

Well-posedness for dislocation based gradient visco-plasticity with isotropic hardening

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Abstract

In this work we establish the well-posedness for infinitesimal dislocation based gradient viscoplasticity with isotropic hardening for general gradient monotone plastic flows. We assume an additive split of the displacement gradient into non-symmetric elastic distortion and non-symmetric plastic distortion. The thermodynamic potential is augmented with a term taking the dislocation density tensor $\text{Curl} p$ into account. The constitutive equations in the models we study are assumed to be of self-controlling type. Based on the generalized version of Korn's inequality for incompatible tensor fields (the non-symmetric plastic distortion) due to Neff/Pauly/Witsch the existence of solutions of quasi-static initial-boundary value problems under consideration is shown using a time-discretization technique and a monotone operator method.

Key words: gradient viscoplasticity, rate dependent response, non-associative flow rule, Rothe's time-discretization method, Korn's inequality for incompatible tensor fields, maximal monotone method, geometrically necessary dislocations, plastic spin, defect energy.

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1 Introduction

Within the framework of the strain gradient plasticity theory we study the existence of solutions of quasistatic initial-boundary value problems arising in gradient viscoplasticity with isotropic hardening. The models we study use

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rate-dependent constitutive equations with internal variables to describe the deformation behaviour of metals at infinitesimally small strain.

In this paper we consider the rate-dependent (viscoplastic) behaviour only. The model we study here has been first presented in [21] and has been inspired by the early work of Menzel and Steinmann [18]. Contrary to more classical strain gradient approaches, the model features a non-symmetric plastic distortion field $p \in \mathcal{M}^3$ (see [3]), a dislocation based energy storage based solely on $|\text{Curl} p|$ and second gradients of the plastic distortion in the form of $\text{Curl} \text{Curl} p$ acting as dislocation based kinematical backstresses. Preliminary works on the problem concern the uniqueness (see [20]), the well-posedness of a rate-independent variant (see [8, 9, 21]) and of a rate-dependent variant without isotropic hardening (see [27, 28]), the possibility of homogenization (see [26]), as well as FEM-implementations (see [25, 31]). In [10, 31] the well-posedness for a rate-independent model of Gurtin and Anand [12] is shown under the decisive assumption that the plastic distortion is symmetric (the irrotational case), in which case we may really speak of a strain gradient plasticity model, since the gradient acts on the plastic strain.

Presentation of the model. For completeness of the work we sketch some of the ingredients of the model. The sets, \mathcal{M}^3 and \mathcal{S}^3 denote the sets of all 3×3 -matrices and of all symmetric 3×3 -matrices, respectively. We recall that the space of all 3×3 -matrices \mathcal{M}^3 can be isomorphically identified with the space \mathbb{R}^9 . Therefore we can define a linear mapping $B : \mathbb{R}^N \rightarrow \mathcal{M}^3$ as a composition of a projector from \mathbb{R}^N onto \mathbb{R}^9 and the isomorphism between \mathbb{R}^9 and \mathcal{M}^3 . The transpose $B^T : \mathcal{M}^3 \rightarrow \mathbb{R}^N$ is given then by

$$B^T \tau = (\hat{z}, 0)^T$$

for $\tau \in \mathcal{M}^3$ and $z = (\hat{z}, \tilde{z})^T \in \mathbb{R}^N$, $\tilde{z} \in \mathbb{R}^{N-9}$. Next, as is usual in plasticity theory, we split the total displacement gradient into non symmetric elastic and plastic distortions

$$\nabla u = e + p.$$

For invariance reasons, the elastic energy contribution may only depend on the elastic strains $\text{sym } e = \text{sym}(\nabla u - p)$. While p is non-symmetric, a distinguishing feature of the model is that, similar to classical approaches, only the symmetric part $\varepsilon_p := \text{sym } p$ of the plastic distortion appears in the local Cauchy stress σ , while the higher order stresses are non-symmetric (see [19, 34] for more details). We assume as well plastic incompressibility $\text{tr } p = 0$. We consider here a free energy of the form

$$\begin{aligned} \Psi(\nabla u, \text{Curl } p, z) : &= \underbrace{\Psi_e^{\text{lin}}(e)}_{\text{elastic energy}} + \underbrace{\Psi_{\text{Curl}}^{\text{lin}}(\text{Curl } p)}_{\text{defect energy (GND)}} \\ &+ \underbrace{\Psi_{\text{hard}}(z)}_{\text{hardening energy (SSD)}} \end{aligned} \quad (1)$$

where

$$\Psi_e^{\text{lin}}(e) := \frac{1}{2} e \cdot \mathbb{C} e, \quad \Psi_{\text{Curl}}^{\text{lin}}(\text{Curl } p) = \frac{C_1}{2} \|\text{Curl } p\|^2 \text{ and } \Psi_{\text{hard}}(z) = \frac{1}{2} L z \cdot z.$$

Here, the linear mapping $L : \mathbb{R}^N \rightarrow \mathbb{R}^N$ corresponds to isotropic hardening effects and is assumed to be positive semi-definite and C_1 is a given non-negative material constant. The positive definite elasticity tensor \mathbb{C} is able to represent the elastic anisotropy of the material. The local free-energy imbalance states that

$$\dot{\Psi} - \sigma \cdot \dot{e} - \sigma \cdot \dot{p} \leq 0. \quad (2)$$

Now we expand the first term, substitute (1) and get

$$(\mathbb{C}e - \sigma) \cdot \dot{e} - \sigma \cdot \dot{p} + C_1 \text{Curl} p \cdot \text{Curl} \dot{p} + Lz \cdot \dot{z} \leq 0, \quad (3)$$

which, using arguments from thermodynamics gives the elastic relation

$$\sigma = \mathbb{C} \text{sym}(\nabla u - p) \quad (4)$$

and the reduced dissipation inequality

$$-\sigma \cdot \dot{p} + C_1 \text{Curl} p \cdot \text{Curl} \dot{p} + Lz \cdot \dot{z} \leq 0. \quad (5)$$

Now we integrate (5) over Ω and get

$$\begin{aligned} 0 &\geq \int_{\Omega} \left[-\sigma \cdot \dot{p} + C_1 \text{Curl} p \cdot \text{Curl} \dot{p} + Lz \cdot \dot{z} \right] \\ &= - \int_{\Omega} \left[\sigma \cdot \dot{p} - C_1 \text{Curl} \text{Curl} p \cdot \dot{p} - Lz \cdot \dot{z} \right. \\ &\quad \left. + \sum_{i=1}^3 \int_{\partial\Omega} C_1 \langle \dot{p}^i \times (\text{Curl} p)^i, \vec{n} \rangle dS \right]. \end{aligned} \quad (6)$$

In order to obtain a dissipation inequality in the spirit of classical plasticity, we assume that the infinitesimal plastic distortion p satisfies the so-called *linearized insulation condition*

$$\sum_{i=1}^3 \int_{\partial\Omega} C_1 \left\langle \frac{d}{dt} p^i \times (\text{Curl} p)^i, \vec{n} \right\rangle dS = 0. \quad (7)$$

We specify a sufficient condition for the linearized insulation boundary condition (see [13]), namely

$$p \times n|_{\partial\Omega} = 0, \quad (8)$$

which is called the micro-hard boundary condition. Under (8), we then obtain the dissipation inequality

$$\int_{\Omega} [(B^T \sigma + \Sigma_{\text{Curl}}^{\text{lin}}) \cdot \dot{z} + \hat{g} \cdot \dot{z}] dV \geq 0, \quad (9)$$

where

$$\Sigma_{\text{Curl}}^{\text{lin}} := -C_1 B^T \text{Curl} \text{Curl} p \quad \text{and} \quad \hat{g} := -Lz.$$

Adapted to our situation, the plastic flow has the form

$$\partial_t z \in g(B^T \sigma - Lz - C_1 B^T \text{Curl} \text{Curl} p), \quad (10)$$

where g is a multivalued monotone flow function which is not necessary the subdifferential of a convex plastic potential (associative plasticity).

We note that the micro-hard boundary (8) is the correct boundary condition for tensor fields in $L^2_{\text{Curl}}(\Omega, \mathcal{M}^3)$ (see Notation for the definition of L^2_{Curl} -space) which admits tangential traces. We combine this with a new inequality extending Korn's inequality to incompatible tensor fields, namely

$$\forall p \in L^2_{\text{Curl}}(\Omega, \mathcal{M}^3) : \quad p \times n|_{\partial\Omega} = 0 : \quad (11)$$

$$\underbrace{\|p\|_{L^2(\Omega)}}_{\text{plastic distortion}} \leq C(\Omega) \left(\underbrace{\|\text{sym } p\|_{L^2(\Omega)}}_{\text{plastic strain}} + \underbrace{\|\text{Curl } p\|_{L^2(\Omega)}}_{\text{dislocation density}} \right).$$

Here, the domain Ω needs to be **sliceable**, i.e. cuttable into finitely many simply connected subdomains with Lipschitz boundaries. This inequality has been derived in [22, 23, 24] and is precisely motivated by the well-posedness question for our model [21]. The inequality (11) expresses the fact that controlling the plastic strain $\text{sym } p$ and the dislocation density $\text{Curl } p$ in $L^2(\Omega, \mathcal{M}^3)$ gives a control of the plastic distortion p in $L^2(\Omega, \mathcal{M}^3)$ provided the micro-hard boundary condition is specified.

It is worthy to note that with g only monotone and not necessarily a subdifferential the powerful energetic solution concept [10, 16, 17] cannot be applied. In this contribution we face the combined challenge of a gradient plasticity model based on the dislocation density tensor $\text{Curl } p$ involving the plastic spin, a general non-associative monotone flow-rule and a rate-dependent response.

Setting of the problem. Let $\Omega \subset \mathbb{R}^3$ be an open bounded set, the set of material points of the solid body, with a C^1 -boundary $\partial\Omega$. By T_e we denote a positive number (time of existence), which can be chosen arbitrarily large, and for $0 < t \leq T_e$

$$\Omega_t = \Omega \times (0, t).$$

Let $\mathfrak{sl}(3)$ be the set of all traceless 3×3 -matrices, i.e.

$$\mathfrak{sl}(3) = \{v \in \mathcal{M}^3 \mid \text{tr } v = 0\}.$$

Unknown in our small strain formulation are the displacement $u(x, t) \in \mathbb{R}^3$ of the material point x at time t and the vector of the internal variables $z = (p, \gamma)$. Here, $p(x, t) \in \mathfrak{sl}(3)$ denotes the non-symmetric infinitesimal plastic distortion and $\gamma(x, t) \in \mathbb{R}$ is the isotropic hardening variable. The model equations of the problem are

$$-\text{div}_x \sigma(x, t) = b(x, t), \quad (12)$$

$$\sigma(x, t) = \mathbb{C}[x](\text{sym}(\nabla_x u(x, t) - Bz(x, t))), \quad (13)$$

$$\begin{aligned} \partial_t z(x, t) &\in g(x, \Sigma^{\text{lin}}(x, t)), \quad \Sigma^{\text{lin}} = \Sigma_e^{\text{lin}} + \Sigma_{\text{sh}}^{\text{lin}} + \Sigma_{\text{curl}}^{\text{lin}}, \\ \Sigma_e^{\text{lin}} &= B^T \sigma, \quad \Sigma_{\text{sh}}^{\text{lin}} = -Lz, \quad \Sigma_{\text{curl}}^{\text{lin}} = -C_1 B^T \text{Curl } \text{Curl}(Bz), \end{aligned} \quad (14)$$

which must be satisfied in $\Omega \times [0, T_e)$. Here, Σ^{lin} is the infinitesimal Eshelby stress tensor driving the evolution of the plastic distortion p . The initial condition and Dirichlet boundary condition are

$$z(x, 0) = 0, \quad x \in \Omega, \quad (15)$$

$$Bz(x, t) \times n(x) = 0, \quad (x, t) \in \partial\Omega \times [0, T_e), \quad (16)$$

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T_e), \quad (17)$$

where n is a normal vector on the boundary $\partial\Omega$. For simplicity we consider only homogeneous boundary condition. The elasticity tensor $\mathbb{C}[x] : \mathcal{S}^3 \rightarrow \mathcal{S}^3$ is a linear, symmetric, uniformly positive definite mapping. The tensor \mathbb{C} has measurable coefficients. Classical linear isotropic hardening is included for $L \neq 0$. We assume that the linear mapping $L : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is positive semi-definite and satisfies the inequality

$$(Lz, z) \geq \alpha \|\gamma\|^2, \quad z = (p, \gamma) \in \mathbb{R}^N, \quad (18)$$

for some positive constant $\alpha \in \mathbb{R}$. In the model equations, the nonlocal back-stress contribution is given by the dislocation density motivated term $\Sigma_{\text{curl}}^{\text{lin}} = -C_1 B^T \text{Curl Curl } p$ together with the corresponding micro-hard boundary condition (16). For the model we require that the nonlinear constitutive mapping $(v \mapsto g(\cdot, v)) : \mathbb{R}^N \rightarrow 2^{\mathbb{R}^N}$ is monotone¹, i.e. it satisfies

$$0 \leq (v_1 - v_2) \cdot (v_1^* - v_2^*), \quad (19)$$

for all $v_i \in \mathbb{R}^N$, $v_i^* \in g(x, v_i)$, $i = 1, 2$, and for a.e. $x \in \Omega$. We also require that

$$0 \in g(x, 0), \quad \text{a.e. } x \in \Omega. \quad (20)$$

The mapping $x \mapsto g(x, \cdot) : \Omega \rightarrow 2^{\mathbb{R}^N}$ is measurable (see [5, 14, 29] for the definition of the measurability of multi-valued maps). Given are the volume force $b(x, t) \in \mathbb{R}^3$. In this work we also assume that the function g possesses **the self-controlling property**, i.e. there exists a continuous function $\mathcal{F} : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that the inequality

$$\|Bg(y)\|_2 \leq \mathcal{F}(\|Lg(y)\|_2, \|y\|_2) \quad (21)$$

holds for all $y \in L^2(\Omega, \mathbb{R}^N)$. The self-controlling property was first introduced by Chelminski in [6] for the study of inelastic models of monotone type and beyond that class.

Remark 1.1. Visco-plasticity is typically included in the former conditions by choosing the function g to be in Norton-Hoff form, i.e.

$$g(\Sigma) = [|\Sigma| - \sigma_y]_+^r \frac{\Sigma}{|\Sigma|}, \quad \Sigma \in \mathcal{M}^3,$$

where σ_y is the flow stress and r is some parameter together with $[x]_+ := \max(x, 0)$. If $g : \mathcal{M}^3 \mapsto \mathcal{S}^3$ then the flow is called irrotational (no plastic spin).

In case of a non-associative flow rule, g is not a subdifferential but may e.g. be written as

$$g(\Sigma) = \mathcal{F}_1(\Sigma) \partial \mathcal{F}_2(\Sigma),$$

where \mathcal{F}_1 describes the yield-function and \mathcal{F}_2 the flow direction.

Remark 1.2. It is well known that classical viscoplasticity (without gradient effects) gives rise to a well-posed problem. We extend this result to our formulation of rate-dependent gradient plasticity. The presence of the classical linear isotropic hardening in our model is related to $L \neq 0$ whereas the presence of the nonlocal gradient term is always related to $C_1 > 0$.

¹Here $2^{\mathbb{R}^N}$ denotes the power set of \mathbb{R}^N .

Notation. Throughout the whole work we choose the numbers q, q^* satisfying the following conditions

$$1 < q, q^* < \infty \text{ and } 1/q + 1/q^* = 1,$$

and $|\cdot|$ denotes a norm in \mathbb{R}^k , $k \in \mathbb{N}$. We also assume for simplicity that $\Gamma_{\text{hard}} = \partial\Omega$. Moreover, the following notations are used in this work. The space $W^{m,q}(\Omega, \mathbb{R}^k)$ with $q \in [1, \infty]$ consists of all functions in $L^q(\Omega, \mathbb{R}^k)$ with weak derivatives in $L^q(\Omega, \mathbb{R}^k)$ up to order m . If m is not integer, then $W^{m,q}(\Omega, \mathbb{R}^k)$ denotes the corresponding Sobolev-Slobodecki space. We set $H^m(\Omega, \mathbb{R}^k) = W^{m,2}(\Omega, \mathbb{R}^k)$. The norm in $W^{m,q}(\Omega, \mathbb{R}^k)$ is denoted by $\|\cdot\|_{m,q,\Omega}$ ($\|\cdot\|_q := \|\cdot\|_{0,q,\Omega}$). The operator Γ_0 defined by

$$\Gamma_0 : v \in W^{1,q}(\Omega, \mathbb{R}^k) \mapsto W^{1-1/q,q}(\partial\Omega, \mathbb{R}^k)$$

denotes the usual trace operator. The space $W_0^{m,q}(\Omega, \mathbb{R}^k)$ with $q \in [1, \infty]$ consists of all functions v in $W^{m,q}(\Omega, \mathbb{R}^k)$ with $\Gamma_0 v = 0$. One can define the bilinear form on the product space $L^q(\Omega, \mathcal{M}^3) \times L^{q^*}(\Omega, \mathcal{M}^3)$ by

$$(\xi, \zeta)_\Omega = \int_\Omega \xi(x) \cdot \zeta(x) dx.$$

The space

$$L_{\text{Curl}}^q(\Omega, \mathcal{M}^3) = \{v \in L^q(\Omega, \mathcal{M}^3) \mid \text{Curl } v \in L^q(\Omega, \mathcal{M}^3)\}$$

is a Banach space with respect to the norm

$$\|v\|_{q, \text{Curl}} = \|v\|_q + \|\text{Curl } v\|_q.$$

The well known result on the generalized trace operator can be easily adopted to the functions with values in \mathcal{M}^3 (see [33, Section II.1.2]). Then, according to this result, there is a bounded operator Γ_n on $L_{\text{Curl}}^q(\Omega, \mathcal{M}^3)$

$$\Gamma_n : v \in L_{\text{Curl}}^q(\Omega, \mathcal{M}^3) \mapsto (W^{1-1/q^*, q^*}(\partial\Omega, \mathcal{M}^3))^*$$

with

$$\Gamma_n v = v \times n|_{\partial\Omega} \text{ if } v \in C^1(\bar{\Omega}, \mathcal{M}^3),$$

where X^* denotes the dual of a Banach space X . Next,

$$L_{\text{Curl},0}^q(\Omega, \mathcal{M}^3) = \{w \in L_{\text{Curl}}^q(\Omega, \mathcal{M}^3) \mid \Gamma_n(w) = 0\}.$$

We also define the space $Z_{\text{Curl}}^q(\Omega, \mathcal{M}^3)$ by

$$Z_{\text{Curl}}^q(\Omega, \mathcal{M}^3) = \{v \in L_{\text{Curl},0}^q(\Omega, \mathcal{M}^3) \mid \text{Curl } \text{Curl } v \in L^q(\Omega, \mathcal{M}^3)\},$$

which is a Banach space with respect to the norm

$$\|v\|_{Z_{\text{Curl}}^q} = \|v\|_{q, \text{Curl}} + \|\text{Curl } \text{Curl } v\|_q.$$

For functions v defined on $\Omega \times [0, \infty)$ we denote by $v(t)$ the mapping $x \mapsto v(x, t)$, which is defined on Ω . The space $L^q(0, T_e; X)$ denotes the Banach space of all Bochner-measurable functions $u : [0, T_e) \rightarrow X$ such that $t \mapsto \|u(t)\|_X^q$ is integrable on $[0, T_e)$. Finally, we frequently use the spaces $W^{m,q}(0, T_e; X)$, which consist of Bochner measurable functions having q -integrable weak derivatives up to order m .

2 Preliminaries

Some properties of the Curl Curl-operator In this subsection we present some results concerning the Curl Curl-operator, which are relevant to the further investigations. For the Curl Curl-operator with a slightly different domain of definition similar results are obtained in [27, Section 4]. Here we adopt the results in [27] to our purposes.

Lemma 2.1 (Self-adjointness of Curl Curl). *Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with a Lipschitz boundary and $A : L^2(\Omega, \mathcal{M}^3) \rightarrow L^2(\Omega, \mathcal{M}^3)$ be the linear operator defined by*

$$Av = \text{Curl Curl } v$$

with $\text{dom}(A) = Z_{\text{Curl}}^2(\Omega, \mathcal{M}^3)$. The operator A is selfadjoint and non-negative.

Proof. Indeed, let us consider first the following linear closed operator $S : L^2(\Omega, \mathcal{M}^3) \rightarrow L^2(\Omega, \mathcal{M}^3)$ defined by

$$Sv = \text{Curl } v, \quad v \in \text{dom}(S) = L_{\text{Curl},0}^2(\Omega, \mathcal{M}^3).$$

It is easily seen that its adjoint is given by

$$S^*v = \text{Curl } v, \quad v \in \text{dom}(S^*) = L_{\text{Curl}}^2(\Omega, \mathcal{M}^3).$$

Then, by Theorem V.3.24 in [15], the operator S^*S with

$$\text{dom}(S^*S) = \{v \in \text{dom}(S) \mid Sv \in \text{dom}(S^*)\},$$

which is exactly the operator A , is self-adjoint in $L^2(\Omega, \mathcal{M}^3)$. The non-negativity of A follows from its representation by the operator S , i.e. $A = S^*S$, and the identity

$$(Av, u)_\Omega = (S^*Sv, u)_\Omega = (Sv, Su)_\Omega,$$

which holds for all $v \in \text{dom}(A)$ and $u \in \text{dom}(S)$. \square

Corollary 2.2. *The operator $A : L^2(\Omega, \mathcal{M}^3) \rightarrow L^2(\Omega, \mathcal{M}^3)$ defined in Lemma 2.1 is maximal monotone.*

Proof. According to the result of Brezis (see [4, Theorem 1]), a linear monotone operator A is maximal monotone, if it is a densely defined closed operator such that its adjoint A^* is monotone. The statement of the corollary follows then directly from Lemma 2.1 and the mentioned result of Brezis. \square

Boundary value problems. Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with a Lipschitz boundary. For every $v \in L^2(\Omega, \mathcal{M}^3)$ we define a functional Ψ on $L^2(\Omega, \mathcal{M}^3)$ by

$$\Psi(v) = \begin{cases} \frac{1}{2} \int_{\Omega} |\text{Curl } v(x)|^2 dx, & v \in L_{\text{Curl},0}^2(\Omega, \mathcal{M}^3) \\ +\infty, & \text{otherwise} \end{cases}.$$

It is easy to check that Ψ is proper, convex, lower semi-continuous. The next lemma gives a precise description of the subdifferential $\partial\Psi$.

Lemma 2.3. *We have that $\partial\Psi = \text{Curl Curl}$ with*

$$\text{dom}(\partial\Psi) = Z_{\text{Curl}}^2(\Omega, \mathcal{M}^3).$$

Proof. Let $A : L^2(\Omega, \mathcal{M}^3) \rightarrow L^2(\Omega, \mathcal{M}^3)$ be the linear operator defined by

$$Av = \text{Curl Curl } v$$

and $\text{dom}(A) = Z_{\text{Curl}}^2(\Omega, \mathcal{M}^3)$. Due to Lemma 2.1, the following identity

$$\int_{\Omega} \text{Curl Curl } v(x) \cdot w(x) dx = \int_{\Omega} \text{Curl } v(x) \cdot \text{Curl } w(x) dx \quad (22)$$

holds for any $v, w \in Z_{\text{Curl}}^2(\Omega, \mathcal{M}^3)$. Therefore, using (22) we obtain

$$\int_{\Omega} \text{Curl Curl } v \cdot (w - v) dx = \int_{\Omega} \text{Curl } v \cdot (\text{Curl } w - \text{Curl } v) dx \leq \Psi(w) - \Psi(v)$$

for every $v, w \in \text{dom}(A)$. This shows that $A \subset \partial\Psi$. Since A is maximal monotone (see Corollary 2.2) we conclude that $A = \partial\Psi$. \square

Helmholtz's projection. In the linear elasticity theory it is well known (see [11, Theorem 10.15]) that a Dirichlet boundary value problem formed by the equations

$$-\text{div}_x \sigma(x) = \hat{b}(x), \quad x \in \Omega, \quad (23)$$

$$\sigma(x) = \mathbb{C}[x](\text{sym}(\nabla_x u(x)) - \hat{\varepsilon}_p(x)), \quad x \in \Omega, \quad (24)$$

$$u(x) = 0, \quad x \in \partial\Omega, \quad (25)$$

to given $\hat{b} \in W^{-1,q}(\Omega, \mathbb{R}^3)$ and $\hat{\varepsilon}_p \in L^q(\Omega, \mathcal{S}^3)$ has a unique weak solution $(u, \sigma) \in W_0^{1,q}(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathcal{S}^3)$ provided the open set Ω has a C^1 -boundary and \mathbb{C} is continuous on $\bar{\Omega}$. Here the number q satisfies $1 < q < \infty$. The solution of (23) - (25) satisfies the inequality

$$\|\text{sym}(\nabla_x u)\|_q \leq C(\|\hat{\varepsilon}_p\|_q + \|\hat{b}\|_{-1,q})$$

with some positive constant C .

Definition 2.4. *For every $\hat{\varepsilon}_p \in L^q(\Omega, \mathcal{S}^3)$ we define a linear operator $P_q : L^q(\Omega, \mathcal{S}^3) \rightarrow L^q(\Omega, \mathcal{S}^3)$ by*

$$P_q \hat{\varepsilon}_p := \text{sym}(\nabla_x u),$$

where $u \in W_0^{1,q}(\Omega, \mathbb{R}^3)$ is the unique weak solution of (23) - (25) to the given function $\hat{\varepsilon}_p$ and $\hat{b} = 0$.

Next, a subset \mathcal{G}^q of $L^q(\Omega, \mathcal{S}^3)$ is defined by

$$\mathcal{G}^q = \{\text{sym}(\nabla_x u) \mid u \in W_0^{1,q}(\Omega, \mathbb{R}^3)\}.$$

The main properties of P_q are stated in the following lemma.

Lemma 2.5. *For every $1 < q < \infty$ the operator P_q is a bounded projector onto the subset \mathcal{G}^q of $L^q(\Omega, \mathcal{S}^3)$. The projector $(P_q)^*$ adjoint with respect to the bilinear form $[\xi, \zeta]_\Omega := (\mathbb{C}\xi, \zeta)_\Omega$ on $L^q(\Omega, \mathcal{S}^3) \times L^{q^*}(\Omega, \mathcal{S}^3)$ satisfies*

$$(P_q)^* = P_{q^*}, \quad \text{where } \frac{1}{q^*} + \frac{1}{q} = 1.$$

Due to Lemma 2.5 the following projection operator

$$Q_q = (I - P_q) : L^q(\Omega, \mathcal{S}^3) \rightarrow L^q(\Omega, \mathcal{S}^3)$$

is well-defined and generalizes the classical Helmholtz projection.

Let $L : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a linear, positive semi-definite mapping. The next result is needed for the subsequent analysis.

Corollary 2.6. *The operator $B^T \text{sym } \mathbb{C}Q_2 \text{sym } B + L : L^2(\Omega, \mathbb{R}^N) \rightarrow L^2(\Omega, \mathbb{R}^N)$ is non-negative and self-adjoint.*

For the proof of this result the reader is referred to [1].

3 Existence of strong solutions

In this section we prove the main existence result for (12) - (17). To show the existence of weak solutions a time-discretization method is used in this work. In the first step, we prove the existence of the solutions of the time-discretized problem in an appropriate Hilbert spaces based on the Helmholtz projection in $L^2(\Omega, \mathcal{S}^3)$ (Section 2) and the monotone operator methods. In order to be able to apply the monotone operator method to the time-discretized problem we regularize it by a linear positive definite term. In the second step, we derive the uniform a priori estimates for the solutions of the time-discretized problem using the polynomial growth of the function g (see Definition 3.1 below) and then we pass to the weak limit in the equivalent formulation of the time-discretized problem employing the weak lower semi-continuity of lower semi-continuous convex functions and the maximal monotonicity of g .

Main result. First, we define a class of maximal monotone functions we deal with in this work.

Definition 3.1. *For $m \in L^1(\Omega, \mathbb{R})$, $\alpha \in \mathbb{R}_+$ and $q > 1$, $\mathbb{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$ is the set of multi-valued functions $h : \Omega \times \mathbb{R}^k \rightarrow 2^{\mathbb{R}^k}$ with the following properties*

- $v \mapsto h(x, v)$ is maximal monotone for almost all $x \in \Omega$,
- the mapping $x \mapsto j_\lambda(x, v) : \Omega \rightarrow \mathbb{R}^k$ is measurable for all $\lambda > 0$, where $j_\lambda(x, v)$ is the inverse of $v \mapsto v + \lambda h(x, v)$,
- for a.e. $x \in \Omega$ and every $v^* \in h(x, v)$

$$\alpha \left(\frac{|v|^q}{q} + \frac{|v^*|^{q^*}}{q^*} \right) \leq (v, v^*) + m(x), \quad (26)$$

where $1/q + 1/q^* = 1$.

Remark 3.2. We note that the condition (26) is equivalent to the following two inequalities

$$|v^*|^{q^*} \leq m_1(x) + \alpha_1|v|^q, \quad (27)$$

$$(v, v^*) \geq m_2(x) + \alpha_2|v|^q, \quad (28)$$

for a.e. $x \in \Omega$, every $v^* \in h(x, v)$, with suitable functions $m_1, m_2 \in L^1(\Omega, \mathbb{R})$ and numbers $\alpha_1, \alpha_2 \in \mathbb{R}_+$.

The main properties of the class $\mathbb{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$ are collected in the following proposition.

Proposition 3.3. *Let \mathcal{H} be a canonical extension² of a function $h : \Omega \times \mathbb{R}^k \rightarrow 2^{\mathbb{R}^k}$, which belongs to $\mathbb{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$. Then \mathcal{H} is maximal monotone, surjective and $D(\mathcal{H}) = L^q(\Omega, \mathbb{R}^k)$.*

Proof. See Corollary 2.15 in [7]. □

Next, we define the following notion of solutions for the initial boundary value problem (12) - (17). Both notions of the solutions for (12) - (17) are introduced without assuming the homogeneity of the the initial condition (15).

Definition 3.4. (Strong solutions) *A function (u, σ, z) with $z = (p, \gamma)$ such that*

$$(u, \sigma) \in H^1(0, T_e; H_0^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3)), \\ p \in H^1(0, T_e; L^2_{\text{Curl}}(\Omega, \mathcal{M}^3)) \cap L^2(0, T_e; Z^2_{\text{Curl}}(\Omega, \mathcal{M}^3))$$

and

$$\gamma \in H^1(0, T_e; L^2(\Omega, \mathbb{R})), \quad \Sigma^{\text{lin}} \in L^q(\Omega_{T_e}, \mathcal{M}^3)$$

is called a strong solution of the initial boundary value problem (12) - (17), if for every $t \in [0, T_e]$ the function $(u(t), \sigma(t))$ is a weak solution of the boundary value problem (23) - (25) with $\hat{\varepsilon}_p = \text{sym } p(t)$ and $\hat{b} = b(t)$, the evolution inclusion (14), the initial condition (15) and the boundary condition (16) are satisfied pointwise.

Next, we state the main result of this work.

Theorem 3.5. *Suppose that $1 < q^* \leq 2 \leq q < \infty$. Assume that $\Omega \subset \mathbb{R}^3$ is a sliceable domain with a C^1 -boundary and the tensor \mathbb{C} has L^∞ -coefficients. Let the function $b \in W^{1,q}(0, T_e; L^q(\Omega, \mathbb{R}^3))$ be given. Assume that the function $g \in \mathbb{M}(\Omega, \mathbb{R}^N, q, \alpha, m)$ is of a subdifferential type, enjoys the self-controlling property (21) and that for a.e. $x \in \Omega$ the relation*

$$0 \in g(x, B^T \sigma^{(0)}(x)) \quad (29)$$

holds, where the function $\sigma^{(0)} \in L^2(\Omega, \mathcal{S}^3)$ is determined by equations (23) - (25) for $\hat{\varepsilon}_p = 0$ and $\hat{b} = b(0)$. Then there exists a strong solution (u, σ, z) of the initial boundary value problem (12) - (17).

² The canonical extension \mathcal{H} of a function $h : \Omega \rightarrow \mathbb{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$ is a monotone graph from $L^q(\Omega, \mathbb{R}^k)$ to $L^{q^*}(\Omega, \mathbb{R}^k)$, with $1/q + 1/q^* = 1$, defined by:

$$\text{Gr}\mathcal{H} = \{[v, v^*] \in L^q(\Omega, \mathbb{R}^k) \times L^{q^*}(\Omega, \mathbb{R}^k) \mid [v(x), v^*(x)] \in \text{Gr } h(x) \text{ for a.e. } x \in \Omega\}.$$

In order to deal with the measurable elasticity tensor \mathbb{C} , we reformulate the problem (12) - (17) as follows:

Let the function $(\hat{v}, \hat{\sigma}) \in W^{1,q}(0, T_e, W_0^{1,q}(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathcal{S}^3))$ be a solution of the linear elasticity problem formed by the equations

$$-\operatorname{div}_x \hat{\sigma}(x, t) = b(x, t), \quad x \in \Omega, \quad (30)$$

$$\hat{\sigma}(x, t) = \hat{\mathbb{C}}(\operatorname{sym}(\nabla_x \hat{v}(x, t))), \quad x \in \Omega, \quad (31)$$

$$\hat{v}(x, t) = 0, \quad x \in \partial\Omega, \quad (32)$$

where $\hat{\mathbb{C}} : \mathcal{S}^3 \rightarrow \mathcal{S}^3$ is any positive definite symmetric linear mapping independent of (x, t) . Such a function $(\hat{v}, \hat{\sigma})$ exists (see [11, Theorem 10.15]). Then the solution (u, σ, z) of the initial boundary value problem (12) - (17) has the following form

$$(u, \sigma, z) = (\tilde{v} + \hat{v}, \tilde{\sigma} + \hat{\sigma}, z),$$

where the function $(\tilde{v}, \tilde{\sigma}, z)$ solves the problem

$$-\operatorname{div}_x \tilde{\sigma}(x, t) = 0, \quad (33)$$

$$\tilde{\sigma}(x, t) = \mathbb{C}[x](\operatorname{sym}(\nabla_x \tilde{v}(x, t) - Bz(x, t))) \quad (34)$$

$$+(\mathbb{C}[x] - \hat{\mathbb{C}})(\operatorname{sym}(\nabla_x \hat{v}(x, t))),$$

$$\partial_t z(x, t) \in g(x, \Sigma^{\operatorname{lin}}(x, t)), \quad \Sigma^{\operatorname{lin}} = \Sigma_e^{\operatorname{lin}} + \Sigma_{\operatorname{sh}}^{\operatorname{lin}} + \Sigma_{\operatorname{curl}}^{\operatorname{lin}}, \quad (35)$$

$$\Sigma_e^{\operatorname{lin}} = B^T(\tilde{\sigma} + \hat{\sigma}), \quad \Sigma_{\operatorname{sh}}^{\operatorname{lin}} = -Lz, \quad \Sigma_{\operatorname{curl}}^{\operatorname{lin}} = -C_1 B^T \operatorname{Curl} \operatorname{Curl} Bz,$$

$$z(x, 0) = 0, \quad x \in \Omega, \quad (36)$$

$$Bz(x, t) \times n(x) = 0, \quad (x, t) \in \partial\Omega \times [0, T_e], \quad (37)$$

$$\tilde{v}(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T_e]. \quad (38)$$

Here, the function $(\hat{v}, \hat{\sigma})$ given as the solution of (30) - (32) is considered as known. Next, we show that the problem (33) - (38) has a solution. This will prove the existence of solutions for (12) - (17).

Proof. We show the existence of solutions using the Rothe method (a time-discretization method, see [32] for details). In order to introduce a time-discretized problem, let us fix any $m \in \mathbb{N}$ and set

$$h := \frac{T_e}{2^m}, \quad z_m^0 := 0, \quad \hat{\sigma}_m^n := \frac{1}{h} \int_{(n-1)h}^{nh} \hat{\sigma}(s) ds \in L^q(\Omega, \mathcal{S}^3), \quad n = 1, \dots, 2^m.$$

Then we are looking for functions $u_m^n \in H^1(\Omega, \mathbb{R}^3)$, $\sigma_m^n \in L^2(\Omega, \mathcal{S}^3)$ and $z_m^n \in Z_{\operatorname{Curl}}^2(\Omega, \mathbb{R}^N)$ with $Bz_m^n(x) \in \mathfrak{sl}(3)$ for a.e. $x \in \Omega$ and

$$\Sigma_{n,m}^{\operatorname{lin}} := B^T(\sigma_m^n + \hat{\sigma}_m^n) - Lz_m^n - \frac{1}{m} z_m^n - C_1 B^T \operatorname{Curl} \operatorname{Curl} Bz_m^n \in L^q(\Omega, \mathbb{R}^N)$$

solving the following problem

$$-\operatorname{div}_x \sigma_m^n(x) = 0, \quad (39)$$

$$\sigma_m^n(x) = \mathbb{C}[x](\operatorname{sym}(\nabla_x u_m^n(x) - Bz_m^n(x))) \quad (40)$$

$$+(\mathbb{C}[x] - \hat{\mathbb{C}})(\hat{\mathbb{C}})^{-1} \hat{\sigma}_m^n(x),$$

$$\frac{z_m^n(x) - z_m^{n-1}(x)}{h} \in g(x, \Sigma_{n,m}^{\operatorname{lin}}(x)), \quad (41)$$

together with the boundary conditions

$$Bz_m^n(x) \times n(x) = 0, \quad x \in \partial\Omega, \quad (42)$$

$$u_m^n(x) = 0, \quad x \in \partial\Omega. \quad (43)$$

Next, we adopt the reduction technique proposed in [1] to the above equations. Suppose that $(u_m^n, \sigma_m^n, z_m^n)$ is a solution of the boundary value problem (39) - (43). The equations (39) - (40), (43) form a boundary value problem for the solution (u_m^n, σ_m^n) of the problem of linear elasticity. Due to linearity of this problem we can write these components of the solution in the form

$$(u_m^n, \sigma_m^n) = (U_m^n, \Sigma_m^n) + (w_m^n, \tau_m^n),$$

with the solution (w_m^n, τ_m^n) of the Dirichlet boundary value problem (23) - (25) to the data $\hat{b} = 0$, $\hat{\varepsilon}_p = \mathbb{C}^{-1}(\mathbb{C} - \hat{\mathbb{C}})^{-1}\hat{\sigma}_m^n$, and with the solution (U_m^n, Σ_m^n) of the problem (23) - (25) to the data $\hat{b} = 0$, $\hat{\varepsilon}_p = \text{sym}(Bz_m^n)$. We thus obtain

$$\text{sym}(\nabla_x u_m^n) - \text{sym}(Bz_m^n) = (P_2 - I)\text{sym}(Bz_m^n) + \text{sym}(\nabla_x w_m^n).$$

where the operator P_2 is defined in Definition 2.4. We insert this equation into (40) and get that (41) can be rewritten in the following form

$$\frac{z_m^n - z_m^{n-1}}{h} \in \mathcal{G}(-M_m z_m^n - C_1 B^T \text{Curl Curl } Bz_m^n + B^T(\hat{\sigma}_m^n + \tau_m^n)), \quad (44)$$

$$Bz_m^n(x) \times n(x) = 0, \quad x \in \partial\Omega, \quad (45)$$

where

$$M_m := (B^T \text{sym } \mathbb{C} Q_2 \text{sym } B + L) + \frac{1}{m} I : L^2(\Omega, \mathcal{M}^3) \rightarrow L^2(\Omega, \mathcal{M}^3)$$

with the Helmholtz projection Q_2 . Here \mathcal{G} denotes the canonical extension of g . Next, the problem (44) - (45) reads

$$\Psi(z_m^n) \ni B^T(\hat{\sigma}_m^n + \tau_m^n), \quad (46)$$

where

$$\Psi(v) = \mathcal{G}^{-1}\left(\frac{v - z_m^{n-1}}{h}\right) + M_m(v) + \partial\Phi(v).$$

Here, the functional $\Phi : L^2(\Omega, \mathcal{M}^3) \rightarrow \bar{\mathbb{R}}$ is given by

$$\Phi(v) := \begin{cases} \frac{1}{2} \int_{\Omega} |\text{Curl } v(x)|^2 dx, & v \in L^2_{\text{Curl},0}(\Omega, \mathcal{M}^3) \\ +\infty, & \text{otherwise} \end{cases},$$

respectively. The facts that Φ is a proper convex lower semi-continuous functional and that $\text{Curl Curl} = \partial\Phi$ are proved in Section 2. Since M_m is bounded, self-adjoint and positive definite (see Corollary 2.6 and the definition of M_m), it is maximal monotone by Theorem II.1.3 in [2]. The last thing which we have to verify is whether the following operator

$$\Psi = \mathcal{G}^{-1} + M_m + \partial\Phi$$

is maximal monotone. Since $g \in \mathbb{M}(\Omega, \mathcal{M}^3, q, \alpha, m)$, using the boundness of M_m we conclude that the domains of \mathcal{G}^{-1} and M_m are equal to the whole

space $L^2(\Omega, \mathcal{M}^3)$. Therefore, Theorem III.3.6 in [30] (or [2, Theorem II.1.7]) guarantees that the sum $\mathcal{G}^{-1} + M_m + \partial\Phi$ is maximal monotone with

$$\text{dom}(\Psi) = \text{dom}(\partial\Phi) := Z_{\text{Curl}}^2(\Omega, \mathcal{M}^3).$$

Since M_m is coercive in $L^2(\Omega, \mathcal{M}^3)$, which obviously yields the coercivity of Ψ , the operator Ψ is surjective by Theorem III.2.10 in [30]. Thus, we conclude that equation (46) as well as the problem (44) - (45) has a solution with the required regularity, i.e. $Bz_m^n \in Z_{\text{Curl}}^2(\Omega, \mathcal{M}^3)$. The solution is unique due to the following estimate

$$0 = (\Psi(z_1) - \Psi(z_2), z_1 - z_2)_\Omega \geq (M_m(z_1 - z_2), z_1 - z_2)_\Omega \geq \frac{1}{m} \|z_1 - z_2\|_\Omega^2, \quad (47)$$

which holds since the operators \mathcal{G}^{-1} and $\partial\Phi$ are monotone. By the constructions this implies that the boundary value problem (39) - (43) is solvable as well (details can be found in [1]). Moreover, $Bz_m^n(x) \in \mathfrak{sl}(3)$ for a.e. $x \in \Omega$.

Rothe approximation functions: For any family $\{\xi_m^n\}_{n=0, \dots, 2m}$ of functions in a reflexive Banach space X , we define *the piecewise affine interpolant* $\xi_m \in C([0, T_e], X)$ by

$$\xi_m(t) := \left(\frac{t}{h} - (n-1)\right) \xi_m^n + \left(n - \frac{t}{h}\right) \xi_m^{n-1} \quad \text{for } (n-1)h \leq t \leq nh \quad (48)$$

and *the piecewise constant interpolant* $\bar{\xi}_m \in L^\infty(0, T_e; X)$ by

$$\bar{\xi}_m(t) := \xi_m^n \quad \text{for } (n-1)h < t \leq nh, \quad n = 1, \dots, 2m, \quad \text{and } \bar{\xi}_m(0) := \xi_m^0. \quad (49)$$

For the further analysis we recall the following property of $\bar{\xi}_m$ and ξ_m :

$$\|\xi_m\|_{L^q(0, T_e; X)} \leq \|\bar{\xi}_m\|_{L^q(-h, T_e; X)} \leq \left(h \|\xi_m^0\|_X^q + \|\bar{\xi}_m\|_{L^q(0, T_e; X)}^q\right)^{1/q}, \quad (50)$$

where $\bar{\xi}_m$ is formally extended to $t \leq 0$ by ξ_m^0 and $1 \leq q \leq \infty$ (see [32]).

A-priori estimates. Multiplying (39) by $(u_m^n - u_m^{n-1})/h$ and then integrating over Ω we get

$$(\sigma_m^n, \text{sym}(\nabla_x(u_m^n - u_m^{n-1})/h))_\Omega = 0.$$

Equations (40), (41) imply that for a.e. $x \in \Omega$

$$\begin{aligned} & \sigma_m^n \cdot \left(\text{sym}(\nabla_x(u_m^n - u_m^{n-1})/h) - \mathbb{C}^{-1}[x](\sigma_m^n - \sigma_m^{n-1})/h \right) \\ & + \sigma_m^n \cdot \left(\mathbb{C}^{-1}[x](\mathbb{C}[x] - \hat{\mathbb{C}})(\hat{\mathbb{C}})^{-1}(\hat{\sigma}_m^n - \hat{\sigma}_m^{n-1})/h \right) \\ & - \frac{z_m^n - z_m^{n-1}}{h} \cdot \left(Lz_m^n + \frac{1}{m}z_m^n + C_2 B^T \text{Curl Curl } Bz_m^n \right) \\ & + \frac{z_m^n - z_m^{n-1}}{h} \cdot B^T \hat{\sigma}_m^n = g^{-1} \left(\frac{z_m^n - z_m^{n-1}}{h} \right) \cdot \left(\frac{z_m^n - z_m^{n-1}}{h} \right). \end{aligned}$$

After integrating the last identity over Ω , the above computations imply

$$\frac{1}{h} \left(\mathbb{C}^{-1} \sigma_m^n, \sigma_m^n - \sigma_m^{n-1} \right)_\Omega + \frac{1}{h} \left(L^{1/2}(z_m^n - z_m^{n-1}), L^{1/2}z_m^n \right)_\Omega$$

$$\begin{aligned}
& + \frac{1}{m} \frac{1}{h} \left(z_m^n - z_m^{n-1}, z_m^n \right)_\Omega + C_1 \frac{1}{h} \left(\text{Curl} B(z_m^n - z_m^{n-1}), \text{Curl} B z_m^n \right)_\Omega \\
& + \frac{\alpha}{q} \left\| \Sigma_{n,m}^{\text{lin}} \right\|_q^q + \frac{\alpha}{q^*} \left\| \frac{z_m^n - z_m^{n-1}}{h} \right\|_{q^*}^{q^*} \leq \int_\Omega m(x) dx \\
& + \frac{1}{h} \left(\sigma_m^n, \bar{\mathbb{C}}(\hat{\sigma}_m^n - \hat{\sigma}_m^{n-1}) \right)_\Omega + \frac{1}{h} \left(B^T \hat{\sigma}_m^n, z_m^n - z_m^{n-1} \right)_\Omega,
\end{aligned}$$

where $\bar{\mathbb{C}} := \mathbb{C}^{-1}(\mathbb{C} - \hat{\mathbb{C}})(\hat{\mathbb{C}})^{-1}$. Multiplying by h and summing the obtained relation for $n = 1, \dots, l$ for any fixed $l \in [1, 2^m]$ we derive the following inequality (here $\mathbb{B} := \mathbb{C}^{-1}$)

$$\begin{aligned}
& \frac{1}{2} \left(\|\mathbb{B}^{1/2} \sigma_m^l\|_2^2 + \|L^{1/2} z_m^l\|_2^2 + \frac{1}{m} \|z_m^l\|_2^2 + C_1 \|\text{Curl} B z_m^l\|_2^2 \right) \\
& + \frac{h\alpha}{q} \sum_{n=1}^l \|\Sigma_{n,m}^{\text{lin}}\|_q^q + \frac{h\alpha}{q^*} \sum_{n=1}^l \left\| \frac{z_m^n - z_m^{n-1}}{h} \right\|_{q^*}^{q^*} \leq C^{(0)} + lh \int_\Omega m(x) dx \quad (51) \\
& + h \sum_{n=1}^l \left(\sigma_m^n, \bar{\mathbb{C}} \frac{\hat{\sigma}_m^n - \hat{\sigma}_m^{n-1}}{h} \right)_\Omega + h \sum_{n=1}^l \left(B^T \hat{\sigma}_m^n, \frac{z_m^n - z_m^{n-1}}{h} \right)_\Omega,
\end{aligned}$$

where³

$$C^{(0)} := \|\mathbb{B}^{1/2} \sigma^{(0)}\|_2^2.$$

Since $\hat{\sigma}_m^n \in L^q(\Omega, \mathcal{S}^3)$, using Young's inequality with $\epsilon > 0$ we get that

$$\begin{aligned}
\left(B^T \hat{\sigma}_m^n, \frac{z_m^n - z_m^{n-1}}{h} \right)_\Omega & \leq \|B^T \hat{\sigma}_m^n\|_q \|(z_m^n - z_m^{n-1})/h\|_{q^*} \\
& \leq C_\epsilon \|B^T\| \|\hat{\sigma}_m^n\|_q + \epsilon \|(z_m^n - z_m^{n-1})/h\|_{q^*}^{q^*}, \quad (52)
\end{aligned}$$

where C_ϵ is a positive constant appearing in the Young's inequality. Analogically, we obtain

$$\left(\sigma_m^n, \bar{\mathbb{C}} \frac{\hat{\sigma}_m^n - \hat{\sigma}_m^{n-1}}{h} \right)_\Omega \leq \epsilon \|\sigma_m^n\|_2^2 + C_\epsilon \|(\hat{\sigma}_m^n - \hat{\sigma}_m^{n-1})/h\|_2^2 \quad (53)$$

with some other constant C_ϵ . Combining the inequalities (51), (52) and (53), and choosing an appropriate value for $\epsilon > 0$ we obtain the following estimate

$$\begin{aligned}
& \frac{1}{2} \left(\|\mathbb{B}^{1/2} \sigma_m^l\|_2^2 + \|L^{1/2} z_m^l\|_2^2 + \frac{1}{m} \|z_m^l\|_2^2 + C_1 \|\text{Curl} B z_m^l\|_2^2 \right) \\
& + h \hat{C}_\epsilon \sum_{n=1}^l \left(\|\Sigma_{n,m}^{\text{lin}}\|_q^q + \left\| \frac{z_m^n - z_m^{n-1}}{h} \right\|_{q^*}^{q^*} \right) \leq C^{(0)} + lh \int_\Omega m(x) dx \quad (54) \\
& + h\epsilon \sum_{n=1}^l \|\sigma_m^n\|_2^2 + h \tilde{C}_\epsilon \sum_{n=1}^l \left(\|\hat{\sigma}_m^n\|_q^q + \|(\hat{\sigma}_m^n - \hat{\sigma}_m^{n-1})/h\|_2^2 \right),
\end{aligned}$$

³Here we use the following inequality

$$\begin{aligned}
\sum_{n=1}^l (\phi_m^n - \phi_m^{n-1}, \phi_m^n)_\Omega & = \frac{1}{2} \sum_{n=1}^l \left(\|\phi_m^n\|_2^2 - \|\phi_m^{n-1}\|_2^2 \right) \\
& + \frac{1}{2} \sum_{n=1}^l \|\phi_m^n - \phi_m^{n-1}\|_2^2 \geq \frac{1}{2} \|\phi_m^l\|_2^2 - \frac{1}{2} \|\phi_m^0\|_2^2
\end{aligned}$$

for any family of functions $\phi_m^0, \phi_m^1, \dots, \phi_m^m$.

where \tilde{C}_ϵ and \hat{C}_ϵ are some positive constants. Now, taking Remark 8.15 in [32] and the definition of Rothe's approximation functions into account we rewrite (54) as follows

$$\begin{aligned} & \|\mathbb{B}^{1/2}\bar{\sigma}_m(t)\|_2^2 + \|L^{1/2}\bar{z}_m(t)\|_2^2 + \frac{1}{m}\|\bar{z}_m(t)\|_2^2 + C_1\|\text{Curl}B\bar{z}_m(t)\|_2^2 \\ & + \hat{C}_\epsilon \int_0^{T_e} \int_\Omega \left(|\partial_t z_m(x,t)|^{q^*} + |\bar{\Sigma}_m^{\text{lin}}(x,t)|^q \right) dxdt \\ & \leq 2C^{(0)} + 2T_e\|m\|_{1,\Omega} + \epsilon\|\sigma_m\|_{2,\Omega \times (0,T_e)}^2 + 2\tilde{C}_\epsilon\|\hat{\sigma}\|_{W^{1,q}(0,T_e;L^q(\Omega,\mathcal{S}^3))}^q. \end{aligned} \quad (55)$$

From (55) we get immediately that

$$\begin{aligned} & \bar{C}_\epsilon\|\sigma_m\|_{2,\Omega \times (0,t)}^2 + \|L^{1/2}\bar{z}_m(t)\|_2^2 + \frac{1}{m}\|\bar{z}_m(t)\|_2^2 + C_1\|\text{Curl}B\bar{z}_m(t)\|_2^2 \\ & + \hat{C}_\epsilon \left(\|\partial_t z_m\|_{q^*,\Omega \times (0,T_e)}^{q^*} + \|\bar{\Sigma}_m^{\text{lin}}\|_{q,\Omega \times (0,T_e)}^q \right) \\ & \leq 2C^{(0)} + 2T_e\|m\|_{1,\Omega} + 2\tilde{C}_\epsilon\|\hat{\sigma}\|_{W^{1,q}(0,T_e;L^q(\Omega,\mathcal{S}^3))}^q, \end{aligned} \quad (56)$$

where \bar{C}_ϵ is some other constant depending on ϵ . Altogether, from estimate (56) we get that

$$\{z_m\}_m \text{ is uniformly bounded in } W^{1,q^*}(0,T_e;L^{q^*}(\Omega,\mathbb{R}^N)), \quad (57)$$

$$\{L^{1/2}\bar{z}_m\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathbb{R}^N)), \quad (58)$$

$$\{\sigma_m\}_m, \text{ is uniformly bounded in } L^2(0,T_e;L^2(\Omega,\mathcal{S}^3)), \quad (59)$$

$$\{\text{Curl}B\bar{z}_m\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathcal{M}^3)), \quad (60)$$

$$\{\bar{\Sigma}_m^{\text{lin}}\}_m \text{ is uniformly bounded in } L^q(0,T_e;L^q(\Omega,\mathbb{R}^N)), \quad (61)$$

$$\left\{ \frac{1}{\sqrt{m}}\bar{z}_m \right\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathbb{R}^N)). \quad (62)$$

In particular, the uniform boundedness of the sequences in (57) - (62) yields

$$\{u_m\}_m \text{ is uniformly bounded in } W^{1,q^*}(0,T_e;W_0^{1,q^*}(\Omega,\mathbb{R}^3)), \quad (63)$$

$$\{\text{Curl}B\bar{z}_m\}_m \text{ is uniformly bounded in } L^2(0,T_e;L^2(\Omega,\mathcal{M}^3)). \quad (64)$$

Employing (50) the estimates (57) - (64) further imply that the sequences $\{\sigma_m\}_m$, $\{L^{1/2}z_m\}_m$, $\{\text{Curl}Bz_m\}_m$, $\{z_m/\sqrt{m}\}_m$, $\{\bar{\Sigma}_m^{\text{lin}}\}_m$ and $\{\text{Curl}Bz_m\}_m$ are also uniformly bounded in the corresponding spaces. As a result, we have

$$\{Bz_m\}_m \text{ is uniformly bounded in } L^{q^*}(0,T_e;Z_{\text{Curl}}^{q^*}(\Omega,\mathcal{M}^3)). \quad (65)$$

Using the assumption that g is of subdifferential type, i.e. there exists a function \tilde{f} such that

$$(g(x,v),v-w) \geq \tilde{f}(x,v) - \tilde{f}(x,w)$$

is satisfied for a.e. $x \in \Omega$ and all $w \in \mathbb{R}^N$, we can improve the estimates (58), (60) and (62). By assumption (29) we obtain for a.e. $x \in \Omega$ and all $w \in L^2(\Omega,\mathbb{R}^N)$

$$\tilde{f}(x,w(x)) \geq \tilde{f}(x,B^T\sigma^{(0)}(x)).$$

This inequality implies that \tilde{f} attains its minimum at the point $B^T \sigma^{(0)}$. We introduce the function f defined by

$$f(x, v) = \tilde{f}(x, v) - \tilde{f}(x, B^T \sigma^{(0)}(x)). \quad (66)$$

Then the functional f satisfies $f \geq 0$ and $g = \partial \tilde{f} = \partial f$ and we can suppose that $g(x, v) = \partial f(x, v)$ with $f \geq 0$.

Next, we show that the sequence $\{f(\bar{\Sigma}_m^{\text{lin}})\}_m$ is uniformly bounded in the space $L^\infty(0, T; L^1(\Omega, \mathbb{R}))$ and the sequences $\{L^{1/2} z_m\}_m$ and $\{\frac{1}{\sqrt{m}} z_m\}_m$ are uniformly bounded in $W^{1,2}(0, T; L^2(\Omega, \mathbb{R}^N))$. Indeed, multiplying (41) by the term $(\Sigma_{n,m}^{\text{lin}} - \Sigma_{n-1,m}^{\text{lin}})/h$ and integrating over Ω we obtain

$$\begin{aligned} \left(\frac{z_m^n - z_m^{n-1}}{h}, \frac{\Sigma_{n,m}^{\text{lin}} - \Sigma_{n-1,m}^{\text{lin}}}{h} \right)_\Omega &= \frac{1}{h} (\partial f(\Sigma_{n,m}^{\text{lin}}), \Sigma_{n,m}^{\text{lin}} - \Sigma_{n-1,m}^{\text{lin}})_\Omega \\ &\geq \frac{1}{h} (F(\Sigma_{n,m}^{\text{lin}}) - F(\Sigma_{n-1,m}^{\text{lin}})), \end{aligned} \quad (67)$$

where $F : L^2(\Omega) \rightarrow \bar{\mathbb{R}}$ is a convex functional defined by

$$F(v) = \begin{cases} \int_\Omega f(x, v(x)) dx, & f(\cdot, v(\cdot)) \in L^1(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

In the following we use the notation $M := -B^T \text{sym} \mathbb{C} Q_2 \text{sym} B$, $\partial \Phi := C_1 B^T \text{Curl} \text{Curl} B$ and $\hat{z}_m^n := B^T(\hat{\sigma}_m^n + \tau_m^n)$ are used. Since M and $\partial \Phi$ are linear positive semi-definite, symmetric operators we can rewrite the left side of (67) as follows

$$\begin{aligned} &\left(\frac{z_m^n - z_m^{n-1}}{h}, \frac{\Sigma_{n,m}^{\text{lin}} - \Sigma_{n-1,m}^{\text{lin}}}{h} \right)_\Omega = - \left(\frac{z_m^n - z_m^{n-1}}{h}, \left(M + L + \frac{1}{m} \right) \frac{z_m^n - z_m^{n-1}}{h} \right)_\Omega \\ &- \left(\frac{z_m^n - z_m^{n-1}}{h}, \frac{\partial \Phi(z_m^n) - \partial \Phi(z_m^{n-1})}{h} \right)_\Omega + \left(\frac{z_m^n - z_m^{n-1}}{h}, \frac{\hat{z}_m^n - \hat{z}_m^{n-1}}{h} \right)_\Omega \quad (68) \\ &= - \left\| M^{1/2} \frac{z_m^n - z_m^{n-1}}{h} \right\|_2^2 - \left\| L^{1/2} \frac{z_m^n - z_m^{n-1}}{h} \right\|_2^2 - \left\| \frac{1}{\sqrt{m}} \frac{z_m^n - z_m^{n-1}}{h} \right\|_2^2 \\ &- C_1 \left\| \text{Curl} B \frac{z_m^n - z_m^{n-1}}{h} \right\|_2^2 + \left(\frac{z_m^n - z_m^{n-1}}{h}, \frac{\hat{z}_m^n - \hat{z}_m^{n-1}}{h} \right)_\Omega. \end{aligned}$$

Now we combine (67) and (68), multiply the obtained relation by h and sum it up for $n = 1, \dots, l$ and any fixed $l \in \{1, \dots, 2^m\}$. We obtain

$$\begin{aligned} &h \sum_{n=1}^l \left(\left\| M^{1/2} \frac{z_m^n - z_m^{n-1}}{h} \right\|_2^2 + \left\| L^{1/2} \frac{z_m^n - z_m^{n-1}}{h} \right\|_2^2 + \left\| \frac{1}{\sqrt{m}} \frac{z_m^n - z_m^{n-1}}{h} \right\|_2^2 \right. \\ &\quad \left. + C_1 \left\| \text{Curl} B \frac{z_m^n - z_m^{n-1}}{h} \right\|_2^2 \right) + \int_\Omega f(x, \Sigma_{l,m}^{\text{lin}}(x)) dx \\ &\leq F(\Sigma_{0,m}^{\text{lin}}) + h \sum_{n=1}^l \left(\frac{z_m^n - z_m^{n-1}}{h}, \frac{\hat{z}_m^n - \hat{z}_m^{n-1}}{h} \right)_\Omega, \end{aligned} \quad (69)$$

which implies the estimate

$$\begin{aligned} & \|M^{1/2}z_{mt}\|_{2,\Omega_t}^2 + \|L^{1/2}z_{mt}\|_{2,\Omega_t}^2 + \left\| \frac{1}{\sqrt{m}}z_{mt} \right\|_{2,\Omega_t}^2 + C_1 \|\text{Curl } Bz_{mt}\|_{2,\Omega_t}^2 \\ & + \|f(\bar{\Sigma}_m^{\text{lin}}(t))\|_{1,\Omega} \leq F(\Sigma^{\text{lin}}(0)) + \|z_{mt}\|_{q^*,\Omega_t} \|\hat{z}_{mt}\|_{q,\Omega_t}. \end{aligned} \quad (70)$$

Since by (57) the right side of (70) is bounded we obtain

$$\{f(\bar{\Sigma}_m^{\text{lin}})\}_m \text{ is uniformly bounded in } L^\infty(0, T_e; L^1(\Omega, \mathbb{R})), \quad (71)$$

$$\{L^{1/2}z_m\}_m \text{ is uniformly bounded in } W^{1,2}(0, T_e; L^2(\Omega, \mathbb{R}^N)), \quad (72)$$

$$\left\{ \frac{1}{\sqrt{m}}z_m \right\}_m \text{ is uniformly bounded in } W^{1,2}(0, T_e; L^2(\Omega, \mathbb{R}^N)), \quad (73)$$

$$\{M^{1/2}z_m\}_m \text{ is uniformly bounded in } W^{1,2}(0, T_e; L^2(\Omega, \mathbb{R}^N)), \quad (74)$$

$$\{\text{Curl } Bz_m\}_m \text{ is uniformly bounded in } W^{1,2}(0, T_e; L^2(\Omega, M^3)). \quad (75)$$

Furthermore, due to the self-controlling property (21) and (72), (61) we obtain that

$$\{B(\partial_t z_m)\}_m \text{ is uniformly bounded in } L^2(0, T_e; Z_{\text{Curl}}^2(\Omega, \mathcal{M}^3)). \quad (76)$$

Moreover, (48) and (49) yield $\{Bz_m(x, t), B\bar{z}_m(x, t)\}_m \in \mathfrak{sl}(3)$ for a.e. $(x, t) \in \Omega_{T_e}$.

Additional regularity of discrete solutions. In order to get the additional a priori estimates, we extend the function b to $t < 0$ by setting $b(t) = b(0)$. The extended function b is in the space $W^{1,p}(-2h, T_e; W^{-1,p}(\Omega, \mathbb{R}^3))$. Then, we set $b_m^0 = b_m^{-1} := b(0)$. Let us further set

$$z_m^{-1} := z_m^0 - h\mathcal{G}(\Sigma_{0,m}^{\text{lin}}).$$

The assumption (29) implies that $z_m^{-1} = 0$. Next, we define functions $(u_m^{-1}, \sigma_m^{-1})$ and (u_m^0, σ_m^0) as solutions of the linear elasticity problem (23) - (25) to the data $\hat{b} = b_m^{-1}$, $\hat{\varepsilon}_p = 0$ and $\hat{b} = b_m^0$, $\hat{\varepsilon}_p = 0$, respectively. Obviously, the following estimate holds

$$\left\{ \left\| \frac{u_m^0 - u_m^{-1}}{h} \right\|_2, \left\| \frac{\sigma_m^0 - \sigma_m^{-1}}{h} \right\|_2 \right\} \leq C, \quad (77)$$

where C is some positive constant independent of m . Taking now the incremental ratio of (41) for $n = 1, \dots, 2^m$, we obtain⁴

$$\text{rt } z_m^n - \text{rt } z_m^{n-1} = \mathcal{G}(\Sigma_{n,m}^{\text{lin}}) - \mathcal{G}(\Sigma_{(n-1),m}^{\text{lin}}).$$

Let us now multiply the last identity by $-(\Sigma_{n,m}^{\text{lin}} - \Sigma_{(n-1),m}^{\text{lin}})/h$. Then using the monotonicity of \mathcal{G} we obtain that

$$\frac{1}{m} \left(\text{rt } z_m^n - \text{rt } z_m^{n-1}, \text{rt } z_m^n \right)_\Omega + \left(\text{rt } z_m^n - \text{rt } z_m^{n-1}, L \text{rt } z_m^n \right)_\Omega$$

⁴For sake of simplicity we use the following notation $\text{rt } \phi_m^n := (\phi_m^n - \phi_m^{n-1})/h$, where $\phi_m^0, \phi_m^1, \dots, \phi_m^m$ is any family of functions.

$$\begin{aligned}
& + (\text{rt } z_m^n - \text{rt } z_m^{n-1}, B^T \text{Curl Curl } B(\text{rt } z_m^n))_\Omega \\
& \leq (\text{rt } z_m^n - \text{rt } z_m^{n-1}, C_1 B^T \text{rt } \sigma_m^n)_\Omega + (\text{rt } z_m^n - \text{rt } z_m^{n-1}, B^T \text{rt } \hat{\sigma}_m^n)_\Omega.
\end{aligned}$$

With (39) and (40) the previous inequality can be rewritten as follows

$$\begin{aligned}
& \frac{1}{m} (\text{rt } z_m^n - \text{rt } z_m^{n-1}, \text{rt } z_m^n)_\Omega + (\text{rt } z_m^n - \text{rt } z_m^{n-1}, L \text{rt } z_m^n)_\Omega \\
& + (\text{rt } z_m^n - \text{rt } z_m^{n-1}, B^T \text{Curl Curl } B(\text{rt } z_m^n))_\Omega + (\text{rt } \sigma_m^n - \text{rt } \sigma_m^{n-1}, \mathbb{C}^{-1} \text{rt } \sigma_m^n)_\Omega \\
& \leq (\text{rt } \hat{\sigma}_m^n - \text{rt } \hat{\sigma}_m^{n-1}, \bar{\mathbb{C}} \text{rt } \sigma_m^n)_\Omega + (\text{rt } z_m^n - \text{rt } z_m^{n-1}, B^T \text{rt } \hat{\sigma}_m^n)_\Omega.
\end{aligned}$$

As in the proof of (51), multiplying the last inequality by h and summing with respect to n from 1 to l for any fixed $l \in [1, 2^m]$ we get the estimate

$$\begin{aligned}
& \frac{h}{m} \|\text{rt } z_m^l\|_2^2 + h \|L^{1/2} \text{rt } z_m^l\|_2^2 + h \|\mathbb{B}^{1/2} \text{rt } \sigma_m^l\|_2^2 + h \|\text{Curl } B \text{rt } z_m^l\|_2^2 \leq 2hC^{(0)} \\
& + 2h \sum_{n=1}^l (B^T \text{rt } \hat{\sigma}_m^n, \text{rt } z_m^n - \text{rt } z_m^{n-1})_\Omega + 2h \sum_{n=1}^l (\text{rt } \hat{\sigma}_m^n - \text{rt } \hat{\sigma}_m^{n-1}, \bar{\mathbb{C}} \text{rt } \sigma_m^n)_\Omega \quad (78)
\end{aligned}$$

where now $C^{(0)}$ denotes

$$2C^{(0)} := \|\mathbb{B}^{1/2} \text{rt } \sigma_m^0\|_2^2.$$

We note that (77) yields the uniform boundness of $C^{(0)}$ with respect to m . Summing now (78) for $l = 1, \dots, 2^m$ we derive the inequality

$$\begin{aligned}
& \frac{1}{m} \|\partial_t z_m\|_{2, \Omega_{T_e}}^2 + \|L^{1/2} (\partial_t z_m)\|_{2, \Omega_{T_e}}^2 + \|\text{Curl } B (\partial_t z_m)\|_{2, \Omega_{T_e}}^2 \quad (79) \\
& + C \|\partial_t \sigma_m\|_{2, \Omega_{T_e}}^2 \leq C \|\partial_t \hat{\sigma}_m\|_{2, \Omega_{T_e}} (\|\partial_t \sigma_m\|_{2, \Omega_{T_e}} + \|B(\partial_t z_m)\|_{2, \Omega_{T_e}}).
\end{aligned}$$

Using the self-controlling property (21) and Young's inequality with $\epsilon > 0$ we obtain that

$$\begin{aligned}
& \frac{1}{m} \|\partial_t z_m\|_{2, \Omega_{T_e}}^2 + \|L^{1/2} (\partial_t z_m)\|_{2, \Omega_{T_e}}^2 + \|\text{Curl } B (\partial_t z_m)\|_{2, \Omega_{T_e}}^2 + C_\epsilon \|\partial_t \sigma_m\|_{2, \Omega_{T_e}}^2 \quad (80) \\
& \leq C \|\partial_t \hat{\sigma}_m\|_{2, \Omega_{T_e}}^2 + \|B(\partial_t z_m)\|_{2, \Omega_{T_e}}.
\end{aligned}$$

Since $\hat{\sigma}_m$ is uniformly bounded in $W^{1,q}(\Omega_{T_e}, \mathcal{S}^3)$ and $B(\partial_t z_m)$ is uniformly bounded in $L^2(0, T_e; L^2(\Omega, \mathcal{M}^3))$, estimates (79) and (80) imply

$$\{L^{1/2}(\partial_t z_m)\}_m \text{ is uniformly bounded in } L^2(0, T_e; L^2(\Omega, \mathbb{R}^N)), \quad (81)$$

$$\{\partial_t \sigma_m\}_m \text{ is uniformly bounded in } L^2(0, T_e; L^2(\Omega, \mathbb{R}^N)), \quad (82)$$

$$\{\text{Curl } B(\partial_t z_m)\}_m \text{ is uniformly bounded in } L^2(0, T_e; L^2(\Omega, \mathcal{M}^3)), \quad (83)$$

$$\left\{ \frac{1}{\sqrt{m}} \partial_t z_m \right\}_m \text{ is uniformly bounded in } L^2(0, T_e; L^2(\Omega, \mathbb{R}^N)), \quad (84)$$

$$\{Bz_m\}_m \text{ is uniformly bounded in } H^1(0, T_e; L^2_{\text{Curl}}(\Omega, \mathcal{M}^3)). \quad (85)$$

Existence of solutions. At the expense of extracting a subsequence, by estimates (57) - (65), (71) - (76) and (81) - (85) we obtain that the sequences in

(57) - (76), (81) - (85) converge with respect to weak and weak-star topologies in corresponding spaces, respectively. Next, we claim that weak limits of $\{\bar{z}_m\}_m$ and $\{z_m\}_m$ coincide. Indeed, using (57) this can be shown as follows

$$\begin{aligned} \|z_m - \bar{z}_m\|_{q^*, \Omega_{T_e}}^{q^*} &= \sum_{n=1}^m \int_{(n-1)h}^{nh} \left\| (z_m^n - z_m^{n-1}) \frac{t - nh}{h} \right\|_{q^*}^{q^*} dt \\ &= \frac{h^{2+1}}{2+1} \sum_{n=1}^m \left\| \frac{z_m^n - z_m^{n-1}}{h} \right\|_{q^*}^{q^*} = \frac{h^2}{2+1} \left\| \frac{dz_m}{dt} \right\|_{q^*, \Omega_{T_e}}^{q^*}, \end{aligned}$$

which implies that $\bar{z}_m - z_m$ converges strongly to 0 in $L^{q^*}(\Omega_{T_e}, \mathbb{R}^N)$. The proof of the fact that the difference $\bar{\sigma}_m - \sigma_m$ converges weakly to 0 in $L^2(\Omega_{T_e}, \mathcal{S}^3)$ can be performed as in [32, p. 210]. For the reader's convenience we reproduce here the reasoning from there. Let us choose some appropriate number $d \in \mathbb{N}$ and then fix any integer $n_0 \in [1, 2^d]$. Let $h_0 = T_e/2^{n_0}$. Consider functions $I_{[h_0(n_0-1), h_0 n_0]} v$ with $v \in L^2(\Omega, \mathcal{S}^3)$, where I_K denotes the indicator function of a set K . We note that, according to [32, Proposition 1.36], the linear combinations of all such functions are dense in $L^2(\Omega_{T_e}, \mathcal{S}^3)$. Then for any $h \leq h_0$ ⁵

$$\begin{aligned} (\sigma_m - \bar{\sigma}_m, I_{[h_0(n_0-1), h_0 n_0]} v)_{\Omega_{T_e}} &= \int_{h_0(n_0-1)}^{h_0 n_0} (\sigma_m(t) - \bar{\sigma}_m(t), v)_\Omega dt \\ &= \sum_{n=h_0(n_0-1)/h+1}^{h_0 n_0/h} \int_{(n-1)h}^{nh} \left((\sigma_m^n - \sigma_m^{n-1}) \frac{t - nh}{h}, v \right)_\Omega dt \\ &= -\frac{h}{2} \left(\sigma_m^{h_0 n_0/h} - \sigma_m^{h_0(n_0-1)/h}, v \right)_\Omega = -\frac{h}{2} (\bar{\sigma}_m(h_0 n_0) - \bar{\sigma}_m(h_0(n_0-1)), v)_\Omega. \end{aligned}$$

Employing (59) we get that $\bar{\sigma}_m - \sigma_m$ converges weakly to 0 in $L^2(\Omega_{T_e}, \mathcal{S}^3)$. Next, by (62) the sequence $\{z_m/m\}_m$ converges strongly to 0 in $L^2(\Omega_{T_e}, \mathbb{R}^N)$. Summarizing all observations made above we may conclude that the limit functions denoted by $\tilde{v}, \tilde{\sigma}, z$ and Σ^{lin} have the following properties

$$(\tilde{v}, \tilde{\sigma}) \in H^1(0, T_e; H_0^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3)),$$

$$\Sigma^{\text{lin}} \in L^q(\Omega_{T_e}, \mathbb{R}^N), \quad z \in W^{1, q^*}(0, T_e; L^{q^*}(\Omega, \mathbb{R}^N)),$$

$$Bz \in H^1(0, T_e; L_{\text{Curl}}^2(\Omega, \mathcal{M}^3)) \cap L^2(0, T_e; Z_{\text{Curl}}^2(\Omega, \mathcal{M}^3)).$$

Moreover, $Bz(x, t) \in \mathfrak{sl}(3)$ holds for a.e. $(x, t) \in \Omega_{T_e}$. Before passing to the weak limit, we note that the Rothe approximation functions satisfy the equations

$$-\text{div}_x \bar{\sigma}_m(x, t) = 0, \tag{86}$$

$$\sigma_m(x, t) = \mathbb{C}(\text{sym}(\nabla_x u_m(x, t) - Bz_m(x, t))) \tag{87}$$

$$+(\mathbb{C}[x] - \hat{\mathbb{C}})(\hat{\mathbb{C}})^{-1} \hat{\sigma}_m(x),$$

$$\partial_t z_m(x, t) \in g(\bar{\Sigma}_m^{\text{lin}}(x, t)), \tag{88}$$

⁵We recall that h is chosen to be equal to $T_e/2^m$ for some $m \in \mathbb{N}$.

together with the initial and boundary conditions

$$z_m(x, 0) = 0, \quad x \in \Omega, \quad (89)$$

$$Bz_m(x, t) \times n(x) = 0, \quad x \in \partial\Omega, \quad (90)$$

$$u_m(x, t) = 0, \quad x \in \partial\Omega. \quad (91)$$

Passing to the weak limit in (86), (87) and (89) - (91) we obtain that the limit functions $\tilde{v}, \tilde{\sigma}, z$ satisfy equations (33), (34) and (36) - (38). To show that the limit functions satisfy also (35) we proceed as follows:

As above, the system (86) - (91) can be rewritten as

$$\begin{aligned} & \int_0^{T_e} \int_{\Omega} (g^{-1}(\partial_t z_m(x, t)) \cdot \partial_t z_m(x, t)) \, dx dt = - \left(\frac{d\sigma_m}{dt}, \mathbb{C}^{-1} \bar{\sigma}_m \right)_{\Omega_{T_e}} \\ & - \left(\frac{dz_m}{dt}, L \bar{z}_m \right)_{\Omega_{T_e}} - \frac{1}{m} \left(\frac{dz_m}{dt}, \bar{z}_m \right)_{\Omega_{T_e}} \\ & - C_1 \left(\frac{dz_m}{dt}, B^T \text{Curl Curl } B \bar{z}_m \right)_{\Omega_{T_e}} + (B^T \hat{\sigma}_m, \partial_t z_m)_{\Omega_{T_e}} + (\bar{\mathbb{C}} \bar{\sigma}_m, \partial_t \hat{\sigma}_m)_{\Omega_{T_e}}. \end{aligned} \quad (92)$$

Due to (81) - (85) we can pass to the weak limit inferior in (92) to get the following inequality

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \int_0^{T_e} \int_{\Omega} (g^{-1}(\partial_t z_m(x, t)) \cdot \partial_t z_m(x, t)) \, dx dt \\ & \leq (\partial_t z, B^T (\tilde{\sigma} + \hat{\sigma}) - Lz - B^T \text{Curl Curl } Bz)_{\Omega_{T_e}}. \end{aligned} \quad (93)$$

Let \mathcal{G} denote the canonical extension of g . Then (93) reads as follows

$$\limsup_{m \rightarrow \infty} (\mathcal{G}^{-1}(\partial_t z_m), \partial_t z_m)_{\Omega_{T_e}} \leq (\partial_t z, B^T (\tilde{\sigma} + \hat{\sigma}) - Lz - B^T \text{Curl Curl } Bz)_{\Omega_{T_e}}. \quad (94)$$

Since \mathcal{G}^{-1} is pseudo-monotone, inequality (94) yields that for a.e. $(x, t) \in \Omega_{T_e}$

$$\partial_t z(x, t) \in g(B^T (\tilde{\sigma}(x, t) + \hat{\sigma}(x, t)) - Lz(x, t) - B^T \text{Curl Curl } Bz(x, t)).$$

Therefore, we conclude that the limit functions $\tilde{v}, \tilde{\sigma}, z$ and Σ^{lin} satisfy equations (33) - (38) and the existence of strong solutions is herewith established.

This completes the proof of Theorem 3.5. \square

4 Uniqueness of strong solutions

In this section we present the uniqueness result for (12) - (17) with a function g satisfying the self-controlling condition (21). A function g having the property (21) is automatically single-valued. Having noticed this we obtain the following uniqueness result.

Theorem 4.1. *Let all assumptions of Theorem 3.5 be satisfied. Then the solution (u, σ, z) of the initial boundary value problem (12) - (17) is unique.*

Proof. Let (u_1, σ_1, z_1) and (u_2, σ_2, z_2) be two solutions of the initial boundary value problem (12) - (17). Next, we argue similarly as in the proof of Theorem 3.5: The monotonicity of g implies that

$$(\partial_t z_1(x, t) - \partial_t z_2(x, t), \Sigma_1^{\text{lin}}(x, t) - \Sigma_2^{\text{lin}}(x, t)) \geq 0$$

for a.e. $(x, t) \in \Omega_{T_e}$, where

$$\Sigma_i^{\text{lin}} = B^T \sigma_i - L z_i - B^T \text{Curl Curl}(B z_i), \quad i = 1, 2.$$

Integrating the last inequality over Ω_t with $t \in (0, T_e)$ and using the equations (12) and (13) we get the following estimate for the difference of the solutions (here $\mathbb{B} := \mathbb{C}^{-1}$)

$$0 \geq \|\text{Curl } p_1(t) - \text{Curl } p_2(t)\|_2^2 + \|\mathbb{B}^{1/2}(\sigma_1(t) - \sigma_2(t))\|_2^2 + \|L^{1/2}(z_1(t) - z_2(t))\|_2^2, \quad (95)$$

which holds for a.e. $t \in (0, T_e)$. The estimate (95) together with the condition (18) imply that $\Sigma_1^{\text{lin}}(x, t) = \Sigma_2^{\text{lin}}(x, t)$ for a.e. $(x, t) \in \Omega_{T_e}$. Since the function g is single-valued and (u_i, σ_i, z_i) , $i = 1, 2$, are the solutions of the problem (12) - (17), the last identity yields that $p_1(x, t) = p_2(x, t)$ for a.e. $(x, t) \in \Omega_{T_e}$. This completes the proof of the theorem. \square

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