by observing that, from (2.9) and (2.30), we have

$$\partial_t u_{\varepsilon,\delta,\gamma} + \partial_x f(u_{\varepsilon,\delta,\gamma}) - \varepsilon\partial_{xx}^2 u_{\varepsilon,\delta,\gamma} \le \gamma \left\| P_{\varepsilon,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} \le \frac{\gamma^{\frac{3}{4}} C(T)}{\delta^{\frac{3}{4}}\varepsilon^{\frac{1}{4}}} \le C(T).$$

Since the map

$$\mathcal{F}(t) := \left\| u_0 \right\|_{L^\infty(\mathbb{R})} + C(T)t,$$

solves the equation

$$\frac{d\mathcal{F}}{dt} = C(T)$$

and

$$\max\{u_{\varepsilon,\delta,\gamma}(0,x), 0\} \le \mathcal{F}(t), \qquad (t,x) \in (0,T)\times\mathbb{R},$$

the comparison principle for parabolic equations implies that

$$u_{\varepsilon,\delta,\gamma}(t,x) \le \mathcal{F}(t), \qquad (t,x)\in(0,T)\times\mathbb{R}.$$

In a similar way, we can prove that

$$u_{\varepsilon,\delta,\gamma}(t,x) \ge -\mathcal{F}(t), \qquad (t,x)\in(0,T)\times\mathbb{R}.$$

Therefore,

$$|u_{\varepsilon,\delta,\gamma}(t,x)| \le \left\| u_0 \right\|_{L^\infty(\mathbb{R})} + C(T)t \le \left\| u_0 \right\|_{L^\infty(\mathbb{R})} + C(T),$$

which gives (2.36). □

To prove Theorem 2.1, the following technical lemma is needed [22].

Lemma 2.6. *Let Ω be a bounded open subset of $\mathbb{R}^2$. Suppose that the sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_n = \mathcal{L}_{1,n} + \mathcal{L}_{2,n},$$

where $\{\mathcal{L}_{1,n}\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n\in\mathbb{N}}$ lies in a bounded subset of $\mathcal{M}_{loc}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$.

Now, we are ready for the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $\eta : \mathbb{R} \to \mathbb{R}$ be any convex C^2 entropy function, and $q : \mathbb{R} \to \mathbb{R}$ be the corresponding entropy flux defined by $q' = f'\eta'$. By multiplying the first equation in (2.7) with $\eta'(u_{\varepsilon,\delta,\gamma})$ and using the chain rule, we get

$$\partial_t \eta(u_{\varepsilon,\delta,\gamma}) + \partial_x q(u_{\varepsilon,\delta,\gamma}) = \underbrace{\varepsilon\partial_{xx}^2\eta(u_{\varepsilon,\delta,\gamma})}_{=:\mathcal{L}_{1,\varepsilon,\delta,\gamma}} \underbrace{-\varepsilon\eta''(u_{\varepsilon,\delta,\gamma})\left(\partial_x u_{\varepsilon,\delta,\gamma}\right)^2}_{=:\mathcal{L}_{2,\varepsilon,\delta,\gamma}} \underbrace{+\gamma\eta'(u_{\varepsilon,\delta,\gamma})P_{\varepsilon,\delta,\gamma}}_{=:\mathcal{L}_{3,\varepsilon,\delta,\gamma}}, \tag{2.37}$$

where $\mathcal{L}_{1,\varepsilon,\delta,\gamma}$, $\mathcal{L}_{2,\varepsilon,\delta,\gamma}$, $\mathcal{L}_{3,\varepsilon,\delta,\gamma}$ are distributions.

Let us show that

$$\mathcal{L}_{1,\varepsilon,\delta,\gamma} \to 0 \text{ in } H^{-1}((0,T)\times\mathbb{R}),\ T>0.$$

Since

$$\varepsilon\partial_{xx}^2\eta(u_{\varepsilon,\delta,\gamma}) = \partial_x(\varepsilon\eta'(u_{\varepsilon,\delta,\gamma})\partial_x u_{\varepsilon,\delta,\gamma}),$$

from Lemmas 2.3 and 2.5,

$$\begin{aligned}\left\|\varepsilon\eta'(u_{\varepsilon,\delta,\gamma})\partial_x u_{\varepsilon,\delta,\gamma}\right\|^2_{L^2((0,T)\times(\mathbb{R}))} &\le \varepsilon^2\left\|\eta'\right\|^2_{L^\infty(I_T)}\int_0^T \left\|\partial_x u_{\varepsilon,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(0,\infty)} ds\\ &\le \varepsilon\left\|\eta'\right\|^2_{L^\infty(I_T)} C_0 \to 0,\end{aligned}$$

where

$$I_T = \left(-\left\|u_0\right\|_{L^\infty(\mathbb{R})} - C(T), \left\|u_0\right\|_{L^\infty(\mathbb{R})} + C(T)\right).$$

We claim that

$$\{\mathcal{L}_{2,\varepsilon,\delta,\gamma}\}_{\varepsilon,\delta,\gamma>0} \text{ is uniformly bounded in } L^1((0,T)\times\mathbb{R})),\ T>0.$$

Again by Lemmas 2.3 and 2.5,

$$\begin{aligned}\left\|\varepsilon\eta''(u_{\varepsilon,\delta,\gamma})(\partial_x u_{\varepsilon,\delta,\gamma})^2\right\|_{L^1((0,T)\times\mathbb{R})} &\le \left\|\eta''\right\|_{L^\infty(I_T)}\varepsilon\int_0^T \left\|\partial_x u_{\varepsilon,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &\le \left\|\eta''\right\|_{L^\infty(I_T)}C(T).\end{aligned}$$

We have that

$$\{\mathcal{L}_{3,\varepsilon,\delta,\gamma}\}_{\varepsilon,\delta>0} \text{ is uniformly bounded in } L^1_{loc}((0,T)\times(0,\infty)),\ T>0.$$

Let K be a compact subset of $(0,T)\times(\mathbb{R})$. From (2.9) and (2.30),

$$\begin{aligned}\left\|\gamma\eta'(u_{\varepsilon,\delta,\gamma})P_{\varepsilon,\delta,\gamma}\right\|_{L^1(K)} &= \gamma\int_K |\eta'(u_{\varepsilon,\delta,\gamma})||P_{\varepsilon,\delta,\gamma}|dtdx\\ &\le \gamma\left\|\eta'\right\|_{L^\infty(I_T)}\left\|P_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})}|K|\\ &\le \frac{\gamma^{\frac{3}{4}}}{\delta^{\frac{3}{4}}\varepsilon^{\frac{1}{4}}}C(T)\left\|\eta'\right\|_{L^\infty(I_T)}|K|\\ &= C(T)\left\|\eta'\right\|_{L^\infty(I_T)}|K|.\end{aligned}$$

Therefore, Lemma 2.6 implies that

$$\{\partial_t\eta(u_{\varepsilon,\delta,\gamma})+\partial_x q(u_{\varepsilon,\delta,\gamma})\}_{\varepsilon,\delta,\gamma>0} \text{ lies in a compact subset of } H^{-1}_{\mathrm{loc}}((0,\infty)\times\mathbb{R}). \tag{2.38}$$

The L^∞ bound stated in Lemma 2.5, (2.38) and the Tartar's compensated compactness method [30] give the existence of a subsequence $\{u_{\varepsilon_k,\delta_k,\gamma_k}\}_{k\in\mathbb{N}}$ and a limit function $u\in L^\infty((0,T)\times\mathbb{R})$ such that

$$u_{\varepsilon_k,\delta_k,\gamma_k}\to u \text{ a.e. and in } L^p_{loc}((0,T)\times\mathbb{R}),\ 1\le p<\infty. \tag{2.39}$$

Hence,

$$u_{\varepsilon_k,\delta_k,\gamma_k}\to u \text{ in } L^\infty((0,T)\times\mathbb{R}). \tag{2.40}$$

We conclude by proving that u is unique entropy solution of (2.6). Let $\phi\in C^\infty(\mathbb{R}^2)$ be a positive text function with compact support. We have to prove that

$$\int_0^\infty\!\!\int_{\mathbb{R}}(\eta(u)\partial_t\phi+q(u)\partial_x\phi)dtdx+\int_{\mathbb{R}}\eta\left(u_0(x)\right)\phi(0,x)dx\ge 0. \tag{2.41}$$

From (2.37), we have

$$\partial_t\eta(u_{\varepsilon_k,\delta_k,\gamma_k})+\partial_x q(u_{\varepsilon_k,\delta_k,\gamma_k})\le\varepsilon_k\partial^2_{xx}\eta(u_{\varepsilon_k,\delta_k,\gamma_k})+\gamma_k\eta'(u_{\varepsilon,\delta,\gamma})P_{\varepsilon,\delta,\gamma}.$$

Multiplying by ϕ and integrating on $(0,\infty)\times\mathbb{R}$, we have that

$$\begin{aligned}&\int_0^\infty\!\!\int_{\mathbb{R}}(\eta(u_{\varepsilon_k,\delta_k,\gamma_k})\partial_t\phi+q(u_{\varepsilon_k,\delta_k,\gamma_k})\partial_x\phi)dtdx+\int_{\mathbb{R}}\eta\left(u_{0,\varepsilon_k,\delta_k,\gamma_k}(x)\right)\phi(0,x)dx\\ &\quad+\varepsilon_k\int_0^\infty\!\!\int_{\mathbb{R}}\eta(u_{\varepsilon_k,\delta_k,\gamma_k})\partial^2_{xx}\phi dtdx+\gamma_k\int_0^\infty\!\!\int_{\mathbb{R}}\eta'(u_{\varepsilon_k,\delta_k,\gamma_k})P_{\varepsilon_k,\delta_k,\gamma_k}\phi dtdx\ge 0.\end{aligned} \tag{2.42}$$

Let us show that

$$\gamma_k\int_0^\infty\!\!\int_{\mathbb{R}}\eta'(u_{\varepsilon_k,\delta_k,\gamma_k})P_{\varepsilon_k,\delta_k,\gamma_k}\phi dtdx\to 0. \tag{2.43}$$

From (2.9), (2.24), (2.36) and the Hölder inequality, we get

$$
\begin{aligned}
\gamma_k &\left|\int_0^\infty\!\!\int_{\mathbb{R}} \eta'(u_{\varepsilon_k,\delta_k,\gamma_k}) P_{\varepsilon_k,\,\delta_k,\,\gamma_k}\phi dtdx\right| \\
&\le \gamma_k \int_0^\infty\!\!\int_{\mathbb{R}} |\eta'(u_{\varepsilon_k,\delta_k,\gamma_k})||P_{\varepsilon_k,\,\delta_k,\,\gamma_k}||\phi| dtdx \\
&\le \gamma_k \left\|\eta'\right\|_{L^\infty(I_T)} \left\|P_{\varepsilon_k,\,\delta_k,\,\gamma_k}\right\|_{L^2(\text{supp}\,(\phi))} \left\|\phi\right\|_{L^2(\text{supp}\,(\phi))} \\
&\le \gamma_k \left\|\eta'\right\|_{L^\infty(I_T)} \left\|P_{\varepsilon_k,\,\delta_k,\,\gamma_k}\right\|_{L^2((0,T)\times\mathbb{R})} \left\|\phi\right\|_{L^2((0,T)\times\mathbb{R})} \\
&\le \frac{\gamma_k^{\frac{1}{2}}}{\delta_k^{\frac{1}{2}}} C(T) \left\|\eta'\right\|_{L^\infty(I_T)} \left\|\phi\right\|_{L^2((0,T)\times\mathbb{R})} \\
&= \varepsilon_k^{\frac{1}{6}} C(T) \left\|\eta'\right\|_{L^\infty(I_T)} \left\|\phi\right\|_{L^2((0,T)\times\mathbb{R})} \to 0,
\end{aligned}
$$

that is (2.43). Therefore, (2.41) follows from (2.8), (2.36), (2.42), (2.43) and the Lebesgue Dominated Convergence Theorem.

Finally, (2.15) and (2.39) give (2.10). □

3. Ostrovsky equation: $\gamma \to 0$.

In this section, we consider the following Cauchy probelm

$$
\begin{cases}
u + \frac{1}{2}\partial_x u^2 - \beta\partial_{xxx}^3 u = \gamma P, & t>0,\ x\in\mathbb{R},\\
\partial_x P = u & t>0,\ x\in\mathbb{R},\\
P(t,-\infty) = 0 & t>0,\\
u(0,x) = u_0(x), & x\in\mathbb{R},
\end{cases}
\tag{3.1}
$$

or equivalently,

$$
\begin{cases}
\partial_t u + \frac{1}{2}\partial_x u^2 - \beta\partial_{xxx}^3 u = \gamma\int_{-\infty}^x u(t,y)dy, & t>0,\ x\in\mathbb{R},\\
u(0,x) = u_0(x), & x\in\mathbb{R}.
\end{cases}
\tag{3.2}
$$

On the initial datum, we assume

$$
u_0 \in L^2(\mathbb{R})\cap L^4(\mathbb{R}), \qquad \int_{\mathbb{R}} u_0(x)dx = 0,
\tag{3.3}
$$

and on the function

$$
P_0(x) = \int_{-\infty}^x u_0(y)dy, \quad x\in\mathbb{R},
\tag{3.4}
$$

we assume that

$$
\int_{\mathbb{R}} P_0(x)dx = \int_{\mathbb{R}}\left(\int_{-\infty}^x u_0(y)dy\right)dx = 0.
\tag{3.5}
$$

We observe that, if $\beta,\gamma\to 0$, then (3.1) reads

$$
\begin{cases}
\partial_t u + \frac{1}{2}\partial_x u^2 = 0, & t>0,\ x\in\mathbb{R},\\
u(0,x) = u_0(x), & x\in\mathbb{R}.
\end{cases}
\tag{3.6}
$$

which is the Burges' equation.

Fix four small numbers $0<\varepsilon,\beta,\delta,\gamma<1$, and let $u_{\varepsilon,\beta,\delta,\gamma}=u_{\varepsilon,\beta,\delta,\gamma}(t,x)$ be the unique classical solution of the following mixed problem:

$$\begin{cases}\partial_t u_{\varepsilon,\beta,\delta,\gamma}+\frac{1}{2}\partial_x u^2_{\varepsilon,\beta,\delta,\gamma}-\beta\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma}\\ \qquad=\gamma P_{\varepsilon,\beta,\delta,\gamma}+\varepsilon\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}, & t>0,\ x\in\mathbb{R},\\ -\delta\partial_t P_{\varepsilon,\beta,\delta,\gamma}+\partial_x P_{\varepsilon,\beta,\delta,\gamma}=u_{\varepsilon,\beta,\delta,\gamma}, & t>0, x\in\mathbb{R},\\ P_{\varepsilon,\beta,\delta,\gamma}(t,-\infty)=0, & t>0,\\ u_{\varepsilon,\beta,\delta,\gamma}(0,x)=u_{\varepsilon,\beta,\delta,\gamma,0}(x), & x\in\mathbb{R},\end{cases}\tag{3.7}$$

where $u_{\varepsilon,\beta,\delta,\gamma,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned}&u_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}\to u_0\quad \text{in } L^p_{loc}(\mathbb{R}),\ 1\le p<4, \text{ as } \varepsilon,\ \beta,\ \delta,\ \gamma\to 0,\\ &\left\|u_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}\right\|^2_{L^2(\mathbb{R})}+\delta\gamma\left\|P_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}\right\|^2_{L^2(\mathbb{R})}+\delta\left\|\partial_x P_{\varepsilon,\,\beta,\,\delta,\gamma,0}\right\|^2_{L^2(\mathbb{R})}\\ &\qquad+\left\|u_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}\right\|^4_{L^4(\mathbb{R})}+(\beta+\varepsilon^2)\left\|\partial_x u_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}\right\|^2_{L^2(\mathbb{R})}\\ &\qquad+\beta^2\left\|\partial^2_{xx}u_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}\right\|^2_{L^2(\mathbb{R})}\le C_0,\quad \varepsilon,\beta,\delta,\gamma>0,\\ &\beta\int_{\mathbb{R}}u_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}(\partial_x u_{\varepsilon,\,\beta,\,\delta,\,\gamma,0})^2\le C_0,\quad \varepsilon,\ \beta,\ \delta,\ \gamma>0,\\ &\int_{\mathbb{R}}u_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}(x)dx=0,\quad \int_{\mathbb{R}}P_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}(x)dx=0,\quad \varepsilon,\ \beta,\ \delta,\ \gamma>0,\end{aligned}\tag{3.8}$$

and C_0 is a constant independent on ε,β, δ and γ.

The main result of this section is the following theorem.

Theorem 3.1. *Assume that* (3.3), (3.4), (3.5), *and* (3.8) *hold. If*

$$\beta=\mathcal{O}(\varepsilon^2),\quad \gamma=\mathcal{O}(\varepsilon\delta)\tag{3.9}$$

then, there exist four sequences $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\{\beta_k\}_{k\in\mathbb{N}}$, $\{\delta_k\}_{k\in\mathbb{N}}$, $\{\gamma_k\}_{k\in\mathbb{N}}$ *with* $\varepsilon_k,\beta_k,\delta_k,\gamma_k\to 0$, *and a limit function* $u\in L^\infty(0,T;L^4(\mathbb{R})\cap L^2(\mathbb{R}))$, $T>0$, *such that*

i) $u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\to u$ *strongly in* $L^p_{loc}((0,T)\times\mathbb{R})$, *for each* $1\le p<4$, $T>0$,

and u *is a distributional solution of* (3.6). *Moreover, if*

$$\beta=o(\varepsilon^2),\quad \gamma=\mathcal{O}(\varepsilon\delta)\tag{3.10}$$

then,

ii) u *is the unique entropy solution of* (3.6).

In particular, we have (2.10).

Let us prove some a priori estimates on $u_{\varepsilon,\beta,\delta,\gamma}$ and $P_{\varepsilon,\beta,\delta,\gamma}$, denoting with C_0 the constants which depend on the initial datum, and $C(T)$ the constants which depend also on T.

Arguing as Section 2, we obtain the following results

Lemma 3.1. *For each* $t>0$,

$$P_{\varepsilon,\beta,\delta,\gamma}(t,\infty)=\partial_x P_{\varepsilon,\beta,\delta,\gamma}(t,-\infty)=\partial_x P_{\varepsilon,\beta,\delta,\gamma}(t,\infty)=0,\tag{3.11}$$

$$\int_{\mathbb{R}}u_{\varepsilon,\beta,\delta,\gamma}(t,x)dx=-\delta\frac{d}{dt}\int_{\mathbb{R}}P_{\varepsilon,\beta,\delta,\gamma}(t,x)dx,\tag{3.12}$$

$$\int_{\mathbb{R}}P_{\varepsilon,\beta,\delta,\gamma}(t,x)dx=0,\tag{3.13}$$

$$\int_{\mathbb{R}}u_{\varepsilon,\beta,\delta,\gamma}(t,x)dx=0.\tag{3.14}$$

In particular, we have that

$$\|u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \delta\gamma \|P_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon \int_0^t \|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\|^2_{L^2(\mathbb{R})} dx \le C_0. \tag{3.15}$$

Moreover, fixed $T > 0$, there exists $C(T) > 0$, independent on ε, β, δ and γ, such that,

$$\|P_{\varepsilon,\beta,\delta,\gamma}\|_{L^2((0,T)\times\mathbb{R})} \le \frac{C(T)}{\delta^{\frac{1}{2}}\gamma^{\frac{1}{2}}}, \tag{3.16}$$

$$\|\partial_x P_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\|_{L^2(\mathbb{R})} \le \frac{C(T)}{\delta\sqrt{\varepsilon}}, \tag{3.17}$$

$$\|P_{\varepsilon,\beta,\delta,\gamma}\|_{L^\infty((0,T)\times\mathbb{R})} \le \frac{C(T)}{\delta^{\frac{3}{4}}\gamma^{\frac{1}{4}}\varepsilon^{\frac{1}{4}}}, \tag{3.18}$$

for every $0 \le t \le T$.

Lemma 3.2. *Fixed $T > 0$. Then,*

$$\|u_{\varepsilon,\beta,\delta,\gamma}\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T)\beta^{-\frac{1}{3}}. \tag{3.19}$$

Moreover, for every $0 \le t \le T$,

$$\beta \|u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta\varepsilon \int_0^t \left\|\partial^2_{xx} u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T)\beta^{-\frac{1}{3}}. \tag{3.20}$$

Proof. Let $0 \le t \le T$. Multiplying (3.7) by $-2\beta\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma} + u^2_{\varepsilon,\beta,\delta,\gamma}$, and arguing as [5, Lemma 2.5], we obtain that

$$\begin{aligned}
&\frac{d}{dt}\left(\beta \|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{3}\int_{\mathbb{R}} u^3_{\varepsilon,\beta,\delta,\gamma} dx\right)\\
&\qquad + 2\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\beta,\delta,\gamma}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2 dx + 2\beta\varepsilon \left\|\partial^2_{xx} u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad = -2\gamma\beta \int_{\mathbb{R}} \partial^2_{xx} u_{\varepsilon,\beta,\delta,\gamma} P_{\varepsilon,\beta,\delta,\gamma} dx + \gamma \int_{\mathbb{R}} u^2_{\varepsilon,\beta,\delta,\gamma} P_{\varepsilon,\beta,\delta,\gamma} dx.
\end{aligned} \tag{3.21}$$

Since $0 < \beta, \varepsilon < 1$, it follows from (3.9), (3.15) and the Young inequality that

$$\begin{aligned}
2\gamma\beta \int_{\mathbb{R}} \partial^2_{xx} u_{\varepsilon,\beta,\delta,\gamma} P_{\varepsilon,\beta,\delta,\gamma} dx &\le 2\int_{\mathbb{R}} \left|\beta\sqrt{\varepsilon}\partial^2_{xx} u_{\varepsilon,\beta,\delta,\gamma}\right| \left|\frac{\gamma}{\sqrt{\varepsilon}} P_{\varepsilon,\beta,\delta,\gamma}\right| dx\\
&\le \beta^2\varepsilon \left\|\partial^2_{xx} u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\gamma^2}{\varepsilon} \|P_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le \beta\varepsilon \left\|\partial^2_{xx} u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\gamma^2}{\delta\gamma} C(T)\\
&\le \beta\varepsilon \left\|\partial^2_{xx} u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\varepsilon\\
&\le \beta\varepsilon \left\|\partial^2_{xx} u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T).
\end{aligned} \tag{3.22}$$

Since $0 < \delta, \varepsilon < 1$, due to (3.9), (3.15) and the Hölder inequality,

$$
\begin{aligned}
\gamma \int_{\mathbb{R}} u^2_{\varepsilon,\beta,\delta,\gamma} P_{\varepsilon,\beta} dx &\leq \gamma \int_{\mathbb{R}} u^2_{\varepsilon,\beta,\delta,\gamma} |P_{\varepsilon,\beta,\delta,\gamma}| dx \\
&\leq \gamma \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |u_{\varepsilon,\beta,\delta,\gamma}| |P_{\varepsilon,\beta,\delta,\gamma}| dx \\
&\leq \frac{\gamma}{\sqrt{\delta}\sqrt{\gamma}} C_0 \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} \leq \varepsilon^{\frac{1}{2}} C(T) \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} \\
&\leq C(T) \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})}.
\end{aligned} \tag{3.23}
$$

Therefore, (3.15), (3.21), (3.22) and (3.23) give

$$
\begin{aligned}
&\frac{d}{dt}\left(\beta \left\| \partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{1}{3} \int_{\mathbb{R}} u^3_{\varepsilon,\beta,\delta,\gamma} dx \right) + \beta\varepsilon \left\| \partial^2_{xx} u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\
&\quad \leq C(T) \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} - 2\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\beta,\delta,\gamma} (\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2 dx + C(T) \\
&\quad \leq C(T) \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} + 2\varepsilon \int_{\mathbb{R}} |u_{\varepsilon,\beta,\delta,\gamma}| (\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2 dx + C(T) \\
&\quad \leq C(T) \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} + 2\varepsilon \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} (\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2 dx + C(T).
\end{aligned}
$$

It follows from (3.8), (3.15) and an integration on $(0,t)$ that

$$
\begin{aligned}
&\beta \left\| \partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \beta\varepsilon \int_0^t \left\| \partial^2_{xx} u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot) \right\|^2_{L^2(\mathbb{R})} ds \\
&\quad \leq C_0 + C(T) \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} \int_0^t ds + C(T) \int_0^t ds \\
&\qquad + 2\varepsilon \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} \int_0^t \left\| \partial_x u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot) \right\|^2_{L^2(\mathbb{R})} ds + \frac{1}{3} \int_{\mathbb{R}} |u_{\varepsilon,\beta,\delta,\gamma}|^3 dx \\
&\quad \leq C(T) + C(T) \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} + C_0 C(T) \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} \\
&\qquad + \frac{1}{3} \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} \left\| u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\
&\quad \leq C(T) + C(T) \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} + \frac{1}{3} C_0 \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\beta \left\| \partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot) \right\|^2_{L^2(\mathbb{R})} &+ \beta\varepsilon \int_0^t \left\| \partial^2_{xx} u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot) \right\|^2_{L^2(\mathbb{R})} ds \\
&\leq C(T) \left(1 + \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} \right).
\end{aligned} \tag{3.24}
$$

Due to (3.15), (3.24) and the Hölder inequality,

$$
\begin{aligned}
u^2_{\varepsilon,\beta,\delta,\gamma}(t,x) &= 2 \int_{-\infty}^x u_{\varepsilon,\beta,\delta,\gamma} \partial_x u_{\varepsilon,\beta,\delta,\gamma} dy \leq 2 \int_{\mathbb{R}} |u_{\varepsilon,\beta,\delta,\gamma} \partial_x u_{\varepsilon,\beta,\delta,\gamma}| dx \\
&\leq \frac{2}{\sqrt{\beta}} \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^2(\mathbb{R})} \sqrt{\beta} \left\| \partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot) \right\|_{L^2(\mathbb{R})} \\
&\leq \frac{2}{\sqrt{\beta}} C_0 \sqrt{C(T) \left(1 + \left\| u_{\varepsilon,\beta,\delta,\gamma} \right\|_{L^\infty((0,T)\times\mathbb{R})} \right)},
\end{aligned}
$$

that is

$$\|u_{\varepsilon,\beta,\delta,\gamma}\|^4_{L^\infty((0,T)\times\mathbb{R})} \le \frac{C(T)}{\beta}\left(1+\|u_{\varepsilon,\beta,\delta,\gamma}\|_{L^\infty((0,T)\times\mathbb{R})}\right). \tag{3.25}$$

Arguing as [5, Lemma 2.7], we have (3.19).

Finally, (3.20) follows from (3.19) and (3.24). □

Lemma 3.3. *Let $T>0$. Assume (3.9) holds true. Then:*

i) the family $\{u_{\varepsilon,\beta,\delta,\gamma}\}_{\varepsilon,\beta,\delta,\gamma}$ is bounded in $L^4((0,T)\times\mathbb{R})$;

ii) the following families $\{\beta\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}\}_{\varepsilon,\beta,\delta,\gamma}$, $\{\sqrt{\varepsilon}u_{\varepsilon,\beta,\delta,\gamma}\partial_x u_{\varepsilon,\beta,\delta,\gamma}\}_{\varepsilon,\beta,\delta,\beta}$, $\{\beta\sqrt{\varepsilon}\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma}\}_{\varepsilon,\beta,\delta,\gamma}$ are bounded in $L^2((0,T)\times\mathbb{R})$.

The proof of the previous lemma is based on the regularity of the functions $u_{\varepsilon,\beta,\delta,\gamma}$ and [5, Lemma 2.5].

Proof of Lemma 3.3. Let $0\le t\le T$. Multiplying (3.7) by

$$u^3_{\varepsilon,\beta,\delta,\gamma} - 3\beta(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2 - 6\beta u_{\varepsilon,\beta,\delta,\gamma}\partial^2_{xx}u_{\varepsilon,\beta} + \frac{18}{5}\beta^2\partial^4_{xxxx}u_{\varepsilon,\beta,\delta,\gamma},$$

and arguing as [5, Lemma 2.6], we obtain that

$$\begin{aligned}
&\frac{d}{dt}G(t) + 3D_1\varepsilon\int_{\mathbb{R}} u^2_{\varepsilon,\beta,\delta,\gamma}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx + \varepsilon\beta^2 D_2\int_{\mathbb{R}}(\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma})^2dx\\
&\quad\le\gamma\int_{\mathbb{R}}|u_{\varepsilon,\beta,\delta,\gamma}|^3|P_{\varepsilon,\beta,\delta,\gamma}|dx + 3\gamma\beta\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2|P_{\varepsilon,\beta,\delta,\gamma}|dx\\
&\qquad + 6\gamma\beta\int_{\mathbb{R}}|u_{\varepsilon,\beta,\delta,\gamma}||\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}||P_{\varepsilon,\beta,\delta,\gamma}|dx + \frac{18}{5}\gamma\beta^2\int_{\mathbb{R}}\partial^4_{xxxx}u_{\varepsilon,\beta,\delta,\gamma}P_{\varepsilon,\beta,\delta,\gamma}dx,
\end{aligned}$$

where

$$G(t) = \frac{1}{4}\int_{\mathbb{R}} u^4_{\varepsilon,\beta,\delta,\gamma}dx + 3\beta\int_{\mathbb{R}} u_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx + \frac{9}{5}\beta^2\int_{\mathbb{R}}(\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma})^2dx, \tag{3.26}$$

while D_1, D_2 are fixed positive constants.

Due to (3.11),

$$\frac{18}{5}\gamma\beta^2\int_{\mathbb{R}}\partial^4_{xxxx}u_{\varepsilon,\beta,\delta,\gamma}P_{\varepsilon,\beta,\delta,\gamma}dx = -\frac{18}{5}\gamma\beta^2\int_{\mathbb{R}}\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma}\partial_x P_{\varepsilon,\beta,\delta,\gamma}dx \tag{3.27}$$

Since $0<\beta<1$, it follows from (3.9), (3.17), (3.27) and the Young inequality that

$$\begin{aligned}
&-\frac{18}{5}\gamma\beta^2\int_{\mathbb{R}}\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma}\partial_x P_{\varepsilon,\beta,\delta,\gamma}dx \le \frac{18}{5}\gamma\beta^2\left|\int_{\mathbb{R}}\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma}\partial_x P_{\varepsilon,\beta,\delta,\gamma}dx\right|\\
&\le\frac{18}{5}\int_{\mathbb{R}}\left|\beta^2\sqrt{A\varepsilon}\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma}\right|\left|\frac{\gamma\partial_x P_{\varepsilon,\beta,\delta,\gamma}}{\sqrt{A\varepsilon}}\right|\\
&\le\frac{9A}{5}\beta^4\varepsilon\int_{\mathbb{R}}(\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma})^2dx + \frac{9}{5A}\frac{\gamma^2}{\varepsilon}\int_{\mathbb{R}}(\partial_x P_{\varepsilon,\beta,\delta,\gamma})^2dx\\
&\le\frac{9A}{5}\beta^2\varepsilon\int_{\mathbb{R}}(\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma})^2dx + \frac{9}{5A}\frac{\gamma^2}{\delta^2\varepsilon^2}C(T)\\
&\le\frac{9A}{5}\beta^2\varepsilon\int_{\mathbb{R}}(\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma})^2dx + \frac{C(T)}{A},
\end{aligned}$$

where A is a positive constant that will be specified later. Therefore,

$$\frac{d}{dt}G(t) + 3D_1\varepsilon\int_{\mathbb{R}} u^2_{\varepsilon,\beta,\delta,\gamma}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx + \varepsilon\beta^2\left(D_2 - \frac{9A}{5}\right)\int_{\mathbb{R}}(\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma})^2dx$$

$$\begin{aligned}
\leq&\gamma\int_{\mathbb{R}}|u_{\varepsilon,\beta,\delta,\gamma}|^3|P_{\varepsilon,\beta,\delta,\gamma}|dx+3\gamma\beta\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2|P_{\varepsilon,\beta,\delta,\gamma}|dx\\
&+6\gamma\beta\int_{\mathbb{R}}|u_{\varepsilon,\beta,\delta,\gamma}||\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma}||P_{\varepsilon,\beta,\delta,\gamma}|dx+\frac{C(T)}{A}.
\end{aligned}$$

Choosing $A<\frac{5D_2}{9}$, we have

$$\begin{aligned}
\frac{d}{dt}G(t)&+3D_1\varepsilon\int_{\mathbb{R}}u_{\varepsilon,\beta,\delta,\gamma}^2(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx+\varepsilon\beta^2D_3\int_{\mathbb{R}}(\partial_{xxx}^3 u_{\varepsilon,\beta,\delta,\gamma})^2dx\\
\leq&\gamma\int_{\mathbb{R}}|u_{\varepsilon,\beta,\delta,\gamma}|^3|P_{\varepsilon,\beta,\delta,\gamma}|dx+3\gamma\beta\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2|P_{\varepsilon,\beta,\delta,\gamma}|dx\\
&+6\gamma\beta\int_{\mathbb{R}}|u_{\varepsilon,\beta,\delta,\gamma}||\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma}||P_{\varepsilon,\beta,\delta,\gamma}|dx+C(T),
\end{aligned}$$

where D_3 is a fixed positive constant.

Since $0<\varepsilon<1$, due to (3.9), (3.15), (3.18) and the Young inequality, we obtain that

$$\begin{aligned}
\gamma&\int_{\mathbb{R}}|u_{\varepsilon,\beta,\delta,\gamma}^3||P_{\varepsilon,\beta,\delta,\gamma}|dxdx+6\gamma\beta\int_{\mathbb{R}}|u_{\varepsilon,\beta,\delta,\gamma}||\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma}||P_{\varepsilon,\beta,\delta,\gamma}|\\
&=\int_{\mathbb{R}}\left|\frac{1}{\sqrt{2}}u_{\varepsilon,\beta,\delta,\gamma}^2\right|\left|\sqrt{2}\gamma u_{\varepsilon,\beta,\delta,\gamma}P_{\varepsilon,\beta,\delta,\gamma}\right|dx\\
&\quad+\int_{\mathbb{R}}\left|\frac{3\sqrt{2}}{\sqrt{5}}\beta\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma}\right|\left|\frac{\sqrt{5}\gamma}{\sqrt{2}}u_{\varepsilon,\beta,\delta,\gamma}P_{\varepsilon,\beta,\delta,\gamma}\right|dx\\
&\leq\frac{1}{4}\int_{\mathbb{R}}u_{\varepsilon,\beta,\delta,\gamma}^4dx+\gamma^2\int_{\mathbb{R}}u_{\varepsilon,\beta,\delta,\gamma}^2P_{\varepsilon,\beta}^2dx\\
&\quad+\frac{9}{5}\beta^2\int_{\mathbb{R}}(\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma})^2dx+\frac{5\gamma^2}{4}\int_{\mathbb{R}}u_{\varepsilon,\beta,\delta,\gamma}^2P_{\varepsilon,\beta,\delta,\gamma}^2dx\\
&=\frac{1}{4}\int_{\mathbb{R}}u_{\varepsilon,\beta,\delta,\gamma}^4dx+\frac{9}{5}\beta^2\int_{\mathbb{R}}(\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma})^2dx+\frac{9\gamma^2}{4}\int_{\mathbb{R}}u_{\varepsilon,\beta,\delta,\gamma}^2P_{\varepsilon,\beta,\delta,\gamma}^2dx\\
&\leq\frac{1}{4}\int_{\mathbb{R}}u_{\varepsilon,\beta,\delta,\gamma}^4dx+\frac{9}{5}\beta^2\int_{\mathbb{R}}(\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma})^2dx\\
&\quad+\frac{9\gamma^2}{4}\left\|P_{\varepsilon,\beta,\delta,\gamma}\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\int_{\mathbb{R}}u_{\varepsilon,\beta,\delta,\gamma}^2(t,x)dx\\
&\leq\frac{1}{4}\int_{\mathbb{R}}u_{\varepsilon,\beta,\delta,\gamma}^4dx+\frac{9}{5}\beta^2\int_{\mathbb{R}}(\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma})^2dx+C(T)\frac{\gamma^2}{\delta^{\frac{3}{2}}\gamma^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}}\\
&\leq\frac{1}{4}\int_{\mathbb{R}}u_{\varepsilon,\beta,\delta,\gamma}^4dx+\frac{9}{5}\beta^2\int_{\mathbb{R}}(\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma})^2dx+\varepsilon C(T)\\
&\leq\frac{1}{4}\int_{\mathbb{R}}u_{\varepsilon,\beta,\delta,\gamma}^4dx+\frac{9}{5}\beta^2\int_{\mathbb{R}}(\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma})^2dx+C(T).
\end{aligned}$$

Again by (3.9) and (3.18),

$$\begin{aligned}
3\gamma\beta\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2|P_{\varepsilon,\beta,\delta,\gamma}|dx\leq&3\gamma\beta\left\|P_{\varepsilon,\beta,\delta,\gamma}\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx\\
\leq&C(T)\frac{\gamma}{\delta^{\frac{3}{4}}\gamma^{\frac{1}{4}}\varepsilon^{\frac{1}{4}}}\beta\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx\\
\leq&C(T)\sqrt{\varepsilon}\beta\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx\leq C(T)\beta\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx.
\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{d}{dt}G(t)+3\varepsilon D_1\int_{\mathbb{R}}u^2_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx+\varepsilon\beta^2D_3\int_{\mathbb{R}}(\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma})^2dx\\ &\quad\le\frac{1}{4}\int_{\mathbb{R}}u^4_{\varepsilon,\beta,\delta,\gamma}dx+\frac{9}{5}\beta^2\int_{\mathbb{R}}(\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma})^2dx+C(T)\beta\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx+C(T).\end{aligned}$$

Arguing as [5, Lemma 2.6], thanks to [5, Lemma 2.7], we have

$$\begin{aligned}\|u_{\varepsilon,\beta,\delta,\gamma}\|_{L^4((0,T)\times\mathbb{R})}&\le C(T),\\ \beta\left\|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}\right\|_{L^2((0,T)\times\mathbb{R})}&\le C(T),\\ \sqrt{\varepsilon}\left\|u_{\varepsilon,\beta,\delta,\gamma}\partial_x u_{\varepsilon,\beta,\delta,\gamma}\right\|_{L^2((0,T)\times\mathbb{R})}&\le C(T),\\ \sqrt{\varepsilon}\beta\left\|\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma}\right\|_{L^2((0,T)\times\mathbb{R})}&\le C(T).\end{aligned}$$

The proof is done. ☐

Lemma 3.4. *Let $T>0$. Assume that (3.9) holds true. Then:*

i) the family $\{\varepsilon\partial_x u_{\varepsilon,\beta,\delta,\gamma}\}_{\varepsilon,\beta,\delta,\gamma}$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}))$;
ii) the family $\{\varepsilon\sqrt{\varepsilon}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}\}_{\varepsilon,\beta,\delta,\gamma}$ is bounded in $L^2((0,T)\times\mathbb{R})$;
iii) the family $\{\beta\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}\}_{\varepsilon,\beta,\delta,\gamma}$ is bounded in $L^1((0,T)\times\mathbb{R})$.

Moreover,

$$\beta^2\int_0^T\left\|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T)\varepsilon. \tag{3.28}$$

Proof. Let $0\le t\le T$. Multiplying (3.7) by $-\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}$, arguing as [5, Lemma 2.8], we have

$$\begin{aligned}&\varepsilon^2\frac{d}{dt}\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx+2\varepsilon^3\int_{\mathbb{R}}(\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma})^2dx\\ &\qquad=2\varepsilon^2\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta,\delta,\gamma}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}dx-2\varepsilon^2\gamma\int_{\mathbb{R}}P_{\varepsilon,\beta,\delta,\gamma}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}dx.\end{aligned}$$

Since $0<\varepsilon<1$, due to (3.9), (3.15) and the Young inequality,

$$\begin{aligned}&2\varepsilon^2\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta,\delta,\gamma}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}dx-2\varepsilon^2\gamma\int_{\mathbb{R}}P_{\varepsilon,\beta,\delta,\gamma}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}dx\\ &\quad\le\left|2\varepsilon^2\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta,\delta,\gamma}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}dx-2\varepsilon^2\gamma\int_{\mathbb{R}}P_{\varepsilon,\beta,\delta,\gamma}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}dx\right|\\ &\quad\le2\varepsilon^2\left|\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta,\delta,\gamma}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}dx\right|+2\varepsilon^2\gamma\left|\int_{\mathbb{R}}P_{\varepsilon,\beta,\delta,\gamma}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}dx\right|\\ &\quad\le2\int_{\mathbb{R}}\varepsilon^{\frac12}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta,\delta,\gamma}|\varepsilon^{\frac32}|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}|dx+\int_{\mathbb{R}}2\varepsilon^{\frac12}\gamma|P_{\varepsilon,\beta,\delta,\gamma}|\varepsilon^{\frac32}|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}|dx\\ &\quad\le\varepsilon\int_{\mathbb{R}}(u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx+\varepsilon^3\int_{\mathbb{R}}(\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma})^2dx+2\gamma^2\varepsilon\int_{\mathbb{R}}P^2_{\varepsilon,\beta,\delta,\gamma}dx\\ &\qquad+\frac{\varepsilon^3}{2}\int_{\mathbb{R}}(\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma})^2dx\\ &\quad\le\varepsilon\int_{\mathbb{R}}(u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx+\frac{3\varepsilon^3}{2}\int_{\mathbb{R}}(\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma})^2dx+\frac{\gamma^2}{\delta\gamma}C(T)\\ &\quad\le\varepsilon\int_{\mathbb{R}}(u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx+\frac{3\varepsilon^3}{2}\int_{\mathbb{R}}(\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma})^2dx+C(T).\end{aligned}$$

Thus,

$$\varepsilon^2\frac{d}{dt}\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx+\frac{\varepsilon^3}{2}\int_{\mathbb{R}}(\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma})^2dx\le\varepsilon\int_{\mathbb{R}}(u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx+C(T).$$

An integration on $(0,t)$, (3.8) and Lemma 3.3 give

$$\varepsilon^2\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\varepsilon^3}{2}\int_0^t\left\|\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T).$$

Hence,

$$\begin{aligned}&\varepsilon^2\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C(T),\\&\varepsilon^3\int_0^t\left\|\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T).\end{aligned}\tag{3.29}$$

Thanks to (3.9), (3.15), (3.29) and the Hölder inequality,

$$\begin{aligned}\beta\int_0^T\!\!\int_{\mathbb{R}}&|\partial_x u_{\varepsilon,\beta,\delta,\gamma}\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma}|dsdx=\frac{\beta}{\varepsilon^2}\int_0^T\!\!\int_{\mathbb{R}}\varepsilon^{\frac12}|\partial_x u_{\varepsilon,\beta,\delta,\gamma}|\varepsilon^{\frac32}|\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma}|dx\\&\le\frac{\beta}{\varepsilon^2}\left(\varepsilon\int_0^T\!\!\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dsdx\right)^{\frac12}\left(\varepsilon^3\int_0^T\!\!\int_{\mathbb{R}}(\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma})^2dsdx\right)^{\frac12}\\&\le C_0C(T)\frac{\beta}{\varepsilon^2}\le C(T).\end{aligned}$$

Due to (3.9) and (3.29), we have

$$\beta^2\int_0^T\left\|\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C_0^2\varepsilon^4\int_0^T\left\|\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T)\varepsilon,$$

which gives (4.21). □

To prove Theorem 3.1, we use Lemma 2.6 and the following definition.

Definition 3.1. *A pair of functions (η,q) is called an entropy–entropy flux pair if $\eta:\mathbb{R}\to\mathbb{R}$ is a C^2 function and $q:\mathbb{R}\to\mathbb{R}$ is defined by*

$$q(u)=\int^u\eta'(\xi)f'(\xi)d\xi.$$

An entropy-entropy flux pair $(\eta,\ q)$ is called convex/compactly supported if, in addition, η is convex/compactly supported.

We begin by proving the following result.

Lemma 3.5. *Assume that (3.3), (3.4), (3.5), (3.8), and (3.9) hold. Then for any compactly supported entropy–entropy flux pair $(\eta,\ q)$, there exist four sequences $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\{\beta_k\}_{k\in\mathbb{N}}$, $\{\delta_k\}_{k\in\mathbb{N}}$, $\{\gamma_k\}_{k\in\mathbb{N}}$, with $\varepsilon_k,\beta_k,\delta_k,\gamma_k\to0$, and a limit function*

$$u\in L^\infty(0,T;L^2(\mathbb{R})\cap L^4(\mathbb{R})),\qquad T>0$$

such that

$$u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\to u\quad\text{in}\quad L^p_{loc}((0,\infty)\times\mathbb{R}),\quad\text{for each}\quad 1\le p<4,\tag{3.30}$$

and u is a distributional solution of (3.6).

Proof. Let us consider a compactly supported entropy–entropy flux pair (η, q). Multiplying (3.7) by $\eta'(u_{\varepsilon,\beta,\delta,\gamma})$, we have

$$\begin{aligned}\partial_t\eta(u_{\varepsilon,\beta,\delta,\gamma}) + \partial_x q(u_{\varepsilon,\beta,\delta,\gamma}) =&\varepsilon\eta'(u_{\varepsilon,\beta,\delta,\gamma})\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma} + \beta\eta'(u_{\varepsilon,\beta,\delta,\gamma})\partial_{xxx}^3 u_{\varepsilon,\beta,\delta,\gamma}\\ &+ \gamma\eta'(u_{\varepsilon,\beta,\delta,\gamma})P_{\varepsilon,\beta,\delta,\gamma}\\ =&I_{1,\varepsilon,\beta,\delta,\gamma} + I_{2,\varepsilon,\beta,\delta,\gamma} + I_{3,\varepsilon,\beta,\delta,\gamma} + I_{4,\varepsilon,\beta,\delta,\gamma} + I_{5,\varepsilon,\beta,\delta,\gamma},\end{aligned}$$

where

$$\begin{aligned}I_{1,\varepsilon,\beta,\delta,\gamma} &= \partial_x(\varepsilon\eta'(u_{\varepsilon,\beta,\delta,\gamma})\partial_x u_{\varepsilon,\beta}),\\ I_{2,\varepsilon,\beta,\delta,\gamma} &= -\varepsilon\eta''(u_{\varepsilon,\beta,\delta,\gamma})(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2,\\ I_{3,\varepsilon,\beta,\delta,\gamma} &= \partial_x(\beta\eta'(u_{\varepsilon,\beta,\delta,\gamma})\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma}),\\ I_{4,\varepsilon,\beta,\delta,\gamma} &= -\beta\eta''(u_{\varepsilon,\beta,\delta,\gamma})\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma},\\ I_{5,\varepsilon,\beta,\delta,\gamma} &= \gamma\eta'(u_{\varepsilon,\beta,\delta,\gamma})P_{\varepsilon,\beta,\delta,\gamma}.\end{aligned} \tag{3.31}$$

Arguing as [5, Lemma 3.2], we have that $I_{1,\varepsilon,\beta,\delta,\gamma} \to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, $\{I_{2,\varepsilon,\beta,\delta,\gamma}\}_{\varepsilon,\beta,\delta,\gamma>0}$ is bounded in $L^1((0,T)\times\mathbb{R})$, $I_{3,\varepsilon,\beta,\delta,\gamma} \to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, $\{I_{4,\varepsilon,\beta,\delta,\gamma}\}_{\varepsilon,\beta,\delta,\gamma>0}$ is bounded in $L^1((0,T)\times\mathbb{R})$.

Let us show that

$$I_{5,\varepsilon,\beta,\delta,\gamma} \to 0 \quad \text{in } L^1_{loc}((0,\infty)\times\mathbb{R}), \quad \text{as} \quad \varepsilon \to 0.$$

Let K be a compact subset of $(0,T)\times\mathbb{R}$. (3.9) and Lemma 3.1 give

$$\begin{aligned}\left\|\gamma\eta'(u_{\varepsilon,\beta,\delta,\gamma})P_{\varepsilon,\beta,\delta,\gamma}\right\|_{L^1(K)} &= \gamma\int_K |\eta'(u_{\varepsilon,\beta,\delta,\gamma})||P_{\varepsilon,\beta,\delta,\gamma}|dtdx\\ &\le \gamma\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|P_{\varepsilon,\beta,\delta,\gamma}\right\|_{L^\infty((0,T)\times\mathbb{R})}|K|\\ &\le \frac{\gamma}{\delta^{\frac{3}{4}}\gamma^{\frac{1}{4}}\varepsilon^{\frac{1}{4}}}C(T)\left\|\eta'\right\|_{L^\infty(\mathbb{R})}|K|\\ &\le \sqrt{\varepsilon}C(T)\left\|\eta'\right\|_{L^\infty(\mathbb{R})}|K| \to 0.\end{aligned}$$

Therefore, Lemma 2.6 and the L^p compensated compactness of [28] give (3.30).

We conclude by proving that u is a distributional solution of (3.6). Let $\phi \in C^\infty(\mathbb{R}^2)$ be a test function with compact support. We have to prove that

$$\int_0^\infty\!\!\int_{\mathbb{R}}\left(u\partial_t\phi + \frac{u^2}{2}\partial_x\phi\right)dtdx + \int_{\mathbb{R}} u_0(x)\phi(0,x)dx = 0. \tag{3.32}$$

We have that

$$\begin{aligned}&\int_0^\infty\!\!\int_{\mathbb{R}}\left(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\partial_t\phi + \frac{u^2_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}}{2}\partial_x\phi\right)dtdx + \int_{\mathbb{R}} u_{0,\varepsilon_k,\beta_k,\delta_k,\gamma_k}(x)\phi(0,x)dx\\ &- \gamma_k\int_0^\infty\!\!\int_{\mathbb{R}} P_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\phi dtdx + \varepsilon_k\int_0^\infty\!\!\int_{\mathbb{R}} u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\partial_{xx}^2\phi dtdx\\ &+ \varepsilon_k\int_0^\infty u_{0,\varepsilon_k,\beta_k,\delta_k,\gamma_k}(x)\partial_{xx}^2\phi(0,x)dx - \beta_k\int_0^\infty\!\!\int_{\mathbb{R}} u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\partial_{xxx}^3\phi dtdx\\ &- \beta_k\int_0^\infty u_{0,\varepsilon_k,\beta_k,\delta_k,\gamma_k}(x)\partial_{xxx}^3\phi(0,x)dx = 0.\end{aligned}$$

Let us show that

$$-\gamma_k\int_0^\infty\!\!\int_{\mathbb{R}} P_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\phi dtdx \to 0. \tag{3.33}$$

From (3.9) and (3.18), we get

$$\begin{aligned}
&\gamma_k \left|\int_0^\infty\!\!\int_{\mathbb{R}} P_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\phi dtdx\right| \\
&\le \gamma_k \int_0^\infty\!\!\int_{\mathbb{R}} |P_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}||\phi| dtdx \\
&\le \gamma_k \left\|P_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\right\|_{L^\infty((0,T)\times\mathbb{R})} \int_0^\infty\!\!\int_{\mathbb{R}} |\phi| dtdx \\
&\le \frac{\gamma_k}{\delta_k^{\frac{3}{4}}\gamma_k^{\frac{1}{4}}\varepsilon_k^{\frac{1}{4}}} C(T)\int_0^\infty\!\!\int_{\mathbb{R}} |\phi| dtdx \\
&\le \sqrt{\varepsilon_k} C(T)\int_0^\infty\!\!\int_{\mathbb{R}} |\phi| dtdx \to 0,
\end{aligned}$$

that is (3.33).

Therefore, (3.32) follows from (3.8), (3.30) and (3.33). □

Arguing as [13], we prove the following result.

Lemma 3.6. *Assume that* (3.3), (3.4), (3.5), (3.8), *and* (3.10) *hold. Then,*

$$u_{\varepsilon_k,\beta_k,\delta_k\gamma_k} \to u \quad in \quad L^p_{loc}((0,\infty)\times\mathbb{R}), \quad for\ each \quad 1 \le p < 4, \tag{3.34}$$

where u is the unique entropy solution of (3.6).

Proof. Let us consider a compactly supported entropy–entropy flux pair (η, q). Multiplying (3.7) by $\eta'(u_{\varepsilon,\beta,\delta,\gamma})$, we obtain that

$$\begin{aligned}
\partial_t\eta(u_{\varepsilon,\beta,\delta,\gamma}) + \partial_x q(u_{\varepsilon,\beta,\delta,\gamma}) =& \varepsilon\eta'(u_{\varepsilon,\beta,\delta,\gamma})\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma} + \beta\eta'(u_{\varepsilon,\beta,\delta,\gamma})\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma} \\
&+ \gamma\eta'(u_{\varepsilon,\beta,\delta,\gamma})P_{\varepsilon,\beta,\delta,\gamma} \\
=& I_{1,\varepsilon,\beta,\delta,\gamma} + I_{2,\varepsilon,\beta,\delta,\gamma} I_{3,\varepsilon,\beta,\delta,\gamma} + I_{4,\varepsilon,\beta,\delta,\gamma} + I_{5,\varepsilon,\beta,\delta,\gamma},
\end{aligned}$$

where $I_{1,\varepsilon,\beta,\delta,\gamma}$, $I_{2,\varepsilon,\beta,\delta,\gamma}$, $I_{3,\varepsilon,\beta,\delta,\gamma}$, $I_{4,\varepsilon,\beta,\delta,\gamma}$ and $I_{5,\varepsilon,\beta,\delta,\gamma}$ are defined in (3.31).

Arguing as [5, Lemma 3.3], we obtain that $I_{1,\varepsilon,\beta,\delta,\gamma} \to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, $\{I_{2,\varepsilon,\beta,\delta,\gamma}\}_{\varepsilon,\beta,\delta,\gamma>0}$ is bounded in $L^1((0,T)\times\mathbb{R})$, $I_{3,\varepsilon,\beta,\delta,\gamma} \to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, $I_{4,\varepsilon,\beta,\delta,\gamma} \to 0$ in $L^1((0,T)\times\mathbb{R})$, while arguing in Lemma 3.5, $I_{5,\varepsilon,\beta,\delta,\gamma} \to 0$ in $L^1_{loc}((0,\infty)\times\mathbb{R})$.

Therefore, Lemma 2.6 gives (3.34).

We conclude by proving that u is the unique entropy solution of (3.6). Let us consider a compactly supported entropy–entropy flux pair (η, q), and $\phi \in C^\infty_c((0,\infty)\times\mathbb{R})$ non–negative. We have to prove that

$$\int_0^\infty\!\!\int_{\mathbb{R}} (\partial_t\eta(u) + \partial_x q(u))\phi dtdx \le 0. \tag{3.35}$$

We have that

$$\begin{aligned}
&\int_0^\infty\!\!\int_{\mathbb{R}} (\partial_x\eta(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}) + \partial_x q(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}))\phi dtdx \\
&\quad = \gamma_k\int_0^\infty\!\!\int_{\mathbb{R}} P_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\eta'(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})\phi dtdx \\
&\qquad + \varepsilon_k\int_0^\infty\!\!\int_{\mathbb{R}} \partial_x(\eta'(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})\partial_x u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})\phi dtdx \\
&\qquad - \varepsilon_k\int_0^\infty\!\!\int_{\mathbb{R}} \eta''(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})(\partial_x u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})^2\phi dtdx
\end{aligned}$$

$$
\begin{aligned}
&+\beta_k\int_0^\infty\!\!\int_{\mathbb{R}}\partial_x(\eta'(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})\partial_{xx}^2u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})\phi dtdx\\
&+\beta_k\int_0^\infty\!\!\int_{\mathbb{R}}\eta''(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})\partial_xu_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\partial_{xx}^2u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\phi dtdx\\
\le&\gamma_k\int_0^\infty\!\!\int_{\mathbb{R}}P_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\eta'(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})\phi dtdx\\
&-\varepsilon_k\int_0^\infty\!\!\int_{\mathbb{R}}\eta'(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})\partial_xu_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\partial_x\phi dtdx\\
&-\beta_k\int_0^\infty\!\!\int_{\mathbb{R}}\eta'(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})\partial_{xx}^2u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\partial_x\phi dtdx\\
&-\beta_k\int_0^\infty\!\!\int_{\mathbb{R}}\eta''(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})\partial_xu_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\partial_{xx}^2u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\phi dtdx\\
\le&\gamma_k\int_0^\infty\!\!\int_{\mathbb{R}}|P_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}||\eta'(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})||\phi|dtdx\\
&+\varepsilon_k\int_0^\infty\!\!\int_{\mathbb{R}}|\eta'(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})||\partial_xu_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}||\partial_x\phi|dtdx\\
&+\beta_k\int_0^\infty\!\!\int_{\mathbb{R}}|\eta'(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})||\partial_{xx}^2u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}||\partial_x\phi|dtdx\\
&+\beta_k\int_0^\infty\!\!\int_{\mathbb{R}}|\eta''(u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k})||\partial_xu_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\partial_{xx}^2u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}||\phi|dtdx\\
\le&\gamma_k\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|P_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\right\|_{L^2(supp(\partial_x\phi))}\left\|\partial_x\phi\right\|_{L^2(supp(\partial_x\phi))}\\
&+\varepsilon_k\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|\partial_xu_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\right\|_{L^2(supp(\partial_x\phi))}\left\|\partial_x\phi\right\|_{L^2(supp(\partial_x\phi))}\\
&+\beta_k\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|\partial_{xx}^2u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\right\|_{L^2(supp(\partial_x\phi))}\left\|\partial_x\phi\right\|_{L^2(supp(\partial_x\phi))}\\
&+\beta_k\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\left\|\phi\right\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\left\|\partial_xu_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\partial_{xx}^2u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\right\|_{L^1(supp(\partial_x\phi))}\\
\le&\gamma_k\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|P_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\right\|_{L^2((0,T)\times\mathbb{R}))}\left\|\partial_x\phi\right\|_{L^2((0,T)\times\mathbb{R}))}\\
&+\varepsilon_k\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|\partial_xu_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\right\|_{L^2((0,T)\times\mathbb{R})}\left\|\partial_x\phi\right\|_{L^2((0,T)\times\mathbb{R})}\\
&+\beta_k\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|\partial_{xx}^2u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\right\|_{L^2((0,T)\times\mathbb{R})}\left\|\partial_x\phi\right\|_{L^2((0,T)\times\mathbb{R})}\\
&+\beta_k\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\left\|\phi\right\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\left\|\partial_xu_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\partial_{xx}^2u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k}\right\|_{L^1((0,T)\times\mathbb{R})}.
\end{aligned}
$$

(3.35) follows from (3.10), (3.34), Lemmas 3.1 and 3.4. □

Proof of Theorem 3.1. Theorem 3.1 follows from Lemmas 3.5, and 3.6, while (2.10) follows from (3.14) (3.30), or (3.34). Therefore, the proof is done. □

4. The regularized short pulse equation: $\gamma \to 0$.

In this section, we consider the following Cauchy probelm

$$
\begin{cases}
u-\frac{1}{6}\partial_xu^3-\beta\partial_{xxx}^3u=\gamma P, & t>0,\ x\in\mathbb{R},\\
\partial_xP=u & t>0,\ x\in\mathbb{R},\\
P(t,-\infty)=0 & t>0,\\
u(0,x)=u_0(x), & x\in\mathbb{R},
\end{cases}
\tag{4.1}
$$

or equivalently,

$$\begin{cases} \partial_t u - \frac{1}{6}\partial_x u^3 - \beta\partial_{xxx}^3 u = \gamma \int_{-\infty}^{x} u(t,y)dy, & t>0,\ x\in\mathbb{R},\\ u(0,x) = u_0(x), & x\in\mathbb{R}. \end{cases} \tag{4.2}$$

On the initial datum, we assume

$$u_0 \in L^2(\mathbb{R}) \cap L^6(\mathbb{R}), \quad \int_{\mathbb{R}} u_0(x)dx = 0, \tag{4.3}$$

and on the function

$$P_0(x) = \int_{-\infty}^{x} u_0(y)dy, \quad x \in \mathbb{R}, \tag{4.4}$$

we assume that

$$\int_{\mathbb{R}} P_0(x)dx = \int_{\mathbb{R}} \left(\int_{-\infty}^{x} u_0(y)dy\right) dx = 0. \tag{4.5}$$

We observe that, if $\beta,\gamma \to 0$, then (4.1) reads

$$\begin{cases} \partial_t u - \frac{1}{6}\partial_x u^3 = 0, & t>0,\ x\in\mathbb{R},\\ u(0,x) = u_0(x), & x\in\mathbb{R}. \end{cases} \tag{4.6}$$

Fix four small numbers $0 < \varepsilon,\beta,\delta,\gamma < 1$, and let $u_{\varepsilon,\beta,\delta,\gamma} = u_{\varepsilon,\beta,\delta,\gamma}(t,x)$ be the unique classical solution of the following mixed problem:

$$\begin{cases} \partial_t u_{\varepsilon,\beta,\delta,\gamma} - \frac{1}{6}\partial_x u^3_{\varepsilon,\beta,\delta,\gamma} - \beta\partial_{xxx}^3 u_{\varepsilon,\beta,\delta,\gamma} & \\ \qquad = \gamma P_{\varepsilon,\beta,\delta,\gamma} + \varepsilon\partial_{xx}^2 u_{\varepsilon,\beta,\delta,\gamma}, & t>0,\ x\in\mathbb{R},\\ -\delta\partial_t P_{\varepsilon,\beta,\delta,\gamma} + \partial_x P_{\varepsilon,\beta,\delta,\gamma} = u_{\varepsilon,\beta,\delta,\gamma}, & t>0, x\in\mathbb{R},\\ P_{\varepsilon,\beta,\delta,\gamma}(t,-\infty) = 0, & t>0,\\ u_{\varepsilon,\beta,\delta,\gamma}(0,x) = u_{\varepsilon,\beta,\delta,\gamma,0}(x), & x\in\mathbb{R}, \end{cases} \tag{4.7}$$

where $u_{\varepsilon,\beta,\delta,\gamma,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned} & u_{\varepsilon,\,\beta,\,\delta,\,\gamma,0} \to u_0 \quad \text{in } L^p_{loc}(\mathbb{R}),\ 1\le p<6, \text{ as } \varepsilon,\ \beta,\ \delta,\ \gamma\to 0,\\ & \|u_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}\|^2_{L^2(\mathbb{R})} + \delta\gamma\,\|P_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}\|^2_{L^2(\mathbb{R})} + \delta\,\|\partial_x P_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}\|^2_{L^2(\mathbb{R})}\\ & \qquad + \|u_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}\|^6_{L^6(\mathbb{R})} + (\beta+\varepsilon^2)\,\|\partial_x u_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}\|^2_{L^2(\mathbb{R})}\\ & \qquad + \beta^2\,\|\partial_{xx}^2 u_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}\|^2_{L^2(\mathbb{R})} \le C_0, \quad \varepsilon,\ \beta,\ \delta,\ \gamma>0,\\ & \int_{\mathbb{R}} u_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}(x)dx = 0, \quad \int_{\mathbb{R}} P_{\varepsilon,\,\beta,\,\delta,\,\gamma,0}(x)dx = 0, \quad \varepsilon,\ \beta,\ \delta,\ \gamma>0, \end{aligned} \tag{4.8}$$

and C_0 is a constant independent on ε,β, δ and γ.

The main result of this section is the following theorem.

Theorem 4.1. *Assume that* (4.3)*,* (4.4)*,* (4.5)*, and* (4.8) *hold. If*

$$\beta = \mathcal{O}(\varepsilon^2), \quad \gamma = \mathcal{O}(\varepsilon\delta) \tag{4.9}$$

then, there exist four sequences $\{\varepsilon_k\}_{k\in\mathbb{N}}$*,* $\{\beta_k\}_{k\in\mathbb{N}}$*,* $\{\delta_k\}_{k\in\mathbb{N}}$*,* $\{\gamma_k\}_{k\in\mathbb{N}}$ *with* $\varepsilon_k,\beta_k,\delta_k,\gamma_k \to 0$*, and a limit function* $u \in L^\infty(0,T;L^2(\mathbb{R})\cap L^6(\mathbb{R}))$*,* $T>0$*, such that*

i) $u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k} \to u$ *strongly in* $L^p_{loc}((0,T)\times\mathbb{R})$*, for each* $1\le p<6$*,* $T>0$*,*

where u *is a distributional solution of* (4.6)*. Moreover, if*

$$\beta = o(\varepsilon^2), \quad \gamma = \mathcal{O}(\varepsilon\delta) \tag{4.10}$$

then,

ii) u is the unique entropy solution of (4.6)*.*
In particular, we have (2.10)*.*

Let us prove some a priori estimates on $u_{\varepsilon,\beta,\delta,\gamma}$ and $P_{\varepsilon,\beta,\delta,\gamma}$, denoting with C_0 the constants which depend on the initial datum, and $C(T)$ the constants which depend also on T.

We begin by observing that Lemma 3.1 holds also for (4.7).

Lemma 4.1. *Fixed $T>0$. There exists $C(T)>0$, independent on ε, β, δ and γ such that*

$$\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|_{L^\infty((0,T)\times\mathbb{R})}\le C(T)\beta^{-\frac{1}{2}}. \tag{4.11}$$

Moreover, for every $0\le t\le T$,

$$\beta\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta\varepsilon\int_0^t\left\|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T)\beta^{-2}. \tag{4.12}$$

Proof. Let $0\le t\le T$. Multiplying (4.7) by $-\beta\partial^2_{xx}u_{\varepsilon,\beta}-\frac{1}{6}u^3_{\varepsilon,\beta}$, we have

$$\begin{aligned}\left(-\beta\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}-\frac{1}{6}u^3_{\varepsilon,\beta,\delta,\gamma}\right)\partial_t u_{\varepsilon,\beta,\delta,\gamma}-\frac{1}{6}\left(-\beta\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}-\frac{1}{6}u^3_{\varepsilon,\beta,\delta,\gamma}\right)\partial_x u^3_{\varepsilon,\beta}&\\ -\beta\left(-\beta\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}-\frac{1}{6}u^3_{\varepsilon,\beta,\delta,\gamma}\right)\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma}&\\ =\gamma\left(-\beta\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}-\frac{1}{6}u^3_{\varepsilon,\beta}\right)P_{\varepsilon,\beta,\delta,\gamma}&\\ +\varepsilon\left(-\beta\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}-\frac{1}{6}u^3_{\varepsilon,\beta,\delta,\gamma}\right)\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}.&\end{aligned} \tag{4.13}$$

Arguing as [6, Lemma 2.3], we have

$$\begin{aligned}&\frac{d}{dt}\left(\beta\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-\frac{1}{12}\int_{\mathbb{R}}u^4_{\varepsilon,\beta,\delta,\gamma}dx\right)\\ &\qquad+2\beta\varepsilon\left\|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad=2\gamma\beta\int_{\mathbb{R}}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}P_{\varepsilon,\beta,\delta,\gamma}dx-\frac{\gamma}{3}\int_{\mathbb{R}}u^3_{\varepsilon,\beta,\delta,\gamma}P_{\varepsilon,\beta,\delta,\gamma}dx\\ &\qquad+\varepsilon\int_{\mathbb{R}}u^2_{\varepsilon,\beta,\delta,\gamma}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx.\end{aligned}$$

Since $0<\varepsilon,\beta<1$, it follows from (3.15), (4.9) and the Young inequality that

$$\begin{aligned}2\gamma\beta\left|\int_{\mathbb{R}}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}P_{\varepsilon,\beta,\delta,\gamma}dx\right|\le&2\int_{\mathbb{R}}\left|\beta\sqrt{\varepsilon}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}\right|\left|\frac{\gamma}{\sqrt{\varepsilon}}P_{\varepsilon,\beta,\delta,\gamma}\right|dx\\ \le&\varepsilon\beta^2\left\|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\gamma^2}{\varepsilon}\left\|P_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \le&\varepsilon\beta\left\|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\gamma}{\varepsilon\delta}C(T)\\ \le&\varepsilon\beta\left\|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T).\end{aligned}$$

Moreover, from (3.15), (3.18), (4.9) and the Young inequality, we have

$$\frac{\gamma}{3}\left|\int_{\mathbb{R}}u^3_{\varepsilon,\beta,\delta,\gamma}P_{\varepsilon,\beta,\delta,\gamma}dx\right|\le\int_{\mathbb{R}}\left|\frac{u_{\varepsilon,\beta,\delta,\gamma}}{3}\right|\left|\gamma u^2_{\varepsilon,\beta,\delta,\gamma}P_{\varepsilon,\beta,\delta,\gamma}\right|dx$$

$$
\begin{aligned}
&\le \frac{1}{6}\left\|u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\gamma^2}{2}\int_{\mathbb{R}} u^4_{\varepsilon,\beta,\delta,\gamma}P^2_{\varepsilon,\beta,\delta,\gamma}dx\\
&\le C(T) + \frac{\gamma^2}{2}\left\|P_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{\gamma}{3}\left\|P_{\varepsilon,\beta,\delta,\gamma}\right\|_{L^\infty((0,T)\times\mathbb{R})}\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|_{L^\infty((0,T)\times\mathbb{R})}\left\|u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C(T) + \frac{\gamma^2}{\delta^{\frac{3}{2}}\gamma^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}}C(T)\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\\
&\le C(T) + \varepsilon C(T)\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\\
&\le C(T) + C(T)\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{d}{dt}\left(\beta\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{1}{12}\int_{\mathbb{R}} u^4_{\varepsilon,\beta,\delta,\gamma}dx\right) + \beta\varepsilon\left\|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T) + C(T)\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \varepsilon\int_{\mathbb{R}} u^2_{\varepsilon,\beta,\delta,\gamma}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx\\
&\quad\le C(T) + C(T)\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \varepsilon\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx.
\end{aligned}
$$

(3.15), (4.8) and an integration on $(0,t)$ gives

$$
\begin{aligned}
&\beta\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta\varepsilon\int_0^t\left\|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\quad\le C_0 + C(T)\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^t ds + C(T)\int_0^t ds + \frac{1}{12}\int_{\mathbb{R}} u^4_{\varepsilon,\beta,\delta,\gamma}dx\\
&\qquad + \varepsilon\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\quad\le C(T) + C(T)\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \frac{1}{12}\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T) + C(T)\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \frac{C_0}{12}\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})},
\end{aligned}
$$

that is

$$
\begin{aligned}
\beta\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ \beta\varepsilon\int_0^t\left\|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\le C(T)\left(1 + \left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned}
\tag{4.14}
$$

Due to (3.15), (4.14) and the Hölder inequality,

$$
\begin{aligned}
u^2_{\varepsilon,\beta,\delta,\gamma}(t,x) &= 2\int_{-\infty}^x u_{\varepsilon,\beta,\delta,\gamma}\partial_x u_{\varepsilon,\beta,\delta,\gamma}dy \le 2\int_{\mathbb{R}}\left|u_{\varepsilon,\beta,\delta,\gamma}\partial_x u_{\varepsilon,\beta,\delta,\gamma}\right|dx\\
&\le \frac{2}{\sqrt{\beta}}\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|_{L^2(\mathbb{R})}\sqrt{\beta}\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
&\le \frac{2}{\sqrt{\beta}}C_0\sqrt{C(T)\left(1 + \left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)},
\end{aligned}
$$

that is

$$
\left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^4_{L^\infty((0,T)\times\mathbb{R})} \le \frac{C(T)}{\beta}\left(1 + \left\|u_{\varepsilon,\beta,\delta,\gamma}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).
\tag{4.15}
$$

Arguing as [6, Lemma 2.3], we have (4.11).

Finally, (4.12) follows from (4.11) and (4.14). □

Lemma 4.2. *Let $T>0$. Assume* (4.9) *holds true. There exists $C(T)>0$, independent on ε, β, δ, and γ such that*

$$\|u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\|_{L^6(\mathbb{R})} \le C(T), \tag{4.16}$$

$$\varepsilon\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\|_{L^2(\mathbb{R})} \le C(T), \tag{4.17}$$

$$\varepsilon e^t\int_0^t\int_{\mathbb{R}} e^{-s}u^4_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)(\partial_x u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot))^2 dsdx \le C(T), \tag{4.18}$$

$$\varepsilon^3 e^t\int_0^t e^{-s}\left\|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{4.19}$$

for every $0<t<T$. Moreover,

$$\beta\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}\right\|_{L^1((0,T)\times\mathbb{R})} \le C(T), \tag{4.20}$$

$$\beta^2\int_0^T\left\|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T)\varepsilon. \tag{4.21}$$

Proof. Let $0\le t\le T$. Multiplying (4.7) by $u^5_{\varepsilon,\beta}-3\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta}$, we have

$$\begin{aligned}(u^5_{\varepsilon,\beta,\delta,\gamma}-3\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma})\partial_t u_{\varepsilon,\beta,\delta,\gamma} &- \frac{1}{6}(u^5_{\varepsilon,\beta,\delta,\gamma}-3\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma})\partial_x u^3_{\varepsilon,\beta,\delta,\gamma}\\ &- (u^5_{\varepsilon,\beta,\delta,\gamma}-3\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma})\beta\partial^3_{xxx}u_{\varepsilon,\beta,\delta,\gamma}\\ &=\gamma(u^5_{\varepsilon,\beta,\delta,\gamma}-3\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma})P_{\varepsilon,\beta,\delta,\gamma}\\ &\quad+\varepsilon(u^5_{\varepsilon,\beta,\delta,\gamma}-3\varepsilon^2\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma})\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}.\end{aligned}$$

Arguing as [6, Lemma 2.4], we get

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{6}\|u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\|^6_{L^6(\mathbb{R})}+\frac{3\varepsilon^2}{2}\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\ &\qquad+5\varepsilon\int_{\mathbb{R}}u^4_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx+3\varepsilon^3\left\|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}(t\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad=\gamma\int_{\mathbb{R}}u^5_{\varepsilon,\beta,\delta,\gamma}P_{\varepsilon,\beta,\delta,\gamma}dx-3\gamma\varepsilon^2\int_{\mathbb{R}}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}P_{\varepsilon,\beta,\delta,\gamma}dx\\ &\qquad-\frac{3\varepsilon^2}{2}\int_{\mathbb{R}}u^2_{\varepsilon,\beta,\delta,\gamma}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}dx-10\beta\int_{\mathbb{R}}u^3_{\varepsilon,\beta,\delta,\gamma}(\partial_x u_{\varepsilon,\beta,\delta,\gamma})^2dx.\end{aligned} \tag{4.22}$$

Since $0<\varepsilon<1$, due to (3.15), (3.18), (4.9), and the Young inequality,

$$
\begin{aligned}
\gamma\left|\int_{\mathbb{R}} u^5_{\varepsilon,\beta,\delta,\gamma}P_{\varepsilon,\beta,\delta,\gamma}dx\right| &\le \int_{\mathbb{R}}\left|\sqrt{6}\gamma u^2_{\varepsilon,\beta,\delta,\gamma}P_{\varepsilon,\beta,\delta,\gamma}\right|\left|\frac{u^3_{\varepsilon,\beta,\delta,\gamma}}{\sqrt{6}}\right|dx\\
&\le 3\gamma^2\int_{\mathbb{R}} P^2_{\varepsilon,\beta,\delta,\gamma}u^4_{\varepsilon,\beta,\delta,\gamma}dx+\frac{1}{12}\left\|u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\\
&\le \int_{\mathbb{R}}\left|3\sqrt{6}\gamma^2P^2_{\varepsilon,\beta,\delta,\gamma}u_{\varepsilon,\beta,\delta,\gamma}\right|\left|\frac{u^3_{\varepsilon,\beta,\delta,\gamma}}{\sqrt{6}}\right|dx+\frac{1}{12}\left\|u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\\
&\le 27\gamma^4\int_{\mathbb{R}} P^4_{\varepsilon,\beta}u^2_{\varepsilon,\beta,\delta,\gamma}dx+\frac{1}{6}\left\|u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\\
&\le 27\gamma^4\left\|P_{\varepsilon,\beta,\delta,\gamma}\right\|^4_{L^\infty((0,T)\times\mathbb{R})}\left\|u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{1}{6}\left\|u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\\
&\le 27\gamma^4C_0\left\|P_{\varepsilon,\beta,\delta,\gamma}\right\|^4_{L^\infty((0,T)\times\mathbb{R})}+\frac{1}{6}\left\|u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\\
&\le \frac{\gamma^4}{\delta^3\gamma\varepsilon}C(T)+\frac{1}{6}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\le\frac{\gamma^3}{\delta^3\varepsilon}C(T)+\frac{1}{6}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\\
&\le \varepsilon^2C(T)+\frac{1}{6}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\le C(T)+\frac{1}{6}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^6_{L^6(\mathbb{R})}.
\end{aligned}\tag{4.23}
$$

Since $0<\varepsilon<1$, it follows from (3.11), (3.17), (4.9) and the Young inequality that

$$
\begin{aligned}
&-3\gamma\varepsilon^2\int_{\mathbb{R}}\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}P_{\varepsilon,\beta,\delta,\gamma}dx=3\gamma\varepsilon^2\int_{\mathbb{R}}\partial_x u_{\varepsilon,\beta,\delta,\gamma}\partial_x P_{\varepsilon,\beta,\delta,\gamma}dx\\
&\le 3\gamma\varepsilon^2\left|\int_{\mathbb{R}}\partial_x u_{\varepsilon,\beta,\delta,\gamma}\partial_x P_{\varepsilon,\beta,\delta,\gamma}dx\right|\le 3\varepsilon^2\int_{\mathbb{R}}\left|\partial_x u_{\varepsilon,\beta,\delta,\gamma}\right|\left|\gamma\partial_x P_{\varepsilon,\beta,\delta,\gamma}\right|dx\\
&\le \frac{3\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{3\gamma^2\varepsilon^2}{2}\left\|\partial_x P_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{3\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\gamma^2\varepsilon^2}{\delta^2\varepsilon}C(T)\le\frac{3\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon^3C(T)\\
&\le \frac{3\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T).
\end{aligned}\tag{4.24}
$$

Arguing as [6, Lemma 2.4], we have

$$
\begin{aligned}
\frac{d}{dt}G_1(t)-G_1(t)+&\frac{15\varepsilon}{4}\int_{\mathbb{R}}u^4_{\varepsilon,\beta,\delta,\gamma}(\partial_x u_{\varepsilon,\beta})^2dx+\frac{9\varepsilon^3}{4}\left\|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C(T)+\varepsilon C(T)\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}\tag{4.25}
$$

where

$$
G_1(t)=\frac{1}{6}\left\|u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^6_{L^6(\mathbb{R})}+\frac{3\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\tag{4.26}
$$

The Gronwall Lemma, (3.15) and (4.8) give

$$
\begin{aligned}
G_1(t)+&\frac{15\varepsilon}{4}e^t\int_0^t\int_{\mathbb{R}}e^{-s}u^4_{\varepsilon,\beta,\delta,\gamma}(\partial_x u_{\varepsilon,\beta})^2dsdx+\frac{9\varepsilon^3}{4}\int_0^t\left\|\partial^2_{xx}u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\le C_0e^t+C(T)e^t\int_0^t e^{-s}ds+\varepsilon C(T)e^t\int_0^t e^{-s}\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\le C(T)+\varepsilon C(T)\int_0^t\left\|\partial_x u_{\varepsilon,\beta,\delta,\gamma}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T).
\end{aligned}\tag{4.27}
$$

(4.26) and (4.27) give (4.16), (4.17), (4.18) and (4.19).

Arguing as [6, Lemma 2.4], we have (4.20) and (4.21). □

Arguing as Lemmas 3.5 and 3.6 we have the following results

Lemma 4.3. *Assume that* (4.3)*,* (4.4)*,* (4.5)*,* (4.8)*, and* (4.9) *hold. Then for any compactly supported entropy–entropy flux pair* (η, q)*, there exist four sequences* $\{\varepsilon_k\}_{k\in\mathbb{N}}$*,* $\{\beta_k\}_{k\in\mathbb{N}}$*,* $\{\delta_k\}_{k\in\mathbb{N}}$*,* $\{\gamma_k\}_{k\in\mathbb{N}}$*, with* $\varepsilon_k, \beta_k, \delta_k, \gamma_k \to 0$*, and a limit function*

$$u \in L^\infty(0,T;L^2(\mathbb{R}) \cap L^6(\mathbb{R})),$$

such that

$$u_{\varepsilon_k,\beta_k,\delta_k,\gamma_k} \to u \quad \text{in} \quad L^p_{loc}((0,T)\times\mathbb{R}), \quad \text{for each} \quad 1 \le p < 6, \tag{4.28}$$

and u *is a distributional solution of* (4.6)*.*

Lemma 4.4. *Assume that* (4.3)*,* (4.4)*,* (4.5)*,* (4.8)*, and* (4.10) *hold. Then,*

$$u_{\varepsilon_k,\beta_k,\delta_k\gamma_k} \to u \quad \text{in} \quad L^p_{loc}((0,T)\times\mathbb{R}), \quad \text{for each} \quad 1 \le p < 6, \tag{4.29}$$

where u *is the unique entropy solution of* (4.6)*.*

Proof of Theorem 4.1. *i*) and *ii*) follows from Lemmas 4.3, and 4.4, while (2.10) follows from (3.14) (4.28), or (4.29). □

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