ON THE DECOMPOSITION OF A 2D-COMPLEX GERM WITH NON-ISOLATED SINGULARITIES

N. C. COMBE

ABSTRACT. The decomposition of a two dimensional complex germ with non-isolated singularity, into semi-algebraic sets is given. This decomposition consists of four classes: Riemannian cones defined over a Seifert fibered manifold, a topological cone over thickened tori endowed with Cheeger-Nagase metric, a topological cone over mapping torus endowed with Hsiang-Pati metric and a topological cone over the tubular neighbourhoods of the link's singularities. In this decomposition there exist semi-algebraic sets that are metrically conical over the manifolds constituting the link. The germ is reconstituted up to bi-Lipschitz equivalence to a model describing its geometric behavior.

1. INTRODUCTION

The purpose of this paper is to study the structure of a complex 2-dimensional surface germ at the origin with one-dimensional singular locus $(\Sigma, 0)$. Our main result gives a splitting of $(X, 0)$ in finite non-overlapping parts, except on the boundary. Let $(X, 0)$ be a germ of surface singularities and $\mathcal{I} : (X, 0) \to (\mathbb{C}^3, 0)$ an embedding. The embeding determines two metrics. The so called "outer metric", which is the induced metric from the hermitian metric on $\mathbb{C}^3$

$$d_{\text{out}}(x_1, x_2) = |\mathcal{I}(x_1) - \mathcal{I}(x_2)|,$$

and the "inner metric" which is the arc length metric

$$d_{\text{in}}(x_1, x_2) = \inf_{\gamma \in \Gamma} \ell(\mathcal{I} \circ \gamma), \quad \Gamma = \{\gamma | \gamma : [0, 1] \to X, \ \gamma(0) = x_1, \gamma(1) = x_2\},$$

where $\ell(\mathcal{I} \circ \gamma)$ is the length of the arc in $\mathbb{C}^3$.

Let the surface germ $(X, 0)$ be defined by a complex analytic germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ and let $\mathcal{B}_\epsilon$ be a ball, for the induced metric, of small radius ϵ centered at the origin. Let $\mathcal{C}(X \cap S_\epsilon)$ be a real cone over the boundary of the ball intersected with the germ, where $S_\epsilon = \partial \mathcal{B}_\epsilon$ is the 5-sphere.

In the case of isolated singularities J. Milnor showed a conical structure lemma [20]. The lemma states that there exists a homeomorphism between the intersection $X \cap \mathcal{B}_\epsilon$ and a real cone $\mathcal{C}(X \cap S_\epsilon)$. Then the Milnor conical structure lemma states a topological equivalence between the two pairs $(\mathcal{B}_\epsilon, (X \cap \mathcal{B}_\epsilon)$ and $(\mathcal{C}(S_\epsilon), \mathcal{C}(X) \cap S_\epsilon))$.

In 1972 D. Burghelea and A. Verona extended J. Milnor's conical structure lemma to the case of non-isolated singularities [7] using the Whitney pre-stratification of a germ [17, 29].

Lemma 1 (Conic structure lemma)**.** *Let M be a complex three dimensional manifold, $X \subset M$ be a closed set and suppose that $\mathfrak{S}$ is a Whitney pre-stratification of X. Assume the strata of $\mathfrak{S}$ are connected. Consider a Riemannian metric on M and let d be the induced distance function. There exists a sufficiently small ϵ_0, $0 < \epsilon_0 \ll 1$ such that for any ϵ, $0 < \epsilon \leq \epsilon_0$, the pair $(\mathcal{B}_\epsilon, X \cap \mathcal{B}_\epsilon)$ is homeomorphic to the real cone on the pair $(S_\epsilon, X \cap S_\epsilon)$.*

The proof of this lemme used the Whitney pre-stratification of a germ [17, 29], and the fact that any analytic set admits a Whitney pre-stratification. Let us recall the notion of the Whitney pre-stratification.

Let M be a n-dimensional C^μ-manifold without boundary and let V_1, V_2 be a pair of C^μ-sub-manifolds of M, such that for each $p \in V_i$ there exists a coordinates chart (φ_i, U_i) of class C^μ such that $p \in U_i$ and $\varphi_i(V_i \cap U_i) = \mathbb{R}^k \cap \varphi_i(U_i)$ for a suitable coordinate subspace $\mathbb{R}^k \subseteq \mathbb{R}^n$.

Whitney condition [19, 29]. Let $r = dim V_1$ and let $y \in V_2$. The pair (V_1, V_2) satisfies the Whitney condition at $y \in V_2$ if given any sequence $\{x_i\}$ of point of V_1 such that $x_i \to y$ and $\{y_i\}$ a sequence of points in V_2, also converging to y. Suppose $TV_{1_{x_i}}$ converges to some r-plane $\tau \subset \mathbb{R}^n$ and that $x_i \neq y_i$ for all i and the secant $\overline{(x_i y_i)}$ to some $\ell \subset \mathbb{R}^n$. Then $\ell \subset \tau$.

The secant $\overline{(xy)}$ refers to the line in $\mathbb{R}^n$ which is parallel to the line joining x and $y \neq x$, and passes through the origin.

Definition 1 (Whitney pre-stratification). *Let M be a C^1 manifold and $X \subset M$ be locally closed. A cover $\mathfrak{S}$ of X by pairwise disjoint C^1 sub-manifolds is called a Whitney pre-stratification of X if and only if :*

(1) *It is locally finite.*
(2) *If $V \in \mathfrak{S}$ then $(\overline{V} - V) \cap X$ is a union of strata.*
(3) *$V_i, V_j \in \mathfrak{S}$ satisfies the Whitney condition.*

Let Σ be the singular locus and we define by induction $\Sigma^j = \Sigma(\Sigma^{j-1})$ be the j-th singular locus of the (j-1)-th-singular locus. The complex algebraic surfaces X defined by a polynomial f have the following stratification :

$$\mathcal{V}_1 := X - \Sigma^1, \mathcal{V}_2 := \Sigma^1 - \Sigma^2, \ldots, \mathcal{V}_j := \Sigma^{j-1} - \Sigma^j, \ldots, \mathcal{V}_n := \Sigma^{n-1} - \Sigma^n.$$

A natural question in the study of the singularities, is to determine whether the homeomorphism defined in the conical structure lemma, $\varphi : \mathcal{M} = X \cap \mathcal{B}_\epsilon \to \mathcal{N} = \mathcal{C}(X \cap S_\epsilon)$, is bi-Lipschitz, i.e. :

$$\frac{1}{K} d_\mathcal{M}(x_1, x_2) \leq d_\mathcal{N}(\varphi(x_1), \varphi(x_2) \leq K d_\mathcal{M}(x_1, x_2), \ K \geq 1.$$

The condition which assures isomorphisms of metric spaces. In most of the cases the answer to this question is negative. For weighted homogeneous isolated singularities only A_1 and D_4 singularities have a bi-Lipschitz homeomorphism [1].

However, for isolated singularities it is possible to decompose a normal complex surface germ into a union of germs of pure dimension 4. These germs are partitioned into two distinct families $(\boldsymbol{\tau}, \tau)$ [3, 4], in a similar way of the decomposition introduced by Margulis [18] for symmetric space and extended by Thurston [27] and Gromov [10].

The so-called thick part $\boldsymbol{\tau}$ is such that the cone on its link $\mathcal{C}(\boldsymbol{\tau} \cap S_\epsilon)$ is bi-Lipschitz equivalent to $\boldsymbol{\tau} \cap \mathcal{B}_\epsilon$ for the inner metric on X.

The thin part τ of pure dimension 4 has a tangent cone of dimension strictly less than 4. Components of the thin part depend on the topology of their link. Non-trivial closed curves in the thin components have bounded lengths [9, 25]. As the curves move to zero in the thin parts, their length reduces polynomially, while, for thick parts the length of closed curves reduces linearly. In the case of singular surfaces of the type $h(z_3) = g(z_1, z_2)$, the classical tool to investigate the isolated singularity is to use the carrousel [14, 15, 16] and the Puiseux data [16, 23]. This Puiseux data has an incidence on the exponents defined in the metrics of the thin and thick parts. We will briefly recall this procedure: let $\pi : (z_1, z_2, z_3) \to (z_1, z_2)$ be a generic projection map and $\Gamma = \left\{ \{h(z_3) = g(z_1, z_2)\} \cap \left\{ \frac{\partial h}{\partial z_3} = 0 \right\} \right\}$ a curve that contains the singular locus of X. The discriminant locus is $\pi(\Gamma) = \Delta$ in the plane $\mathbb{C}^2$. The following definition and proposition are devoted to define the genericity of maps.

Definition 2. *Let $(X, 0) \subset (\mathbb{C}^n, 0)$, with $dim(X) = p$. Consider the plane in the set of Grassmannian $H \in \mathfrak{G}(n - p, n)$ such that 0 is an isolated point of the intersection between H and X.*

Then there exists an open set U of 0 in $\mathbb{C}^n$ such that $X \cap H \cap U = \{0\}$ and such that the projection $\pi : X \cap U \to U' \subset H^\perp$ along H is k-sheeted, $k \in \mathbb{N}$.

Proposition 1. *Let $f : (\mathbb{C}^3, 0) \to \mathbb{C}$ be a complex germ and let $H \in \mathfrak{G}(1,3)$. The multiplicity of $\pi_H|_X$ is equal to $\mathrm{ord}_0(f)$ if and only if $H \cap \mathcal{C}_3(X,0) = \{0\}$ where $\mathcal{C}_3(X,0)$ is the Zariski tangent cone.*

By a suitable change of coordinates we choose to have transversality of $z_1 = 0$ with the curves of Δ in $\mathbb{C}^2$. Denote the equation of Δ by $q(z_1, z_2) = \prod_{i \geq 1} q_i(z_1, z_2)$.

Let $q_i(z_1, z_2)$ be an irreducible component of $q(z_1, z_2)$. We consider the Puiseux developpement of $q_i^{-1}(0)$ given by:

$$\Phi_p^i(z) = \sum_{k \geq 1} a_k^i z^{\eta_k}, \ a_k^i \in \mathbb{C}, \ \eta_k \in \mathbb{Q}.$$

Let $v(q_i)$ be the valuation of q_i.

  - If $q_i = 0$ then $v(q_i) = -\infty$.
  - If $\forall k, a_k^i \neq 0, k \in \mathbb{N}^*$ then q_i does not have Puiseux pairs.
  - If the precedent cases do not hold, let $r_1 = \inf\{\eta_k | a_k^i \neq 0, \eta_k \in \mathbb{Q} \setminus \mathbb{N}\}$. The number $r_1 = \frac{m_1}{n_1}$, with $gcd(m_1, n_1) = 1$ defines the first Puiseux pair: (m_1, n_1).
  - Let $r_{j+1} = \frac{m_{j+1}}{n_1 \cdots n_{j+1}}$ with $gcd(m_{j+1}, n_{j+1}) = 1$, $n_{j+1} > 1$. The pair (m_{j+1}, n_{j+1}) is the $(j+1)$-Puiseux pair of q_i.
  - The Puiseux characteristic exponents of q_i are the rational numbers $r_j = \frac{m_j}{n_1 \cdots n_j}$ where $1 < r_j < r_{j+1}$ and $j \in \{1, 2, ..., s\}$.

The Puiseux characteristic data of q is the union of sets constituting the Puiseux characteristic data of each irreducible component q_i of q.

The Puiseux exponents allows to describe a Waldhausen splitting of the Milnor fiber by a carousel diagram. Since we have Seifert-fibered spaces, we can have a decomposition as in Theorem 9.1.3 in [28]. Therefore, for a 2-dimensional compact oriented manifold Y, given a diffeomorphism h on Y, one can define h as the monodromy of a fibration over a circle. To define a 3-manifold fibered over a circle with monodromy h, we define the following mapping torus.

Definition 3 (Mapping torus)**.** *By mapping torus MY over the manifold Y we mean the quotient topological space :*

(1)
$$MY := ([0,1] \times Y)/\mathcal{R},$$

such that

  (1) *$h : Y \to Y$ is an orientation preserving homeomorphism*
  (2) *$\mathcal{R}$ is the equivalence relation given by : $(1, x) \sim (0, h(x))$.*

In a case of germs with non-isolated singularities, the thick/thin decomposition presents some difficulties. The difficulties are due to presence of singularities on the link $\mathcal{L}$ and, contrarily to the case of isolated singularities, the link $\mathcal{L}$ is not diffeomorphic to the boundary ∂F of its Milnor fiber. In the case of non-isolated singularities we have follow two steps.

  (1) Firstly, it is necessary to make a normalization of the "non-normal" algebraic variety X. The normalization map $n : \overline{X} \to X$ allows to introduce the normal algebraic variety $\overline{X}$, which has at most an isolated singularity. If $\overline{X}$ has an isolated singularity, we assume, without loss of generality, that the isolated singularity is at 0.
  (2) Secondly, we perform a good resolution $\rho : \widetilde{X} \to \overline{X}$ if $\overline{X}$ has an isolated singularity. By good resolution we mean that exceptional divisor $\rho^{-1}(0) = \cup_j E_j$, where $E_j \cong \mathbb{P}^1$ are the irreducible components and $\rho^{-1}(0)$ has normal crossings. We assume that the proper transform of the singular locus Σ of X intersects $\cup_i E_i$ transversely, outside the intersections of the irreducible components E_i.

Nevertheless, there exists some analogy between the boundary of the Milnor fiber ∂F and the link $\mathcal{L}$. Similarly to the decomposition proposed by Siersma [26] of the boundary of the Milnor fiber into two parts: $\partial F = \partial F_1 \cup \partial F_2$, one can decompose the link $\mathcal{L} = X \cap S_\epsilon$ into two sets $\mathcal{L}_1, \mathcal{L}_2$ [23],

$$\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2. \tag{2}$$

The first set, $\mathcal{L}_1$ is the link without the interior of the tubular neighbourhoods of the singularities of the link. The second set, $\mathcal{L}_2$ is the intersection of these tubular neighbourhoods with the link $\mathcal{L}$. The set $\mathcal{L}_1$ is equivalent to the boundary ∂F_1, in Siersma's decomposition of the Milnor fiber. The existence of a plumbed graph manifold for $\mathcal{L}_1$ is obtained by the Nemethi-Szilard algorithm [23] for the boundray ∂F_1 and, hence, by equivalence between ∂F_1 and $\mathcal{L}_1$, for $\mathcal{L}_1$. To precise the set $\mathcal{L}_2$, we consider the decomposition of the singular locus of X in irreducible components Σ_k:

$$\Sigma = \bigcup_{k=1}^{n} \Sigma_n. \tag{3}$$

Hence the singular locus of the link $\mathcal{L}$ is

$$\boldsymbol{\sigma} = \bigcup_{k=1}^{n} \Sigma_k \cap \mathcal{L}. \tag{4}$$

Let us consider an irreducible component $\boldsymbol{\sigma}_k = \Sigma_k \cap \mathcal{L}$ and denote the tubular neighbourhood of $\boldsymbol{\sigma}_k$ by $\mathcal{T}^k$. Observe that one can consider the tubular neighbourhood $\mathcal{T}^k$ as a total space with base space $\boldsymbol{\sigma}_k$ of a fibration. The fiber will be determined by so-called transversal germs of Σ_k defined at a point $p \in \Sigma_k$ with the condition that $p \neq 0$. The transversal germ at each non singular point $p \in \Sigma_k$, is obtained by intersecting $(X,0)$ by a linear space $\mathfrak{H}$, transverse to Σ_k. The dimension of this linear space is : $dim(\mathfrak{H}) = codim(\Sigma_k)$. This transversal germ will be denoted by: $(\mathfrak{H}, p)$. To define the fiber in the total space $\mathcal{T}^k$, one has to determine the Milnor fiber of f restricted to the transversal slice $(\mathfrak{H}, p)$, i.e: $\phi_{\epsilon,\delta} : B_\epsilon \cap f^{-1}|_{(\mathfrak{H},p)}(\partial B_\delta) \to \partial B_\delta$, for $\delta < \epsilon \ll 1$. This Milnor fiber does not depend on the point p neither on the radius δ. This Milnor fiber will be denoted by M_k. Finally, the fibration: $\mathcal{T}^k \to \boldsymbol{\sigma}_k$ has, up to homeomorphism equivalence, a cone over ∂M_k as fiber.

The aim is to study, from a geometrical and topological point of view, the rate at which systoles, contained in the link $\mathcal{L}$ converge to 0 in $X \cap \mathcal{B}_\epsilon$. The answer to this question leads to a splitting of the neighbourhood of the origine: $X \cap \mathcal{B}_\epsilon$. The subspaces in $X \cap \mathcal{B}_\epsilon$ are Riemannian manifolds with metrics determining the rate of convergence of systoles. These Riemannian manifolds have, topologically, the aspect of topological cones glued together by isometry. The proof will rely on the conical structure lemma.

2. THE DECOMPOSITION MODEL OF $(X, 0)$

In this section, the singular spaces $(X, 0)$ will be described in terms of a model. This model provides a decomposition of $X \cap \mathcal{B}_\epsilon$ and enables to study the rates at which systoles in the boundary $X \cap \mathcal{B}_\epsilon$ converge to zero. These Riemannian manifolds are topological cones endowed with Riemannian metrics.

Theorem 1 (Splitting thorem)**.** *Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be a two dimensional complex germ defined by the form $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ with one-dimensional singular locus Σ. Let us assume that $0 \in \Sigma$. Then there exists a splitting of $X \cap \mathcal{B}_\epsilon$ into a finite family of compact and connected semi-algebraic sets A_i:*

$$X \cap \mathcal{B}_\epsilon = \bigcup_{i=1}^{k} A_i. \tag{5}$$

Each of these semi-algebraic sets is one of the following type:

   (1) *Riemannian cones defined over a Seifert fibered manifold with $0 < t \leq \epsilon$.*

(2) *Topological cones over thickened tori endowed with a Nagase metric.*
(3) *Topological cones over mapping torus endowed with a Hsiang-Pati metric.*
(4) *Topological cones over the tubular neighbourhoods of the link's singularities*

Corollary 1 (Semi-algebraic partition). *Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be a two dimensional complex germ with a one-dimensional singular locus Σ, with $0 \in \Sigma$. There exists a decomposition of $(X, 0)$ into a finite family of semi-algebraic sets, which can be:*

(1) *Semi-algebraic germs which are a union of metrically conical sets, subsets in $X \cap \mathcal{B}_\epsilon$*
(2) *Semi-algebraic germs of pure dimension 4 which have a tangent cone of dimension strictly less than 4.*

Remark 1. *This splitting of which we show the existence, holds for any complex analytic surface germ with non-isolated singularities of complex dimension one. Somehow, if the surface is of type $f(z_1, z_2, z_3) = h(z_3) - g(z_1, z_2) = 0$, where h is a polynomial function in z_3 and g a polynomial function in variables z_1, z_2, then the Puiseux exponents in the carousel decomposition play a role in the determination of the rates in the metrics.*

Remark 2. *Let us come back to the notion of metrically conical. For semi-algebraic set $\mathcal{J}$ in the germ $(X, 0)$, consider a closed curve γ in $\mathcal{J}$ which represents a non-trivial element of $\pi_1(\mathcal{J} - \{0\})$. Define the action on the semi-algebraic set $\mathcal{J}$:*

$$\mathbb{R}_+^* \times \mathcal{J} \to \mathcal{J} : (t, \gamma) \longmapsto t\gamma,$$

restricted to $0 \leq t \leq \epsilon$, with $\epsilon > 0$ is small.

Let h be a homeomorphism between $\mathcal{J} \cap \mathcal{B}_\epsilon$ and the cone over the link $\mathcal{C}(\mathcal{J} \cap \mathcal{S}_\epsilon)$. Suppose that the length of the curve γ is of order t^ν, $\nu \geq 1$. If h is a bi-Lipschitz map, then $h(t\gamma)$ has also length of order t^ν. On the other side the $\mathbb{R}_+^*$ action on the cone induces a bound, say l, on the length of the loop $h(t\gamma)$. Since the distance given by the bi-Lipschitz constant is $1/K$, therefore $l \cdot h(t\gamma)$ is never closer to 0 than $1/K$. So, if this condition is not fulfilled, the map is not bi-Lipschitz and thus the semi-algebraic set is not metrically conical.

A topological cone $\mathbb{R}_+^* \times \mathfrak{M}$ over a Riemannian manifold $(\mathfrak{M}, g)$ equipped with the metric :

(6) $$g_c = dt^2 + t^2 g,$$

is said metrically conical.

The metrically conical set we are interested in is a cone over the link $\mathfrak{M} = X \cap \mathcal{S}_\epsilon$ with induced distance function bounded by $\epsilon > 0$. A germ which is bi-Lipschitz equivalent to the cone over its link is called metrically conical.

Non-metrically conical algebraic sets have the following property:

Proposition 2. *Let Ξ be the intersection of the non-metrically conical semi-algebraic set with the sphere $\mathcal{S}_\epsilon$. Then, the fiber of $\Xi \to S^1$ has diameter of order $\alpha\epsilon^\nu$, where $\alpha > 0$ is a constant, $\epsilon_0 \geq \epsilon$ and $\nu > 1$ is a rationnal number.*

Corollary 2. *Let A_i be a non-metrically conical semi-algebraic set, and let $0 < t \leq \epsilon$. If there exists a systole in the link of A_i, that is in Ξ, then this systole converges with order $O(t^\nu)$ to 0, as t tends to 0.*

Corollary 3. *Let $A_i \subset \mathbb{R} \times \mathbb{R}^n$ and $A_{i,t} := A \cap (\{t\} \times \mathbb{R}^4)$. For each non-metrically conical set A_i there exists a Gromov-Hausdorff limit $\lim_{t \to 0}(A_t, d_{in}(A_t))$ where d_{in} is the arc length metric.*

Recall that a systole of a compact metric space M is a metric invariant of M, defined to be the least length of a free loops representing nontrivial conjugacy classes in the fundamental group of M.

The proof follows from Bernig-Lytchak's Theorem 1 in [2]. Let $X \subset \mathbb{R} \times \mathbb{R}^n$ be a 1-parameter family of subsets of $\mathbb{R}^4$. Suppose that each fiber $X_t := X \cap (\{t\} \times \mathbb{R}^n)$ is connected. Then the Gromov-Hausdorff limit $\lim_{t \to 0}(X_t, d_{in}(X_t))$ exists.

3. Proof of the decomposition theorem

3.1. **Idea of proof.** To show that $X \cap \mathcal{B}_\epsilon$ splits into a finite family of semi-algebraic sets which can be classified along four types, we use the existence [7] of a homeomorphism $h : X \cap \mathcal{B}_\epsilon \to \mathcal{C}(X \cap \mathcal{S}_\epsilon)$. In the following part, we are interested in a small neighbourhood of 0 in X. For the sake of simplicity, we denote by X the ϵ-neighbourhood instead of $X \cap \mathcal{S}_\epsilon$). Let us give the draft of the proof:

(1) As was said before, we have to normalize X and solve the singularity of the normalized surface $\overline{X}$ (if there exists an isolated singularity).

(2) Construct a well chosen tubular neighbourhood of the exceptional divisor $\rho^{-1}(0) = \cup_j E_j$, where E_j are the irreducible components and $\rho^{-1}(0)$ has normal crossings.

(3) Apply Nagase and Hsiang-Pati procedure, on the tubular neighbourhood of the irreducible components of $\rho^{-1}(0)$:

(a) On a neighbourhood of the irreducible components E_i, without the intersection points, following [12], there exists a Hsiang-Pati metric. For each neighbourhood of E_i, up to bi-Lipschitz, one can investigate the Hsiang-Pati metric on the manifold $(0, 1] \times [0, 1] \times Y$, where Y is a compact polygon in $\mathbb{R}^2$ with $\tilde{g}$ a standard metric:

(7) $$g_{HP} = dt^2 + t^2 d\theta^2 + t^{2\nu} \tilde{g}(y), \text{ where } \nu \geq 1, \ (t, \theta, y) \in (0, 1] \times [0, 1] \times Y.$$

(b) Concerning the neighbourhood of the intersection points of the E_i, there exists a Cheeger-Nagase metric on $(0, 1] \times [0, 1]^3$:

(8) $$g_{CN} = dt^2 + t^2 d\theta^2 + t^{2\nu}(ds^2 + h^2(r, s)d\Theta^2), \ (t, \theta, s, \Theta) \in (0, 1] \times [0, 1]^3,$$

where $h(r, s)$ is the product of smooth functions [21].

(4) Finally, it will be necessary to go back to the link of X, so as to characterize the splitting. The procedure allows to recover the topological cone over the link $\mathcal{L}$ of X as a topological cone on the quotient of the normalization's link $\overline{\mathcal{L}}$ of $\overline{X}$ by some equivalence relation.

3.2. **Local metrics in the components of $\overline{X}$.** In this section, first we use a plumbing process to choose a tubular neighbourhood of the exceptional divisor in $\widetilde{X}$. An introduction of a continuous function, defined on the neighbourhood of $\rho^{-1}(0)$, allows to construct metrics.

Consider the normal surface $\overline{X}$ with isolated singularity at 0. Take into account that, for a surface with isolated singularity, the boundary of its Milnor fiber is diffeomorphic to its link $\overline{\mathcal{L}}$. A description of the boundary of the Milnor fiber can be obtained taking the boundary of the tubular neighbourhood of the exceptional divisor $\rho^{-1}(0) = \cup_j E_j \subset \widetilde{X}$ of $\overline{X}$. More precisely, this tubular neighbourhood is the boundary of a 4-manifold, obtained by plumbing together the tubular neighbourhoods of each irreducible component E_j [6, 22, 24].

Take the tubular neighbourhood N_i of each irreducible components $E_i \cong \mathbb{P}^1$. This tubular neighbourhood is diffeomorphic to a disk-bundle $(\mathfrak{E}_i, \mathfrak{F}_i, \mathfrak{p}, D^2)$ where $\mathfrak{F}_i$ is the base space, $\mathfrak{E}_i$ the oriented total space, $\mathfrak{p}$ the continuous surjection satisfying local triviality conditions. In addition, we associate to this quadruplet its Euler number e_i. Note that $\mathfrak{F}_i$ is a compact surface of genus $\mathfrak{g}$, with boundary components. For each pair (E_i, E_j) of intersecting irreducible components, it is necessary to plumb the respective disk bundles. This procedure gives a tubular neighbourhood of $\rho^{-1}(0)$. Let us recall the plumbing process. Consider two irreducible components E_i, E_j that intersect at a point q and consider their respective disk-bundles $(\mathfrak{E}_i, \mathfrak{F}_i, \mathfrak{p}, D^2)$ and $(\mathfrak{E}_j, \mathfrak{F}_j, \mathfrak{p}, D^2)$.

Let $\mu : \mathfrak{E}_i|_{D_i^2} \to \mathfrak{E}_j|_{D_j^2}$, where the disks $D_i^2 \subset \mathfrak{F}_i$ and $D_j^2 \subset \mathfrak{F}_j$, be a map between the restricted total spaces of these bundles.

Let us introduce two trivializations of the restricted bundles $f_i : D_i^2 \times D^2 \to \mathfrak{E}_i|_{D_i^2}$, $f_j : D_j^2 \times D^2 \to \mathfrak{E}_j|_{D_j^2}$. The way to obtain the plumbing of the two disk-bundles is to take the disjoint union of the total spaces $\mathfrak{E}_i$ and $\mathfrak{E}_j$ and to identify the images of f_i and f_j. This identification is attained by pasting the component $D_i^2 \times D^2$ of $\mathfrak{E}_i$ to the component $D_j^2 \times D^2$ of $\mathfrak{E}_j$ by the diffeomorphism $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) : D_i^2 \times D^2 \to D_j^2 \times D^2$. So, the map $\mu : \mathfrak{E}_i|_{D_i^2} \to \mathfrak{E}_j|_{D_j^2}$ is, in other words, defined

by the composition $\mu = f_j \circ J \circ f_i^{-1} : \mathfrak{E_i}|_{D_i^2} \to \mathfrak{E_j}|_{D_j^2}$. By this procedure, the disk in the base space and the fiber are exchanged. So, this identification, defines the plumbing. Applying this procedure to all the intersection points of the irreducible components, one has the construction of a neighbourhood of the exceptional divisor $\rho^{-1}(0)$. Thus, referring to [22, 24] one can describe the link in terms of graph manifold. In the crudest sense, a graph manifold is a set of elementary pieces homeomorphic to $S^1 \times D^2$ or to $S^1 \times (S^2 \setminus \sqcup_{i=1}^n D_i^2)$ that are glued together along some boundary components [5]. Moreover, it is worth stressing that to each graph manifold there exists a corresponding plumbing diagram of the plumbed manifold. This plumbing diagram is a weighted graph. The vertices correspond to the Seifert-fibered manifolds, the edges correspond to the gluing tori and the weights are given by the Euler numbers e_i.

The plumbed manifold is a four dimensional manifold with boundary. For an isolated singularity, the boundary of the plumbed manifold is the graph manifold and defines the topology of the link [22]. We will choose a closed neighbourhood of $\rho^{-1}(0) \subset \widetilde{X}$ such that its boundary is the boundary of the graph manifold. On this neighbourhood, we will define a function $\mathfrak{R}$ with range $[0, +\infty)$. This function will be the key to define the metrics

Definition 4 (Nagase [21]). *Let $\mathfrak{R}$ be a function defined on a neighbourhood $\mathcal{W} \subset \widetilde{X}$ of $\rho^{-1}(0) = \cup_j E_j$ and with range $[0, +\infty)$. We require that this function satisfies the following properties:*

(1) $\mathfrak{R}|_{\rho^{-1}(0)} = 0$
(2) $\mathfrak{R}|_{\mathcal{W} \setminus \rho^{-1}(0)}$ *is smooth and positive*
(3) $\mathcal{W} \setminus \rho^{-1}(0) = \mathfrak{R}^{-1}(0,1]$
(4) $\mathfrak{R}^{-1}(0, \epsilon] = (0, \epsilon] \times \mathfrak{R}^{-1}(\epsilon)$

The last property is defined using $\mathfrak{R}$ and appropriate flow lines in $\mathcal{W} \setminus \rho^{-1}(0)$. We split suitably $\mathfrak{R}^{-1}(\epsilon) = \cup_i W_i \cup_j V_j$ into finite parts, which are non-overlapping except on the boundaries. These parts parts will be classified into two types:

- the tubular neighbourhood of an irreducible component E_i, outside the intersection points of the irreducible components, denoted by W_i.
- the neighbourhood, at the intersection points of the irreducible components, denoted by V_j.

Using Property 4 of Definition 4, we split $\mathfrak{R}^{-1}(0, \epsilon] = \cup_i \mathfrak{W}_i \cup_j \mathfrak{V}_j$ into finite parts accordingly to the previous decomposition of $\mathfrak{R}^{-1}(\epsilon)$. The part $\mathfrak{W}_i$ corresponds, in the splitting of $\mathfrak{R}^{-1}(\epsilon)$, to the neighbourhood of the irreducible component E_i outside of the gluing tori. As for the part $\mathfrak{V}_j$, it corresponds to the gluing tori. There is a bi-Lipschitz map from each $\mathfrak{W}_j$ with induced metric to a model $(0, \epsilon] \times [0, 1] \times Y$, where Y is a compact polygon in $\mathbb{R}^2$. This polygon carries a Riemannian metric $\tilde{g}$. Moreover, this model has Hsiang-Pati metric (7). Whereas, each part $\mathfrak{V}_j$ with induced metric is bi-Lipschitz equivalent to $(0, \epsilon] \times [0, 1]^3$ with a Cheeger-Nagase metric (8). The exponents $\nu \geq 1$, in the metrics, correspond to the exponents in the divisor $\rho^{-1}(0)$.

In the graph manifold, each Seifert-fibered manifold has a well defined Riemannian metric. In the case where there exists a generic map $\pi : \mathbb{C}^3 \to \mathbb{C}^2$ defined over X, one can construct a carousel decomposition. The Seifert-fibered manifolds in the link have topological data, given by the Puiseux exponents, so it is the same for its metric. Finally, for $\mathfrak{W}_i$ the metric will depend on the Puiseux exponent, associated to W_j. For $\mathfrak{V}_i$, notice that, according to Lê's description of the Waldhausen decomposition, by means of the carousel [16], the $\mathfrak{V}_i$ need to be defined over thickened tori, i.e $\mathbb{T} \times I$ where $I = [a, b], b > a > 0$ is an interval in $\mathbb{R}$ and $\mathbb{T}$ is a torus. The real cone over the link $\overline{\mathcal{L}}$ results then, from gluing the boundary of the $\mathfrak{W}_j$ with the boundary of the neighbouring topological cones $\mathfrak{V}_i$ and by using isometry so as to keep consistency of the metrics. The order in which they are glued depends on the order in which the manifolds W_j and V_i are glued in the plumbing diagram.

A thickened torus is the Cartesian product of an annulus and a circle. Let $A := \{(r, \psi)|1 \leq r \leq 2, \ 0 \leq \psi \leq 2\pi\}$, be an Euclidean annulus in polar coordinates, this annulus can be equipped of a family of metric $g_{a,b} = (b - a)^2 dr^2 + ((r - 1)b + (2 - r)a)^2 d\psi^2$ and for $o < a < b < 1$ we can write

$a = t^{\nu'}$ and $b = t^{\nu}$ for some $t \in (0,1], \nu, \nu' > 1$. We denote this metric by:

$$(9) \qquad\qquad g^t_{\nu,\nu'} := (t^{\nu} - t^{\nu'})^2 dr^2 + ((r-1)t^{\nu} + (2-r)t^{\nu'})^2 d\psi^2.$$

Notice that $g^t_{\nu,\nu'}$ is isometric to the metric of an Euclidean annulus with inner radius $t^{\nu'}$ and outer radius t^{ν}. The metric $dt^2 + t^2 d\theta^2 + g^t_{\nu,\nu'}$ on $(0,1] \times S^1 \times A$ is bi-Lipschitz equivalent to a Cheeger-Nagase metric on $(0,1] \times [0,1]^3$,

$$
\begin{aligned}
(10) \qquad g_{HP} &= dt^2 + t^2 d\theta^2 + g^t_{\nu,\nu'}, \\
&= dt^2 + t^2 d\theta^2 + t^{2\nu}(ds^2 + (s + t^{\nu-\nu'})^2 d\psi^2), \quad s = (r-1)(1 - t^{\nu'-\nu}), \\
&= dt^2 + t^2 d\theta^2 + t^{2\nu}(ds^2 + h(t,s)^2 d\Theta^2), \quad h(t,s) = \frac{s + t^{\nu-\nu'}}{2\pi}, \; \Theta \in [0,1],
\end{aligned}
$$

where, we call $g_{HP} = dt^2 + t^2 d\theta^2 + g^t_{\nu,\nu'}$ the Cheeger-Nagase metric on $(0,1] \times S^1 \times A$.

Secondly, let us define the Hsiang-Pati metric on $\mathfrak{W}_j$. The mapping torus MY which was announced in the introduction, has a Riemannian metric g_θ.

Let $\theta \in [0, 2\pi]$, then for a small δ:

$$(11) \qquad\qquad g_\theta := \begin{cases} g_0, & \text{if } \theta \in [0, \delta] \\ h^* g_0 & \text{if } \theta \in [2\pi - \delta, 2\pi]. \end{cases}$$

Therefore, for any $t \in [0,1]$, the metric on $(0,1] \times MY$ is:

$$(12) \qquad\qquad g_N := dt^2 + t^2 d\theta^2 + t^{2\nu} g_\theta, \quad t \in (0,1].$$

The compactification of the metric structure $(0,1] \times MY$ is done by adding the singleton $\{t = 0\}$.

Finally, we define a metric cone . Let $(\mathfrak{M}, g)$ be a compact 3-manifold with a Riemannian metric g. A metric cone on $\mathfrak{M}$ can be obtained by defintion of a metric on $[0,1] \times \mathfrak{M}$. This metric is the completion of the metric (6) $g_c = dt^2 + t^2 g$ on $t \in (0,1] \times \mathfrak{M}$, by adding the singleton at $\{t = 0\}$.

Remark 3. *Consider the mapping torus MY over Y. Fix $t \in (0,1]$ and $\theta \in [0, 2\pi]$. Then the diameter of $\{t\} \times \{\theta\} \times Y$ is of order t^ν. The rational number ν characterizes the rate at which $\{t\} \times \{\theta\} \times Y$ shrinks as t tends to zero.*

3.3. **Local metrics on the subsets of X.** In this section we show that the metrics defined for $\bar{\mathcal{L}}$ hold for $\mathcal{L}$. We assume that we have a plumbed graph obtained by the algorithm introduced by Nemethi and Szilard in [23]. We will use the conditions showed in section 7.4 and Proposition 7.5.10, to show the splitting.

Proposition 3 (Nemethi-Szilard). *Let $\overline{\mathcal{L}}$ be the link of the normalized space. Then, there exists an equivalence relation $\sim$ such that: $\mathcal{L} = \overline{\mathcal{L}}/\sim$.*

A normalization of X is a normal surface $\overline{X}$ and a holomorphic map $n : \overline{X} \to X$ such that :

(1) $n : \overline{X} \to X$ is proper and has finite fibers.
(2) Let $\overline{\Sigma} = n^{-1}(\Sigma)$, then $\overline{X} - \overline{\Sigma}$ is dense in $\overline{X}$ and $n|_{\overline{X} - \overline{\Sigma}}$ is a bi-holomorphic map.

Recall that the normalized surface $\overline{X}$ is a disjoint union of irreducible components. So, the link $\overline{\mathcal{L}}$ of $\overline{X}$ is a disjoint union of the links of each of these irreducible components.

Using the normalization map, for every j, one can establish a correspondence between the strict transform of Σ_j and a finite set $\{1, ..., k\}$ of irreducible components in $\overline{X}$ which will be denoted $S^1_j, ..., S^k_j$.

Notice that property (1) of the normalization implies that the degree of the map $n : S^i_j \to \Sigma_j$ is d. On the other hand $\overline{S_{j,i}} := S^i_j \cap \overline{\mathcal{L}}$ is diffeomorphic to a circle. Therefore, for every $j \in \{1, 2, ..., n\}$, we define a degree d-map from $\overline{S_{j,i}}$ to a circle S^1_j:

$$(13) \qquad\qquad \mu_i : \overline{S_{j,i}} \to S^1_j \quad .$$

Now we can define the equivalence relation as:

$$(14) \qquad x \sim x' \iff \exists \ x \in \overline{S_{j,i}}, \ x' \in \overline{S_{j,l}} \quad \text{such that } \mu_i(x) = \mu_l(x'),$$

For more ample explanations of Proposition 3 we refer to Nemethi and Szilard [23].

Let us come back to the partition of $\overline{\mathcal{L}}$ into a union of manifolds W_i and V_j, $\overline{\mathcal{L}} = \cup_j W_j \cup_i V_i$. Recall that we have defined topological cones over each of the non-overlapping parts. These topological cones are labelled $\mathfrak{W}_i$ and $\mathfrak{V}_j$. The equivalence relation leaves invariant a subset of $\overline{\mathcal{L}}$. So, the topological cones $\mathfrak{W}_i, \mathfrak{V}_j$ that are defined over the manifolds of the link, invariant under the equivalence relation, are also defined in $\overline{X}$.

Recall, from introduction, that $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ where $\mathcal{L}_1 := \mathcal{L} \setminus \cup_{i=1}^n \overset{\circ}{\mathcal{T}}{}^i$ and $\overset{\circ}{\mathcal{T}}{}^i$ is the interior of the tubular neighbourhood. The second set $\mathcal{L}_2 = (\cup_{i=1}^n \overset{\circ}{\mathcal{T}}{}^i) \cap \mathcal{L}$ is the intersection of the tubular neighbourhoods $\overset{\circ}{\mathcal{T}}{}^i$ with the link $\mathcal{L}$. Thus we consider the cone $\mathcal{C}(\mathcal{L}) = \mathcal{C}(\mathcal{L}_1 \cup \mathcal{L}_2)$. As was already said, $\mathcal{T}^i$ is a total space over $L_i = \Sigma_i \cap \mathcal{S}_\epsilon$ with fiber a cone over ∂M_i, where M_i is the Milnor fiber of f restricted to $(\mathfrak{h}, p)$ for Σ_i. Moreover, the tubular neighbourhood $\mathcal{T}^i$ has the homotopy type of L_i.

Let us define the topological cone over the tubular neighbourhood of each singularity L_i of the link $\mathcal{L}$.

Note that from section 7.4 in Nemethi-Szilard [23], there exists a normal crossing divisor $\rho'^{-1}(\Sigma_i)$. Moreover, defining a small tubular neighbourhood V of the discriminant curve for each point $p \in \Sigma_i, p \neq 0$, the resolution ρ' restricted to $\rho'^{-1}(V \cap (\{x\} \times \mathbb{C}^2))$ is an embedded resolution of the plane curve germ, at p.

Assume that the proper transform of Σ_i, intersects the union of irreducible components $\cup_j E_j$ transversally and not in the intersection points of the irreducible components. In order to have the link $\mathcal{L}$, it will be necessary to apply the equivalence relation in Nemethi-Szilard's proposition and glue together the circles $\overline{S_{i,j}}$ and $\overline{S_{i,l}}$, when the condition $\mu_i(\overline{S_{i,j}}) = \mu_l(\overline{S_{i,l}})$ is fulfilled. The components $\overline{S_{i,j}}$ in the dual graph defined in section 7.4 of [23], are situated on the irreducible components E_r, outside the intersection points of the irreducible components, in the resolution. So, $\overline{S_{i,j}}$ is on a Riemannian manifold with Hsiang-Pati metric. Thus, the metric of the topological cone over the tubular neighbourhood $\mathcal{T}^i$ is of Hsiang-Pati type, with exponent ν depending on the exponent of the exceptional divisor. Indeed, consider the resolution of the normalized space $\overline{X}$ and consider the irreducible component which contains the pre-images of the strict transform of Σ_i. In order to define a type of Hsiang-Pati metric, take certain appropriate flow lines in the neighbourhood of $\overline{S_{i,k}}$. Proceed to the gluing of the circles $\overline{S_{i,k}}$ and $\overline{S_{i,l}}$ that have same image under μ. Then, outside S_i^1 the type of Hsiang-Pati metric will be conserved on the topological cone defined over the tubular neighbourhood $\mathcal{T}^i$ in $\mathcal{L}$. The weight in the Hsiang-Pati metric of the topological cone defined over $\mathcal{T}^i$ is determined by the exponents in the resolution.

The boundary of this topological cone is glued to a boundary of the neighbouring Riemannian manifold: a topological cone over a thickened torus. Indeed, the neighbouring manifolds of the tubular neighbourhood $\mathcal{T}^i$ are gluing tori. Let $(\mathcal{C}_{top}(S^1 \times A), g_A)$ and $(\mathcal{C}_{top}(\mathcal{T}^i), g_{\mathcal{T}^i})$ be Riemannian manifolds where $\mathcal{C}_{top}$ denotes a topological cone. Let $f : \partial C_{top}(\mathcal{T}^i) \to \partial \mathcal{C}_{top}(S^1 \times A)$ be a differentiable mapping and $\mathcal{C}_{top}(\mathcal{T}^i)$ be a Riemannian manifold with metric tensor $g_{\mathcal{T}^i}$. The reciprocal image $f^* g_A$ of g_A is a 2-covariant symmetric tensor on $\partial \mathcal{C}_{top}(\mathcal{T}^i)$ defined by:

$$(15) \qquad g_{\mathcal{T}^i} = (f^* g_A)_x(v, w) = (g_A)_y(f'v, f'w) \ \forall v, w \in T_x \partial \mathcal{C}_{top}(\mathcal{T}^i).$$

3.4. The exponents in the metrics for $h(z_3) = g(z_1, z_2)$ surfaces.

In this section we will use the well known method of carousel. This method will be applied to the case of the normalized surface $\overline{X}$. We give an outline of the method.

Let Δ be the discriminant curve in $\mathbb{C}^2$ and proceed to a Puiseux parametrization of each branch of Δ. For each branch, it is necessary to truncate the Puiseux series in such a way that this truncation does not change the topology of the curve. If one supposes that $\{z_1 = \omega\}$, then one obtains a decomposition of the neighbourhood of Δ in different regions, inversely proportional to

the norm of the equation defining the singular locus. For each tangent line to Δ, $\{t = \alpha_i x\}$ we proceed as follows. Suppose, without loss of generality, that there is only one tangent line $\{t = \alpha x\}$ to Δ.

Let us restrain the study of this curve to a set, centered at the tangent line:

$$A := \{(x, y) \mid |x| \leq \epsilon, |y - \alpha x| \leq \mu |x|\} \subset \mathbb{C}^2.$$

Let ν_r be the biggest characteristic exponent in the truncated series. For each $k \leq r$ let $\alpha_k, \beta_k, \gamma_k > 0$ be positive constants, such that $\alpha_k < \beta_k$.

The regions in A are:

- $\Upsilon_k := \left\{ (x, y) : \alpha_k |x^{\nu_k}| \leq \left| y - \sum_{i=1}^{k-1} a_i x^{\nu_i} \right| \leq \beta_k |x^{\nu_k}|, \ \left| y - \sum_{i=1}^{k-1} a_i x^{\nu_k} \right| \geq \gamma_k |x^{\nu_k}| \right\}, \ \forall k \in \{1, ..., m\}.$

- $\Omega_i := \overline{(\Upsilon_{i-1} - \Upsilon_i)}, i \in \{2, ..., m\}.$

- $\Omega_1 := \overline{(\partial A - \Upsilon_1)}, i \in \{2, ..., m\}$

- $\Lambda := \overline{(A - \cup \Omega_i \cup \Upsilon_i)}, i \in \{1, ..., q\} = \bigcup_{i=1}^{m+1} \Lambda_i.$

Consider the normalized surface $\overline{X}$ and $\pi : \overline{X} \to \mathbb{C}^2$ a generic projection with $(\pi(\Gamma)) = \Delta \subset \mathbb{C}^2$. The regions $\Omega, \Lambda, \Upsilon$ are lifted by π to the boundary of the Milnor fiber of the surface.

Moreover, there exists a bi-Lipschitz map between the regions $\Omega_i, \Upsilon_k, \Lambda$ and the topological cones defined on the gluing tori and the mapping tori.

- Ω_i is bi-Lipschitz equivalent to the topological cone over the gluing torus. The exponent in the metric of this topological cone is determined by the Puiseux exponents ν_i and ν_{i-1};
- Υ_k is bi-Lipschitz equivalent to the topological cone over the mapping torus defined over a planar surface Y and with a finite order morphism h; The exponent in the metric is determined by the Puiseux exponent ν_k.
- Λ is bi-Lipschitz equivalent to a topological cone over the mapping torus defined on the manifold $Y = D^2$. [4].

We will investigate the part of $\overline{\mathcal{L}}$, which is invariant under the equivalence relation (14). We recall that this equivalence relation is defined such that: $\mathcal{L} = \overline{\mathcal{L}}/\sim$. Define the topological cones, over the manifolds in $\overline{\mathcal{L}}$. Each topological cone carries a given metric with a given exponent ν. By carousel method, there exists a correlation between the zones in the carousel and the manifolds defining the link $\overline{\mathcal{L}}$. Moreover, a given Puiseux exponent gives a topological description of the corresponding manifold.

Now take into consideration the topological cones with Puiseux exponents $\nu \geq 1$, defined over the manifolds of $\bar{\mathcal{L}}$ and invariant under the equivalence relation (14). Then, from the invariance of topological cones it follows that these manifolds are also defined on $\mathcal{L}$. It is worth noticing that the invariant part of $\mathcal{L}$ is the set of plumbed manifolds equivalent to ∂F_1. In other words, the topological cones defined over the manifolds in $\mathcal{L}_1$ exist in the link $\mathcal{L}$.

Take into account that if the Puiseux exponents of the pieces in $\mathcal{L}_1$ are greater than one, then the topological cones that are endowed with $\nu > 1$ define non-metrically conical zones. In addition, if, in a manifold of $\mathcal{L}_1$, there exists a non-trivial loop, this loop will shrink at a rate greater than one. Thus we have defined a non-metrically conical zone in X.

As for the metrically conical zone, it corresponds to a Puiseux exponent $\nu = 1$. So, the non-trivial loops defined in the Seifert manifold with rate 1, shrink linearly, that is $O(t)$ to 0.

First of all, one has the following non-commutative diagram between the discriminants of the normalized and non-normal surface:

$$
\begin{array}{ccc}
\overline{X} & \xrightarrow{\;n\;} & X \\
\downarrow{\overline{\pi}} & & \downarrow{\pi} \\
\overline{\Delta} & & \Delta
\end{array}
$$

Notice that the discriminant curves of $\overline{X}$ and X differ by their Puiseux exponents. Indeed, not all of Δ's Puiseux data is preserved during the normalization process. In order to have a description of $\mathcal{T}^i$ we consider first $\partial F_{2,i}$. Recall that for each i, $\partial F_{2,i}$ is a graph manifold. To show this assertion, it is sufficient to follow the Nemethi-Szilard algorithm [23]. The decomposition into Seifert-fibered pieces of the three-manifolds $\partial F_{2,i}$ contains the data of f, restricted to the transversal slice germ of Σ_i.

3.5. **A property on the loops in $\partial \mathcal{L}_2^i$.** Now, we will examine loops in the boundary $\partial \mathcal{L}_2^i$.

First, consider the set $\partial G_\beta := \{z \in \mathbb{C}^3 | \, \|K(z)|^2 = \beta\}$ where $K(z)$ generates the ideal defining the singular locus and let β be a positive, small enough constant. Let us consider the map:

(16) $$\phi : (\mathcal{L} := S_\epsilon \cap X) \cap \partial G_\beta \to S^1.$$

The fibers of this map are a set of m manifolds. These manifolds will be denoted $R_j, j \in \{1, 2, ..., m\}$. Assume that the surface is such that the intersection of a hyperplane with the singular locus is transversal and contains 0. Let l be the projection operator onto the first coordinate axis z_1. Notice that in the case of $l : \Sigma \to 0z_1$, one has a covering with finite multiplicity [13]. Then, for a small $c \neq 0, l^{-1}(c) \cap \Sigma$ consists of a set of m points that we will label $\{p_1, p_2, ..., p_m\}$. Consider the inverse image of these points $\{p_i\}_{i=1}^m$ in the surface. Then, we call R_j the intersection of a sphere of small radius η, centered at a point p_j, and the pre-image of c in the surface.

Going back to the previous map (16), if β is sufficiently small then $S_\epsilon \cap G_\beta$ splits into a disjoint union of sets that are neighbourhoods of $S_\epsilon \cap \Sigma_i$. Thus one has to consider $\mathcal{L}^i := (\mathcal{L} \cap \partial G_\beta^i)$ where ∂G_β^i is a neighborhood of the Σ_i component.

Proposition 4. *If $i \in \{1, ..., n\}$, then $\mathcal{L}^i$ stratifies over a circle with fiber R_i*

An interesting topological information is given by I. Iomdin [13]:

Proposition 5. $\forall z \in \mathcal{L}^i \subset \mathbb{C}^3, i \in \{1, ..., n\}$:

$$\lambda : \mathcal{L}^i \to V_{3,3} \, , \; z \mapsto [grad \, f, grad \, g, z]$$

where $V_{3,3}$ is a complex Stiefel manifold.

Let $\lambda : \mathcal{L}^i \to V_{3,3}$ and let $i : R_i \to \mathcal{L}^i$ be an embedding. Let r_j be the fundamental cycle of the manifold R_i and s a generator of the first homology group $H_1(V_{3,3})$. Then ,

$$\lambda_* \circ i_*(r_i) = \kappa \cdot s,$$

where κ is an integer.

So, using this procedure, we have defined a splitting of the germ $(X, 0)$ into a finite family of semi-algebraic sets. Each of these sets correspond to a type of topological cone, endowed with Nagase, modified Nagase or Hsiang-Pati type of metric over the elements of the link of the singularity. Therfore this shows the existence of such a splitting of $X \cap \mathcal{B}_\epsilon$. $\qquad\square$

Proof. Propostion 2. Let $n : \overline{X} \to X$ be the normalization map and consider $X \setminus \Sigma$ with normalization $\overline{X} \setminus n^{-1}(\Sigma)$. Recall that outside the singular locus, the map is bi-holomorphic. So, outside the singular locus, the metric on $\overline{\mathcal{L}} \setminus n^{-1}(\Sigma \cap \mathcal{S}_\epsilon)$ gives an equivalent metric, topologically speaking, on: $\mathcal{L} \setminus (\Sigma \cap \mathcal{S}_\epsilon)$. For normal surfaces, the boundary of its Milnor fiber is diffeomorphic to its link. The discriminant curve of $\overline{X}$, provides the zones $\Omega_k, \Upsilon_i, \Lambda$, delimited by the Puiseux-exponents of

the discriminat curve. These zones give a filtration of the fibration: $\mathcal{B}_\epsilon \cap f^{-1}(\partial \mathcal{B}_\eta) \to \partial \mathcal{B}_\eta$, $\eta < \epsilon$, and provide a decomposition in terms of graph manifolds of the link $\overline{\mathcal{L}}$. We use the fact that there exists a bi-Lipschitz map between the set Υ_k with Puiseux exponent ν_k, and the Reimannian manifold defined as a topological cone over a preimage $\pi^{-1}(\Upsilon_k)$, with Riemannian metric $dt^2 + t^2 d\theta + t^{2\nu_k} g_\theta, t \in (0,1]$. So, applying Cheeger-Kleiner theorem [8]: let $\pi : \mathfrak{W}_j \to \Upsilon_j$ be a Lipschitz map between metric spaces. We assume that for all $r \in \mathbb{R}^+$, if $U \subset \Upsilon_j$ and diam(U)$\leq r$, then the r-components of $\pi^{-1}(U)$ have diameter at most Cr. It follows that if there exists a systol in the manifold MY, which is a fiber of $\Xi \to S^1$ has diameter of order αt^ν, where $\alpha > 0$ is a constant and ν is a Puiseux exponent. Note that $0 < \alpha < \infty$ by section 3 [2]. $\qquad\square$

Proof. Corollary 2. Let Ξ be the region defined in the previous proposition. Ξ is the region in $\mathcal{L}$ that corresponds to a zone of type Υ, defined by a Puiseux exponent ν. Thus, a systole, in this region, has a diameter of order αt^ν. Recall that the metric on the Riemannian manifold is of type : $dt^2 + t^2 d\theta + t^{2\nu_k} g_\theta, t \in (0,1]$. From a topological point of view, the topological cone can be studied in terms of a compact 1-parameter family of subsets of $\mathbb{R}^n$. That is, if A_i is the topological cone over Ξ, $A_i \subset \mathbb{R} \times \mathbb{R}^n$ and we define $A_{i,t} := A_i \cap (\{t\} \times \mathbb{R})$. Notice that this set is connected. So, as t tends to 0, the systole shrinks at a rate of order $O(t^\nu)$. $\qquad\square$

3.6. **Example.** Consider the complex germ defined by:

$$(17) \qquad f(z_1, z_2, z_3) = z_3^2 - (z_1^3 - z_2^2)^2 (z_1^4 - z_2^3) = 0,$$

with singular locus $\Sigma = \{z_3 = 0, z_1^3 - z_2^2 = 0\}$. The map $l : \Sigma \to z_1$ has finite multiplicity 2. So, the pre-image of $w \in \mathbb{C}^1$ is composed of two points p_1 and p_2. Hence, the intersection in X: $l^{-1}(w) \cap X$, has an isolated singularity at p_1 and at p_2. Let us now define the following manifolds $R_1 := \mathcal{S}_\epsilon(c_1) \cap l^{-1}(w) \cap X$ and $R_2 := \mathcal{S}_\epsilon(c_2) \cap l^{-1}(w) \cap X$. These two manifolds are diffeomorphic since, p_1 and p_2 are in the same branch of the singular curve defined by Σ. We will denote this manifold for short R. As it has been showed above, the set $X \cap \mathcal{S}_\epsilon \cap \partial G_\beta$ stratifies over a circle S^1_γ, $|z_1| = \gamma$ with fiber corresponding to the manifold R.

Let $n : \overline{X} \to X$ be the normalization map. The normalized space is defined by the following equation:

$$(18) \qquad \bar{f} = z_3^2 - z_1^4 - z_2^3 = 0.$$

By the Newton-Puiseux algorithm, one determines the characteristic Puiseux exponent of the branch. In this example, the Puiseux characteristic exponent of the discriminant curve is $4/3$. The delimited regions in this examples are:

(1) let $\Upsilon_1 := \{(z_1, z_2)\mathbb{C}^2 : \alpha|z_1^{\frac{4}{3}}| \leq |z_2 - z_1^{\frac{4}{3}}| \leq \beta|z_1^{\frac{4}{3}}|\}$, $\alpha \leq \beta \in \mathbb{R}$
(2) $A := \{(z_1, z_2) \in \mathbb{C}^2 | |z_1| \leq \epsilon, |z_2^- z_1^|| \leq \mu|z_1|\} \subset \mathbb{C}^2$
(3) $\Omega_1 := \overline{(\partial A - \Upsilon_1)}$,
(4) $\Lambda := \overline{(A - \Omega_1 \cup \Upsilon_1)}, = \cup_{k=1}^2 \Lambda_k$

On the other side, applying Hirzebruch-Jung resolution of singularities, one obtains the resolution of this singularity and hence its dual representation graph. On this graph there exists a metrically conical zone, situated at the end vertex of the graph and a non-metrically zone, with metric rate $\nu = \frac{4}{3}$. In order to have the link of the non-isolated singularity, one must consider the inverse image of the singular locus $\Sigma \subset X$ by n in the link $\overline{\mathcal{L}}$. Then, apply the equivalence relation in Proposition 3. In the neighbourhood of the glued points, we define a cone over the boundary of M_j and take the topological cone over this manifold. Recall that M_j is the Milnor fiber of f restricted to a transversal slice germ corresponding to $z_1^3 - z_2^2$. This corresponds to the $\mathcal{L}_2$ part of the link. In $\mathcal{L}$, the adjacent manifolds are tori. These tori join $\mathcal{L}_2$ to the part $\mathcal{L}_1$. Notice that $\mathcal{L}_1$ is equivalent to the invariant part of $\overline{\mathcal{L}}$ under equivalence relation. Since the metric rate, for the non-metrically conical part of the link, is $4/3$, we have the four types of topological cones in the splitting of $(X,0)$: There exists a semi-algebraic set in $(X,0)$ which is metrically conical,

a semi-algebraic set non-metrically conical with rate 4/3 and semi-algebraic set equivalent to a topological cone defined over $\mathcal{C}\partial M_j$.

References

[1] L. Birbrair, A. Fernandes, W. Neumann. *Bi-Lipschitz geometry of weighted homogeneous surface singularities.* Math. Ann. **342**, (2008), 139-144.

[2] A. Bernig, A. Lytchak. *Tangent spaces and Gromov-Hausdorff limits of subanalytic spaces* J. Reine Angew Math. **608**, (2007), 1-15. DOI:10.15 15/CRELLE.2007.50

[3] L. Birbrair, T. Mostowski. *Normal Embedding of Semialgebraic Sets.* Michigan Math. J. **47**, (2000), 125-132.

[4] L. Birbrair, W. Neumann, A. Pichon. *The thick-thin decomposition and the bilipschitz classification of normal surface singularities* Acta Math. 212 (2014), 199-256.

[5] M. Boileau *Thick/thin decomposition of a 3-manifold and a geometrization conjecture*, Preliminary notes on Geometry and Topology of 3-dimensionale manifolds (May 2012).

[6] E. Brieskorn, H. Knörrer. *Plane Algebraic Curves.* Birkhäuser, Boston, (1986).

[7] D. Burghelea, A.Verona. *Local homological properties of analytic sets.* Manuscripta Math.**7**, (1972), 55-66.

[8] J. Cheeger, B. Kleiner. *Realization of metric spaces as inverse limits, and bilipschitz embedding in L1*, Geom. Funct. Analysis, DOI:10.1007/s00039-012-0201-8, Springer Basel (2013).

[9] M. Gromov. *Systoles and intersystolic inequalities.* Actes de la table ronde de géométrie différentielle, Luminy 1992, 291-362, Sémin. Congr, 1, Soc. Math. France, Paris (1996).

[10] M. Gromov. *Manifolds of negative curvature.* J. Diff. Geometry **13**, (1978), 223-230.

[11] H. A. Hamm. *Lokale topologische Eigenschaften komplexer Raume,* Math. Ann. **191** No.3, (1971), 235-252.

[12] W. C. Hsiang, V. Pati. *L^2-cohomology of normal algebraic surfaces. I.* Inventiones Mathematicae **81**, (1985), 395-412.

[13] I. Iomdin. *Complex surfaces with a one-dimensional set of singularities,* Sibirsk. Matem. Zb.,**15** No.5, (1974), 1061-1082.

[14] Lê Dung-Trang.*The Geometry of the Monodromy Theorem*, in C.P Ramanujan - a tribute, ed. K. G. Ramanathan, Tata Inst. Studies in Math. **8**, (1978).

[15] Lê Dung-Trang.*La Monodromy n'a pas de Points Fixes*, J. Fac. Sci. Univ. Tokyo, Sec. 1A **22**,(1975), 409-427.

[16] Lê Dung-Trang.*Exposants critiques et resolution des surfaces*, Acta mathematica Vietnamica, Vol.II, No.2,(1986), 127-135.

[17] S. Lojasiewicz. *Ensembles semi- analitiques.* Cours de la Faculté des Sciences d'Orsay, (1965). ⟨ https://perso.univ-rennes1.fr/michel.coste/Lojasiewicz.pdf ⟩

[18] D. A. Kazhdan, G. A. Margulis, *A proof of Selberg's conjecture*, Mat. Sb. (N.S.), **75 (117)** (1968), 163-168. (In russian),

[19] J. Mather. *Notes on Topological Stability*, Harvard University (1970).

[20] J. Milnor. *J. Milnor Singular points of complex hypersurfaces*, Ann. Math. Studies Princeton Univ .Press, (1968).

[21] M. Nagase. *Remarks on the L^2-cohomology*, J. Math. Soc. Japan **41**, (1989), 97-1016.

[22] W. Neumann. *A calculus for plumbing applied to the topology of complex surface singulari- ties and degenerating complex curves*, Trans. Amer. Math. Soc. **268**, (1981), 299-343.

[23] A. Nemethi, A. Szilard. *Milnor fiber boundary of a non-isolated surface singularity*, Lecture Notes in Math, **2037**, Springer (2012).

[24] P. Orlik. *Seifert Manifolds*, Lecture Notes in Mathematics **260**, Spinger-Verlag 1972.

[25] P. M. Pu. *Some inequalities in certain non-orientable Riemannian manifolds*, Pacific J. Math. $\underline{2}$. (1952), 55-71.

[26] D. Siersma. *Variation mappings of singularities with 1-dimensional critical locus.* Topology **30**, (1991), 445-469.

[27] W. Thurston. *The geometry and topology of 3-manifolds.* Princeton Math. Dept., (1979).

[28] C. T. C. Wall. *Singular Points of Plane Curves*, London Mathematical Society Student Texts 63, Cambridge University press (2004).

[29] H. Whitney. *Tangents to an analytic variety.* Ann. of Math. **81**, (1964), 496-549.

Aix-Marseille University, IML, Campus de Luminy case 907 13288 Marseille Cedex-9 France
E-mail address: `noemie.combe@univ-amu.fr`