#### MIXED VARIATIONAL APPROACH TO FINDING GUARANTEED ESTIMATES FROM SOLUTIONS AND RIGHT-HAND SIDES OF THE SECOND-ORDER LINEAR ELLIPTIC EQUATIONS UNDER INCOMPLETE DATA

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#### Abstract

We investigate the problem of guaranteed estimation of values of linear continuous functionals defined on solutions to mixed variational equations generated by linear elliptic problems from indirect noisy observations of these solutions.

We assume that right-hand sides of the equations, as well as the second moments of noises in observations are not known; the only available information is that they belong to given bounded sets in the appropriate functional spaces.

We are looking for linear with respect to observations optimal estimates of solutions of aforementioned equations called minimax or guaranteed estimates. We develop constructive methods for finding these estimates and estimation errors which are expressed in terms of solutions to special mixed variational equations and prove that Galerkin approximations of the obtained variational equations converge to their exact solutions.

We study also the problem of guaranteed estimation of right-hand sides of mixed variational equations.

#### Introduction

Estimation theory for systems with lumped and distributed parameters under uncertainty conditions was developed intensively during the last 30 years. That was motivated by the fact that the realistic setting of boundary value problems describing physical processes often contains perturbations of unknown (or partially unknown) nature. In such cases the minimax estimation method proved to be useful, making it possible to obtain optimal estimates both for the unknown solutions (or right-hand sides of equations appearing in the boundary value problems) and for linear functionals from them, that is estimates looked for in the class of linear estimates with respect to observations Here we understand observations of unknown solutions as the functions that are linear transformations of same solutions distorted by additive random noises. for which the maximal mean square error taken over all the realizations of perturbations from certain given sets takes its minimal value.

The above estimation method was investigated in the works by N. N. Krasovsky, A. B. Kurzhansky, A. G. Nakonechny, and others (see [3], [4], [5]–[7], [18]). This approach makes it possible to find optimal estimates of parameters of boundary value problems reckoning on the "worst" realizations of perturbations.

A. G. Nakonechny used traditional variational formulations of boundary value problems (their solvability is based on the Lax-Milgram lemma), to obtain systems of variational equations whose solutions generate the minimax mean square estimates.

At the same time many physical processes of the real world are described by mixed variational problems. Among such processes, there are flows of viscous fluids, propagation of electromagnetic and acoustical waves. In addition, many classical boundary value problems admit mixed variational formulations. The mixed method consists of simultaneous finding, from systems of variational equations, both solutions and certain expression generated by solutions taken as new auxiliary unknowns. As a rule, these unknowns are related to derivatives of the solutions and have important physical meaning (such as flux, bending moment etc), and their calculation or estimation often has even greater practical significance.

The theory of mixed variational methods of solving boundary value problems and their numerical implementation, the mixed finite element methods, was developed by Babuška, Brezzi, Fortin, Raviard, Glowinski and others (see [10] -[13]). In particular, Brezzi and Fortin proved solvability theorems for a wide class of mixed variational problems and their discrete analogs.

In this paper we show that mixed variational formulations of boundary value problems can be used also for a guaranteed estimation of linear functionals from unknown solutions and their gradients, as well as functionals from unknown right-hand sides of second order linear elliptic equations. It is proved that guaranteed estimates of these functionals and estimation errors are expressed explicitly from the solutions of special systems of mixed variational equations, for which the unique solvability is established. We develop, on the basis of the Galerkin method, numerical methods of finding these solutions and prove the convergence of the approximate solutions to exact ones.

The estimation methods proposed here yield, for example, in stationary and non-stationary heat conduction problems, estimates of heat flux from temperature observations, or conversely, estimates of temperature from heat flux observations, as well as estimates of the unknown distribution of density of sources from heat flux observations. The theory of guaranteed estimation developed in this work provides an essential generalization of well-known results in this direction by the authors mentioned above.

Note that the available estimation methods do not provide solution of such estimation problems, so that the methods developed here are essentially new.

#### **1** Preliminaries and auxiliary results

Let us introduce the notations and definitions that will be used in this work.

We denote matrices and vectors by bold letters;  $x = (x_1, \ldots, x_n)$  denotes a spatial variable in an open domain  $D \subset \mathbb{R}^n$  with Lipschitzian boundary  $\Gamma$ ;  $dx = dx_1 \cdots dx_n$  is Lebesgue measure in  $\mathbb{R}^n$ ;  $H^1(D)$  and  $H^1_0(D)$  are standard Sobolev spaces of the first order in the domain D with corresponding norm.

If X is a Hilbert space over  $\mathbb{R}$  with inner product  $(\cdot, \cdot)_X$  and norm  $\|\cdot\|_X$ , then  $J_X \in \mathcal{L}(X, X')$  denotes the Riesz operator acting from X to its adjoint X' and determined by the equality (we note that this operator exists according to the Riesz theorem)  $(v, u)_X = \langle v, J_X u \rangle_{X \times X'}$  $\forall u, v \in X$ , where  $\langle x, f \rangle_{X \times X'} := f(x)$  for  $x \in X$ ,  $f \in X'$ .

Below random variable  $\xi$  with values in a separable Hilbert space X is considered as a function  $\xi : \Omega \to X$  mapping random events  $E \in \mathcal{B}$  to Borel sets in H (Borel  $\sigma$ -algebra in X is generated by open sets in X). By  $L^2(\Omega, X)$  we denote the Bochner space composed of random variables  $\xi = \xi(\omega)$  defined on a certain probability space  $(\Omega, \mathcal{B}, P)$  with values in a separable Hilbert space X such that

$$\|\xi\|_{L^{2}(\Omega,X)}^{2} = \int_{\Omega} \|\xi(\omega)\|_{X}^{2} dP(\omega) < \infty.$$
(1.1)

In this case there exists the Bochner integral

$$\mathbb{E}\xi := \int_{\Omega} \xi(\omega) \, dP(\omega) \in X \tag{1.2}$$

called the mathematical expectation or the mean value of random variable  $\xi(\omega)$  which satisfies the condition

$$(h, \mathbb{E}\xi)_X = \int_{\Omega} (h, \xi(\omega))_X \, dP(\omega) \quad \forall h \in X.$$
(1.3)

Being applied to random variable  $\xi$  with values in  $\mathbb{R}$  this expression leads to a usual definition of its mathematical expectation because the Bochner integral (1.2) reduces to a Lebesgue integral with probability measure  $dP(\omega)$ .

In  $L^2(\Omega, X)$  one can introduce the inner product

$$(\xi,\eta)_{L^2(\Omega,X)} := \int_{\Omega} (\xi(\omega),\eta(\omega))_X dP(\omega) \quad \forall \xi,\eta \in L^2(\Omega,X).$$
(1.4)

Applying the sign of mathematical expectation, one can write relationships (1.1)-(1.4) as

$$\|\xi\|_{L^{2}(\Omega,X)}^{2} = \mathbb{E}\|\xi(\omega)\|_{X}^{2}, \qquad (1.5)$$

$$(h, \mathbb{E}\xi)_X = \mathbb{E}(h, \xi(\omega))_X \quad \forall h \in X,$$
 (1.6)

$$(\xi,\eta)_{L^2(\Omega,X)} := \mathbb{E}(\xi(\omega),\eta(\omega))_X \quad \forall \xi,\eta \in L^2(\Omega,X).$$
(1.7)

 $L^{2}(\Omega, X)$  equipped with norm (1.5) and inner product (1.7) is a Hilbert space.

## 2 Statement of the estimation problem of linear functionals from solutions to mixed variational equations

Let the state of a system be characterized by the function  $\varphi(x)$  which is defined as a solution of the Dirichlet boundary value problem:

$$-\operatorname{div}\left(\mathbf{A}\operatorname{\mathbf{grad}}\varphi\right) + c\varphi = f \quad \text{in} \quad D, \tag{2.1}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma. \tag{2.2}$$

Introducing the additional unknown  $\mathbf{j} = -\mathbf{A} \operatorname{\mathbf{grad}} \varphi$  in D, rewrite this problem as the first-order system

$$\mathbf{A}^{-1}\mathbf{j} = -\mathbf{grad}\,\varphi \quad \text{in} \quad D, \tag{2.3}$$

div 
$$\mathbf{j} + c\varphi = f$$
  $D, \quad \varphi = 0 \quad \text{on} \quad \Gamma,$  (2.4)

where  $\mathbf{A} = \mathbf{A}(x) = (a_{ij}(x))$  is an  $n \times n$  matrix with entries  $a_{ij} \in L^{\infty}(D)$  for which there exists a positive number  $\mu$  such that

$$\mu \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \quad \forall x \in D \quad \forall \xi = (\xi_1, \dots, \xi_n)^T \in \mathbb{R}^n,$$

 $\mathbf{A}^{-1}$  is the inverse matrix of  $\mathbf{A}$ , and c is a piecewise continuous function satisfying for  $x \in D$  the inequality  $c_0 \leq c(x) \leq c_1, c_0, c_1 = \text{const}, 0 \leq c_0 \leq c_1$ .

According to [10] and [17], by a solution of problem (2.3), (2.4) we will mean a pair of functions  $(\mathbf{j}, \varphi) \in H(\operatorname{div}; D) \times L^2(\Omega)$  such that

$$\int_{D} ((\mathbf{A}(x))^{-1} \mathbf{j}(x), \mathbf{q}(x))_{\mathbb{R}^{n}} dx - \int_{D} \varphi(x) \operatorname{div} \mathbf{q}(x) dx = 0 \quad \forall \mathbf{q} \in H(\operatorname{div}; D)$$
(2.5)

$$\int_{D} v(x) \operatorname{div} \mathbf{j}(x) dx + \int_{D} c(x) \varphi(x) v(x) dx = \int_{D} f(x) v(x) dx \ \forall v \in L^{2}(D).$$
(2.6)

Note that from equations (2.5) and (2.6) it follows that  $\varphi \in H_0^1(D)$ , i. e. the boundary condition  $\varphi|_{\Gamma} = 0$  is implicitly contained in these equations.

Problem (2.5), (2.6) is commonly referred to as the mixed formulation of (2.3), (2.4).

From physical point of view problem (2.3), (2.4) simulates a stationary process of the propagation of heat in the domain D, and the functions  $\varphi(x)$ ,  $\mathbf{j}(x)$ , and f(x) have the sense of temperature, heat flux, and volume density of heat sources, respectively, at the point x.

Introduce the bilinear forms a, b, c and the functional l given by

$$a(\mathbf{q}_1, \mathbf{q}_2) := \int_D ((\mathbf{A}(x))^{-1} \mathbf{q}_1, \mathbf{q}_2)_{\mathbb{R}^n} dx \quad \forall \mathbf{q}_1, \mathbf{q}_2 \in H(\operatorname{div}; \mathbf{D}),$$
(2.7)

$$b(\mathbf{q}, v) := -\int_{D} v \operatorname{div} \mathbf{q} \, dx \quad \forall \mathbf{q} \in H(\operatorname{div}; \mathbf{D}), v \in \mathrm{L}^{2}(\mathbf{D}),$$
(2.8)

$$c(v,v) := \int_D c(x)v_1(x)v_2(x) \, dx \quad \forall v_1, v_2 \in L^2(D),$$
(2.9)

$$l(v) := -\int_D f v \, dx \quad \forall v \in L^2(D).$$
(2.10)

Then the problem under study may be stated as follows.

Find  $(\mathbf{j}, \varphi) \in H(\operatorname{div}; D) \times L^2(\Omega)$  such that

$$a(\mathbf{j}, \mathbf{q}) + b(\mathbf{q}, \varphi) = 0 \quad \forall \mathbf{q} \in H(\operatorname{div}; D),$$
(2.11)

$$b(\mathbf{j}, v) - c(\varphi, v) = l(v) \quad \forall v \in L^2(D).$$
(2.12)

Denote by  $B : H(\operatorname{div}; D) \to L^2(D)$  the operator associated with the bilinear form b. It is easy to see that a, b and c are continuous bilinear forms with a being coercive on Ker B, cbeing symmetric, positive semidefinite and b satisfying the standard inf-sup condition (Brezzi condition). Since  $\operatorname{Im} B = L^2(D)$ , we have  $\operatorname{Ker} B^t = \{\emptyset\}$ , where  $B^t : L^2(D) \to H(\operatorname{div}; D)'$  is the transpose operator of B defined by

$$\langle v, B^t q \rangle_{H(\operatorname{div},D) \times H(\operatorname{div},D)'} = b(v,q) \quad \forall v \in H(\operatorname{div},D), \quad \forall q \in L^2(D).$$

Consequently, it follows from Theorem 1.2 of [10] that problem (2.11), (2.12) has a unique solution and the following *a priori* estimate is valid

$$\|\mathbf{j}\|_{H(\operatorname{div};D)} + \|\varphi\|_{L^2(D)} \le C \|f\|_{L^2(D)}$$
 (C = const).

Further we assume that the function f(x) in equations (2.4) and (2.6) is unknown and belongs to the set

$$G_0 := \left\{ \tilde{f} \in L^2(D) : \left( Q(\tilde{f} - f_0), \tilde{f} - f_0 \right)_{L^2(D)} \le \epsilon_1 \right\},$$
(2.13)

where  $f_0 \in L^2(D)$  is a given function,  $\epsilon_1 > 0$  is a given constant, and  $Q : L^2(D) \to L^2(D)$  is a bounded positive selfadjoint operator for which there exists the inverse bounded operator  $Q^{-1}$ . It is known that the operator  $Q^{-1}$  is positive and selfadjoint.

In this paper we focus on the following estimation problem:

From observations of random variables

$$y_1 = C_1 \mathbf{j} + \eta_1, \quad y_2 = C_2 \varphi + \eta_2,$$
 (2.14)

with values in separable Hilbert spaces  $H_1$  and  $H_2$  over  $\mathbb{R}$ , respectively, it is necessary to estimate the value of the linear functional

$$l(\mathbf{j},\varphi) := \int_D (\mathbf{l}_1(x), \mathbf{j}(x))_{\mathbb{R}^n} \, dx + \int_D l_2(x)\varphi(x) \, dx \tag{2.15}$$

in the class of the estimates linear with respect to observations, which have the form

$$\widehat{l(\mathbf{j},\varphi)} := (y_1, u_1)_{H_1} + (y_2, u_2)_{H_2} + c, \qquad (2.16)$$

where  $(\mathbf{j}, \varphi)$  is a solution of problem (2.5), (2.6),  $\mathbf{l}_1$  and  $l_2$  are given functions from  $L^2(D)^n$ and  $L^2(D)$ ,  $u_1 \in H_1$ ,  $u_2 \in H_2$ ,  $c \in \mathbb{R}$ ,  $C_1 \in \mathcal{L}(L^2(D)^n, H_1)$ , and  $C_2 \in \mathcal{L}(L^2(D), H_2)$  are linear bounded operators,

$$\eta := (\eta_1, \eta_2) \in G_1;$$
 (2.17)

by  $G_1$  we denote the set of pairs  $\{(\tilde{\eta}_1, \tilde{\eta}_2)\}$  of uncorrelated random variables  $\tilde{\eta}_1 \in L^2(\Omega, H_1)$  and  $\tilde{\eta}_2 \in L^2(\Omega, H_2)$  with zero expectations satisfying the condition

$$\mathbb{E}(\tilde{Q}_1\tilde{\eta}_1, \tilde{\eta}_1)_{H_1} \le \epsilon_2, \quad \mathbb{E}(\tilde{Q}_2\tilde{\eta}_2, \tilde{\eta}_2)_{H_2} \le \epsilon_3, \tag{2.18}$$

where  $\tilde{Q}_1$  and  $\tilde{Q}_2$  are bounded positive-definite selfadjoint operators in  $H_1$  and  $H_2$ , respectively, for which there exist the inverse bounded operators  $\tilde{Q}_1^{-1}$  and  $\tilde{Q}_2^{-1}$ . We note that random variables  $\xi_1 \in H_1$  and  $\xi_2 \in H_2$  are called uncorrelated if

$$\mathbb{E}(\xi_1, u_1)_{H_1}(\xi_2, u_2)_{H_2} = 0 \quad \forall u_1 \in H_1, u_2 \in H_2$$
(2.19)

(see, for example, [2], p. 146).

It is known that operators  $\tilde{Q}_1^{-1}$  and  $\tilde{Q}_2^{-1}$  are positive definite and selfadjoint, that is there exists a positive number  $\alpha$  such that

$$(\tilde{Q}_1^{-1}u_1, u_1)_{H_1} \ge \alpha \|u_1\|_{H_1}^2 \ \forall u_1 \in H_1, \ (\tilde{Q}_2^{-1}u_2, u_2)_{H_2} \ge \alpha \|u_2\|_{H_2}^2 \ \forall u_2 \in H_2.$$
(2.20)

Set  $u := (u_1, u_2) \in H := H_1 \times H_2$ .

**Definition 1.** An estimate

$$\widehat{\overline{l(\mathbf{j},\varphi)}} = (y_1, \hat{u}_1)_{H_1} + (y_2, \hat{u}_2)_{H_2} + \hat{c}$$

is called a guaranteed (or minimax) estimate of  $l(\mathbf{j}, \varphi)$ , if elements  $\hat{u}_1 \in H_1$ ,  $\hat{u}_2 \in H_2$  and a number  $\hat{c}$  are determined from the condition

$$\inf_{u \in H, c \in \mathbb{R}} \sigma(u, c) = \sigma(\hat{u}, \hat{c}),$$

where

$$\sigma(u,c) := \sup_{\tilde{f} \in G_0, (\tilde{\eta}_1, \tilde{\eta}_2) \in G_1} \mathbb{E} |l(\tilde{\mathbf{j}}, \tilde{\varphi}) - \widehat{l(\tilde{\mathbf{j}}, \tilde{\varphi})}|^2$$

and  $(\tilde{\mathbf{j}}, \tilde{\varphi})$  is a solution of problem (2.3), (2.4) at  $f(x) = \tilde{f}(x)$ ,  $\widehat{l(\tilde{\mathbf{j}}, \tilde{\varphi})} := (\tilde{y}_1, u_1)_{H_1} + (\tilde{y}_2, u_2)_{H_2} + c, \tilde{y}_1 = C_1 \tilde{\mathbf{j}} + \tilde{\eta}_1, \quad \tilde{y}_2 = C_2 \tilde{\varphi} + \tilde{\eta}_2.$ The quantity

$$\sigma := [\sigma(\hat{u}, \hat{c})]^{1/2} \tag{2.21}$$

is called the error of the guaranteed estimation of  $l(\mathbf{j}, \varphi)$ .

Thus, the guaranteed estimate is an estimate minimizing the maximal mean-square estimation error calculated for the "worst" implementation of perturbations.

Further, without loss of generality, we may set  $\epsilon_k = 1, k = 1, 2, 3$ , in (2.13) and (2.18).

## 3 Reduction of the estimation problem to the optimal control problem of a system governed by mixed variational equations

Introduce a pair of functions  $(\mathbf{z}_1(\cdot; u), z_2(\cdot; u)) \in H(\operatorname{div}; D) \times L^2(D)$  as a solution of the problem:

$$\int_{D} (((\mathbf{A}(x))^{-1})^{T} \mathbf{z}_{1}(x; u), \mathbf{q}(x))_{\mathbb{R}^{n}} dx - \int_{D} z_{2}(x; u) \operatorname{div} \mathbf{q}(x) dx$$
$$= \int_{D} (\mathbf{l}_{1}(x) - (C_{1}^{t} J_{H_{1}} u_{1})(x), \mathbf{q}(x))_{\mathbb{R}^{n}} dx \quad \forall \mathbf{q} \in H(\operatorname{div}, D), \quad (3.1)$$

$$-\int_{D} v(x) \operatorname{div} \mathbf{z}_{1}(x; u) \, dx - \int_{D} c(x) z_{2}(\cdot; u) v(x) \, dx$$
$$= \int_{D} (l_{2}(x) - (C_{2}^{t} J_{H_{2}} u_{2})(x)) v(x) \, dx \quad \forall v \in L^{2}(D), \quad (3.2)$$

where  $u \in H$ ,  $C_1^t : H_1' \to L^2(D)^n$  and  $C_2^t : H_2' \to L^2(D)$  are the transpose operators of  $C_1$  and  $C_2$ , respectively, defined by  $\int_D (v(x), C_1^t w(x))_{\mathbb{R}^n} dx = \langle Cv, w \rangle_{H_1 \times H_1'}$  for all  $v \in L^2(D)^n$ ,  $w \in H_1'$  and  $\int_D v(x) C_2^t w(x) dx = \langle Cv, w \rangle_{H_2 \times H_2'}$  for all  $v \in L^2(D)$ ,  $w \in H_2'$ .

From the theory of mixed variational problems it is known that the pair  $(\mathbf{z}_1(x; u), z_2(x; u))$ is uniquely determined <sup>1</sup>

**Lemma 1.** The problem of guaranteed estimation of the functional  $l(\mathbf{j}, \varphi)$  (i.e. the determination of  $\hat{u} = (\hat{u}_1, \hat{u}_2)$  and  $\hat{c}$ ) is equivalent to the problem of optimal control of the system described by mixed variational problem (3.1), (3.2) with a cost function

$$I(u) = (Q^{-1}z_2(\cdot; u), z_2(\cdot; u))_{L^2(D)} + (\tilde{Q}_1^{-1}u_1, u_1)_{H_1} + (\tilde{Q}_2^{-1}u_2, u_2)_{H_2} \to \inf_{u \in H}.$$
 (3.4)

*Proof.* From relations (2.14)–(2.16) at  $\mathbf{j} = \tilde{\mathbf{j}}, \varphi = \tilde{\varphi}, \eta_1 = \tilde{\eta}_1, \eta_2 = \tilde{\eta}_2$ , we have

$$l(\tilde{\mathbf{j}}, \tilde{\varphi}) - l(\tilde{\mathbf{j}}, \tilde{\varphi}) = (\mathbf{l}_{1}, \tilde{\mathbf{j}})_{L^{2}(D)^{n}} + (l_{2}, \tilde{\varphi})_{L^{2}(D)}$$

$$-(\tilde{y}_{1}, u_{1})_{H_{1}} - (\tilde{y}_{2}, u_{2})_{H_{2}} - c = (\mathbf{l}_{1}, \tilde{\mathbf{j}})_{L^{2}(D)^{n}} + (l_{2}, \tilde{\varphi})_{L^{2}(D)}$$

$$-(u_{1}, C_{1}\tilde{\mathbf{j}} + \tilde{\eta}_{1})_{H_{1}} - (u_{2}, C_{2}\tilde{\varphi} + \tilde{\eta}_{2})_{H_{2}} - c = (\mathbf{l}_{1}, \tilde{\mathbf{j}})_{L^{2}(D)^{n}} + (l_{2}, \tilde{\varphi})_{L^{2}(D)}$$

$$- \langle J_{H_{1}}u_{1}, C_{1}\tilde{\mathbf{j}} \rangle_{H_{1}' \times H_{1}} - \langle J_{H_{2}}u_{2}, C_{2}\tilde{\varphi} \rangle_{H_{2}' \times H_{2}}$$

$$-(u_{1}, \tilde{\eta}_{1})_{H_{1}} - (u_{2}, \tilde{\eta}_{2})_{H_{2}} - c = (\mathbf{l}_{1}, \tilde{\mathbf{j}})_{L^{2}(D)^{n}} + (l_{2}, \tilde{\varphi})_{L^{2}(D)}$$

$$-(C_{1}^{t}J_{H_{1}}u_{1}, \tilde{\mathbf{j}})_{L^{2}(D)^{n}} - (C_{2}^{t}J_{H_{2}}u_{2}, \tilde{\varphi})_{L^{2}(D)} - (u_{1}, \tilde{\eta}_{1})_{H_{1}} - (u_{2}, \tilde{\eta}_{2})_{H_{2}} - c$$

$$= (\mathbf{l}_{1} - C_{1}^{t}J_{H_{1}}u_{1}, \tilde{\mathbf{j}})_{L^{2}(D)^{n}} + (l_{2} - C_{2}^{t}J_{H_{2}}u_{2}, \tilde{\varphi})_{L^{2}(D)}$$

$$-(u_{1}, \tilde{\eta}_{1})_{H_{1}} - (u_{2}, \tilde{\eta}_{2})_{H_{2}} - c.$$
(3.5)

Futher, taking into account system of variational equations

$$\int_{D} ((\mathbf{A}(x))^{-1} \tilde{\mathbf{j}}(x), \mathbf{q}(x))_{\mathbb{R}^{n}} dx - \int_{D} \tilde{\varphi}(x) \operatorname{div} \mathbf{q}(x) dx = 0 \ \forall \mathbf{q} \in H(\operatorname{div}; D),$$
(3.6)

$$\int_{D} v(x) \operatorname{div} \tilde{\mathbf{j}}(x) \, dx + \int_{D} c(x) \tilde{\varphi}(x) v(x) \, dx = \int_{D} \tilde{f}(x) v(x) \, dx \quad \forall v \in L^{2}(D), \tag{3.7}$$

<sup>1</sup>In fact, note that problem (3.1), (3.2) can be rewritten in the form

 $a^*(\mathbf{z}_1, \mathbf{q}) + b(\mathbf{q}, z_2) = l_1(\mathbf{q}) \quad \forall \mathbf{q} \in H(\text{div}; \mathbf{D}),$  $b(\mathbf{z}_1; v) - c(z_2; v) = l_2(v) \quad \forall v \in L^2(D),$ 

where  $a^*(\mathbf{z}_1, \mathbf{q}) = a(\mathbf{q}, \mathbf{z}_1)$ , the bilinear forms a, b, and c, are defined by (2.7), (2.8), and (2.9), respectively,  $l_1(\mathbf{q}) = (\mathbf{l}_1 - C_1^t J_{H_1} u_1, \mathbf{q})_{L^2(D)^n}, l_2(v) = (l_2 - C_2^t J_{H_2} u_2, v)_{L^2(D)}$ . Since  $a^*(\mathbf{q}, \mathbf{q}) = a(\mathbf{q}, \mathbf{q})$  then the bilinear form  $a^*(\mathbf{z}_1, \mathbf{q})$  is also coercive on Ker B and, hence, by Theorem 1.2 from [10] problem (3.1), (3.2) is uniquely solvable. Moreover we have:

$$\|\mathbf{z}_1\|_{H(\operatorname{div};D)} + \|z_2\|_{L^2(D)} \le C(\|l_1\|_{H(\operatorname{div};D)'} + \|l_2\|_{L^2(D)}) \le C((l_1 - C_1^t J_{H_1} u_1)_{L^2(D)^n} + (l_2 - C_2^t J_{H_2} u_2)_{L^2(D)}), \quad (3.3)$$

where C = const.

which follows from (2.5)–(2.6) if we set there  $\mathbf{f} = \tilde{\mathbf{f}}$ , and (3.1), (3.2), transform the third and the fourth summands in (3.5). By setting  $\mathbf{q} = \tilde{\mathbf{j}}$  in (3.1) and  $v = \tilde{\varphi}$  in (3.2), we have

$$\int_{D} ((\mathbf{A}(x))^{-1} \tilde{\mathbf{j}}(x), \mathbf{z}_{1}(x; u))_{\mathbb{R}^{n}} dx - \int_{D} z_{2}(x; u) \operatorname{div} \tilde{\mathbf{j}}(x) dx$$
$$= \int_{D} (\mathbf{l}_{1}(x) - (C_{1}^{t} J_{H_{1}} u_{1})(x), \tilde{\mathbf{j}}(x))_{\mathbb{R}^{n}} dx, \quad (3.8)$$

$$-\int_{D} \tilde{\varphi}(x) \operatorname{div} \mathbf{z}_{1}(x; u) \, dx - \int_{D} c(x) z_{2}(x; u) \tilde{\varphi}(x) \, dx$$
$$= \int_{D} (l_{2}(x) - (C_{2}^{t} J_{H_{2}} u_{2})(x)) \tilde{\varphi}(x) \, dx. \quad (3.9)$$

On the other hand, putting  $\mathbf{q} = \mathbf{z}_1(\cdot; u)$  in (3.6) and  $v = z_2(\cdot; u)$  in (3.7), we find

$$\int_{D} ((\mathbf{A}(x))^{-1} \tilde{\mathbf{j}}(x), \mathbf{z}_1(x; u))_{\mathbb{R}^n} dx - \int_{D} \tilde{\varphi}(x) \operatorname{div} \mathbf{z}_1(x; u) \, dx = 0, \qquad (3.10)$$

$$\int_{D} z_2(x;u) \operatorname{div} \tilde{\mathbf{j}}(x) \, dx + \int_{D} c(x) \tilde{\varphi}(x) z_2(x;u) \, dx = \int_{D} \tilde{f}(x) z_2(x;u) \, dx. \tag{3.11}$$

From (3.8)-(3.11), we get

$$(\mathbf{l}_{1} - C_{1}^{t} J_{H_{1}} u_{1}, \tilde{\mathbf{j}})_{L^{2}(D)^{n}} + (l_{2} - C_{2}^{t} J_{H_{2}} u_{2}, \tilde{\varphi})_{L^{2}(D)}$$

$$= \int_{D} ((\mathbf{A}(x))^{-1} \tilde{\mathbf{j}}(x), \mathbf{z}_{1}(x; u))_{\mathbb{R}^{n}} dx - \int_{D} z_{2}(x; u) \operatorname{div} \tilde{\mathbf{j}}(x) dx$$

$$- \int_{D} \tilde{\varphi}(x) \operatorname{div} \mathbf{z}_{1}(x; u) dx - \int_{D} c(x) z_{2}(x; u) \tilde{\varphi}(x) dx = \int_{D} ((\mathbf{A}(x))^{-1} \tilde{\mathbf{j}}(x), \mathbf{z}_{1}(x; u))_{\mathbb{R}^{n}} dx$$

$$- \int_{D} \tilde{\varphi}(x) \operatorname{div} \mathbf{z}_{1}(x; u) dx - \int_{D} z_{2}(x; u) \operatorname{div} \tilde{\mathbf{j}}(x) dx - \int_{D} c(x) z_{2}(x; u) \tilde{\varphi}(x) dx =$$

$$= 0 - (\tilde{f}, z_{2}(\cdot; u))_{L^{2}(D)} = -(\tilde{f}, z_{2}(\cdot; u))_{L^{2}(D)}. \qquad (3.12)$$

Equalities (3.12) and (3.5) imply

$$\widehat{l(\mathbf{j}, \tilde{\varphi})} - \widehat{l(\mathbf{j}, \tilde{\varphi})} = -(\tilde{f}, z_2(\cdot; u_1, u_2)_{L^2(D)} - (u_1, \tilde{\eta}_1)_{H_1} - (u_2, \tilde{\eta}_2)_{H_2} - c$$

$$= -(\tilde{f} - f_0, z_2(\cdot; u_1, u_2))_{L^2(D)} - (f_0, z_2(\cdot; u_1, u_2))_{L^2(D)}$$

$$- (u_1, \tilde{\eta}_1)_{H_1} - (u_2, \tilde{\eta}_2)_{H_2} - c =: \xi,$$
(3.13)

where by  $\xi$  we denote the random variable defined by the right-hand side of the latter equality. It is obvious that

$$\mathbb{E}\xi = -(\tilde{f} - f_0, z_2(\cdot; u_1, u_2))_{L^2(D)} - (f_0, z_2(\cdot; u_1, u_2))_{L^2(D)} - c,$$
  
$$\xi - \mathbb{E}\xi = -(u_1, \tilde{\eta}_1)_{H_1} - (u_2, \tilde{\eta}_2)_{H_2}.$$

Taking into consideration the relationship

$$\mathbf{D}\xi = \mathbb{E}(\xi - \mathbb{E}\xi)^2 = \mathbb{E}\xi^2 - (\mathbb{E}\xi)^2$$
(3.14)

that couples dispersion  $\mathbf{D}\xi$  of the random variable  $\xi$  and its expectation  $\mathbb{E}\xi$ , we obtain from (3.13)

$$\mathbb{E} \left| l(\tilde{\mathbf{j}}, \tilde{\varphi}) - (\widehat{l(\tilde{\mathbf{j}}, \tilde{\varphi})} \right|^2 = \left| (\tilde{f}_2 - f_0, z_2(\cdot; u))_{L^2(D)} \right|^2 \\ (f_0, z_2(\cdot; u))_{L^2(D)} + c \Big|^2 + \mathbb{E} [(u_1, \tilde{\eta}_1)_{H_1} + (u_2, \tilde{\eta}_2)_{H_2}]^2,$$

whence we get

+

$$\inf_{c \in \mathbb{R}} \sup_{\tilde{f} \in G_{0}, (\tilde{\eta}_{1}, \tilde{\eta}) \in G_{1}} \mathbb{E} |l(\tilde{\mathbf{j}}, \tilde{\varphi}) - l(\tilde{\mathbf{j}}, \tilde{\varphi})|^{2} = \\
= \inf_{c \in \mathbb{R}} \sup_{\tilde{f} \in G_{0}} \left[ (\tilde{f} - f_{0}, z_{2}(\cdot; u))_{L^{2}(D)} + (f_{0}, z_{2}(\cdot; u))_{L^{2}(D)} + c \right]^{2} \\
+ \sup_{(\tilde{\eta}_{1}, \tilde{\eta}_{2}) \in G_{1}} \mathbb{E} [(\tilde{\eta}_{1}, u_{1})_{H_{1}} + (\tilde{\eta}_{2}, u_{2})_{H_{2}}]^{2} \\
= \sup_{\tilde{f} \in G_{0}} \left[ (\tilde{f}_{2} - f_{2}^{(0)}, z_{2}(\cdot; u))_{L^{2}(D)} \right]^{2} + \sup_{(\tilde{\eta}_{1}, \tilde{\eta}_{2}) \in G_{1}} \mathbb{E} [(\tilde{\eta}_{1}, u_{1})_{H_{1}} + (\tilde{\eta}_{2}, u_{2})_{H_{2}}]^{2}, \quad (3.15)$$

with

$$c = -(f_0, z_2(\cdot; u))_{L^2(D)}$$

In order to calculate the first term on the right-hand side of (3.15) make use of the Cauchy–Bunyakovsky inequality (see [8], p. 186) and (2.13). We have

$$|(\tilde{f} - f_0, z_2(\cdot; u))_{L^2(D)}|^2 \le \le (Q^{-1}z_2(\cdot; u), z_2(\cdot; u))_{L^2(D)}(Q(\tilde{f} - f_0), \tilde{f} - f_0)_{L^2(D)} \le (Q^{-1}z_2(\cdot; u), z_2(\cdot; u))_{L^2(D)}.$$

The direct substitution shows that last inequality is transformed to an equality on the element

$$\tilde{f} = f_0 + \frac{Q^{-1}z_2(\cdot; u)}{(Q^{-1}z_2(\cdot; u), z(\cdot; u))_{L^2(D)}^{1/2}}$$

Hence,

$$\sup_{\tilde{f}\in G_0} \left[ (\tilde{f}_2 - f_2^{(0)}, z_2(\cdot; u))_{L^2(D)} \right]^2 = (Q^{-1}z_2(\cdot; u), z_2(\cdot; u))_{L^2(D)}.$$
(3.16)

In order to calculate the second term on the right-hand side of (3.15), note that the Cauchy–Bunyakovsky inequality, (2.18), (1.6), and (2.19) yields

$$\sup_{(\tilde{\eta}_{1},\tilde{\eta}_{2})\in G_{1}} \mathbb{E}[(\tilde{\eta}_{1},u_{1})_{H_{1}} + (\tilde{\eta}_{2},u_{2})_{H_{2}}]^{2}$$

$$\leq \sup_{(\tilde{\eta}_{1},\tilde{\eta}_{2})\in G_{1}} [(\tilde{Q}_{1}^{-1}u_{1},u_{1})_{H_{1}} \mathbb{E}(\tilde{Q}_{1}\tilde{\eta}_{1},\tilde{\eta}_{1})_{H_{1}} + (\tilde{Q}_{2}^{-1}u_{2},u_{2})_{H_{2}} \mathbb{E}(\tilde{Q}_{2}\tilde{\eta}_{2},\tilde{\eta}_{2})_{H_{2}}]$$

$$\leq (\tilde{Q}_{1}^{-1}u_{1},u_{1})_{H_{1}} + (\tilde{Q}_{2}^{-1}u_{2},u_{2})_{H_{2}}$$
(3.17)

It is easy to see that (3.17) takes the form at

$$\tilde{\eta}_1 = \nu_1 \tilde{Q}_1^{-1} u_1 / (\tilde{Q}_1^{-1} u_1, u_1)^{1/2}, \quad \tilde{\eta}_2 = \nu_2 \tilde{Q}_2^{-1} u_1 / (\tilde{Q}_2^{-1} u_1, u_1)^{1/2},$$

where  $\nu_1$  and  $\nu_2$  are uncorrelated random variables with  $\mathbb{E}\nu_1 = \mathbb{E}\nu_2 = 0$ ,  $\mathbb{E}\nu_1^2 = \mathbb{E}\nu_2^2 = 1$ . Therefore,

$$\sup_{(\tilde{\eta}_1,\tilde{\eta}_2)\in G_1} \mathbb{E}[(\tilde{\eta}_1,u_1)_{H_1} + (\tilde{\eta}_2,u_2)_{H_2}]^2 = (\tilde{Q}_1^{-1}u_1,u_1)_{H_1} + (\tilde{Q}_2^{-1}u_2,u_2)_{H_2}.$$
(3.18)

From (3.18), (3.16), and (3.15), we find

$$\inf_{c \in \mathbb{R}} \sup_{\tilde{f} \in G_0, (\tilde{\eta}_1, \tilde{\eta}) \in G_1} \mathbb{E} |l(\tilde{\mathbf{j}}, \tilde{\varphi}) - \widehat{l(\tilde{\mathbf{j}}, \tilde{\varphi})}|^2 = I(u),$$

at  $c = -(z_2(\cdot; u), f_0)_{L^2(D)}$ , where I(u) is determined by (3.4). This proves the required assertion.

## 4 Representation for guaranteed estimates and errors of estimation via solutions of mixed variational equations

Solving optimal control problem (3.1)–(3.4), we come to the following result.

**Theorem 1.** There exists a unique guaranteed estimate of  $l(\mathbf{j}, \varphi)$  which has the form

$$\widehat{\hat{l}(\mathbf{j},\varphi)} = (y_1, \hat{u}_1)_{H_1} + (y_2, \hat{u}_2)_{H_2} + \hat{c}, \qquad (4.1)$$

where

$$\hat{c} = -\int_D \hat{z}_2(x) f_0(x) \, dx, \quad \hat{u}_1 = \tilde{Q}_1 C_1 \mathbf{p}_1, \quad \hat{u}_2 = \tilde{Q}_2 C_2 p_2,$$
(4.2)

and the functions  $\mathbf{p}_1 \in H(\operatorname{div}, D)$  and  $\hat{z}_2, p_2 \in L^2(D)$  are determined as a solution of the following uniquely solvable problem:

$$\int_{D} (((\mathbf{A}(x))^{-1})^{T} \hat{\mathbf{z}}_{1}(x), \mathbf{q}_{1}(x))_{\mathbb{R}^{n}} dx - \int_{D} \hat{z}_{2}(x) \operatorname{div} \mathbf{q}_{1}(x) dx$$
$$= \int_{D} (\mathbf{l}_{1}(x) - C_{1}^{t} J_{H_{1}} \tilde{Q}_{1} C_{1} \mathbf{p}_{1}(x), \mathbf{q}_{1}(x))_{\mathbb{R}^{n}} dx \quad \forall \mathbf{q}_{1} \in H(\operatorname{div}, D), \quad (4.3)$$

$$-\int_{D} v_{1}(x) \operatorname{div} \hat{\mathbf{z}}_{1}(x) \, dx - \int_{D} c(x) z_{2}(x) v_{1}(x) \, dx$$
$$= \int_{D} (l_{2}(x) - C_{2}^{t} J_{H_{2}} \tilde{Q}_{2} C_{2} p_{2}(x)) v_{1}(x) \, dx \, \forall v_{1} \in L^{2}(D), \quad (4.4)$$

$$\int_{D} ((\mathbf{A}(x))^{-1} \mathbf{p}_1(x), \mathbf{q}_2(x))_{\mathbb{R}^n} dx - \int_{D} p_2(x) \operatorname{div} \mathbf{q}_2(x) \, dx = 0 \ \forall \mathbf{q}_2 \in H(\operatorname{div}, D),$$
(4.5)

$$-\int_{D} v_2(x) \operatorname{div} \hat{\mathbf{p}}_1(x) \, dx - \int_{D} c(x) p_2(x) v_2(x) dx$$
$$= \int_{D} v_2(x) Q^{-1} \hat{z}_2(x) \, dx \quad \forall v_2 \in L^2(D), \quad (4.6)$$

where  $\hat{\mathbf{z}}_1 \in H(\operatorname{div}, D)$ . The error of estimation  $\sigma$  is given by an expression

$$\sigma = l(\mathbf{p}_1, p_2)^{1/2}.$$
(4.7)

*Proof.* Let us prove that the solution to the optimal control problem (3.1)–(3.4) can be reduced to the solution of system (4.3)-(4.6).

Note first that functional I(u), where  $u \in H$  can be represented in the form

$$I(u) = \tilde{I}(u) + L(u) + \int_D Q^{-1} \tilde{z}_2^{(0)}(x) \tilde{z}_2^{(0)}(x) \, dx,$$

where

$$\tilde{I}(u) = \int_D Q^{-1} \tilde{z}_2(x; u) \tilde{z}_2(x; u) \, dx + (\tilde{Q}_1^{-1} u_1, u_1)_{H_1} + (\tilde{Q}_2^{-1} u_2, u_2)_{H_2},$$
$$L(u) = 2 \int_D Q^{-1} \tilde{z}_2(x; u) \tilde{z}_2^{(0)}(x) \, dx,$$

 $\tilde{z}_2(x; u)$  is the second component of the pair  $(\tilde{\mathbf{z}}_1(x; u), \tilde{z}_2(x; u))$  which the unique solution to problem (3.1), (3.2) at  $\mathbf{l}_0^{(1)}(x) = 0$ ,  $l_0^{(2)}(x) = 0$ , and  $\tilde{z}_2^{(0)}(x)$  is the second component of the pair  $(\tilde{\mathbf{z}}_1^{(0)}(x), \tilde{z}_2^{(0)}(x))$  which the unique solution to the same problem at u = 0.

Show that  $\tilde{I}(u)$  is a quadratic form corresponding to a symmetric continuous bilinear form

$$\pi(u,v) := \int_D Q^{-1} \tilde{z}_2(x;u) \tilde{z}_2(x;v) \, dx + (\tilde{Q}_1^{-1}u_1,v_1)_{H_1} + (\tilde{Q}_2^{-1}u_2,v_2)_{H_2} \tag{4.8}$$

on  $H \times H$  and L(u) is a linear continuous functional defined on H.

The continuity of form  $\pi(u, v)$  on  $H \times H$  means that for all  $u, v \in H$  the inequality

$$|\pi(u,v)| \le C ||u||_H ||v||_H \tag{4.9}$$

must be valid, where C = const.

To prove (4.9), we use the estimate

$$\int_{D} \tilde{z}_{2}^{2}(x; u) \, dx \le c_{1} \Big( \left\| C_{1}^{t} J_{H_{1}} u_{1} \right\|_{L^{2}(D)^{n}}^{2} + \left\| C_{2}^{t} J_{H_{2}} u_{2} \right\|_{L^{2}(D)}^{2} \Big), \quad c_{1} = \text{const}, \tag{4.10}$$

which follows from the inequality (3.3) at  $l_1 = 0$  and  $l_2 = 0$ . For the first term in the right-hand side of (4.8), due to the Cauchy–Bunyakovsky inequality and (4.10) we have

$$\left| \int_{D} Q^{-1} \tilde{z}_{2}(x; u) \tilde{z}_{2}(x; v) dx \right| \leq$$

$$\leq c_{2} \left( \int_{D} \tilde{z}_{2}^{2}(x; u) dx \right)^{1/2} \left( \int_{D} \tilde{z}_{2}^{2}(x; v) dx \right)^{1/2},$$

$$\leq c_{2} c_{3} \left( \left\| C_{1}^{t} J_{H_{1}} u_{1} \right\|_{L^{2}(D)^{n}}^{2} + \left\| C_{2}^{t} J_{H_{2}} u_{2} \right\|_{L^{2}(D)}^{2} \right)^{1/2}$$

$$\times c_{3} \left( \left\| C_{1}^{t} J_{H_{1}} v_{1} \right\|_{L^{2}(D)^{n}}^{2} + \left\| C_{2}^{t} J_{H_{2}} v_{2} \right\|_{L^{2}(D)}^{2} \right)^{1/2}$$

$$\leq c_{4} \left( \left\| u_{1} \right\|_{H_{1}}^{2} + \left\| u_{2} \right\|_{H_{2}}^{2} \right)^{1/2} \left( \left\| v_{1} \right\|_{H_{1}}^{2} + \left\| v_{2} \right\|_{H_{2}}^{2} \right)^{1/2} = c_{4} \| u \|_{H} \| v \|_{H},$$

$$(4.12)$$

where  $c_2, c_3, c_4 = \text{const.}$ 

Analogously,

$$(\tilde{Q}_1^{-1}u_1, v_1)_{H_1} + (\tilde{Q}_2^{-1}u_2, v_2)_{H_2} \le c_5 ||u||_H ||v||_H, \quad c_5 = \text{const.}$$

From this estimate and (4.12) it follows the validity of the inequality (4.9).

The continuity of linear functional L(u) on H can be proved similary.

It is obvious that

$$\tilde{I}(u) = \pi(u, u) \ge (Q_1^{-1}u_1, u_1)_{H_1} + (\tilde{Q}_2^{-1}u_2, u_2)_{H_2} \ge \alpha \|u\|_H^2 \quad \forall u \in H,$$

where  $\alpha$  is a constant from (2.20). In line with Theorem 1.1 proved in [1], p. 11, the latter statements imply the existence of the unique element  $\hat{u} := (\hat{u}_1, \hat{u}_2) \in H$  such that

$$I(\hat{u}) = \inf_{u \in H} I(u).$$

Therefore, for any fixed  $w \in H$  and  $\tau \in \mathbb{R}$  the function  $s(\tau) := I(\hat{u} + \tau w)$  reaches its minimum at a unique point  $\tau = 0$ , so that,

$$\frac{d}{d\tau}I(\hat{u}+\tau w)\mid_{\tau=0} = 0.$$
(4.13)

Since

$$z_2(x; \hat{u} + \tau w) = z_2(x; \hat{u}) + \tau \tilde{z}_2(x; w),$$

relation (4.13) yields

$$\frac{1}{2} \frac{d}{dt} I(\hat{u} + \tau w) \Big|_{\tau=0}$$
  
=  $(Q^{-1} z_2(\cdot; \hat{u}), \tilde{z}_2(\cdot; w))_{L^2(D)} + (\tilde{Q}_1^{-1} \hat{u}_1, w_1)_{H_1} + (\tilde{Q}_2^{-1} \hat{u}_2, w_2)_{H_2} = 0.$  (4.14)

Introduce a pair of functions  $(\mathbf{p}_1, p_2) \in H(\operatorname{div}, D) \times L^2(D)$  as the unique solution of the problem

$$\int_{D} ((\mathbf{A}(x))^{-1} \mathbf{p}_1(x), \mathbf{q}_2(x))_{\mathbb{R}^n} dx$$
$$-\int_{D} p_2(x) \operatorname{div} \mathbf{q}_2(x) dx = 0 \quad \forall \mathbf{q}_2 \in H(\operatorname{div}, D), \quad (4.15)$$

$$-\int_{D} v_2(x) \operatorname{div} \mathbf{p}_1(x) \, dx - \int_{D} c(x) p_2(x) v_2(x) dx$$
$$= \int_{D} v_2(x) Q^{-1} z_2(x; \hat{u}) \, dx \quad \forall v_2 \in L^2(D). \quad (4.16)$$

Setting in (4.15)  $\mathbf{q}_2 = \tilde{\mathbf{z}}_1(\cdot; w)$  and in (4.16)  $v_2 = \tilde{z}_2(\cdot; w)$ , we obtain

$$\int_{D} ((\mathbf{A}(x))^{-1} \mathbf{p}_1(x), \tilde{\mathbf{z}}_1(x; w))_{\mathbb{R}^n} dx - \int_{D} p_2(x) \operatorname{div} \tilde{\mathbf{z}}_1(x; w) \, dx = 0, \tag{4.17}$$

$$-\int_{D} \tilde{z}_{2}(x;w) \operatorname{div} \mathbf{p}_{1}(x) \, dx - \int_{D} c(x) p_{2}(x) \tilde{z}_{2}(x;w) \, dx$$
$$= \int_{D} v_{2}(x) Q^{-1} z_{2}(x;\hat{u}) \, dx. \quad (4.18)$$

From (4.17) and (4.18), we find

$$(Q^{-1}z_{2}(\cdot;\hat{u}),\tilde{z}_{2}(\cdot;w))_{L^{2}(D)} = -\int_{D}\tilde{z}_{2}(x;w)\operatorname{div}\mathbf{p}_{1}(x)\,dx - \int_{D}c(x)p_{2}(x)\tilde{z}_{2}(x;w)dx$$
$$+\int_{D}((\mathbf{A}(x))^{-1}\mathbf{p}_{1}(x),\tilde{\mathbf{z}}_{1}(x;w))_{\mathbb{R}^{n}}dx - \int_{D}p_{2}(x)\operatorname{div}\tilde{\mathbf{z}}_{1}(x;w)\,dx$$
$$=\int_{D}(((\mathbf{A}(x))^{-1})^{T}\tilde{\mathbf{z}}_{1}(x;w),\mathbf{p}_{1}(x))_{\mathbb{R}^{n}}dx - \int_{D}\tilde{z}_{2}(x;w)\operatorname{div}\mathbf{p}_{1}(x)\,dx$$
$$-\int_{D}p_{2}(x)\operatorname{div}\tilde{\mathbf{z}}_{1}(x;w)\,dx - \int_{D}c(x)p_{2}(x)\tilde{z}_{2}(x;w)dx$$
$$= -\int_{D}(C_{1}^{t}J_{H_{1}}w_{1},\mathbf{p}_{1}(x))_{\mathbb{R}^{n}}dx - \int_{D}(C_{2}^{t}J_{H_{2}}w_{2})p_{2}\,dx$$
$$= -(w_{1},C_{1}\mathbf{p}_{1})_{H_{1}} - (w_{2},C_{2}p_{2})_{H_{2}}.$$

Last relation and (4.14) imply

$$(w_1, C_1\mathbf{p}_1)_{H_1} + (w_2, C_2p_2)_{H_2} = (\tilde{Q}_1^{-1}\hat{u}_1, w_1)_{H_1} + (\tilde{Q}_2^{-1}\hat{u}_2, w_2)_{H_2}.$$

Hence,

$$\hat{u}_1 = \tilde{Q}_1 C_1 \mathbf{p}_1, \quad \hat{u}_2 = \tilde{Q}_2 C_2 p_2.$$
 (4.19)

Setting these expressions into (3.1), (3.2) and and denoting  $\mathbf{z}_1(x;\hat{u}) =: \hat{\mathbf{z}}_1(x), z_2(x;\hat{u}) =:$  $\hat{z}_2(x)$ , we establish that functions  $(\hat{z}_1, \hat{z}_2)$ ,  $(\mathbf{p}_1, \text{ and } p_2)$  satisfy (4.3) – (4.6); the unique solvability of the problem (4.3) – (4.6) follows from the existence of the unique minimum point  $\hat{u}$  of functional I(u).

Now let us establish the validity of formula (4.7). From (3.4) at  $u = \hat{u}$  and (4.19), it follows

$$\sigma^{2} = I(\hat{u}) = (Q^{-1}z_{2}(\cdot;\hat{u}), z_{2}(\cdot;\hat{u}))_{L^{2}(D)} + (\tilde{Q}_{1}^{-1}\hat{u}_{1}, \hat{u}_{1})_{H_{1}} + (\tilde{Q}_{2}^{-1}\hat{u}_{2}, \hat{u}_{2})_{H_{2}}$$
$$= (Q^{-1}\hat{z}_{2}, \hat{z}_{2})_{L^{2}(D)} + (C_{1}\mathbf{p}_{1}, \tilde{Q}_{1}C_{1}\mathbf{p}_{1})_{H_{1}} + (C_{2}p_{2}, \tilde{Q}_{2}C_{2}p_{2})_{H_{2}}.$$
(4.20)

Transform the first term in (4.20). Setting in (4.15) and (4.16)  $\mathbf{q}_2 = \hat{\mathbf{z}}_1$  and  $v_2 = \hat{z}_2$ , we find

$$\int_{D} ((\mathbf{A}(x))^{-1} \mathbf{p}_{1}(x), \hat{\mathbf{z}}_{1}(x))_{\mathbb{R}^{n}} dx - \int_{D} p_{2}(x) \operatorname{div} \hat{\mathbf{z}}_{1}(x) dx = 0,$$
$$-\int_{D} \hat{z}_{2}(x) \operatorname{div} \mathbf{p}_{1}(x) dx - \int_{D} c(x) p_{2}(x) \hat{z}_{2}(x) dx = \int_{D} \hat{z}_{2}(x) Q^{-1} \hat{z}_{2}(x) dx.$$

From the latter relations and from equations (4.3) and (4.4) with  $\mathbf{q}_1 = \mathbf{p}_1$  and  $v_1 = p_2$ , we have

$$(Q^{-1}\hat{z}_{2},\hat{z}_{2})_{L^{2}(D)} = -\int_{D}\hat{z}_{2}(x)\operatorname{div}\mathbf{p}_{1}(x)\,dx - \int_{D}c(x)p_{2}(x)\hat{z}_{2}(x)\,dx \\ + \int_{D}((\mathbf{A}(x))^{-1}\mathbf{p}_{1}(x),\hat{\mathbf{z}}_{1}(x))_{\mathbb{R}^{n}}dx - \int_{D}p_{2}(x)\operatorname{div}\hat{\mathbf{z}}_{1}(x)\,dx \\ = \int_{D}(((\mathbf{A}(x))^{-1})^{T}\hat{\mathbf{z}}_{1}(x),\mathbf{p}_{1}(x))_{\mathbb{R}^{n}}dx - \int_{D}z_{2}(x)\operatorname{div}\mathbf{p}_{1}(x)\,dx \\ - \int_{D}p_{2}(x)\operatorname{div}\hat{\mathbf{z}}_{1}(x)\,dx - \int_{D}c(x)p_{2}(x)\hat{z}_{2}(x)\,dx \\ = \int_{D}(\mathbf{l}_{1}(x) - C_{1}^{t}J_{H_{1}}\tilde{Q}_{1}C_{1}\mathbf{p}_{1}(x),\mathbf{p}_{1}(x))_{\mathbb{R}^{n}}\,dx + \int_{D}(l_{2}(x) - C_{2}^{t}J_{H_{2}}\tilde{Q}_{2}C_{2}p_{2}(x))p_{2}(x)\,dx \\ = \int_{D}(\mathbf{l}_{1}(x),\mathbf{p}_{1}(x),\mathbf{p}_{1}(x))_{\mathbb{R}^{n}}\,dx + \int_{D}l_{2}(x)p_{2}(x)\,dx \\ - (C_{1}\mathbf{p}_{1},\tilde{Q}_{1}C_{1}\mathbf{p}_{1})_{H_{1}} - (C_{2}p_{2},\tilde{Q}_{2}C_{2}p_{2})_{H_{2}}.$$
(4.21)  
(4.20) and (4.21), we otain (4.7). Theorem is proved.

From (4.20) and (4.21), we obtain (4.7). Theorem is proved.

Note that the pair of functions  $(\hat{\mathbf{z}}(x), \hat{z}_2(x)) = (\mathbf{z}_1(x; \hat{u}), z_2(x; \hat{u}))$  and the element  $u = \hat{u} \in H$ is a solution of optimal control problem (3.1), (3.2), (3.4).

In the following theorem we obtain an alternative representation for the guaranteed estimate of quantity  $l(\mathbf{j}, \varphi)$  which is expressed via a solution of certain system of mixed variational equations not depending on  $\mathbf{l}_1$  and  $l_2$ .

**Theorem 2.** The guaranteed estimate of  $l(\mathbf{j}, \varphi)$  has the form

$$\widehat{\widehat{l(\mathbf{j},\varphi)}} = l(\hat{\mathbf{j}},\hat{\varphi}), \qquad (4.22)$$

where the pair  $(\hat{\mathbf{j}}, \hat{\varphi}) \in H(\operatorname{div}, D) \times L^2(D)$  is a solution to the following problem:

$$\int_{D} (((\mathbf{A}(x))^{-1})^{T} \hat{\mathbf{p}}_{1}(x), \mathbf{q}_{1}(x))_{\mathbb{R}^{n}} dx - \int_{D} \hat{p}_{2}(x) \operatorname{div} \mathbf{q}_{1}(x) dx$$
$$= \int_{D} (C_{1}^{t} J_{H_{1}} \tilde{Q}_{1}(y_{1} - C_{1} \hat{\mathbf{j}})(x), \mathbf{q}_{1}(x))_{\mathbb{R}^{n}} dx \ \forall \mathbf{q}_{1} \in H(\operatorname{div}, D), \quad (4.23)$$

$$-\int_{D} v_{1}(x) \operatorname{div} \hat{\mathbf{p}}_{1}(x) \, dx - \int_{D} c(x) \hat{p}_{2}(x) v_{1}(x) \, dx$$
$$= \int_{D} C_{2}^{t} J_{H_{2}} \tilde{Q}_{2}(y_{2} - C_{2} \hat{\varphi})(x) v_{1}(x) \, dx \quad \forall v_{1} \in L^{2}(D), \quad (4.24)$$

$$\int_{D} ((\mathbf{A}(x))^{-1} \hat{\mathbf{j}}(x), \mathbf{q}_2(x))_{\mathbb{R}^n} dx - \int_{D} \hat{\varphi}(x) \operatorname{div} \mathbf{q}_2(x) dx = 0 \quad \forall \mathbf{q}_2 \in H(\operatorname{div}, D),$$
(4.25)

$$-\int_{D} v_{2}(x) \operatorname{div} \hat{\mathbf{j}}(x) \, dx - \int_{D} c(x) \hat{\varphi}(x) v_{2}(x) \, dx$$
$$= \int_{D} v_{2}(x) (Q^{-1} \hat{p}_{2}(x) - f_{0}(x)) \, dx \quad \forall v_{2} \in L^{2}(D), \quad (4.26)$$

where equalities (4.23)-(4.26) are fulfilled with probability 1. Problem (4.23) - (4.26) is uniquely solvable.

The random fields  $\hat{\mathbf{j}}$ ,  $\hat{\mathbf{p}}_1$  and  $\hat{\varphi}$ ,  $\hat{p}_2$ , whose realizations satisfy problem (4.23)–(4.26), belong to the spaces  $L^2(\Omega, H(\text{div}, D))$  and  $L^2(\Omega, L^2(D))$ , respectively.

*Proof.* Note that unique solvability of problem (4.23)-(4.26) at realizations  $y_1$  and  $y_2$  that belong with probability 1 to the spaces  $H_1$  and  $H_2$ , respectively, can be proved similarly as to the problem (4.3)-(4.6).

Namely, consider optimal control problem of the system described by  $^2$ 

$$\hat{\mathbf{p}}_1 \in L^2(\Omega, H(\operatorname{div}, D)) \quad \hat{p}_2 \in L^2(\Omega, L^2(D)),$$
(4.27)

$$\mathbb{E}\Big[\int_{D} (((\mathbf{A}(x))^{-1})^{T} \hat{\mathbf{p}}_{1}(x; u), \mathbf{q}_{1}(x))_{\mathbb{R}^{n}} dx\Big] - \mathbb{E}\Big[\int_{D} \hat{p}_{2}(x; u) \operatorname{div} \mathbf{q}_{1}(x) dx\Big] \\ = \mathbb{E}\Big[\int_{D} (\mathbf{d}_{1}(x) - (C_{1}^{t} J_{H_{1}} u_{1})(x)), \mathbf{q}_{1}(x))_{\mathbb{R}^{n}} dx\Big] \ \forall \mathbf{q}_{1} \in L^{2}(\Omega, H(\operatorname{div}, D)), \quad (4.28)$$

$$-\mathbb{E}\left[\int_{D} v_{1}(x) \operatorname{div} \hat{\mathbf{p}}_{1}(x; u) \, dx\right] - \mathbb{E}\left[\int_{D} c(x) \hat{p}_{2}(x; u) v_{1}(x) \, dx\right]$$
$$= \mathbb{E}\left[\int_{D} (d_{2}(x) - (C_{2}^{t} J_{H_{2}} u_{2})(x)) v_{1}(x) \, dx\right] \quad \forall v_{1} \in L^{2}(\Omega, L^{2}(D)), \quad (4.29)$$

<sup>2</sup> Unique solvability of problem (4.27)–(4.29) for every fixed  $u = (u_1, u_2)$  follows from correctness of stochastic statement of mixed variational problem (2.2) on page 1427 in [9].

with cost function

$$I(u) = \mathbb{E}\left[\int_{D} Q^{-1}(\hat{p}_{2}(\cdot; u) - Qf_{0})(x)(\hat{p}_{2}(\cdot; u) - Qf_{0})(x) dx\right]$$
  
+ $(\tilde{Q}_{1}^{-1}u_{1}, u_{1})_{L^{2}(\Omega, H_{1})} + (\tilde{Q}_{2}^{-1}u_{2}, u_{2})_{L^{2}(\Omega, H_{2})} \rightarrow \inf_{u=(u_{1}, u_{2})\in L^{2}(\Omega, H)=L^{2}(\Omega, H_{1}\times H_{2})},$ 

where

$$\mathbf{d}_{1}(x) = C_{1}^{t} J_{H_{1}} \tilde{Q}_{1} y_{1}(x),$$
  
$$d_{2}(x) = C_{2}^{t} J_{H_{2}} \tilde{Q}_{2} y_{2}(x).$$

Functional I(u) is quadratic and coercive on the space  $L^2(\Omega, H)$ . Therefore, there exists a unique element  $\hat{u} \in L^2(\Omega, H)$  such that

$$I(\hat{u}) = \inf_{u \in L^2(\Omega, H)} I(u).$$

Next, denoting by  $(\hat{\mathbf{j}}, \hat{\varphi}) \in L^2(\Omega, H(\operatorname{div}, D)) \times L^2(\Omega, L^2(D))$  a unique solution of the problem:

$$\mathbb{E}\left[\int_{D} ((\mathbf{A}(x))^{-1} \hat{\mathbf{j}}(x), \mathbf{q}_{2}(x))_{\mathbb{R}^{n}} dx\right] - \mathbb{E}\left[\int_{D} \hat{\varphi}(x) \operatorname{div} \mathbf{q}_{2}(x) dx\right] = 0 \quad \forall \mathbf{q}_{2} \in L^{2}(\Omega, H(\operatorname{div}, D)),$$

$$-\mathbb{E}\left[\int_{D} v_{2}(x) \operatorname{div} \hat{\mathbf{j}}(x) \, dx\right] - \mathbb{E}\left[\int_{D} c(x)\hat{\varphi}(x)v_{2}(x) \, dx\right]$$
$$= \mathbb{E}\left[\int_{D} v_{2}(x)(Q^{-1}\hat{p}_{2}(x;\hat{u}) - f_{0}(x)) \, dx\right] \quad \forall v_{2} \in L^{2}(\Omega, L_{2}),$$

and making use of virtually the same reasoning that led to the proof of Theorem 1, we arrive at the equalities  $\hat{u}_1 = \tilde{Q}_1 C_1 \hat{\mathbf{j}}$  and  $\hat{u}_2 = \tilde{Q}_2 C_2 \hat{\varphi}$ . Denoting  $\hat{\mathbf{p}}_1(x) = \hat{\mathbf{p}}_1(x; \hat{u})$ ,  $\hat{p}_2(x) = \hat{p}_2(x; \hat{u})$ , we deduce from the latter statement the unique solvability of problem

$$\mathbb{E}\left[\int_{D} (((\mathbf{A}(x))^{-1})^{T} \hat{\mathbf{p}}_{1}(x), \mathbf{q}_{1}(x))_{\mathbb{R}^{n}} dx\right] - \mathbb{E}\left[\int_{D} \hat{p}_{2}(x) \operatorname{div} \mathbf{q}_{1}(x) dx\right]$$
$$= \mathbb{E}\left[\int_{D} (C_{1}^{t} J_{H_{1}} \tilde{Q}_{1}(y_{1} - C_{1} \hat{\mathbf{j}})(x), \mathbf{q}_{1}(x))_{\mathbb{R}^{n}} dx\right] \forall \mathbf{q}_{1} \in L^{2}(\Omega, H(\operatorname{div}, D)),$$

$$-\mathbb{E}\left[\int_{D} v_1(x) \operatorname{div} \hat{\mathbf{p}}_1(x) \, dx\right] - \mathbb{E}\left[\int_{D} c(x) \hat{p}_2(x) v_1(x) \, dx\right]$$
$$= \mathbb{E}\left[\int_{D} C_2^t J_{H_2} \tilde{Q}_2(y_2 - C_2 \hat{\varphi})(x) v_1(x) \, dx\right] \quad \forall v_1 \in L^2(\Omega, L^2(D)),$$

$$\mathbb{E}\left[\int_{D} ((\mathbf{A}(x))^{-1} \hat{\mathbf{j}}(x), \mathbf{q}_{2}(x))_{\mathbb{R}^{n}} dx\right] - \mathbb{E}\left[\int_{D} \hat{\varphi}(x) \operatorname{div} \mathbf{q}_{2}(x) dx\right] = 0 \quad \forall \mathbf{q}_{2} \in L^{2}(\Omega, H(\operatorname{div}, D)),$$

$$-\mathbb{E}\left[\int_{D} v_{2}(x) \operatorname{div} \hat{\mathbf{j}}(x) \, dx\right] - \mathbb{E}\left[\int_{D} c(x)\hat{\varphi}(x)v_{2}(x) \, dx\right]$$
$$= \mathbb{E}\left[\int_{D} v_{2}(x)(Q^{-1}\hat{p}_{2}(x) - f_{0}(x)) \, dx\right] \quad \forall v_{2} \in L^{2}(\Omega, L^{2}(D)).$$

From here following the argument of paper [9], we conclude that problem (4.23)-(4.26) is uniquely solvable.

Now let us prove the representation (4.22). By virtue of (2.16) and (4.2),

$$\widehat{\hat{l}(\mathbf{j},\varphi)} = (y_1, \hat{u}_1)_{H_1} + (y_2, \hat{u}_2)_{H_2} + \hat{c}$$
$$= (y_1, \tilde{Q}_1 C_1 \mathbf{p}_1)_{H_1} + (y_2, \tilde{Q}_2 C_2 p_2)_{H_2} - (\hat{z}_2, f_0)_{L^2(D)}.$$
(4.30)

Putting in (4.23) and (4.24)  $\mathbf{q}_1 = \mathbf{p}_1$  and  $v_1 = p_2$ , we obtain

$$\int_{D} (((\mathbf{A}(x))^{-1})^{T} \hat{\mathbf{p}}_{1}(x), \mathbf{p}_{1}(x))_{\mathbb{R}^{n}} dx - \int_{D} \hat{p}_{2}(x) \operatorname{div} \mathbf{p}_{1}(x) dx$$
$$= \int_{D} (C_{1}^{t} J_{H_{1}} \tilde{Q}_{1}(y_{1} - C_{1} \hat{\mathbf{j}})(x), \mathbf{p}_{1}(x))_{\mathbb{R}^{n}} dx, \quad (4.31)$$

$$-\int_{D} p_{2}(x) \operatorname{div} \hat{\mathbf{p}}_{1}(x) \, dx - \int_{D} c(x) \hat{p}_{2}(x) p_{2}(x) \, dx$$
$$= \int_{D} (C_{2}^{t} J_{H_{2}} \tilde{Q}_{2}(y_{2} - C_{2} \hat{\varphi})(x) p_{2}(x) \, dx. \quad (4.32)$$

Putting in (4.5) and (4.6)  $\mathbf{q}_2 = \hat{\mathbf{p}}_1$  and  $v_2 = \hat{p}_2$ , we find

$$\int_{D} ((\mathbf{A}(x))^{-1} \mathbf{p}_1(x), \hat{\mathbf{p}}_1(x))_{\mathbb{R}^n} dx - \int_{D} p_2(x) \operatorname{div} \hat{\mathbf{p}}_1(x) \, dx = 0, \tag{4.33}$$

$$-\int_{D} \hat{p}_{2}(x) \operatorname{div} \mathbf{p}_{1}(x) \, dx - \int_{D} c(x) p_{2}(x) \hat{p}_{2}(x) \, dx = \int_{D} \hat{p}_{2}(x) Q^{-1} \hat{z}_{2}(x) \, dx.$$
(4.34)

Since the sum of the left-hand sides of equalities (4.31) and (4.32) is equal to the sum of the left-hand sides of (4.33) and (4.34), we find from (4.30)

$$\widehat{\hat{l}(\mathbf{j},\varphi)} = (C_1 \hat{\mathbf{j}}, \tilde{Q}_1 C_1 \mathbf{p}_1)_{H_1} + (C_2 \hat{\varphi}, \tilde{Q}_2 C_2 p_2)_{H_2} + (Q^{-1} \hat{p}_2 - f_0, \hat{z}_2)_{L^2(D)}.$$
(4.35)

Next, putting in (4.25), (4.26)  $\mathbf{q}_2 = \hat{\mathbf{z}}_1$ ,  $v_2 = \hat{z}_2$  and in (4.3), (4.4)  $\mathbf{q}_1 = \hat{\mathbf{j}}$ ,  $v_1 = \hat{\varphi}$ , we obtain

$$\int_{D} ((\mathbf{A}(x))^{-1} \hat{\mathbf{j}}(x), \hat{\mathbf{z}}_{1}(x))_{\mathbb{R}^{n}} dx - \int_{D} \hat{\varphi}(x) \operatorname{div} \hat{\mathbf{z}}_{1}(x) dx = 0, \qquad (4.36)$$

$$-\int_{D} \hat{z}_{2}(x) \operatorname{div} \hat{\mathbf{j}}(x) \, dx - \int_{D} c(x) \hat{\varphi}(x) \hat{z}_{2}(x) \, dx$$
$$= \int_{D} \hat{z}_{2}(x) (Q^{-1} \hat{p}_{2}(x) - f_{0}(x)) \, dx, \quad (4.37)$$

and

$$\int_{D} (((\mathbf{A}(x))^{-1})^{T} \hat{\mathbf{z}}_{1}(x), \hat{\mathbf{j}}(x))_{\mathbb{R}^{n}} dx - \int_{D} \hat{z}_{2}(x) \operatorname{div} \hat{\mathbf{j}}(x) dx$$
$$= \int_{D} (\mathbf{l}_{1}(x) - C_{1}^{t} J_{H_{1}} \tilde{Q}_{1} C_{1} \mathbf{p}_{1}(x), \hat{\mathbf{j}}(x))_{\mathbb{R}^{n}} dx, \quad (4.38)$$

$$-\int_{D} \hat{\varphi}(x) \operatorname{div} \hat{\mathbf{z}}_{1}(x) \, dx - \int_{D} c(x) z_{2}(x) \hat{\varphi}(x) \, dx$$
$$= \int_{D} (l_{2}(x) - C_{2}^{t} J_{H_{2}} \tilde{Q}_{2} C_{2} p_{2}(x)) \hat{\varphi}(x) \, dx. \quad (4.39)$$

Relations (4.36)–(4.39) imply

$$\int_D \hat{z}_2(x) (Q^{-1} \hat{p}_2(x) - f_0(x)) \, dx = (\mathbf{l}_1 - \tilde{Q}_1 C_1 \hat{\mathbf{p}}_1, C_1 \hat{\mathbf{j}}(x))_{H_1} + (l_2 - \tilde{Q}_2 C_2 \hat{p}_2, C_2 \hat{\varphi}(x))_{H_2}.$$

By virtue of (4.35), it follows from here representation (4.22).

**Remark 1.** Notice that in representation  $l(\hat{\mathbf{j}}, \hat{\varphi})$  for minimax estimate  $\widehat{l(\mathbf{j}, \varphi)}$  the functions  $\hat{\mathbf{j}}, \hat{\varphi}$  which are defined from equations (4.23)–(4.26) do not depend on specific form of functional l and hence can be taken as a good estimate for unknown solution  $\mathbf{j}, \varphi$  of Dirichlet problem (2.3), (2.4).

## 5 Approximate Guaranteed Estimates: The Theorems on Convergence

In this section we introduce the notion of approximate guaranteed estimates of  $l(\mathbf{j}, \varphi)$  and prove their convergence to  $\widehat{l(\mathbf{j}, \varphi)}$ . To do this, we use the mixed finite element method for solving the aforementioned problems (4.3)-(4.6) and (4.23)–(4.26) and obtain approximate estimates via solutions of linear algebraic equations. We show their convergence to the optimal estimates.

In this section D is supposed to be bounded and connected domain of  $\mathbb{R}^n$  with polyhedral boundary  $\Gamma$ . First, we note that according to the mixed finite element method, an approximation  $(\mathbf{j}^h, \varphi^h)$  to the solution  $(\mathbf{j}, \varphi)$  of the problem (2.11), (2.12) is sought in the finite element space  $V_1^h \times V_2^h$  given by

$$V_1^h = \{ \mathbf{q}^h \in H(\operatorname{div}; D) : \mathbf{q}^h |_K \in (P^k(K))^n + \mathbf{x} P^k(K) \quad \forall K \in \mathcal{T}_h \},$$
$$V_2^h = \{ v^h \in L^2(D) : v^h |_K \in P^k(K) \quad \forall K \in \mathcal{T}_h \},$$

where  $\mathcal{T}_h$  is a simplicial triangulation of D,  $P^k(K)$  denotes the space of polynomials on K of degree at most  $k, k \ge 0, \mathbf{x} := (x_1, \ldots, x_n)$ , and is defined by requiring that

$$a(\mathbf{j}^h, \mathbf{q}^h) + b(\mathbf{q}^h, \varphi^h) = 0 \quad \forall \mathbf{q}^h \in V_1^h,$$
(5.1)

$$b(\mathbf{j}^{h}, v^{h}) - c(\varphi^{h}, v^{h}) = (f, v^{h})_{L^{2}(D)} \quad \forall v^{h} \in V_{2}^{h}$$
(5.2)

Here the bilinear forms  $a(\cdot, \cdot)$ ,  $b(\cdot, \cdot)$ , and  $c(\cdot, \cdot)$  are defined by (2.7)–(2.9). Hence system (5.1), (5.2) can be rewritten in the form

$$\int_{D} ((\mathbf{A}(x))^{-1} \mathbf{j}^{h}(x), \mathbf{q}^{h}(x))_{\mathbb{R}^{n}} dx - \int_{D} \varphi^{h}(x) \operatorname{div} \mathbf{q}^{h}(x) dx = 0 \quad \forall \mathbf{q}^{h} \in V_{1}^{h}$$
(5.3)

$$\int_D v(x) \operatorname{div} \mathbf{j}^h(x) dx + \int_D c(x) \varphi^h(x) v^h(x) dx = \int_D f(x) v^h(x) dx \ \forall v^h \in V_2^h.$$
(5.4)

It can be easily verified that the bilinear form  $a|_{V_1^h \times V_1^h}$  is uniformly coercive on Ker  $B|_{V_1^h}$ and that the bilinear form  $b|_{V_1^h \times V_2^h}$  satisfies the inf-sup condition (Babuska-Brezzi condition). Moreover, we have Ker  $B^t|_{V_2^h} = \emptyset$  and therefore, the mixed discretization (5.1), (5.2) (or what is the same (5.3), (5.4)) is uniquely solvable and the following estimates are valid

$$\|\mathbf{j} - \mathbf{j}^{h}\|_{H(\operatorname{div},D)} + \|\varphi - \varphi^{h}\|_{L^{2}(D)} \leq \tilde{c} \left( \inf_{\mathbf{q}^{h} \in V_{1}^{h}} \|\mathbf{j} - \mathbf{q}^{h}\|_{H(\operatorname{div},D)} + \inf_{v^{h} \in V_{2}^{h}} \|\varphi - v^{h}\|_{L^{2}(D)} \right), \quad (5.5)$$

$$\|\mathbf{j}^{h}\|_{H(\operatorname{div},D)} + \|\varphi^{h}\|_{L^{2}(D)} \leq \tilde{\tilde{c}}\|f\|_{L^{2}(D)},$$
(5.6)

where  $\tilde{c}$  and  $\tilde{\tilde{c}}$  are constant not depending on h (cf. e.g. [10]; §II, Prop. 2.11]) and [13], page 102).

Note that since div  $\mathbf{q}_h|_K \in P^k(K)$ ,  $K \in \mathcal{T}_h$ , then a natural choice for the approximation of the variable  $\varphi$  is to use piecewise polynomials of degree at most k leading to the space  $V_2^h$  defined above. Due to Proposition 3.9 of [10], p. 132, it follows that the sequences of the subspaces  $\{V_1^h\}$  and  $\{V_2^h\}$  are complete in H(div; D) and  $L^2(D)$ , respectively, in the following sense.

**Definition 2.** Let V be a Hilbert space. Introduce a sequence of finite-dimensional subspaces  $V^h$  in V, defined by an infinite set of parameters  $h_1, h_2, \ldots$  with  $\lim_{k\to\infty} h_k = 0$ .

We say that sequence  $\{V^h\}$  is complete in V, if for any  $v \in V$  and  $\epsilon > 0$  there exists an  $\hat{h} = \hat{h}(v, \epsilon) > 0$  such that  $\inf_{w \in V^h} ||v - w||_H < \varepsilon$  for any  $h < \hat{h}$ . In other words, the completeness of sequence  $\{V^h\}$  means that any element  $v \in V$  may be approximated with any degree of accuracy by elements of  $\{V^h\}$ .

Completeness of  $\{V_1^h\}$  and  $\{V_2^h\}$  in H(div; D) and  $L^2(D)$  together with estimate (5.5) imply that

$$\lim_{h \to 0} \left( \|\mathbf{j} - \mathbf{j}^{h}\|_{H(\operatorname{div},D)} + \|\varphi - \varphi^{h}\|_{L^{2}(D)} \right) = 0.$$
(5.7)

Now we are in a position to give the following definition.

Take an approximate guaranteed estimate of  $l(\mathbf{j}, \varphi)$  as

$$\widehat{l^{h}(\mathbf{j},\varphi)} = (u_{1}^{h}, y_{1})_{H_{1}} + (u_{2}^{h}, y_{2})_{H_{1}} + c^{h},$$
(5.8)

where  $u_1^h = \tilde{Q}_1 C_1 \mathbf{p}_1^h$ ,  $u_2^h = \tilde{Q}_2 C_2 p_2^h$ ,  $c^h = \int_D \hat{z}_2^h(x) f_0(x) dx$ , and functions  $\hat{\mathbf{z}}_1^h, \mathbf{p}_1^h \in V_1^h$  and  $\hat{z}_2^h, p_2^h \in V_2^h$  are determined from the following uniquely solvable system of variational equalities

$$\int_{D} (\mathbf{A}^{-1}(x) \, \hat{\mathbf{z}}_{1}^{h}(x), \mathbf{q}_{1}^{h}(x))_{\mathbb{R}^{n}} dx + \int_{D} \hat{z}_{2}^{h}(x) \operatorname{div} \mathbf{q}_{1}^{h}(x) \, dx$$
$$= \int_{D} (\mathbf{l}_{1}(x) - C_{1}^{t} J_{H_{1}} \tilde{Q}_{1} C_{1} \mathbf{p}_{1}^{h}(x), \mathbf{q}_{1}^{h}(x))_{\mathbb{R}^{n}} \, dx \quad \forall \mathbf{q}_{1}^{h} \in V_{1}^{h}, \quad (5.9)$$

$$\int_{D} v_1^h(x) \operatorname{div} \hat{\mathbf{z}}_1^h(x) \, dx = \int_{D} (l_2(x) - C_2^t J_{H_2} \tilde{Q}_2 C_2 p_2^h(x)) v_1^h(x) \, dx \quad \forall v_1^h \in V_2^h, \tag{5.10}$$

$$\int_{D} (\mathbf{A}^{-1}(x) \, \mathbf{p}_{1}^{h}(x), \mathbf{q}_{2}^{h}(x))_{\mathbb{R}^{n}} dx + \int_{D} p_{2}^{h}(x) \operatorname{div} \mathbf{q}_{2}^{h}(x) \, dx = 0 \quad \forall \mathbf{q}_{2}^{h} \in V_{1}^{h}, \tag{5.11}$$

$$\int_{D} v_{2}^{h}(x) \operatorname{div} \mathbf{p}_{1}^{h}(x) \, dx = \int_{D} v_{2}^{h}(x) Q^{-1} \hat{z}_{2}^{h}(x) \, dx \quad \forall v_{2}^{h} \in V_{2}^{h}.$$
(5.12)

The unique solvability of system (5.9)–(5.12) follows from the same reasoning of the previous sections which led to the proof of Theorem 1 with  $H(\operatorname{div}, D)$  and  $L^2(D)$  being replaced by  $V_1^h$  and  $V_2^h$ , respectively.

**Theorem 3.** Let  $\hat{\mathbf{z}}_1, \mathbf{p}_1 \in H(\text{div}, D)$ ,  $\hat{z}_2, p_2 \in L^2(D)$  and  $\hat{\mathbf{z}}_1^h, \mathbf{p}_1^h \in V_1^h$ ,  $\hat{z}_2^h, p_2^h \in V_2^h$  be solutions of problems (4.3)–(4.6) and (5.9)–(5.12), respectively.

Then the following hold:

$$\|\hat{\mathbf{z}}_1 - \hat{\mathbf{z}}_1^h\|_{H(\operatorname{div},D)} + \|\hat{z}_2 - \hat{z}_2^h\|_{L^2(D)} \to 0 \quad \text{as} \quad h \to 0,$$
 (5.13)

i)

$$\|\mathbf{p}_1 - \mathbf{p}_1^h\|_{H(\operatorname{div},D)} + \|p_2 - p_2^h\|_{L^2(D)} \to 0 \quad \text{as} \quad h \to 0.$$
 (5.14)

ii) Approximate guaranteed estimate  $\widehat{l^{h}(\mathbf{j},\varphi)}$  of  $l(\mathbf{j},\varphi)$  tends to a guaranteed estimate  $\widehat{l(\mathbf{j},\varphi)}$  of this expression as  $h \to 0$  in the sense that

$$\lim_{h \to 0} \mathbb{E} |\widehat{l^h(\mathbf{j},\varphi)} - \widehat{\widehat{l(\mathbf{j},\varphi)}}|^2 = 0$$

Moreover,

$$\lim_{h \to 0} \sup_{\tilde{f} \in G_0, (\tilde{\eta}_1, \tilde{\eta}_2) \in G_1} \mathbb{E} |\widehat{l^h(\tilde{\mathbf{j}}, \tilde{\varphi})} - \widehat{\tilde{l(\tilde{\mathbf{j}}, \tilde{\varphi})}}|^2 = 0,$$
(5.15)

and

$$\lim_{h\to 0} \sup_{\tilde{f}\in G_0, (\tilde{\eta}_1, \tilde{\eta}_2)\in G_1} \mathbb{E}|\widehat{l^h(\tilde{\mathbf{j}}, \tilde{\varphi})} - l(\tilde{\mathbf{j}}, \tilde{\varphi})|^2 = \sup_{\tilde{f}\in G_0, (\tilde{\eta}_1, \tilde{\eta}_2)\in G_1} \mathbb{E}|\widehat{l(\tilde{\mathbf{j}}, \tilde{\varphi})} - l(\tilde{\mathbf{j}}, \tilde{\varphi}))|^2,$$
(5.16)

where  $\tilde{f}$ ,  $\tilde{\mathbf{j}}$ , and  $\tilde{\varphi}$  have the same sence as in the definition 1,  $\widehat{l^h(\mathbf{j}, \tilde{\varphi})} = (u_1^h, \tilde{y}_1)_{H_1} + (u_2^h, \tilde{y}_2)_{H_1} + c^h$ ,  $\tilde{y}_1 = C_1 \tilde{\mathbf{j}} + \tilde{\eta}_1$ ,  $\tilde{y}_2 = C_2 \tilde{\varphi} + \tilde{\eta}_2$ .

*Proof.* Denote by  $\{h_n\}$  any sequence of positive numbers such that  $h_n \to 0$  when  $n \to \infty$ . Let  $\mathbf{z}_1^{h_n}(\cdot; u) \in V_1^{h_n}$ , i  $z_2^{h_n}(\cdot; u) \in V_2^{h_n}$  be a solution of the problem

$$\int_{D} (((\mathbf{A}(x))^{-1})^{T} \mathbf{z}_{1}^{h_{n}}(x; u), \mathbf{q}^{h_{n}}(x))_{\mathbb{R}^{n}} dx - \int_{D} z_{2}^{h_{n}}(x; u) \operatorname{div} \mathbf{q}^{h_{n}}(x) dx = \int_{D} (\mathbf{l}_{1}(x) - (C_{1}^{t} J_{H_{1}} u_{1})(x), \mathbf{q}^{h_{n}}(x))_{\mathbb{R}^{n}} dx \quad \forall \mathbf{q}^{h_{n}} \in V_{1}^{h_{n}}, \quad (5.17)$$

$$-\int_{D} v^{h_n}(x) \operatorname{div} \mathbf{z}_1^{h_n}(x; u) \, dx - \int_{D} c(x) z_2^{h_n}(x; u) v^{h_n}(x) \, dx$$
$$= \int_{D} (l_2(x) - (C_2^t J_{H_2} u_2)(x)) v^{h_n}(x) \, dx \quad \forall v^{h_n} \in V_2^{h_n}.$$
(5.18)

Then

$$\hat{\mathbf{z}}_{1}^{h_{n}}(x) = \mathbf{z}_{1}^{h_{n}}(x; u^{h_{n}}), \quad \hat{z}_{2}^{h_{n}}(x) = z_{2}^{h_{n}}(x; u^{h_{n}}).$$
(5.19)

Problem (5.17), (5.18) can be rewritten as

$$a^*(\mathbf{j}^h, \mathbf{q}^h) + b(\mathbf{q}^h, \varphi^h) = 0 \quad \forall \mathbf{q}^h \in V_1^h,$$
$$b(\mathbf{j}^h, v^h) - c(\varphi^h, v^h) = (f, v^h)_{L^2(D)} \quad \forall v^h \in V_2^h,$$

where

$$a^*(\mathbf{j}^h, \mathbf{q}^h) = a(\mathbf{q}^h, \mathbf{j}^h) = \int_D (((\mathbf{A}(x))^{-1})^T \mathbf{j}^h(x), \mathbf{q}^h(x))_{\mathbb{R}^n} dx$$

and the bilinear forms  $a(\cdot, \cdot)$ ,  $b(\cdot, \cdot)$ , and  $c(\cdot, \cdot)$  are defined by (2.7), (2.8), and (2.9) respectively.

Since the bilinear form  $a(\mathbf{j}^h, \mathbf{q}^h)$  is uniformly coercive on Ker  $B|_{V_1^h}$  then the form is also uniformly coercive on Ker  $B|_{V_1^h}$  with the same constant and hence system (5.17), (5.18) is uniquely solvable. Theorem 1.2, Prop. 2.11 in §2 from [10] (see also [13], page 102), and uniform coerciveness of the form  $a^*(\mathbf{j}^h, \mathbf{q}^h)$  on Ker  $B|_{V_1^h}$  imply that the following estimates are valid

$$\|\mathbf{z}_{1}(\cdot; u) - \mathbf{z}_{1}^{h_{n}}(\cdot; u)\|_{H(\operatorname{div}, D)} + \|z_{2}(\cdot; u) - z_{2}^{h_{n}}(\cdot; u)\|_{L^{2}(D)}$$

$$\leq \tilde{c} \left(\inf_{\mathbf{q}^{h_{n}} \in V_{1}^{h_{n}}} \|\mathbf{z}_{1}(\cdot; u) - \mathbf{q}^{h_{n}}\|_{H(\operatorname{div}, D)} + \inf_{v^{h_{n}} \in V_{2}^{h}} \|z_{2}(\cdot; u) - v^{h_{n}}\|_{L^{2}(D)}\right), \quad (5.20)$$

$$\begin{aligned} \|\mathbf{z}_{1}^{h_{n}}(\cdot; u)\|_{H(\operatorname{div},D)} + \|z_{2}^{h_{n}}(\cdot; u)\|_{L^{2}(D)} \\ &\leq \tilde{\tilde{c}}\left(\|\mathbf{l}_{1} - C_{1}^{t}J_{H_{1}}u_{1}\|_{L^{2}(D)^{n}} + \|l_{2} - C_{2}^{t}J_{H_{2}}u_{2}\|_{L^{2}(D)}\right), \end{aligned}$$
(5.21)

where  $\tilde{c}, \tilde{\tilde{c}}$  are constants not depending on h and  $(\mathbf{z}_1(\cdot; u), z_2(\cdot; u))$  is a solution of system of variational equations (3.1), (3.2).

From estimate (5.20) and completeness of  $\{V_1^h\}$  and  $\{V_2^h\}$  in H(div; D) and  $L^2(D)$ , it follows that

$$\|\mathbf{z}_{1}(\cdot; u) - \mathbf{z}_{1}^{h_{n}}(\cdot; u)\|_{H(\operatorname{div}, D)} + \|z_{2}(\cdot; u) - z_{2}^{h_{n}}(\cdot; u)\|_{L^{2}(D)} \to 0$$
(5.22)

as  $n \to \infty$ .

Prove now that

$$\lim_{n \to \infty} \|u^{h_n} - \hat{u}\|_H = \lim_{n \to \infty} \left( \|u_1^{h_n} - \hat{u}_1\|_{H_1}^2 + \|u_2^{h_n} - \hat{u}_2\|_{H_2}^2 \right)^{1/2} = 0,$$

where  $u^{h_n} = (u_1^{h_n}, u_2^{h_n}), \ \hat{u} = (\hat{u}_1, \hat{u}_2), \ H = H_1 \times H_2.$ Set

$$I_n(u) = (Q^{-1} z_2^{h_n}(\cdot; u), z_2^{h_n}(\cdot; u))_{L^2(D)} + (\tilde{Q}_1^{-1} u_1, u_1)_{H_1} + (\tilde{Q}_2^{-1} u_2, u_2)_{H_2}.$$

It is clear that

$$\inf_{u \in H} I_n(u) = I_n(u^{h_n})$$

and

 $I_n(u^{h_n}) \le I_n(\hat{u}).$ 

From strong convergence of the sequence  $\{(\mathbf{z}_1^{h_n}(\cdot; \hat{u}), z_2^{h_n}(\cdot; \hat{u}))\}$  to  $(\mathbf{z}_1(\hat{u}), z_2(\hat{u}))$  in the space  $H(\operatorname{div}, D) \times L^2(D)$ , which follows from (5.22), we have

$$\lim_{n \to \infty} I_n(\hat{u}) = I(\hat{u}),$$

and, hence  $\overline{\lim}_{n\to\infty} I_n(u^{h_n}) \leq I(\hat{u})$ . Since

$$I_n(u^{h_n}) \ge (\tilde{Q}_1^{-1}u_1^{h_n}, u_1^{h_n})_{H_1} + (\tilde{Q}_2^{-1}u_2^{h_n}, u_2^{h_n})_{H_2} \ge \alpha \|u^{h_n}\|_H^2,$$

where  $\alpha > 0$  is the constant from (2.20), then  $||u^{h_n}||_H \leq C$  (C = const) and we can extract from the sequence  $\{u^{h_n}\}$  a subsequence  $\{u^{h_{n_k}}\}$  such that  $u^{h_{n_k}} \to \tilde{u}$  weakly in H (see [16], Theorem 1, p. 180).

Prove that the sequence  $\{(\mathbf{z}_1^{h_{n_k}}(\cdot; u^{h_{n_k}}), z_2^{h_{n_k}}(\cdot; u^{h_{n_k}}))\}$  weakly converges to  $(\mathbf{z}_1(\tilde{u}), z_2(\tilde{u}))$  in  $H(\operatorname{div}, D) \times L^2(D)$ .

In fact, take a subsequence  $\{(\mathbf{z}_1^{h_{n_{k_i}}}(\cdot; u^{h_{n_{k_i}}}), z_2^{h_{n_{k_i}}}(\cdot; u^{h_{n_{k_i}}}))\}$  of the sequence  $\{(\mathbf{z}_1^{h_{n_k}}(\cdot; u^{h_{n_k}}), z_2^{h_{n_k}}(\cdot; u^{h_{n_k}}))\}$  which weakly converges to some  $(\tilde{\mathbf{z}}_1, \tilde{z}_2)$  in  $H(\operatorname{div}, D) \times L^2(D)$  and for an arbitrary  $(\mathbf{q}, v)$  from  $H(\operatorname{div}, D) \times L^2(D)$  take a sequence  $\{(\mathbf{q}^{h_{n_{k_i}}}, v^{h_{n_{k_i}}})\}, (\mathbf{q}^{h_{n_{k_i}}}, v^{h_{n_{k_i}}})\}$ 

 $V_1^{h_{n_{k_i}}} \times V_2^{h_{n_{k_i}}}$  which strongly converges to  $(\mathbf{q}, v)$  in  $H(\operatorname{div}, D) \times L^2(D)^{-3}$  and pass to the limit in both sides of equations

$$\int_{D} (((\mathbf{A}(x))^{-1})^{T} \mathbf{z}_{1}^{h_{n_{k_{i}}}}(x; u^{h_{n_{k_{i}}}}), \mathbf{q}^{h_{n_{k_{i}}}}(x))_{\mathbb{R}^{n}} dx - \int_{D} z_{2}^{h_{n_{k_{i}}}}(x; u^{h_{n_{k_{i}}}}) \operatorname{div} \mathbf{q}^{h_{n_{k_{i}}}}(x) dx$$
$$= \int_{D} (\mathbf{l}_{1}(x) - (C_{1}^{t} J_{H_{1}} u_{1}^{h_{n_{k_{i}}}})(x), \mathbf{q}^{h_{n_{k_{i}}}}(x))_{\mathbb{R}^{n}} dx, \quad (5.23)$$

$$-\int_{D} v^{h_{n_{k_i}}}(x) \operatorname{div} \mathbf{z}_1^{h_{n_{k_i}}}(x; u^{h_{n_{k_i}}}) \, dx - \int_{D} c(x) z_2^{h_{n_{k_i}}}(x; u^{h_{n_{k_i}}}) v^{h_{n_{k_i}}}(x) \, dx$$
$$= \int_{D} (l_2(x) - (C_2^t J_{H_2} u_2^{h_{n_{k_i}}})(x)) v^{h_{n_{k_i}}}(x) \, dx \quad (5.24)$$

(which follows from (5.17), (5.18)), when  $i \to \infty$ . Taking into account that <sup>4</sup>

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$$\lim_{i \to \infty} \left( \int_{D} (((\mathbf{A}(x))^{-1})^{T} \mathbf{z}_{1}^{h_{n_{k_{i}}}}(x; u^{h_{n_{k_{i}}}}), \mathbf{q}^{h_{n_{k_{i}}}}(x))_{\mathbb{R}^{n}} dx - \int_{D} z_{2}^{h_{n_{k_{i}}}}(x; u^{h_{n_{k_{i}}}}) \operatorname{div} \mathbf{q}^{h_{n_{k_{i}}}}(x) dx \right)$$

$$= \lim_{i \to \infty} a(\mathbf{q}^{h_{n_{k_{i}}}}, \mathbf{z}_{1}^{h_{n_{k_{i}}}}(\cdot; u^{h_{n_{k_{i}}}})) + \lim_{i \to \infty} b(\mathbf{q}^{h_{n_{k_{i}}}}, z_{2}^{h_{n_{k_{i}}}}(\cdot; u^{h_{n_{k_{i}}}}))$$

$$= \lim_{i \to \infty} < A \mathbf{q}_{1}^{h_{n_{k_{i}}}}, \mathbf{z}_{1}^{h_{n_{k_{i}}}}(\cdot; u^{h_{n_{k_{i}}}}) >_{H(\operatorname{div},D)' \times H(\operatorname{div},D)}$$

$$+ \lim_{i \to \infty} < B \mathbf{q}^{h_{n_{k_{i}}}}, z_{2}^{h_{n_{k_{i}}}}(\cdot; u^{h_{n_{k_{i}}}}) >_{L^{2}(D)' \times L^{2}(D)}$$

$$= < A \mathbf{q}, \tilde{\mathbf{z}}_{1} >_{H(\operatorname{div},D)' \times H(\operatorname{div},D)} + < B \mathbf{q}, \tilde{z}_{2} >_{L^{2}(D)' \times L^{2}(D)} = a(\mathbf{q}, \tilde{\mathbf{z}}_{1}) + b(\mathbf{q}, \tilde{z}_{2})$$

$$= \int_{D} (((\mathbf{A}(x))^{-1})^{T} \tilde{\mathbf{z}}_{1}(x), \mathbf{q}(x))_{\mathbb{R}^{n}} dx - \int_{D} \tilde{z}_{2}(x) \operatorname{div} \mathbf{q}(x) dx, \qquad (5.25)$$

where by  $A : H(\operatorname{div}, D) \to H(\operatorname{div}, D)'$  we denote the bounded operator associated with the bilinear form  $a(\cdot, \cdot)$ , defined by  $a(u, v) = \langle Au, v \rangle \forall u, v \in H(\operatorname{div}, D)$ ,

$$-\lim_{i \to \infty} \int_{D} v^{h_{n_{k_i}}}(x) \operatorname{div} \mathbf{z}_{1}^{h_{n_{k_i}}}(x; u^{h_{n_{k_i}}}) dx = \lim_{i \to \infty} b(\mathbf{z}_{1}^{h_{n_{k_i}}}(\cdot; u^{h_{n_{k_i}}}), v^{h_{n_{k_i}}})$$

$$= \lim_{i \to \infty} \langle B^{t} v^{h_{n_{k_i}}}, \mathbf{z}_{1}^{h_{n_{k_i}}}(\cdot; u^{h_{n_{k_i}}})) \rangle_{H(\operatorname{div}, D)' \times H(\operatorname{div}, D)}$$

$$= b(\tilde{\mathbf{z}}_{1}, v) = -\int_{D} v(x) \operatorname{div} \tilde{\mathbf{z}}_{1}(x) dx, \qquad (5.26)$$

$$\lim_{i \to \infty} \int_{D} c(x) z_{2}^{h_{n_{k_i}}}(x; u^{h_{n_{k_i}}}) v^{h_{n_{k_i}}}(x) dx = \lim_{i \to \infty} (z_{2}^{h_{n_{k_i}}}(\cdot; u^{h_{n_{k_i}}}), cv^{h_{n_{k_i}}})_{L^{2}(D)}$$

<sup>3</sup>Such sequences exist due to the boundedness of the sequence  $\{(\mathbf{z}_1^{h_{n_k}}(\cdot; u^{h_{n_k}}), z_2^{h_{n_k}}(\cdot; u^{h_{n_k}}))\}$  in the space  $H(\operatorname{div}; \mathbf{D}) \times \mathbf{L}^2(\mathbf{D})$ , which follows from inequality (5.21) and the boundedness of the sequence  $\{u^{h_{n_k}}\}$  in the space H, and from completeness of the sequence of the subspaces  $\{V_1^h \times V_2^h\}$  in  $H(\operatorname{div}; \mathbf{D}) \times \mathbf{L}^2(\mathbf{D})$ .

$$\lim_{n \to \infty} < F_n, u_n >_{X' \times X} = < F_0, u_0 >_{X' \times X}.$$

<sup>&</sup>lt;sup>4</sup>Passage to the limit in (5.25)–(5.29) is justified by the following assertion (see, for example [15], page 12): Let a sequence  $\{v_n\}$  weakly converge to  $v_0$  in some linear normed space X and a sequence  $\{F_n\}$  strongly converge to  $F_0$  in the space X', dual of X. Then

$$= (\tilde{z}_{2}, cv)_{L^{2}(D)} = \int_{D} c(x)\tilde{z}_{2}(x)v(x) dx, \qquad (5.27)$$
$$\lim_{i \to \infty} \int_{D} (\mathbf{l}_{1}(x) - (C_{1}^{t}J_{H_{1}}u_{1}^{h_{n_{k_{i}}}})(x), \mathbf{q}^{h_{n_{k_{i}}}}(x))_{\mathbb{R}^{n}} dx$$
$$= \lim_{i \to \infty} \left( (\mathbf{l}_{1}, \mathbf{q}^{h_{n_{k_{i}}}})_{L^{2}(D)^{n}} - \langle J_{H_{1}}C_{1}\mathbf{q}^{h_{n_{k_{i}}}}, u_{1}^{h_{n_{k_{i}}}} \rangle_{H_{1}'} \times H_{1} \right)$$
$$= (\mathbf{l}_{1}, \mathbf{q})_{L^{2}(D)^{n}} - \langle J_{H_{1}}C_{1}\mathbf{q}, \tilde{u}_{1} \rangle_{H_{1}'} \times H_{1}} = \int_{D} (\mathbf{l}_{1}(x) - (C_{1}^{t}J_{H_{1}}\tilde{u}_{1})(x), \mathbf{q}(x))_{\mathbb{R}^{n}} dx, \qquad (5.28)$$

$$\lim_{i \to \infty} \int_D (l_2(x) - (C_2^t J_{H_2} u_2^{h_{n_{k_i}}})(x)) v^{h_{n_{k_i}}}(x) dx$$
$$= \int_D (l_2(x) - (C_2^t J_{H_2} \tilde{u}_2)(x)) v(x) dx, \quad (5.29)$$

we see, from (5.23) – (5.29), that  $(\tilde{\mathbf{z}}_1, \tilde{z}_2) \in H(\operatorname{div}, D) \times L^2(D)$  satisfy equations (3.1) and (3.2) at  $u = \tilde{u}$ . But problem (3.1), (3.2) has a unique solution  $(\mathbf{z}_1(\tilde{u}), z_2(\tilde{u}))$  at  $u = \tilde{u}$ . Hence  $(\tilde{\mathbf{z}}_1, \tilde{z}_2) = (\mathbf{z}_1(\tilde{u}), z_2(\tilde{u}))$  and

$$(\mathbf{z}_1^{h_{n_k}}(\cdot; u^{h_{n_k}}), z_2^{h_{n_k}}(\cdot; u^{h_{n_k}})) \to (\mathbf{z}_1(\tilde{u}), z_2(\tilde{u})) \quad \text{weakly in} \quad H(\operatorname{div}, D) \times L^2(D)$$

Then, since the functionals  $F_1(z_2) := (Q^{-1}z_2, z_2)_{L^2(D)}$  and  $F_2(u) := (\tilde{Q}^{-1}u, u)_H := (\tilde{Q}_1^{-1}u_1, u_1)_{H_1} + (\tilde{Q}_2^{-1}u_2, u_2)_{H_2}$  are weakly lower semicontinuous in the spaces  $L^2(D)$  and H, respectively,<sup>5</sup> we obtain

$$I(\tilde{u}) = (Q^{-1}z_{2}(\cdot;\tilde{u}), z_{2}(\cdot;\tilde{u}))_{L^{2}(D)} + (\tilde{Q}_{1}\tilde{u}, \tilde{u})_{H}$$

$$\leq \underline{\lim}_{k \to \infty} (Q^{-1}z_{2}^{h_{n_{k}}}(\cdot; u^{h_{n_{k}}}), z_{2}^{h_{n_{k}}}(\cdot; u^{h_{n_{k}}}))_{L^{2}(D)} + \underline{\lim}_{k \to \infty} (\tilde{Q}^{-1}u^{h_{n_{k}}}, u^{h_{n_{k}}})_{H}$$

$$\leq \underline{\lim}_{k \to \infty} \Big[ (Q^{-1}z_{2}^{h_{n_{k}}}(\cdot; u^{h_{n_{k}}}), z_{2}^{h_{n_{k}}}(\cdot; u^{h_{n_{k}}}))_{L^{2}(D)} + (\tilde{Q}^{-1}u^{h_{n_{k}}}, u^{h_{n_{k}}})_{H} \Big]$$

$$= \underline{\lim}_{k \to \infty} I_{n_{k}}(u^{h_{n_{k}}}) \leq \overline{\lim}_{k \to \infty} I_{n_{k}}(u^{h_{n_{k}}}) \leq I(\hat{u}).$$
(5.30)

Here  $\tilde{Q}^{-1}: H \to H$  is the bounded selfadjoint positive definite operator defined by

$$\tilde{Q}^{-1}u = \tilde{Q}_1^{-1}u_1 + \tilde{Q}_2^{-1}u_2, \quad u = (u_1, u_2) \in H = H_1 \times H_2,$$

satisfying the inequality

$$(\tilde{Q}^{-1}u, u)_H \ge \alpha \|u\|_H^2 \quad \forall u \in H,$$
(5.31)

where  $\alpha$  is a constant from (2.20). Taking into account the uniqueness of an element on which the minimum of functional I(u) is attained, we find from (5.30) that  $\tilde{u} = \hat{u}$ . This implies that

$$\lim_{n \to \infty} I_n(u^{h_n}) = I(\hat{u}) \tag{5.32}$$

and  $u^{h_n} \xrightarrow{\text{weakly}} \hat{u}$  in H,  $\hat{z}_2^{h_n} = z_2^{h_n}(\cdot; u^{h_n}) \xrightarrow{\text{weakly}} z_2(\cdot; \hat{u}) = \hat{z}_2$  in  $L^2(D)$  as  $n \to \infty$ . Hence,

$$(Q^{-1}z_2(\cdot;\hat{u}), z_2(\cdot;\hat{u}))_{L^2(D)} \le \underline{\lim}_{n \to \infty} (Q^{-1}z_2^{h_n}(\cdot; u^{h_n}), z_2^{h_n}(\cdot; u^{h_n}))_{L^2(D)},$$
(5.33)

$$(\tilde{Q}_1\hat{u},\hat{u})_H \le \underline{\lim}_{n \to \infty} (\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H$$
(5.34)

<sup>&</sup>lt;sup>5</sup>These assertions are the corollary of a more general statement (that can be found, for example, in [15], p. 41): Let X be a reflexive Banach space, and  $B: X \to X^*$  a linear bounded nonnegative selfadjoint operator. Then the functional  $F(u) := \langle Bu, u \rangle_{X^* \times X}$  is a weakly lower semicontinuous on X.

and from (5.33), (5.34), we have

$$\underline{\lim}_{n\to\infty} (Q^{-1}z_2^{h_n}(\cdot; u^{h_n}), z_2^{h_n}(\cdot; u^{h_n}))_{L^2(D)} + \underline{\lim}_{n\to\infty} (\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H$$

$$\geq (Q^{-1}z_2(\cdot; \hat{u}), z_2(\cdot; \hat{u}))_{L^2(D)} + (\tilde{Q}_1\hat{u}, \hat{u})_H = I(\hat{u})$$

$$= \lim_{n\to\infty} \left[ Q^{-1}z_2^{h_n}(\cdot; u^{h_n}), z_2^{h_n}(\cdot; u^{h_n}))_{L^2(D)} + (\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H \right]$$

$$= \overline{\lim}_{n\to\infty} \left[ Q^{-1}z_2^{h_n}(\cdot; u^{h_n}), z_2^{h_n}(\cdot; u^{h_n}))_{L^2(D)} + (\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H \right]$$

$$\geq \underline{\lim}_{n\to\infty} Q^{-1}z_2^{h_n}(\cdot; u^{h_n}), z_2^{h_n}(\cdot; u^{h_n}))_{L^2(D)} + \overline{\lim}_{n\to\infty} (\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H.$$

Whence

$$\underline{\lim}_{n \to \infty} (\tilde{Q}^{-1} u^{h_n}, u^{h_n})_H \ge \overline{\lim}_{n \to \infty} (\tilde{Q}^{-1} u^{h_n}, u^{h_n})_H$$

The last inequality shows that the sequence  $\{(\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H\}$  is convergent. This fact and (5.32) also imply convergence of the sequence  $\{(Q^{-1}z_2^{h_n}(\cdot; u^{h_n}), z_2^{h_n}(\cdot; u^{h_n}))_{L^2(D)}\}$  and equality

$$I(\hat{u}) = \lim_{n \to \infty} (Q^{-1} z_2^{h_n}(\cdot; u^{h_n}), z_2^{h_n}(\cdot; u^{h_n}))_{L^2(D)} + \lim_{n \to \infty} (\tilde{Q}^{-1} u^{h_n}, u^{h_n})_H.$$
(5.35)

It is easy to see that

$$\lim_{n \to \infty} (\tilde{Q}^{-1} u^{h_n}, u^{h_n})_H = (\tilde{Q}^{-1} \hat{u}, \hat{u})_H.$$
(5.36)

In fact, if we suppose that (5.36) does not hold, i.e.

$$\lim_{n \to \infty} (\tilde{Q}^{-1} u^{h_n}, u^{h_n})_H = (\tilde{Q}^{-1} \hat{u}, \hat{u})_H + a,$$

where a is a certain positive number, then (due to (5.35)) there must be valid

$$\underline{\lim}_{n \to \infty} (Q^{-1} z_2^{h_n}(\cdot; u^{h_n}), z_2^{h_n}(\cdot; u^{h_n}))_{L^2(D)} = \lim_{n \to \infty} (Q^{-1} z_2^{h_n}(\cdot; u^{h_n}), z_2^{h_n}(\cdot; u^{h_n}))_{L^2(D)} = (Q^{-1} z_2(\cdot; \hat{u}, z_2(\cdot; \hat{u}))_{L^2(D)} - a.$$
(5.37)

But this is impossible since (5.37) leads to the contradictory inequality

$$\underline{\lim}_{n \to \infty} (Q^{-1} z_2^{h_n}(\cdot; u^{h_n}), z_2^{h_n}(\cdot; u^{h_n}))_{L^2(D)} < (Q^{-1} z_2(\cdot; \hat{u}, z_2(\cdot; \hat{u}))_{L^2(D)}.$$

Hence, (5.36) is proved.

Now let us show that  $u^{h_n} \to \hat{u}$  strongly in H. To this end introduce Hilbert space  $\tilde{H}$  consisting of elements of H endowed with norm

$$\|v\|_{\tilde{H}} := (\tilde{Q}^{-1}v, v)_{H}^{1/2}$$

Then from weak convergence of the sequence  $\{u^{h_n}\}$  to  $\hat{u}$  as  $n \to \infty$ , it follows, obviously, that

$$u^{h_n} \to \hat{u}$$
 weakly in  $\tilde{H}$  as  $n \to \infty$ . (5.38)

Since (5.36) means that

$$\|u^{h_n}\|_{\tilde{H}} \to \|\hat{u}\|_{\tilde{H}} \text{ as } n \to \infty,$$
(5.39)

we obtain from (5.38) and (5.39) that  $u^{h_n} \to \hat{u}$  strongly in  $\tilde{H}$  i.e.,<sup>6</sup>

$$\lim_{n \to \infty} \|u^{h_n} - \hat{u}\|_{\tilde{H}} = \lim_{n \to \infty} (\tilde{Q}^{-1}(u^{h_n} - \hat{u}), u^{h_n} - \hat{u})_H^{1/2} = 0.$$

<sup>&</sup>lt;sup>6</sup>Here we use the following statement (see, for example [16], p. 124). Let  $\{f_n\}$  be a sequence in Hilbert space X. If  $f_n \to f$  weakly in X and  $||f_n||_X \to ||f||_X$  as  $n \to \infty$  then  $f_n \to f$  strongly in X.

From here, due to the inequality

$$\|u^{h_n} - \hat{u}\|_H \le \frac{1}{\alpha} (\tilde{Q}^{-1}(u^{h_n} - \hat{u}), u^{h_n} - \hat{u})_H^{1/2},$$

following from (5.31), we find that

$$\lim_{n \to \infty} \|u^{h_n} - \hat{u}\|_H = 0,$$

i.e. the sequence  $\{u^{h_n}\}$  strongly converges to  $\hat{u}$  in H.

In order to get estimate (5.13), we note that

$$(\mathbf{z}_1^{h_n}(\cdot;\hat{u}) - \mathbf{z}_1^{h_n}(\cdot;u^{h_n}), z_2^{h_n}(\cdot;\hat{u}) - z_2^{h_n}(\cdot;u^{h_n}))$$

is a solution of the following problem

$$\int_{D} (((\mathbf{A}(x))^{-1})^{T} (\mathbf{z}_{1}^{h_{n}}(x;\hat{u}) - \mathbf{z}_{1}^{h_{n}}(x;u^{h_{n}})), \mathbf{q}^{h_{n}}(x))_{\mathbb{R}^{n}} dx 
- \int_{D} (z_{2}^{h_{n}}(x;\hat{u}) - z_{2}^{h_{n}}(x;u^{h_{n}})) \operatorname{div} \mathbf{q}^{h_{n}}(x) dx 
= \int_{D} ((C_{1}^{t} J_{H_{1}}(u_{1}^{h_{n}} - \hat{u}_{1}))(x), \mathbf{q}^{h_{n}}(x))_{\mathbb{R}^{n}} dx \quad \forall \mathbf{q}^{h_{n}} \in V_{1}^{h_{n}}, \quad (5.40)$$

$$-\int_{D} v^{h_{n}}(x) \operatorname{div}\left(\mathbf{z}_{1}^{h_{n}}(x;\hat{u}) - \mathbf{z}_{1}^{h_{n}}(x;u^{h_{n}})\right) dx$$
$$-\int_{D} c(x)(z_{2}^{h_{n}}(x;\hat{u}) - z_{2}^{h_{n}}(x;u^{h_{n}}))v^{h_{n}}(x) dx$$
$$=\int_{D} (C_{2}^{t}J_{H_{2}}(u_{2}^{h_{n}} - \hat{u}_{2}))(x)v^{h_{n}}(x) dx \quad \forall v^{h_{n}} \in V_{2}^{h_{n}}.$$
(5.41)

Applying estimate (5.21) to the solution of problem (5.40), (5.41), we obtain

$$\|\mathbf{z}_{1}^{h_{n}}(\cdot;\hat{u}) - \mathbf{z}_{1}^{h_{n}}(\cdot;u^{h_{n}})\|_{H(\operatorname{div},D)} + \|z_{2}^{h_{n}}(\cdot;\hat{u}) - z_{2}^{h_{n}}(\cdot;u^{h_{n}})\|_{L^{2}(D)} \leq C\|u_{1}^{h_{n}} - \hat{u}_{1}\|_{H}.$$
 (5.42)

From triangle inequality, (5.19), (5.42), and the fact that the sequence  $\{(\mathbf{z}_1^{h_n}(\cdot; \hat{u}), z_2^{h_n}(\cdot; \hat{u}))\}$ strongly converges to  $(\mathbf{z}_1(\hat{u}), z_2(\hat{u}))$  in the space  $H(\operatorname{div}, D) \times L^2(D)$ , we have

$$\begin{aligned} \|\hat{\mathbf{z}}_{1} - \hat{\mathbf{z}}_{1}^{h_{n}}\|_{H(\operatorname{div},D)} + \|\hat{z}_{2} - \hat{z}_{2}^{h_{n}}\|_{L^{2}(D)} \\ &\leq \|\mathbf{z}_{1}(\cdot;\hat{u}) - \mathbf{z}_{1}^{h_{n}}(\cdot;\hat{u})\|_{H(\operatorname{div},D)} + \|z_{2}(\cdot;\hat{u}) - z_{2}^{h_{n}}(\cdot;\hat{u})\|_{L^{2}(D)} \end{aligned}$$

$$+ \|\mathbf{z}_{1}^{h_{n}}(\cdot;\hat{u}) - \mathbf{z}_{1}^{h_{n}}(\cdot;u^{h_{n}})\|_{H(\operatorname{div},D)} + \|z_{2}^{h_{n}}(\cdot;\hat{u}) - z_{2}^{h_{n}}(\cdot;u^{h_{n}})\|_{L^{2}(D)} \to 0 \quad \text{as} \quad n \to \infty.$$
(5.43)

Analogously, in order to obtain estimate (5.14), we note that

$$\|\mathbf{p}_{1} - \mathbf{p}_{1}^{h_{n}}\|_{H(\operatorname{div},D)} + \|p_{2} - p_{2}^{h_{n}}\|_{L^{2}(D)}$$
  
$$\leq \|\mathbf{p}_{1} - \mathbf{p}_{1}^{h_{n}}(\cdot;\hat{u})\|_{H(\operatorname{div},D)} + \|p_{2} - p_{2}^{h_{n}}(\cdot;\hat{u})\|_{L^{2}(D)}$$

+ 
$$\|\mathbf{p}_{1}^{h_{n}}(\cdot;\hat{u}) - \mathbf{p}_{1}^{h_{n}}\|_{H(\operatorname{div},D)} + \|p_{2}^{h_{n}}(\cdot;\hat{u}) - p_{2}^{h_{n}}\|_{L^{2}(D)},$$
 (5.44)

where  $(\mathbf{p}_1^{h_n}(\cdot;\hat{u}), p_2^{h_n}(\cdot;\hat{u}))$  is a solution of the problem

$$\int_{D} ((\mathbf{A}(x))^{-1} \mathbf{p}_{1}^{h_{n}}(x;\hat{u}), \mathbf{q}_{2}^{h_{n}}(x))_{\mathbb{R}^{n}} dx - \int_{D} p_{2}^{h_{n}}(x;\hat{u}) \operatorname{div} \mathbf{q}_{2}^{h}(x) dx = 0 \quad \forall \mathbf{q}_{2}^{h_{n}} \in V_{1}^{h_{n}}, \quad (5.45)$$

$$-\int_{D} v_{2}^{h_{n}}(x) \operatorname{div} \mathbf{p}_{1}^{h_{n}}(x;\hat{u}) \, dx - \int_{D} c(x) p_{2}^{h_{n}}(x;\hat{u}) v_{2}^{h_{n}}(x) \, dx$$
$$= \int_{D} v_{2}^{h_{n}}(x) Q^{-1} z_{2}(x;\hat{u}) \, dx \quad v_{2}^{h_{n}} \in V_{2}^{h_{n}}.$$
(5.46)

Taking into account that, due to (5.11), (5.12) and (5.45), (5.46),

$$(\mathbf{p}_1^{h_n}(\cdot;\hat{u}) - \mathbf{p}_1^{h_n}, p_2^{h_n}(\cdot;\hat{u}) - p_2^{h_n})$$

is a solution of the following problem

$$\int_{D} ((\mathbf{A}(x))^{-1}(\mathbf{p}_{1}^{h_{n}}(x;\hat{u}) - \mathbf{p}_{1}^{h_{n}}(x)), \mathbf{q}_{2}^{h_{n}}(x))_{\mathbb{R}^{n}} dx - \int_{D} (p_{2}^{h_{n}}(x;\hat{u}) - p_{2}^{h_{n}}(x)) \operatorname{div} \mathbf{q}_{2}^{h_{n}}(x) dx = 0 \quad \forall \mathbf{q}_{2}^{h_{n}} \in V_{1}^{h_{n}}, \quad (5.47)$$

$$-\int_{D} v_{2}^{h_{n}}(x) \operatorname{div}\left(\mathbf{p}_{1}^{h_{n}}(x;\hat{u})-\mathbf{p}_{1}^{h_{n}}(x)\right) dx - \int_{D} c(x)(p_{2}^{h_{n}}(x;\hat{u})-p_{2}^{h_{n}}(x))v_{2}^{h_{n}}(x) dx$$
$$=\int_{D} v_{2}^{h_{n}}(x)Q^{-1}(z_{2}(\cdot;\hat{u})-z_{2}^{h_{n}}(\cdot;u^{h_{n}}))(x) dx \quad v_{2}^{h_{n}} \in V_{2}^{h_{n}}, \quad (5.48)$$

and applying relationship (5.7) to the solution of problem (5.45), (5.46) and estimate (5.6) to the solution of problem (5.47), (5.48), respectively, we obtain, in view of (5.43), that

$$\|\mathbf{p}_{1} - \mathbf{p}_{1}^{h_{n}}(\cdot;\hat{u})\|_{H(\operatorname{div},D)} + \|p_{2} - p_{2}^{h_{n}}(\cdot;\hat{u})\|_{L^{2}(D)} \to 0 \quad \text{as} \quad n \to \infty,$$
(5.49)

$$\begin{aligned} \|\mathbf{p}_{1}^{h_{n}}(\cdot;\hat{u}) - \mathbf{p}_{1}^{h_{n}})\|_{H(\operatorname{div},D)} + \|p_{2}^{h_{n}}(\cdot;\hat{u}) - p_{2}^{h_{n}})\|_{L^{2}(D)} \\ &\leq C \|z_{2}(\cdot;\hat{u}) - \hat{z}_{2}^{h_{n}}\|_{L^{2}(D)} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$
(5.50)

From (5.50), (5.49), and (5.44), we find

$$\|\mathbf{p}_1 - \mathbf{p}_1^{h_n}\|_{H(\operatorname{div},D)} + \|p_2 - p_2^{h_n}\|_{L^2(D)} \to 0 \quad \text{as} \quad n \to \infty.$$
 (5.51)

Relationships (5.51) and (5.43) mean that (5.13) and (5.14) are proved.

Now show the validity of (5.15) and (5.16).

Let  $(\tilde{\mathbf{j}}, \tilde{\varphi})$  be a solution of problem (2.5), (2.6) at  $f(x) = \tilde{f}(x)$ . Then from (5.8) and (4.1), we have

$$\widehat{\mathbb{E}|\hat{l^{h_n}(\tilde{\mathbf{j}},\tilde{\varphi})} - \hat{l(\tilde{\mathbf{j}},\tilde{\varphi})}|^2} = \mathbb{E}[(u_1^{h_n},\tilde{y}_1)_{H_1} + (u_2^{h_n},\tilde{y}_2)_{H_1} + c^{h_n} - (\hat{u}_1,\tilde{y}_1)_{H_1} - (\hat{u}_2,\tilde{y}_2)_{H_2} - \hat{c}]^2 \\
= \mathbb{E}[(u_1^{h_n} - \hat{u}_1,\tilde{y}_1)_{H_1} + (u_2^{h_n} - \hat{u}_2,\tilde{y}_2)_{H_2} + c^{h_n} - \hat{c}]^2$$

$$= [(u_1^{h_n} - \hat{u}_1, C_1 \tilde{\mathbf{j}})_{H_1} + (u_2^{h_n} - \hat{u}_2, C_2 \tilde{\varphi})_{H_2} + c^{h_n} - \hat{c}]^2 + \mathbb{E}[(u_1^{h_n} - \hat{u}_1, \tilde{\eta}_1)_{H_1} + (u_2^{h_n} - \hat{u}_2, \tilde{\eta}_2)_{H_2}].$$
(5.52)

Weak convergence of the sequence  $\{\hat{z}_2^{h_n}\}$  to  $\hat{z}_2$  in the space  $L^2(D)$  implies that  $c^{h_n} \to \hat{c}$  as  $n \to \infty$ . Then from the fact that  $\tilde{f} \in G_0$  and the inequality

$$\begin{split} [(u_1^{h_n} - \hat{u}_1, C_1 \tilde{\mathbf{j}})_{H_1} + (u_2^{h_n} - \hat{u}_2, C_2 \tilde{\varphi})_{H_2} + c^{h_n} - \hat{c}]^2 \\ &\leq C \left( \|u_1^{h_n} - \hat{u}_1\|_{H_1}^2 + \|u_2^{h_n} - \hat{u}_2\|_{H_2}^2 + (c^{h_n} - \hat{c})^2 \right) \left( \|\tilde{\mathbf{j}}\|_{H(\operatorname{div},D)}^2 + \|\tilde{\varphi}\|_{L^2(D)}^2 \right) \\ &\leq \tilde{C} \left( \|u^{h_n} - \hat{u}\|_{H}^2 + (c^{h_n} - \hat{c})^2 \right) \|\tilde{f}\|_{L^2(D)}^2 \\ &\leq \tilde{\tilde{C}} \left( \|u^{h_n} - \hat{u}\|_{H}^2 + (c^{h_n} - \hat{c})^2 \right) \quad (C, \tilde{C}, \tilde{\tilde{C}} = \operatorname{const}), \end{split}$$

we see that the fist term in the r.h.s of (5.52) tends to 0 as  $n \to \infty$ . Analogously, we may show that for the last term in the r.h.s of (5.52) the following estimate is valid

$$\mathbb{E}[(u_1^{h_n} - \hat{u}_1, \tilde{\eta}_1)_{H_1} + (u_2^{h_n} - \hat{u}_2, \tilde{\eta}_2)_{H_2}] \le C \|u^{h_n} - \hat{u}\|_H^2 \quad (C = \text{const})$$

and therefore this term also tends to 0 as  $n \to \infty$ . From here and the inequality

$$\begin{aligned} &\widehat{\mathbb{E}|l^{h_n}(\tilde{\mathbf{j}},\tilde{\varphi}) - l(\tilde{\mathbf{j}},\tilde{\varphi})|^{1/2}} = \mathbb{E}[\widehat{l^{h_n}(\tilde{\mathbf{j}},\tilde{\varphi})} - \widehat{\overline{l(\tilde{\mathbf{j}},\tilde{\varphi})}} + \widehat{\overline{l(\mathbf{j},\varphi)}} - l(\tilde{\mathbf{j}},\tilde{\varphi})|^{1/2} \\ &\leq \left\{ \mathbb{E}|\widehat{l^{h_n}(\tilde{\mathbf{j}},\tilde{\varphi})} - \widehat{\widehat{l(\tilde{\mathbf{j}},\tilde{\varphi})}}|^2 \right\}^{1/2} + \left\{ \mathbb{E}[\widehat{\overline{l(\tilde{\mathbf{j}},\tilde{\varphi})}} - l(\tilde{\mathbf{j}},\tilde{\varphi})]^2| \right\}^{1/2},
\end{aligned}$$

it follows the validity of the conclusion of the theorem.

Let us formulate a similar result in the case when an estimate  $(\hat{\mathbf{j}}, \hat{\varphi})$  of the state  $(\mathbf{j}, \varphi)$  is directly determined from the solution to problem (4.23)–(4.26).

**Theorem 4.** Let  $(\hat{\mathbf{j}}^h, \hat{\varphi}^h) \in V_1^h \times V_2^h$  be an approximate estimate of  $(\hat{\mathbf{j}}, \hat{\varphi})$  determined from the solution to the variational problem

$$\int_{D} (((\mathbf{A}(x))^{-1})^{T} \hat{\mathbf{p}}_{1}^{h}(x), \mathbf{q}_{1}^{h}(x))_{\mathbb{R}^{n}} dx - \int_{D} \hat{p}_{2}^{h}(x) \operatorname{div} \mathbf{q}_{1}^{h}(x) dx$$
$$= \int_{D} (C_{1}^{t} J_{H_{1}} \tilde{Q}_{1}(y_{1}(x) - C_{1} \hat{\mathbf{j}}^{h}), \mathbf{q}_{1}^{h}(x))_{\mathbb{R}^{n}} dx \ \forall \mathbf{q}_{1}^{h} \in V_{1}^{h}, \quad (5.53)$$

$$-\int_{D} v_{1}^{h}(x) \operatorname{div} \hat{\mathbf{p}}_{1}^{h}(x) \, dx - \int_{D} c(x) \hat{p}_{2}^{h}(x) v_{1}^{h}(x) \, dx$$
$$= \int_{D} (C_{2}^{t} J_{H_{2}} \tilde{Q}_{2}(y_{2}(x) - C_{2} \hat{\varphi}^{h}(x)) v_{1}^{h}(x) \, dx \quad \forall v_{1}^{h} \in V_{2}^{h}, \quad (5.54)$$

$$\int_{D} ((\mathbf{A}(x))^{-1} \hat{\mathbf{j}}^{h}(x), \mathbf{q}_{2}^{h}(x))_{\mathbb{R}^{n}} dx - \int_{D} \hat{\varphi}^{h}(x) \operatorname{div} \mathbf{q}_{2}^{h}(x) dx = 0 \quad \forall \mathbf{q}_{2}^{h} \in V_{1}^{h},$$
(5.55)

$$-\int_{D} v_{2}^{h}(x) \operatorname{div} \hat{\mathbf{j}}^{h}(x) \, dx - \int_{D} c(x) \hat{\varphi}^{h}(x) v_{2}^{h}(x) \, dx$$
$$= \int_{D} v_{2}(x) (Q^{-1} \hat{p}_{2}^{h}(x) + f_{0}(x)) \, dx \quad \forall v_{2}^{h} \in V_{2}^{h}.$$
(5.56)

Then

$$\|\hat{\mathbf{j}} - \hat{\mathbf{j}}^h\|_{H(\operatorname{div},D)} + \|\hat{\varphi} - \hat{\varphi}^h\|_{L^2(D)} \to 0 \quad \text{as} \quad h \to 0$$

and

$$\|\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_1^h\|_{H(\operatorname{div},D)} + \|\hat{p}_2 - \hat{p}_2^h\|_{L^2(D)} \to 0 \quad \text{as} \quad h \to 0$$

Introducing the bases in the spaces  $V_1^h$  and  $V_2^h$ , problem (5.9)–(5.12) can be rewritten as a system of liner algebraic equations. To do this, let us denote the elements of the bases of  $V_1^h$  and  $V_2^h$  by  $\boldsymbol{\xi}_i$   $(i = 1, \ldots, n_1)$  and  $\eta_i$   $(i = 1, \ldots, n_2)$ , respectively, where  $n_1 = \dim V_1^h$ ,  $n_2 = \dim V_2^h$ . The fact that  $\hat{\mathbf{z}}_1^h$ ,  $\mathbf{p}_1^h$  and  $\hat{z}_2^h$ ,  $p_2^h$  belong to the spaces  $V_1^h$  and  $V_2^h$  means the existence of constants  $\hat{z}_i^{(1)}$ ,  $p_i^{(1)}$  and  $\hat{z}_i^{(2)}$ ,  $p_i^{(2)}$  such that

$$\hat{\mathbf{z}}_{1}^{h} = \sum_{j=1}^{n_{1}} \hat{z}_{j}^{(1)} \boldsymbol{\xi}_{j}, \quad \mathbf{p}_{1}^{h} = \sum_{j=1}^{n_{1}} p_{j}^{(1)} \boldsymbol{\xi}_{j}$$
(5.57)

and

$$\hat{z}_2^h = \sum_{j=1}^{n_2} \hat{z}_j^{(2)} \eta_j, \quad p_2^h = \sum_{j=1}^{n_2} p_j^{(2)} \eta_j.$$
(5.58)

Setting in (5.9), (5.11)  $\mathbf{q}_1^h = \mathbf{q}_2^h = \boldsymbol{\xi}_i$   $(i = 1, ..., n_1)$  and in (5.10), (5.12)  $v_1^h = v_2^h = \eta_i$   $(i = 1, ..., n_2)$  respectively, we obtain that finding  $\hat{\mathbf{z}}_1^h$ ,  $\mathbf{p}_1^h$ ,  $z_2^h$  and  $p_2^h$  from (5.9)–(5.12) is equivalent to solving the following system of linear algebraic equations with respect to coefficients  $\hat{z}_j^{(1)}$ ,  $p_j^{(1)}$ ,  $\hat{z}_j^{(2)}$ , and  $p_j^{(2)}$  of expansions (5.57), (5.58):

$$\sum_{j=1}^{n_1} \bar{a}_{ij}^{(1)} \hat{z}_j^{(1)} + \sum_{j=1}^{n_2} a_{ji}^{(2)} \hat{z}_j^{(2)} + \sum_{j=1}^{n_1} a_{ij}^{(3)} p_j^{(1)} = b_i^{(1)}, \quad i = 1, \dots, n_1,$$
(5.59)

$$\sum_{j=1}^{n_1} a_{ij}^{(2)} \hat{z}_j^{(1)} + \sum_{j=1}^{n_2} a_{ij}^{(6)} \hat{z}_j^{(2)} + \sum_{j=1}^{n_2} a_{ij}^{(4)} p_j^{(2)} = b_i^{(1)}, \quad i = 1, \dots, n_2,$$
(5.60)

$$\sum_{j=1}^{n_1} a_{ij}^{(1)} p_j^{(1)} + \sum_{j=1}^{n_2} a_{ji}^{(2)} p_j^{(2)} = 0, \quad i = 1, \dots, n_1,$$
(5.61)

$$\sum_{j=1}^{n_1} a_{ij}^{(2)} p_j^{(1)} + \sum_{j=1}^{n_2} a_{ij}^{(6)} p_j^{(2)} + \sum_{j=1}^{n_2} a_{ij}^{(5)} \hat{z}_j^{(2)} = 0, \quad i = 1, \dots, n_2,$$
(5.62)

where

$$\begin{split} \bar{a}_{ij}^{(1)} &= \int_{D} (((\mathbf{A}(x))^{-1})^{T} \boldsymbol{\xi}_{i}(x), \boldsymbol{\xi}_{j}(x))_{\mathbb{R}^{n}} dx, \quad i, j = 1, \dots, n_{1}, \\ a_{ij}^{(1)} &= \int_{D} ((\mathbf{A}(x))^{-1} \boldsymbol{\xi}_{i}(x), \boldsymbol{\xi}_{j}(x))_{\mathbb{R}^{n}} dx, \quad i, j = 1, \dots, n_{1}, \\ a_{ij}^{(2)} &= -\int_{D} \eta_{i}(x) \operatorname{div} \boldsymbol{\xi}_{j}(x) dx, \quad i = 1, \dots, n_{2}, \quad j = 1, \dots, n_{1}, \\ a_{ij}^{(3)} &= \int_{D} (C_{1}^{t} J_{H_{1}} \tilde{Q}_{1} C_{1} \boldsymbol{\xi}_{i}(x), \boldsymbol{\xi}_{j}(x))_{\mathbb{R}^{n}} dx, \quad i, j = 1, \dots, n_{1}, \\ a_{ij}^{(4)} &= \int_{D} C_{2}^{t} J_{H_{2}} \tilde{Q}_{2} C_{2} \eta_{i}(x) \eta_{j}(x) dx, \quad i, j = 1, \dots, n_{2}, \\ a_{ij}^{(5)} &= -\int_{D} \eta_{j}(x) Q^{-1} \eta_{i}(x) dx, \quad i, j = 1, \dots, n_{2}, \\ a_{ij}^{(6)} &= -\int_{D} c(x) \eta_{i}(x) \eta_{j}(x) dx, \quad i, j = 1, \dots, n_{2}, \\ b_{i}^{(1)} &= \int_{D} (\mathbf{l}_{1}(x), \boldsymbol{\xi}_{i}(x))_{\mathbb{R}^{n}} dx, \quad i = 1, \dots, n_{1}, \\ b_{i}^{(2)} &= \int_{D} l_{2}(x) \eta_{i}(x) dx, \quad i = 1, \dots, n_{2}. \end{split}$$

Analogous system of linear algebraic equations can be also obtained for problem (5.53)-(5.56).

#### The case of integral observation operators 6

As an example, we consider the case when  $H_1 = L^2 \left( D_1^{(1)} \right)^n \times \cdots \times L^2 \left( D_{i_1}^{(1)} \right)^n \times \cdots \times L^2 \left( D_{n_1}^{(1)} \right)^n$ ,  $H_2 = L^2(D_1^{(2)}) \times \cdots \times L^2(D_{i_2}^{(2)}) \times \cdots \times L^2(D_{n_2}^{(2)})$ . Then  $J_{H_1} = I_{H_1}, J_{H_2} = I_{H_2}$ , where  $I_{H_1}$  and  $I_{H_2}$  are the identity operators in  $H_1$  and  $H_2$ , respectively,

$$y_1(x) = \left(\mathbf{y}_1^{(1)}(x), \dots, \mathbf{y}_{i_1}^{(1)}(x), \dots, \mathbf{y}_{n_1}^{(1)}(x)\right),$$
$$\eta_1(x) = \left(\boldsymbol{\eta}_1^{(1)}(x), \dots, \boldsymbol{\eta}_{i_1}^{(1)}(x), \dots, \boldsymbol{\eta}_{n_1}^{(1)}(x)\right),$$

where  $\mathbf{y}_{i_1}^{(1)}(x) = (y_{i_1,1}^{(1)}(x), \dots, y_{i_1,n}^{(1)}(x))^T \in L^2(D_{i_1}^{(1)})^n, \ \boldsymbol{\eta}_{i_1}^{(1)}(x) = (\eta_{i_1,1}^{(1)}(x), \dots, \eta_{i_1,n}^{(1)}(x))^T$  is a stochastic vector process with components  $\eta_{i_1,j}^{(1)}(x)$   $(j = 1, \ldots, n, i_1 = 1, \ldots, n_1)$  that are stochastic processes with zero expectations and finite second moments,

$$y_2(x) = \left(y_1^1(x), \dots, y_{i_2}^{(2)}(x), \dots, y_{n_2}^{(2)}(x)\right),$$
  
$$\eta_2(x) = \left(\eta_1^{(2)}(x), \dots, \eta_{i_2}^{(2)}(x), \dots, \eta_{n_2}^{(2)}(x)\right),$$
(6.1)

where  $y_{i_2}^{(2)} \in L^2(D)$ ,  $\eta_{i_2}^{(2)}(x)$   $(i_2 = 1, \ldots, n_2)$  is a stochastic process with zero expectation and finite second moment.

Let in observations (2.14) the operators  $C_1: L^2(D)^n \to H_1$  and  $C_2: L^2(D) \to H_2$  be defined by

$$C_{1}\mathbf{j}(x) = \left(C_{1}^{(1)}\mathbf{j}(x), \dots, C_{i_{1}}^{(1)}\mathbf{j}(x), \dots, C_{n_{1}}^{(1)}\mathbf{j}(x)\right),$$
$$C_{2}\varphi(x) = \left(C_{1}^{(2)}\varphi(x), \dots, C_{i_{2}}^{(2)}\varphi(x), \dots, C_{n_{2}}^{(2)}\varphi(x)\right),$$

where  $C_{i_1}^{(1)}: L^2(D)^n \to L^2(D_{i_1}^{(1)})^n$  and  $C_{i_2}^{(2)}: L^2(D) \to L^2(D_{i_2}^{(2)})$  are integral operators defined by

$$C_{i_1}^{(1)}\mathbf{j}(x) := \int_{D_{i_1}^{(1)}} \mathbf{K}_{i_1}^{(1)}(x,\xi) \mathbf{j}(\xi) \, d\xi,$$

and

$$C_{i_2}^{(2)}\varphi(x) := \int_{D_{i_2}^{(2)}} K_{i_2}^{(2)}(x,\xi)\varphi(\xi) \,d\xi,$$

correspondingly,  $\mathbf{K}_{i_1}^{(1)}(x,\xi) = \{k_{i_s}^{(i_1)}(x,\xi)\}_{i,j=1}^n$  is a matrix with entries  $k_{i_s}^{(i_1)} \in L^2(D_{i_1}^{(1)}) \times L^2(D_{i_1}^{(1)})$ ,  $i_1 = 1, \dots, n_1, \ K_{i_2}^{(2)}(x,\xi) \in L^2(D_{i_2}^{(2)}) \times L^2(D_{i_2}^{(2)})$  is a given function,  $i_2 = 1, \dots, n_2$ . As a result, observations  $y_1$  and  $y_2$  in (2.14) take the form

$$y_1 = (\mathbf{y}_1^{(1)}(x), \dots, \mathbf{y}_{i_1}^{(1)}(x), \dots, \mathbf{y}_{n_1}^{(1)}(x)),$$
$$y_2 = (y_1^{(2)}(x), \dots, y_{i_2}^{(2)}(x), \dots, \mathbf{y}_{n_2}^{(2)}(x)),$$

where

$$\mathbf{y}_{i_1}^{(1)}(x) = \int_{D_{i_1}^{(1)}} \mathbf{K}_{i_1}^{(1)}(x,\xi) \mathbf{j}(\xi) \, d\xi + \boldsymbol{\eta}_{i_1}^{(1)}(x), \quad i_1 = \overline{1, n_1}, \tag{6.2}$$

$$y_{i_2}^{(2)}(x) = \int_{D_{i_2}^{(2)}} K_{i_2}^{(2)}(x,\xi)\varphi(\xi) \,d\xi + \eta_{i_2}^{(2)}(x), \quad i_2 = \overline{1, n_2}, \tag{6.3}$$

and the operators

$$\tilde{Q}_{1} \in \mathcal{L}\left(L^{2}\left(D_{1}^{(1)}\right)^{n} \times \dots \times L^{2}\left(D_{i_{1}}^{(1)}\right)^{n} \times \dots \times L^{2}\left(D_{n_{1}}^{(1)}\right)^{n}, L^{2}\left(D_{1}^{(1)}\right)^{n} \times \dots \times L^{2}\left(D_{i_{1}}^{(1)}\right)^{n} \times \dots \times L^{2}\left(D_{n_{1}}^{(1)}\right)^{n}\right)$$

and

$$\tilde{Q}_{2} \in \mathcal{L}\left(L^{2}(D_{1}^{(2)}) \times \cdots \times L^{2}(D_{i_{2}}^{(2)}) \times \cdots \times L^{2}(D_{n_{2}}^{(2)}), L^{2}(D_{1}^{(2)}) \times \cdots \times L^{2}(D_{i_{2}}^{(2)}) \times \cdots \times L^{2}(D_{n_{2}}^{(2)})\right)$$

in (2.18), which is contained in the definition of set  $G_1$ , are given by

$$\tilde{Q}_1 \tilde{\eta}_1 = (\tilde{\mathbf{Q}}_1^{(1)} \tilde{\boldsymbol{\eta}}_1^{(1)}, \dots, \tilde{\mathbf{Q}}_{r_1}^{(1)} \tilde{\boldsymbol{\eta}}_{r_1}^{(1)}, \dots, \tilde{\mathbf{Q}}_{n_1}^{(1)} \tilde{\boldsymbol{\eta}}_{n_1}^{(1)})$$

and

$$\tilde{Q}_2 \tilde{\eta}_2 = (\tilde{Q}_2^{(2)} \tilde{\eta}_2^{(2)}, \dots, \tilde{Q}_{r_2}^{(2)} \tilde{\eta}_{r_2}^{(2)}, \dots, \tilde{Q}_{n_2}^{(2)} \tilde{\eta}_{n_2}^{(2)}),$$

where  $\tilde{\mathbf{Q}}_{r_1}^{(1)}(x)$  is a symmetric positive definite  $n \times n$ -matrix with entries  $\tilde{q}_{ij}^{(r_1)} \in C(\bar{D}_{r_1}^{(1)}), \quad i, j = 1, \ldots, n, \quad \tilde{\boldsymbol{\eta}}_{r_1}^{(1)} \in L^2(\Omega, L^2(D_{r_1}^{(1)})^n), \quad r_1 = 1, \ldots, n_1, \quad \tilde{Q}_{r_2}^{(2)}(x)$  is a continuous positive function defined in the domain  $\bar{D}_{r_2}^{(2)}, \quad \tilde{\eta}_{r_2}^{(2)} \in L^2(\Omega, L^2(D_{r_2}^{(2)})), \quad r_2 = 1, \ldots, n_2.$ 

In this case condition (2.18) takes the form<sup>8</sup>

$$\sum_{r_1=1}^{n_1} \int_{D_{r_1}^{(1)}} \operatorname{Sp}(\tilde{\mathbf{Q}}_{r_1}^{(1)}(x)\tilde{\mathbf{R}}_{r_1}^{(1)}(x,x)) \, dx \le 1, \quad \sum_{r_2=1}^{n_2} \int_{D_{r_2}^{(2)}} \tilde{Q}_{r_2}^{(2)}(x)\tilde{R}_{r_2}^{(2)}(x,x) \, dx \le 1,$$

where by  $\tilde{\mathbf{R}}_{r_1}^{(1)}(x,y) = [\tilde{b}_{i,j}^{(r_1)}(x,y)]_{i,j=1}^n$  we denote the correlation matrix of vector process  $\tilde{\boldsymbol{\eta}}_{r_1}^{(1)}(x) = (\tilde{\eta}_{r_1,1}^{(1)}(x), \dots, \tilde{\eta}_{r_1,n}^{(1)}(x))$  with components

$$\tilde{b}_{i,j}^{(r_1)}(x,y) = \mathbb{E}\Big(\tilde{\eta}_{r_1,i}^{(1)}(x)\tilde{\eta}_{r_1,j}^{(1)}(y)\Big), \quad (x,y) \in D_{i_1}^{(1)} \times D_{i_1}^{(1)},$$

and by  $\tilde{R}_{r_2}^{(2)}(x,y) = \mathbb{E}\tilde{\eta}_{r_2}^{(2)}(x)\tilde{\eta}_{r_2}^{(2)}(y)$  we denote the correlation function of process  $\tilde{\eta}_{r_2}^{(2)}(x)$ ,  $(x,y) \in D_{r_2}^{(2)} \times D_{r_2}^{(2)}$ . In fact,

$$\mathbb{E}(\tilde{Q}_{1}\tilde{\eta}_{1},\tilde{\eta}_{1})_{H_{1}} = \sum_{r_{1}=1}^{n_{1}} \mathbb{E}(\tilde{\mathbf{Q}}_{r_{1}}^{(1)}(x)\boldsymbol{\eta}_{r_{1}}^{(1)}(x),\boldsymbol{\eta}_{r_{1}}^{(1)}(x))_{L^{2}(D_{r_{1}}^{(1)})}$$
$$= \sum_{r_{1}=1}^{n_{1}} \mathbb{E}\left(\int_{D_{r_{1}}^{(1)}} \tilde{\mathbf{Q}}_{r_{1}}^{(1)}(x)\boldsymbol{\eta}_{r_{1}}^{(1)}(x),\boldsymbol{\eta}_{r_{1}}^{(1)}(x))_{\mathbb{R}^{n}}dx\right)$$
$$= \sum_{r_{1}=1}^{n_{1}} \sum_{i=1}^{n} \int_{D_{r_{1}}^{(1)}} \sum_{j=1}^{n} \mathbb{E}(\tilde{q}_{ij}^{(r_{1})}(x)\boldsymbol{\eta}_{j,r_{1}}^{(1)}(x)\boldsymbol{\eta}_{i,r_{1}}^{(1)}(x)) dx$$
$$= \sum_{r_{1}=1}^{n_{1}} \sum_{i=1}^{n} \int_{D_{r_{1}}^{(1)}} \sum_{j=1}^{n} \tilde{q}_{ij}^{(r_{1})}(x) \mathbb{E}(\boldsymbol{\eta}_{j,r_{1}}^{(1)}(x)\boldsymbol{\eta}_{i,r_{1}}^{(1)}(x)) dx$$

<sup>7</sup>Here and below we denote by  $C(\bar{D})$  a class of functions continuous in the domain  $\bar{D}$ . <sup>8</sup>Bv

$$\operatorname{Sp}(\tilde{\mathbf{Q}}_{r_1}^{(1)}(x)\tilde{\mathbf{R}}_{r_1}^{(1)}(x,x))$$

we denote the trace of matrix  $\tilde{\mathbf{Q}}_{r_1}^{(1)}(x)\tilde{\mathbf{R}}_{r_1}^{(1)}(x,x)$ , i.e. the sum of diagonal elements of this matrix.

$$=\sum_{r_1=1}^{n_1}\int_{D_{r_1}^{(1)}}\sum_{i=1}^n\sum_{j=1}^n \tilde{q}_{ij}^{(r_1)}(x)\tilde{b}_{j,i}^{(r_1)}(x,x)\,dx=\sum_{r_1=1}^{n_1}\int_{D_{r_1}^{(1)}}\operatorname{Sp}(\tilde{\mathbf{Q}}_{r_1}^{(1)}(x)\tilde{\mathbf{R}}_{r_1}^{(1)}(x,x))\,dx$$

Analogously,

$$\mathbb{E}(\tilde{Q}_{2}\tilde{\eta}_{2},\tilde{\eta}_{2})_{H_{2}} = \sum_{r_{2}=1}^{n_{2}} \mathbb{E}(\tilde{Q}_{r_{2}}^{(2)}(x)\eta_{r_{2}}^{(2)}(x),\eta_{r_{2}}^{(2)}(x))_{L^{2}(D_{r_{2}}^{(2)})}$$
$$= \sum_{r_{2}=1}^{n_{2}} \int_{D_{r_{2}}^{(2)}} \tilde{Q}_{r_{2}}^{(2)}(x)\mathbb{E}(\eta_{r_{2}}^{(2)}(x)\eta_{r_{2}}^{(2)}(x)) \, dx = \sum_{r_{2}=1}^{n_{2}} \int_{D_{r_{2}}^{(2)}} \tilde{Q}_{r_{2}}^{(2)}(x)\tilde{R}_{r_{2}}^{(2)}(x,x) \, dx.$$

Uncorrelatedness of random variables  $\tilde{\eta}_1$  and  $\tilde{\eta}_2$  reduces in this case to the condition of uncorrelatedness of the componets  $\tilde{\eta}_{i_1,j}^{(1)}$  of random vector fields  $\tilde{\eta}_{i_1}^{(1)}$ ,  $i_1 = \overline{1, n_1}$ ,  $j = \overline{1, n}$ , with random fields  $\tilde{\eta}_{i_2}^{(2)}$ ,  $i_2 = \overline{1, n_2}$ , and hence the set  $G_1$  is described by the formula

$$G_{1} = \left\{ \tilde{\eta} = (\tilde{\eta}_{1}, \tilde{\eta}_{2}) : \tilde{\eta}_{1} = (\tilde{\eta}_{1}^{(1)}, \dots, \tilde{\eta}_{i_{1},1}^{(1)}, \dots, \tilde{\eta}_{i_{1}}^{(1)}), \tilde{\eta}_{i_{1}}^{(1)} = (\tilde{\eta}_{i_{1},1}^{(1)}, \dots, \tilde{\eta}_{i_{1},n}^{(1)}) \right.$$

$$\in L^{2}(\Omega, L^{2}(D_{i_{1}}^{(1)})^{n}), \tilde{\eta}_{2} = (\tilde{\eta}_{1}^{(2)}, \dots, \tilde{\eta}_{i_{2}}^{(2)}, \dots, \tilde{\eta}_{i_{2}}^{(2)}), \tilde{\eta}_{i_{2}}^{(2)} \in L^{2}(\Omega, L^{2}(D_{i_{2}}^{(2)})),$$

$$\mathbb{E}\tilde{\eta}_{i_{1}}^{(1)} = 0, \mathbb{E}\tilde{\eta}_{i_{2}}^{(2)} = 0, \tilde{\eta}_{i_{1},j}^{(1)} \text{ and } \tilde{\eta}_{i_{2}}^{(2)} \text{ are uncorrelated, } j = \overline{1,n}, i_{1} = \overline{1,n_{1}}, i_{2} = \overline{1,n_{2}};$$

$$\sum_{i_{1}=1}^{n_{1}} \int_{D_{i_{1}}^{(1)}} \operatorname{Sp}\left[\tilde{\mathbf{Q}}_{i_{1}}^{(1)}(x)\tilde{\mathbf{R}}_{i_{1}}^{(1)}(x,x)\right] dx \leq 1, \sum_{i_{1}=1}^{n_{1}} \int_{D_{i_{2}}^{(2)}} \tilde{Q}_{i_{2}}^{(2)}(x)\tilde{R}_{i_{2}}^{(2)}(x,x) dx \leq 1 \right\}. \tag{6.4}$$

It is easily verified that the operator  $C_1^t : L^2(D_1^{(1)})^n \times \cdots \times L^2(D_{l_1}^{(1)})^n \times \cdots \times L^2(D_{n_1}^{(1)})^n \to L^2(D)^n$ , transpose of  $C_1$ , is defined by  $C_1^t\psi_1(x) = \sum_{l_1=1}^{n_1} \chi_{D_{l_1}^{(1)}}(x) \int_{D_{l_1}^{(1)}} [\mathbf{K}_{l_1}^{(1)}(\xi, x)]^T \psi_{l_1}^{(1)}(\xi) d\xi$ , where

$$\psi_1 = (\psi_1^{(1)}, \dots, \psi_{l_1}^{(1)}, \dots, \psi_{n_1}^{(1)}), \quad \psi_{l_1} \in L^2(D_{l_1}^{(1)})^n, \quad l_1 = 1, \dots, n_1,$$

and the operator  $C_2^t : L^2(D_1^{(2)}) \times \cdots \times L^2(D_{l_2}^{(2)}) \times \cdots \times L^2(D_{n_2}^{(2)}) \to L^2(D)$ , transpose of  $C_2$ , is defined by  $C_2^t \psi_2(x) = \sum_{l_2=1}^{n_2} \chi_{D_{l_2}^{(2)}}(x) \int_{D_{l_2}^{(2)}} K_{l_2}^{(2)}(\xi, x) \psi_{l_2}^{(2)}(\xi) d\xi$ , where

$$\psi_2 = (\psi_1^{(2)}, \dots, \psi_{l_2}^{(2)}, \dots, \psi_{n_2}^{(2)}), \quad \psi_2 \in L^2(D_{l_2}^{(2)}), \quad l_2 = 1, \dots, n_2,$$

and  $\chi(M)$  is a characteristic function of the set  $M \subset \mathbb{R}^n$ .

Since

$$\hat{u}_1 = \tilde{Q}_1 C_1 \mathbf{p}_1 = (\hat{\mathbf{u}}_1^1, \dots, \hat{\mathbf{u}}_{l_1}^1, \dots, \hat{\mathbf{u}}_{n_1}^1), \ \hat{\mathbf{u}}_{l_1}^1 \in L^2 (D_{l_1}^{(1)})^n, \ l_1 = 1, \dots, n_1,$$
$$\hat{u}_2 = \tilde{Q}_2 C_2 p_2 = (\hat{u}_1^2, \dots, \hat{u}_{l_2}^2, \dots, \hat{u}_{n_2}^2), \ \hat{u}_{l_2}^2 \in L^2 (D_{l_2}^{(2)}), \ l_2 = 1, \dots, n_2,$$

where

$$\hat{\mathbf{u}}_{i_1}^{(1)}(\cdot) = \tilde{\mathbf{Q}}_{i_1}^{(1)}(\cdot) \int_{D_{i_1}^{(1)}} \mathbf{K}_{i_1}^{(1)}(\cdot,\eta) \mathbf{p}_1(\eta) \, d\eta, \quad i_1 = 1, \dots, n_1,$$
(6.5)

$$\hat{u}_{i_2}^{(2)}(\cdot) = \tilde{Q}_{i_2}^{(2)}(\cdot) \int_{D_{l_2}^{(2)}} K_{i_2}^{(2)}(\cdot,\eta) p_2(\eta) \, d\eta, \quad i_2 = 1, \dots, n_2,$$
(6.6)

we find

$$C_{1}^{t}J_{H_{1}}\tilde{Q}_{1}C_{1}\mathbf{p}_{1}(\cdot) = C_{1}^{t}\hat{u}_{1}(\cdot) = \sum_{l_{1}=1}^{n_{1}}\chi_{D_{l_{1}}^{(1)}}(\cdot)\int_{D_{l_{1}}^{(1)}}[\mathbf{K}_{l_{1}}^{(1)}(\xi,\cdot)]^{T}\hat{\mathbf{u}}_{l_{1}}(\xi)\,d\xi$$
$$= \sum_{l_{1}=1}^{n_{1}}\chi_{D_{l_{1}}^{(1)}}(\cdot)\int_{D_{l_{1}}^{(1)}}[\mathbf{K}_{l_{1}}^{(1)}(\xi,\cdot)]^{T}\tilde{\mathbf{Q}}_{l_{1}}^{(1)}(\xi)\int_{D_{l_{1}}^{(1)}}\mathbf{K}_{l_{1}}^{(1)}(\xi,\xi_{1})\mathbf{p}_{1}(\xi_{1})\,d\xi_{1}\,d\xi =$$

$$=\sum_{l_1=1}^{n_1} \chi_{D_{l_1}^{(1)}}(\cdot) \int_{D_{l_1}^{(1)}} \tilde{\mathbf{K}}_{l_1}^{(1)}(\cdot,\xi_1) \mathbf{p}_1(\xi_1) \, d\xi_1, \tag{6.7}$$

$$C_2^t J_{H_2} \tilde{Q}_2 C_2 p_2(\cdot) = \sum_{l_2=1}^{n_2} \chi_{D_{l_2}^{(2)}}(\cdot) \int_{D_{l_2}^{(2)}} \tilde{K}_{l_2}^{(2)}(\cdot,\xi_1) p_2(\xi_1) \, d\xi_1, \tag{6.8}$$

where<sup>9</sup>

$$\tilde{\mathbf{K}}_{l_{1}}^{(1)}(\cdot,\xi_{1}) = \int_{D_{l_{1}}^{(1)}} (\mathbf{K}_{l_{1}}^{(1)}(\xi,\cdot))^{T} \tilde{\mathbf{Q}}_{l_{1}}^{(1)}(\xi) \mathbf{K}_{l_{1}}^{(1)}(\xi,\xi_{1}) d\xi,$$
  
$$\tilde{K}_{l_{2}}^{(2)}(\cdot,\xi_{1}) = \int_{D_{l_{1}}^{(1)}} K_{l_{2}}^{(2)}(\xi,\cdot)) \tilde{Q}_{l_{2}}^{(1)}(\xi) K_{l_{2}}^{(2)}(\xi,\xi_{1}) d\xi.$$

A class of linear with respect of observations (6.2) and (6.3) estimates  $\widehat{l(\mathbf{j},\varphi)}$  will take the form

$$\widehat{l(\mathbf{j},\varphi)} = \sum_{i_1=1}^{n_1} \int_{D_{i_1}^{(1)}} (\mathbf{u}_{i_1}^{(1)}(x), \mathbf{y}_{i_1}^{(1)}(x))_{\mathbb{R}^n} \, dx + \sum_{i_2=1}^{n_2} \int_{D_{i_2}^{(2)}} u_{i_2}^{(2)}(x) y_{i_2}^{(2)}(x) \, dx + c.$$
(6.9)

Thus, taking into account (6.5)–(6.9), we obtain that, under assumptions (2.13), (6.4), and (2.17), the following result is valid for integral observation operators as a corollary from Theorems 1 and 2.

**Theorem 5.** The guaranteed estimate  $\widehat{\overline{l(\mathbf{j},\varphi)}}$  of  $l(\mathbf{j},\varphi)$  is determined by the formula

$$\widehat{\hat{l(\mathbf{j},\varphi)}} = \sum_{i_1=1}^{n_1} \int_{D_{i_1}^{(1)}} (\hat{\mathbf{u}}_{i_1}^{(1)}(x), \mathbf{y}_{i_1}^{(1)}(x))_{\mathbb{R}^n} \, dx + \sum_{i_2=1}^{n_2} \int_{D_{i_2}^{(2)}} \hat{u}_{i_2}^{(2)}(x) y_{i_2}^{(2)}(x) \, dx + \hat{c} = l(\hat{\mathbf{j}}, \hat{\varphi}),$$

where

$$\hat{c} = -\int_{D} \hat{z}_{2}(x) f_{0}(x) dx,$$
$$\hat{\mathbf{u}}_{i_{1}}^{(1)}(x) = \tilde{\mathbf{Q}}_{i_{1}}^{(1)}(x) \int_{D_{i_{1}}^{(1)}} \mathbf{K}_{i_{1}}^{(1)}(x,\eta) \mathbf{p}_{1}(\eta) d\eta, \quad i_{1} = \overline{1, n_{1}},$$
$$\hat{u}_{i_{2}}^{(2)}(x) = \tilde{Q}_{i_{2}}^{(2)}(x) \int_{D_{i_{2}}^{(2)}} K_{i_{2}}^{(2)}(x,\eta) p_{2}(\eta) d\eta, \quad i_{2} = \overline{1, n_{2}},$$

and functions  $\mathbf{p}_1 \in H(\operatorname{div}, D)$ ,  $\hat{z}_2, p_2 \in L^2(D)$  and  $\hat{\mathbf{j}} \in H(\operatorname{div}, D)$ ,  $\hat{\varphi} \in L^2(D)$  are found from solution to systems of variational equations

$$\begin{split} \int_{D} (((\mathbf{A}(x))^{-1})^{T} \hat{\mathbf{z}}_{1}(x), \mathbf{q}_{1}(x))_{\mathbb{R}^{n}} dx &- \int_{D} \hat{z}_{2}(x) \operatorname{div} \mathbf{q}_{1}(x) \, dx \\ &= \int_{D} \Big( \mathbf{l}_{1}(x) - \sum_{i_{1}=1}^{n_{1}} \chi_{D_{i_{1}}^{(1)}}(x) \int_{D_{i_{1}}^{(1)}} \tilde{\mathbf{K}}_{i_{1}}^{(1)}(x, \xi_{1}) \mathbf{p}_{1}(\xi_{1}) \, d\xi_{1}, \mathbf{q}_{1}(x) \Big)_{\mathbb{R}^{n}} \, dx \,\, \forall \mathbf{q}_{1} \in H(\operatorname{div}, D), \end{split}$$

<sup>9</sup>We use the following notation: if  $\mathbf{A}(\xi) = [a_{ij}(\xi)]_{i,j=1}^N$  is a matrix dependig on variable  $\xi$  that varies on measurable set  $\Omega$ , then we define  $\int_{\Omega} \mathbf{A}(\xi) d\xi$  by the equality

$$\int_{\Omega} \mathbf{A}(\xi) \, d\xi = \left[ \int_{\Omega} a_{ij}(\xi) \, d\xi \right]_{i,j=1}^{N}.$$

$$\begin{split} &-\int_{D} v_{1}(x) \operatorname{div} \hat{\mathbf{z}}_{1}(x) \, dx - \int_{D} c(x) \hat{z}_{2}(x) v_{1}(x) \, dx \\ &= \int_{D} \Big( l_{2}(x) - \sum_{i_{2}=1}^{n_{2}} \chi_{D_{i_{2}}^{(2)}}(x) \int_{D_{i_{2}}^{(2)}} \tilde{K}_{i_{2}}^{(2)}(x, \xi_{1}) p_{2}(\xi_{1}) \, d\xi_{1} \Big) v_{1}(x) \, dx \, \, \forall v_{1} \in L^{2}(D), \\ &\int_{D} ((\mathbf{A}(x))^{-1} \mathbf{p}_{1}(x), \mathbf{q}_{2}(x))_{\mathbb{R}^{n}} dx - \int_{D} p_{2}(x) \operatorname{div} \mathbf{q}_{2}(x) \, dx = 0 \, \, \, \forall \mathbf{q}_{2} \in H(\operatorname{div}, D), \\ &- \int_{D} v_{2}(x) \operatorname{div} \hat{\mathbf{p}}_{1}(x) \, dx - \int_{D} c(x) \hat{p}_{2}(x) v_{2}(x) \, dx \\ &= \int_{D} v_{2}(x) Q^{-1} \hat{z}_{2}(x) \, dx \, \, \, \forall v_{2} \in L^{2}(D). \end{split}$$

and

$$\begin{split} &\int_{D} (((\mathbf{A}(x))^{-1})^{T} \hat{\mathbf{p}}_{1}(x), \mathbf{q}_{1}(x))_{\mathbb{R}^{n}} dx - \int_{D} \hat{p}_{2}(x) \operatorname{div} \mathbf{q}_{1}(x) \, dx \\ &= \int_{D} \left( \mathbf{d}_{1}(x) - \sum_{i_{1}=1}^{n_{1}} \chi_{D_{i_{1}}^{(1)}}(x) \int_{D_{i_{1}}^{(1)}} \tilde{\mathbf{K}}_{i_{1}}^{(1)}(x, \xi_{1}) \hat{\mathbf{j}}_{1}(\xi_{1}) \, d\xi_{1}, \mathbf{q}_{1}(x) \right)_{\mathbb{R}^{n}} dx \,\, \forall \mathbf{q}_{1} \in H(\operatorname{div}, D), \\ &- \int_{D} v_{1}(x) \operatorname{div} \hat{\mathbf{p}}_{1}(x) \, dx - \int_{D} c(x) \hat{p}_{2}(x) v_{1}(x) \, dx \\ &= \int_{D} \left( d_{2}(x) - \sum_{i_{2}=1}^{n_{2}} \chi_{D_{i_{2}}^{(2)}}(x) \int_{D_{i_{2}}^{(2)}} \tilde{K}_{i_{2}}^{(2)}(x, \xi_{1}) \hat{\varphi}(\xi_{1}) \, d\xi_{1} \right) v_{1}(x) \, dx \,\,\, \forall v_{1} \in L^{2}(D), \\ &\int_{D} ((\mathbf{A}(x))^{-1} \hat{\mathbf{j}}(x), \mathbf{q}_{2}(x))_{\mathbb{R}^{n}} \, dx - \int_{D} \hat{\varphi}(x) \operatorname{div} \mathbf{q}_{2}(x) \, dx = 0 \quad \forall \mathbf{q}_{2} \in H(\operatorname{div}, D), \\ &- \int_{D} v_{2}(x) \operatorname{div} \hat{\mathbf{j}}(x) \, dx - \int_{D} c(x) \hat{\varphi}(x) v_{2}(x) \, dx \\ &= \int_{D} v_{2}(x) (Q^{-1} \hat{p}_{2}(x) + f_{0}(x)) \, dx \quad \forall v_{2} \in L^{2}(D), \end{split}$$

respectively. Here  $\hat{\mathbf{z}}_1, \hat{\mathbf{p}}_1 \in H(\operatorname{div}, D), \, \hat{p}_2 \in L^2(D)$  and

$$\mathbf{d}_{1}(x) = \sum_{i_{1}=1}^{n_{1}} \chi_{D_{i_{1}}^{(1)}}(x) \int_{D_{i_{1}}^{(1)}} (\mathbf{K}_{i_{1}}^{(1)}(\xi, x))^{T} \tilde{\mathbf{Q}}_{i_{1}}^{(1)}(\xi) \mathbf{y}_{i_{1}}^{(1)}(\xi) \, d\xi,$$
$$d_{2}(x) = \sum_{i_{2}=1}^{n_{2}} \chi_{D_{i_{2}}^{(2)}}(x) \int_{D_{i_{2}}^{(2)}} K_{i_{2}}^{(2)}(\xi, x) \tilde{Q}_{i_{2}}^{(2)}(\xi) y_{i_{2}}^{(2)}(\xi) \, d\xi.$$

The estimation error  $\sigma$  is given by the expression

$$\sigma = l(\mathbf{p}_1, p_2)^{1/2}.$$

# 7 Minimax estimation of linear functionals from righthand sides of elliptic equations: Representations for guaranteed estimates and estimation errors

The problem is to determine a minimax estimate of the value of the functional

$$l(f) := \int_{D} l_0(x) f(x) \, dx \tag{7.1}$$

from observations (2.14) in the class of estimates, linear with respect to observations,

$$\widehat{l(f)} := (y_1, u_1)_{H_1} + (y_2, u_2)_{H_2} + c, \tag{7.2}$$

where  $u_1$  and  $u_2$  are elements from Hilbert spaces  $H_1$  and  $H_2$ , respectively,  $c \in \mathbb{R}$ ,  $l_0 \in L^2(D)$ is a given function, under the assumption that  $f \in G_0$  and  $\eta \in G_1$ , where sets  $G_0$  and  $G_1$  are defined on page 6.

**Definition 3.** The estimate of the form

$$\widehat{l(f)} = (y_1, \hat{u}_1)_{H_1} + (y_2, \hat{u}_2)_{H_2} + \hat{c}$$
(7.3)

will be called the guaranteed estimate of l(f) if the elements  $\hat{u}_1 \in H_1$ ,  $\hat{u}_2 \in H_2$  and a number  $\hat{c}$  are determined from the condition

$$\inf_{u \in H, c \in \mathbb{R}} \sigma(u, c) = \sigma(\hat{u}, \hat{c}),$$

where  $u = (u_1, u_2) \in H = H_1 \times H_2$ ,  $\hat{u} = (\hat{u}_1, \hat{u}_2) \in H$ ,

$$\sigma(u,c) := \sup_{\tilde{f} \in G_0, (\tilde{\eta}_1, \tilde{\eta}_2) \in G_1} \mathbb{E} |l(\tilde{f}) - \tilde{l}(\tilde{f})|^2,$$

$$\widehat{l(\tilde{f})} := (\tilde{y}_1, u_1)_{H_1} + (\tilde{y}_2, u_2)_{H_2} + c, \tag{7.4}$$

 $\tilde{y}_1 = C_1 \tilde{\mathbf{j}} + \tilde{\eta}_1, \quad \tilde{y}_2 = C_2 \tilde{\varphi} + \tilde{\eta}_2, \text{ and } (\tilde{\mathbf{j}}, \tilde{\varphi}) \text{ is a solution to problem (2.3)-(2.4) when } f(x) = \tilde{f}(x).$ The quantity

 $\sigma := [\sigma(\hat{u}, \hat{c})]^{1/2}$ 

is called the error of the guaranteed estimation of l(f).

For any fixed  $u := (u_1, u_2) \in H$ , introduce a pair of functions  $(\mathbf{z}_1(\cdot; u), z_2(\cdot; u)) \in H(\operatorname{div}; D) \times L^2(D)$  as a unique solution of the following problem:

$$\int_{D} (((\mathbf{A}(x))^{-1})^{T} \mathbf{z}_{1}(x; u), \mathbf{q}(x))_{\mathbb{R}^{n}} dx - \int_{D} z_{2}(x; u) \operatorname{div} \mathbf{q}(x) dx = -\int_{D} ((C_{1}^{t} J_{H_{1}} u_{1})(x), \mathbf{q}(x))_{\mathbb{R}^{n}} dx \quad \forall \mathbf{q} \in H(\operatorname{div}, D), \quad (7.5)$$

$$\int_{D} v(x) \operatorname{div} \mathbf{z}_{1}(x; u) \, dx + \int_{D} c(x) z_{2}(x; u) v(x) \, dx$$
$$= \int_{D} (C_{2}^{t} J_{H_{2}} u_{2})(x) v(x) \, dx \quad \forall v \in L^{2}(D). \quad (7.6)$$

**Lemma 2.** Finding the guaranteed estimate of l(f) is equivalent to the problem of optimal control of a system described by the problem (7.5), (7.6) with cost function

$$I(u) = \left(Q^{-1}(l_0 - z_2(\cdot; u)), l_0 - z_2(\cdot; u)\right)_{L^2(D)} + (\tilde{Q}_1^{-1}u_1, u_1)_{H_1} + (\tilde{Q}_2^{-1}u_2, u_2)_{H_2} \to \inf_{u \in H}.$$
 (7.7)

*Proof.* Taking into account (7.1) at  $f = \tilde{f}$  and (7.4), we have

$$l(\tilde{f}) - \widehat{l(\tilde{f})} = (l_0, \tilde{f})_{L^2(D)} - (\tilde{y}_1, u_1)_{H_1} - (\tilde{y}_2, u_2)_{H_2} - c$$
  

$$= (l_0, \tilde{f})_{L^2(D)} - (u_1, C_1 \tilde{\mathbf{j}} + \tilde{\eta}_1)_{H_1} - (u_2, C_2 \tilde{\varphi} + \tilde{\eta}_2)_{H_2} - c$$
  

$$= (l_0, \tilde{f})_{L^2(D)} - \langle J_{H_1} u_1, C_1 \tilde{\mathbf{j}} \rangle_{H_1' \times H_1} - \langle J_{H_2} u_2, C_2 \tilde{\varphi} \rangle_{H_2' \times H_2} - (u_1, \tilde{\eta}_1)_{H_1} - (u_2, \tilde{\eta}_2)_{H_2} - c$$
  

$$= -(C_1^t J_{H_1} u_1, \tilde{\mathbf{j}})_{L^2(D)^n} - (C_2^t J_{H_2} u_2, \tilde{\varphi})_{L^2(D)} + (l_0, \tilde{f})_{L^2(D)} - (u_1, \tilde{\eta}_1)_{H_1} - (u_2, \tilde{\eta}_2)_{H_2} - c.$$
(7.8)

Using a similar argument as in the proof of Lemma 1 in which the solution of problem (3.1), (3.2) is substituted by the solution of problem (7.5), (7.6), we obtain from (7.8) the following representation

$$\begin{split} l(\tilde{f}) - \widehat{l(\tilde{f})} &= (\tilde{f}, l_0 - z_2(\cdot; u)_{L^2(D)} - (u_1, \tilde{\eta}_1)_{H_1} - (u_2, \tilde{\eta}_2)_{H_2} - c \\ &= (\tilde{f} - f_0, l_0 - z_2(\cdot; u))_{L^2(D)} + (f_0, l_0 - z_2(\cdot; u))_{L^2(D)} \\ &- (u_1, \tilde{\eta}_1)_{H_1} - (u_2, \tilde{\eta}_2)_{H_2} - c. \end{split}$$

By virtue of (3.14), we find from here

$$\mathbb{E} \left| l(\tilde{f}) - \widehat{(l(\tilde{f}))} \right|^2 = \left| (\tilde{f}_2 - f_0, l_0 - z_2(\cdot; u))_{L^2(D)} + (f_0, l_0 - z_2(\cdot; u))_{L^2(D)} - c \right|^2 \\ + \mathbb{E} [(u_1, \tilde{\eta}_1)_{H_1} + (u_2, \tilde{\eta}_2)_{H_2}]^2.$$

From the latter equality, we obtain

=

$$\inf_{c \in \mathbb{R}} \sup_{\tilde{f} \in G_{0}, (\tilde{\eta}_{1}, \tilde{\eta}) \in G_{1}} \mathbb{E} |l(\tilde{f}) - l(\tilde{f})|^{2} = \\
= \inf_{c \in \mathbb{R}} \sup_{\tilde{f} \in G_{0}} \left[ (\tilde{f} - f_{0}, l_{0} - z_{2}(\cdot; u))_{L^{2}(D)} + (f_{0}, l_{0} - z_{2}(\cdot; u))_{L^{2}(D)} - c \right]^{2} \\
+ \sup_{(\tilde{\eta}_{1}, \tilde{\eta}_{2}) \in G_{1}} \mathbb{E} [(\tilde{\eta}_{1}, u_{1})_{H_{1}} + (\tilde{\eta}_{2}, u_{2})_{H_{2}}]^{2} \\
\sup_{\tilde{f} \in G_{0}} \left[ (\tilde{f} - f^{(0)}, l_{0} - z_{2}(\cdot; u))_{L^{2}(D)} \right]^{2} + \sup_{(\tilde{\eta}_{1}, \tilde{\eta}_{2}) \in G_{1}} \mathbb{E} [(\tilde{\eta}_{1}, u_{1})_{H_{1}} + (\tilde{\eta}_{2}, u_{2})_{H_{2}}]^{2}, \quad (7.9)$$

where infimum over c is attained at  $c = (f_0, l_0 - z_2(\cdot; u))_{L^2(D)}$ . Cauchy–Bunyakovsky inequality and (2.13) imply

$$|(\tilde{f} - f_0, l_0 - z_2(\cdot; u))_{L^2(D)}|^2 \le (Q^{-1}(l_0 - z_2(\cdot; u)), l_0 - z_2(\cdot; u))_{L^2(D)}(Q(\tilde{f} - f_0), \tilde{f} - f_0)_{L^2(D)})$$

$$\leq (Q^{-1}(l_0 - z_2(\cdot; u)), l_0 - z_2(\cdot; u))_{L^2(D)},$$

where inequality becomes an equality at

$$\tilde{f} = f_0 + \frac{Q^{-1}(l_0 - z_2(\cdot; u))}{(Q^{-1}(l_0 - z_2(\cdot; u)), l_0 - z(\cdot; u))_{L^2(D)}^{1/2}}$$

Hence

$$\sup_{\tilde{f}\in G_0} \left[ (\tilde{f}_2 - f_2^{(0)}, l_0 - z_2(\cdot; u))_{L^2(D)} \right]^2 = (Q^{-1}(l_0 - z_2(\cdot; u)), l_0 - z_2(\cdot; u))_{L^2(D)}.$$

Analoguosly, due to (2.18), (1.6), and (2.19), we have

$$\sup_{(\tilde{\eta}_1,\tilde{\eta}_2)\in G_1} \mathbb{E}[(\tilde{\eta}_1,u_1)_{H_1} + (\tilde{\eta}_2,u_2)_{H_2}]^2 = (\tilde{Q}_1^{-1}u_1,u_1)_{H_1} + (\tilde{Q}_2^{-1}u_2,u_2)_{H_2}.$$

From two latter relations and (7.9), we get

$$\inf_{c \in \mathbb{R}} \sup_{\tilde{f} \in G_0, (\tilde{\eta}_1, \tilde{\eta}) \in G_1} \mathbb{E} |l(\tilde{f}) - \tilde{l(\tilde{f})}|^2 = I(u),$$

at  $c = (l_0 - z_2(\cdot; u), f_0)_{L^2(D)}$ , where I(u) is determined by (7.7).

As a result of solving of optimal control problem (7.5) - (7.7), we come to the following assertion.

**Theorem 6.** There exists a unique estimate of l(f) which has the form

$$\widehat{\widehat{l(f)}} = (y_1, \hat{u}_1)_{H_1} + (y_2, \hat{u}_2)_{H_2} + \hat{c}, \qquad (7.10)$$

where

$$\hat{c} = \int_{D} (l_0(x) - \hat{z}_2(x)) f_0(x) \, dx, \quad \hat{u}_1 = \tilde{Q}_1 C_1 \mathbf{p}_1, \quad \hat{u}_2 = \tilde{Q}_2 C_2 p_2, \tag{7.11}$$

and the functions  $\mathbf{p}_1 \in H(\text{div}, D)$  and  $\hat{z}_2, p_2 \in L^2(D)$  are found from solution of the following variational problem:

$$\int_{D} (((\mathbf{A}(x))^{-1})^{T} \hat{\mathbf{z}}_{1}(x), \mathbf{q}_{1}(x))_{\mathbb{R}^{n}} dx - \int_{D} \hat{z}_{2}(x) \operatorname{div} \mathbf{q}_{1}(x) dx$$
  
=  $-\int_{D} (C_{1}^{t} J_{H_{1}} \tilde{Q}_{1} C_{1} \mathbf{p}_{1}(x), \mathbf{q}_{1}(x))_{\mathbb{R}^{n}} dx \quad \forall \mathbf{q}_{1} \in H(\operatorname{div}, D), \quad (7.12)$ 

$$\int_{D} v_1(x) \operatorname{div} \hat{\mathbf{z}}_1(x) \, dx + \int_{D} c(x) \hat{z}_2(x) v_1(x) \, dx$$
$$= \int_{D} (C_2^t J_{H_2} \tilde{Q}_2 C_2 p_2)(x) v_1(x) \, dx \quad \forall v_1 \in L^2(D), \quad (7.13)$$

$$\int_{D} ((\mathbf{A}(x))^{-1} \mathbf{p}_{1}(x), \mathbf{q}_{2}(x))_{\mathbb{R}^{n}} dx - \int_{D} p_{2}(x) \operatorname{div} \mathbf{q}_{2}(x) dx = 0 \quad \forall \mathbf{q}_{2} \in H(\operatorname{div}, D), \quad (7.14)$$

$$\int_{D} v_2(x) \operatorname{div} \mathbf{p}_1(x) \, dx + \int_{D} c(x) p_2(x) v_2(x) \, dx$$
$$= \int_{D} v_2(x) Q^{-1} (l_0 - \hat{z}_2(\cdot))(x) \, dx \quad \forall v_2 \in L^2(D), \quad (7.15)$$

where  $\hat{\mathbf{z}}_1 \in H(\text{div}, D)$ . Problem (7.12)–(7.15) is uniquely solvable. The error of estimation  $\sigma$  is given by the expression

$$\sigma = \left( l(Q^{-1}(l_0 - \hat{z}_2)) \right)^{1/2}.$$
(7.16)

*Proof.* Show that the solution to the optimal control problem (7.5)-(7.7) can be reduced to the solution of system (7.12)-(7.15).

First, we note that functional I(u), defined by (7.7), can be represented in the form

$$I(u) = \tilde{I}(u) - L(u) + \int_D Q^{-1} l_0(x) l_0(x) \, dx, \qquad (7.17)$$

where

$$\tilde{I}(u) = \int_D Q^{-1} z_2(x; u) z_2(x; u) \, dx + (\tilde{Q}_1^{-1} u_1, u_1)_{H_1} + (\tilde{Q}_2^{-1} u_2, u_2)_{H_2}$$

is a quadratic form corresponding to a symmetric continuous bilinear form

$$\pi(u,v) := \int_D Q^{-1} z_2(x;u) z_2(x;v) \, dx + (\tilde{Q}_1^{-1} u_1, v_1)_{H_1} + (\tilde{Q}_2^{-1} u_2, v_2)_{H_2},$$

defined on  $H \times H$  and

$$L(u) = 2 \int_D Q^{-1} z_2(x; u) l_0(x) \, dx$$

is a linear continuous functional defined on H.

The representation of in the form (7.17) follows from the reasoning similar to that in the proof of Theorem 1 (replacing  $\tilde{z}_2(x; u)$  by  $z_2(x; u)$  and  $z_2^{(0)}(x)$  by  $l_0(x)$ , correspondingly).

Since

$$\tilde{I}(u) = \pi(u, u) \ge (Q_1^{-1}u_1, u_1)_{H_1} + (\tilde{Q}_2^{-1}u_2, u_2)_{H_2} \ge \alpha \|u\|_H^2 \quad \forall u \in H,$$

where  $\alpha$  is a constant from (2.20), then the bilinear form  $\pi(u, v)$  and the linear functional L(u) satisfy the condition of Theorem 1.1 from [1]. Therefore, by this theorem, there exists a unique element  $\hat{u} := (\hat{u}_1, \hat{u}_2) \in H$  on which the minimum of the functional I(u) is attained, i.e.  $I(\hat{u}) = \inf_{u \in H} I(u)$ . This implies that for any fixed  $w = (w_1, w_2) \in H$  and  $\tau \in \mathbb{R}$  the function  $s(\tau) := I(\hat{u} + \tau w)$  reaches its minimum at the point  $\tau = 0$ , so that

$$\frac{d}{d\tau}I(\hat{u}+\tau w)\mid_{\tau=0} = 0.$$
(7.18)

Taking into account that

$$z_2(x; \hat{u} + \tau w) = z_2(x; \hat{u}) + \tau z_2(x; w),$$

we obtain from (7.18)

$$0 = \frac{1}{2} \frac{d}{dt} I(\hat{u} + \tau w) \Big|_{\tau=0}$$
  
=  $-(Q^{-1}(l_0 - z_2(\cdot; \hat{u})), z_2(\cdot; w))_{L^2(D)} + (\tilde{Q}_1^{-1} \hat{u}_1, w_1)_{H_1} + (\tilde{Q}_2^{-1} \hat{u}_2, w_2)_{H_2}.$  (7.19)

Further, introducing a pair of functions  $(\mathbf{p}_1, p_2) \in H(\operatorname{div}, D) \times L^2(D)$  as a unique solution of the problem

$$\int_{D} ((\mathbf{A}(x))^{-1} \mathbf{p}_1(x), \mathbf{q}_2(x))_{\mathbb{R}^n} dx$$
$$-\int_{D} p_2(x) \operatorname{div} \mathbf{q}_2(x) dx = 0 \quad \forall \mathbf{q}_2 \in H(\operatorname{div}, D), \quad (7.20)$$

$$\int_{D} v_2(x) \operatorname{div} \mathbf{p}_1(x) \, dx + \int_{D} c(x) p_2(x) v_2(x) \, dx$$
$$= \int_{D} v_2(x) Q^{-1} (l_0 - z_2(\cdot; \hat{u}))(x) \, dx \quad \forall v_2 \in L^2(D) \quad (7.21)$$

and reasoning analogously as in the proof of Theorem 1, we arrive at the following relation

$$-(Q^{-1}(l_0 - z_2(\cdot; \hat{u})), z_2(\cdot; w))_{L^2(D)} = -(w_1, C_1\mathbf{p}_1)_{H_1} - (w_2, C_2p_2)_{H_2}.$$

By using (4.14), we find from the latter equality

$$(w_1, C_1\mathbf{p}_1)_{H_1} + (w_2, C_2p_2)_{H_2} = (\tilde{Q}_1^{-1}\hat{u}_1, w_1)_{H_1} + (\tilde{Q}_2^{-1}\hat{u}_2, w_2)_{H_2},$$

Whence, it follows that  $\hat{u}_1 = \tilde{Q}_1 C_1 \mathbf{p}_1$ ,  $\hat{u}_2 = \tilde{Q}_2 C_2 p_2$ . Substituting these expressions into (7.5) and (7.6) and setting  $\mathbf{z}_1(x; \hat{u}) =: \hat{\mathbf{z}}_1(x), z_2(x; \hat{u}) =: \hat{z}_2(x)$ , we establish that functions  $\hat{\mathbf{z}}_1, \hat{z}_2$  and  $\mathbf{p}_1, p_2$  satisfy system of variational equations (7.12)–(7.15) and the validity of equalities (7.10), (7.11). The unique solvability of this system follows from the existence of the unique minimum point  $\hat{u}$  of functional I(u).

Now let us find the error of estimation. From (7.7) at  $u = \hat{u}$  and (7.11), it follows

$$\sigma^{2} = I(\hat{u}) = (Q^{-1}(l_{0} - z_{2}(\cdot; \hat{u})), l_{0} - z_{2}(\cdot; \hat{u}))_{L^{2}(D)} + (\tilde{Q}_{1}^{-1}\hat{u}_{1}, \hat{u}_{1})_{H_{1}} + (\tilde{Q}_{2}^{-1}\hat{u}_{2}, \hat{u}_{2})_{H_{2}} = (Q^{-1}(l_{0} - \hat{z}_{2}), l_{0} - \hat{z}_{2})_{L^{2}(D)} + (C_{1}\mathbf{p}_{1}, \tilde{Q}_{1}C_{1}\mathbf{p}_{1})_{H_{1}} + (C_{2}p_{2}, \tilde{Q}_{2}C_{2}p_{2})_{H_{2}}.$$

$$(7.22)$$

Setting in (7.14) and (7.15)  $\mathbf{q}_2 = \hat{\mathbf{z}}_1$  and  $v_2 = \hat{z}_2$ , we find

$$\int_{D} ((\mathbf{A}(x))^{-1} \mathbf{p}_{1}(x), \hat{\mathbf{z}}_{1}(x))_{\mathbb{R}^{n}} dx - \int_{D} p_{2}(x) \operatorname{div} \hat{\mathbf{z}}_{1}(x) dx = 0,$$
$$\int_{D} \hat{z}_{2}(x) \operatorname{div} \mathbf{p}_{1}(x) dx + \int_{D} c(x) p_{2}(x) \hat{z}_{2}(x) dx = \int_{D} \hat{z}_{2}(x) Q^{-1} (l_{0} - \hat{z}_{2})(x) dx.$$

Setting in equations (7.12) and (7.13)  $\mathbf{q}_1 = \mathbf{p}_1$  and  $v_1 = p_2$ , we derive from two latter relations

$$(Q^{-1}(l_0 - \hat{z}_2), l_0 - \hat{z}_2)_{L^2(D)} = (Q^{-1}l_0, l_0 - \hat{z}_2)_{L^2(D)} - \int_D \hat{z}_2(x) \operatorname{div} \hat{\mathbf{p}}_1(x) \, dx$$
  

$$- \int_D c(x)p_2(x)\hat{z}_2(x) \, dx + \int_D ((\mathbf{A}(x))^{-1}\mathbf{p}_1(x), \hat{\mathbf{z}}_1(x))_{\mathbb{R}^n} dx - \int_D p_2(x) \operatorname{div} \hat{\mathbf{z}}_1(x) \, dx$$
  

$$= (l_0, Q^{-1}(l_0 - \hat{z}_2))_{L^2(D)} + \int_D (((\mathbf{A}(x))^{-1})^T \hat{\mathbf{z}}_1(x), \mathbf{p}_1(x))_{\mathbb{R}^n} dx - \int_D z_2(x) \operatorname{div} \mathbf{p}_1(x) \, dx$$
  

$$- \int_D p_2(x) \operatorname{div} \hat{\mathbf{z}}_1(x) \, dx - \int_D c(x)p_2(x)\hat{z}_2(x) \, dx = (l_0, Q^{-1}(l_0 - \hat{z}_2))_{L^2(D)}$$
  

$$- \int_D (C_1^t J_{H_1} \tilde{Q}_1 C_1 \mathbf{p}_1(x), \mathbf{p}_1(x))_{\mathbb{R}^n} \, dx - \int_D C_2^t J_{H_2} \tilde{Q}_2 C_2 p_2(x) p_2(x) \, dx$$
  

$$= l(Q^{-1}(l_0 - \hat{z}_2)) - (C_1 \mathbf{p}_1, \tilde{Q}_1 C_1 \mathbf{p}_1)_{H_1} - (C_2 p_2, \tilde{Q}_2 C_2 p_2)_{H_2}.$$

From here and (7.22), it follows representation (7.16) for the estimation error.

In the following theorem we obtain another representation for the guaranteed estimate  $\widehat{l(f)}$  of quantity l(f) similar to (4.22).

**Theorem 7.** The guaranteed estimate of l(f) has the form

$$\widehat{\widehat{l(f)}} = l(\widehat{f}), \tag{7.23}$$

where  $\hat{f}(x) = f_0(x) - Q^{-1}\hat{p}_2(x)$  and  $\hat{p}_2 \in L^2(\Omega, L^2(D))$  is determined from solution of problem (4.23)-(4.26).

*Proof.* From (7.10) and (7.11), we have

$$\widehat{\widehat{l(f)}} = (y_1, \hat{u}_1)_{H_1} + (y_2, \hat{u}_2)_{H_2} + \hat{c}$$
$$= (y_1, \tilde{Q}_1 C_1 \mathbf{p}_1)_{H_1} + (y_2, \tilde{Q}_2 C_2 p_2)_{H_2} + (l_0 - \hat{z}_2, f_0)_{L^2(D)}.$$
(7.24)

Putting in (4.23) and (4.24),  $\mathbf{q}_1 = \mathbf{p}_1$  and  $v_1 = p_2$ , respectively, we come to the relations

$$\int_{D} (((\mathbf{A}(x))^{-1})^{T} \hat{\mathbf{p}}_{1}(x), \mathbf{p}_{1}(x))_{\mathbb{R}^{n}} dx - \int_{D} \hat{p}_{2}(x) \operatorname{div} \mathbf{p}_{1}(x) dx$$
$$= \int_{D} (C_{1}^{t} J_{H_{1}} \tilde{Q}_{1}(y_{1} - C_{1} \hat{\mathbf{j}})(x), \mathbf{p}_{1}(x))_{\mathbb{R}^{n}} dx, \quad (7.25)$$

$$-\int_{D} p_{2}(x) \operatorname{div} \hat{\mathbf{p}}_{1}(x) \, dx - \int_{D} c(x) \hat{p}_{2}(x) p_{2}(x) \, dx$$
$$= \int_{D} (C_{2}^{t} J_{H_{2}} \tilde{Q}_{2}(y_{2} - C_{2} \hat{\varphi})(x) p_{2}(x) \, dx, \quad (7.26)$$

Putting  $\mathbf{q}_2 = \hat{\mathbf{p}}_1$  and  $v_2 = \hat{p}_2$  in (7.14) and (7.15), we have

$$\int_{D} ((\mathbf{A}(x))^{-1} \mathbf{p}_1(x), \hat{\mathbf{p}}_1(x))_{\mathbb{R}^n} dx - \int_{D} p_2(x) \operatorname{div} \hat{\mathbf{p}}_1(x) \, dx = 0,$$
(7.27)

$$-\int_{D} \hat{p}_{2}(x) \operatorname{div} \mathbf{p}_{1}(x) \, dx - \int_{D} c(x) p_{2}(x) \hat{p}_{2}(x) \, dx$$
$$= -\int_{D} \hat{p}_{2}(x) Q^{-1} (l_{0} - \hat{z}_{2})(x) \, dx. \quad (7.28)$$

Relations (7.25)-(7.28), and (7.24) imply

$$\widehat{\hat{l(f)}} = (C_1 \hat{\mathbf{j}}, \tilde{Q}_1 C_1 \mathbf{p}_1)_{H_1} + (C_2 \hat{\varphi}, \tilde{Q}_2 C_2 p_2)_{H_2} - (Q^{-1} \hat{p}_2 - f_0, (l_0 - \hat{z}_2)_{L^2(D)}).$$
(7.29)

Setting  $\mathbf{q}_2 = \hat{\mathbf{z}}_1$ ,  $v_2 = \hat{z}_2$  and  $\mathbf{q}_1 = \hat{\mathbf{j}}$ ,  $v_1 = \hat{\varphi}$  in equations (4.25), (4.26) and (7.12), (7.13), respectively, we obtain

$$\int_{D} ((\mathbf{A}(x))^{-1} \hat{\mathbf{j}}(x), \hat{\mathbf{z}}_{1}(x))_{\mathbb{R}^{n}} dx - \int_{D} \hat{\varphi}(x) \operatorname{div} \hat{\mathbf{z}}_{1}(x) dx = 0,$$
(7.30)

$$-\int_{D} \hat{z}_{2}(x) \operatorname{div} \hat{\mathbf{j}}(x) \, dx - \int_{D} c(x) \hat{\varphi}(x) \hat{z}_{2}(x) \, dx = \int_{D} \hat{z}_{2}(x) (Q^{-1} \hat{p}_{2}(x) - f_{0}(x)) \, dx, \qquad (7.31)$$

and

$$\int_{D} (((\mathbf{A}(x))^{-1})^{T} \hat{\mathbf{z}}_{1}(x), \hat{\mathbf{j}}(x))_{\mathbb{R}^{n}} dx - \int_{D} \hat{z}_{2}(x) \operatorname{div} \hat{\mathbf{j}}(x) dx$$
  
=  $-\int_{D} (C_{1}^{t} J_{H_{1}} \tilde{Q}_{1} C_{1} \mathbf{p}_{1}(x), \hat{\mathbf{j}}(x))_{\mathbb{R}^{n}} dx, \quad (7.32)$ 

$$-\int_{D}\hat{\varphi}(x)\operatorname{div}\hat{\mathbf{z}}_{1}(x)\,dx - \int_{D}c(x)\hat{z}_{2}(x)\hat{\varphi}(x)\,dx = -\int_{D}(C_{2}^{t}J_{H_{2}}\tilde{Q}_{2}C_{2}p_{2}(x)\hat{\varphi}(x)\,dx.$$
(7.33)

From (7.30) and (7.33), we deduce

$$\int_D \hat{z}_2(x) (Q^{-1} \hat{p}_2(x) - f_0(x)) \, dx = -(\tilde{Q}_1 C_1 \hat{\mathbf{p}}_1, C_1 \hat{\mathbf{j}}(x))_{H_1} - (\tilde{Q}_2 C_2 \hat{p}_2, C_2 \hat{\varphi}(x))_{H_2},$$

whence, by virtue of (7.29), it follows representation (7.23).

**Remark 2**. Notice that in representation  $l(\hat{f})$  for minimax estimate  $\widehat{l(f)}$  the function  $\hat{f}(x) = f_0(x) - Q^{-1}\hat{p}_2(x)$ , where  $\hat{p}_2$  is defined from equations (4.23)–(4.26), can be taken as a good estimate for unknown function f entering the right-hand side of equation (2.4) (for explanations, see Remark 1).

## 8 Approximate guaranteed estimates of linear functionals from right-sides of elliptic equations

In this section we introduce the notion of approximate guaranteed estimates of  $l(\mathbf{j}, \varphi)$  and prove their convergence to  $\widehat{l(\mathbf{j}, \varphi)}$ .

Further, as in section 6, the domain D is supposed to be bounded and connected domain of  $\mathbb{R}^n$  with polyhedral boundary  $\Gamma$ .

Take an approximate minimax estimate of l(f) as

$$\widehat{l^h(f)} = (u_1^h, y_1)_{H_1} + (u_2^h, y_2)_{H_1} + c^h,$$
(8.1)

where  $u_1^h = \tilde{Q}_1 C_1 \mathbf{p}_1^h$ ,  $u_2^h = \tilde{Q}_2 C_2 p_2^h$ ,  $c^h = \int_D (l_0(x) - \hat{z}_2^h(x)) f_0(x) dx$ , and functions  $\hat{\mathbf{z}}_1^h, \mathbf{p}_1^h \in V_1^h$ and  $\hat{z}_2^h, p_2^h \in V_2^h$  are determined<sup>10</sup> from the following uniquely solvable system of variational equalities:

$$\int_{D} (((\mathbf{A}(x))^{-1})^{T} \hat{\mathbf{z}}_{1}^{h}(x), \mathbf{q}_{1}^{h}(x))_{\mathbb{R}^{n}} dx - \int_{D} \hat{z}_{2}^{h}(x) \operatorname{div} \mathbf{q}_{1}^{h}(x) dx$$
$$= -\int_{D} (C_{1}^{t} J_{H_{1}} \tilde{Q}_{1} C_{1} \mathbf{p}_{1}^{h}(x), \mathbf{q}_{1}^{h}(x))_{\mathbb{R}^{n}} dx \quad \forall \mathbf{q}_{1}^{h} \in V_{1}^{h}, \quad (8.2)$$

$$\int_{D} v_{1}^{h}(x) \operatorname{div} \hat{\mathbf{z}}_{1}^{h}(x) \, dx + \int_{D} c(x) \hat{z}_{2}^{h}(x) v_{1}^{h}(x) \, dx$$
$$= \int_{D} (C_{2}^{t} J_{H_{2}} \tilde{Q}_{2} C_{2} p_{2}^{h}(x) v_{1}^{h}(x) \, dx \quad \forall v_{1}^{h} \in V_{2}^{h}, \quad (8.3)$$

$$\int_{D} ((\mathbf{A}(x))^{-1} \mathbf{p}_{1}^{h}(x), \mathbf{q}_{2}^{h}(x))_{\mathbb{R}^{n}} dx - \int_{D} p_{2}^{h}(x) \operatorname{div} \mathbf{q}_{2}^{h}(x) dx = 0 \quad \forall \mathbf{q}_{2}^{h} \in V_{1}^{h},$$
(8.4)

$$\int_{D} v_{2}^{h}(x) \operatorname{div} \mathbf{p}_{1}^{h}(x) \, dx + \int_{D} c(x) p_{2}^{h}(x) v_{2}^{h}(x) \, dx$$
$$= \int_{D} v_{2}^{h}(x) Q^{-1}(l_{0} - \hat{z}_{2}^{h}(\cdot))(x) \, dx \quad \forall v_{2}^{h} \in V_{2}^{h}.$$
(8.5)

<sup>10</sup>The spaces  $V_1^h$  and  $V_2^h$  are described on page 18.

The quantity  $\sigma^h = (I(u^h))^{1/2}$ , where

$$I(u^{h}) = (Q^{-1}(l_{0} - \hat{z}_{2}^{h}), l_{0} - \hat{z}_{2}^{h})_{L^{2}(D)} + (\tilde{Q}_{1}^{-1}u_{1}^{h}, u_{1}^{h})_{H_{1}} + (\tilde{Q}_{2}^{-1}u_{2}^{h}, u_{2}^{h})_{H_{2}},$$

is called the approximate error of the guaranteed estimation of l(f).

**Theorem 8.** Approximate guaranteed estimate  $\widehat{l^h(f)}$  of l(f) which is defined by (8.1) can be represented in the form  $\widehat{l^h(f)} = l(\widehat{f^h})$ , where  $\widehat{f^h} = f_0(x) - Q^{-1}\widehat{p}_2^h(x)$ , and function  $\widehat{p}_2^h \in Q^h$  is determined from solution of problem (5.53)–(5.56). Approximate error of estimation has the form

$$\sigma^h = \left( l(Q^{-1}(l_0 - \hat{z}_2^h)) \right)^{1/2}$$

In addition,

$$\lim_{h \to 0} \mathbb{E} |\widehat{l^h(f)} - \widehat{\widehat{l(f)}}|^2 = 0, \quad \lim_{h \to \infty} \sigma^h = \sigma,$$

and

$$\begin{aligned} \|\hat{\mathbf{z}}_{1} - \hat{\mathbf{z}}_{1}^{h}\|_{H(\operatorname{div},D)} + \|\hat{z}_{2} - \hat{z}_{2}^{h}\|_{L^{2}(D)} &\to 0 \quad \text{as} \quad h \to 0, \\ \|\mathbf{p}_{1} - \mathbf{p}_{1}^{h}\|_{H(\operatorname{div},D)} + \|p_{2} - p_{2}^{h}\|_{L^{2}(D)} &\to 0 \quad \text{as} \quad h \to 0, \\ \|\hat{\mathbf{j}} - \hat{\mathbf{j}}^{h}\|_{H(\operatorname{div},D)} + \|\hat{\varphi} - \hat{\varphi}^{h}\|_{L^{2}(D)} &\to 0 \quad \text{as} \quad h \to 0, \\ \|\hat{\mathbf{p}}_{1} - \hat{\mathbf{p}}_{1}^{h}\|_{H(\operatorname{div},D)} + \|\hat{p}_{2} - \hat{p}_{2}^{h}\|_{L^{2}(D)} &\to 0 \quad \text{as} \quad h \to 0. \end{aligned}$$

*Proof.* The proof of this theorem is similar to the proofs of Theorems 4 and 5.

System of linear algebraic equations with respect to coefficients of expansions (5.57), (5.58) of functions  $\hat{\mathbf{z}}_1^h$ ,  $\hat{z}_2^h$ ,  $\mathbf{p}_1^h$ , and  $p_2^h$ , analogous to (5.59)–(5.62), can be also obtained for problem (8.2)–(8.5).

#### 9 Corollary from the obtained results

Note in conclusion that the above results generalize, for the class of estimation problems for systems described by boundary value problems considered in this work, the results by A. G. Nakonechnyi [5], [6].

To do this, suppose, as in these papers, that from observations of random variable of the form

$$y_2 = C_2 \varphi + \eta_2, \tag{9.1}$$

it is necessary to estimate the expression

$$l(\varphi) := \int_D l_2(x)\varphi(x)\,dx \tag{9.2}$$

in the class of estimates of the form

$$\widehat{l(\varphi)} := (y_2, u_2)_{H_2} + c,$$
(9.3)

where  $\varphi$  is a solution to the problem (2.1), (2.2),  $l_2$  is a given function from  $L^2(D)$ ,  $u_2 \in H_2$ ,  $c \in \mathbb{R}$ ,  $C_2 \in \mathcal{L}(L^2(D), H_2)$  is a linear operator.

The case considered here corresponds to setting  $C_1 = 0$ ,  $\eta_1 = 0$ ,  $\mathbf{l}_1 = 0$ ,  $u_1 = 0$ , respectively in (2.14), (2.15), (2.16), (2.18), and Lemma 1 can be stated as follows.

**Lemma 3.** Finding the minimax estimate of  $l(\varphi)$  is equivalent to the problem of optimal control of the system described by the equations

$$\int_{D} (((\mathbf{A}(x))^{-1})^T \mathbf{z}_1(x; u), \mathbf{q}(x))_{\mathbb{R}^n} dx - \int_{D} z_2(x; u) \operatorname{div} \mathbf{q}(x) \, dx = 0$$
$$\forall \mathbf{q} \in H(\operatorname{div}, D), \quad (9.4)$$

$$-\int_{D} v(x) \operatorname{div} \mathbf{z}_{1}(x; u) \, dx - \int_{D} c(x) z_{2}(\cdot; u) v(x) \, dx$$
$$= \int_{D} (l_{2}(x) - (C_{2}^{t} J_{H_{2}} u_{2})(x)) v(x) \, dx \quad \forall v \in L^{2}(D) \quad (9.5)$$

with the cost function

$$I(u) = (Q^{-1}z_2(\cdot; u), z_2(\cdot; u))_{L^2(D)} + (\tilde{Q}_2^{-1}u_2, u_2)_{H_2} \to \inf_{u \in H_2}.$$
(9.6)

It is easy to see that the second component  $z_2(\cdot; u)$  of the solution  $(\mathbf{z}_1(\cdot; u), z_2(\cdot; u))$  to this problem belongs to the space  $H_0^1(D)$  and is a weak solution to problem (2.1)–(2.2), i.e. it satisfies the integral identity

$$- (\mathbf{A}^T \operatorname{\mathbf{grad}} z_2, \operatorname{\mathbf{grad}} v)_{L^2(D)^n} - (cz_2, v)_{L^2(D)} = ((l_2 - (C_2^t J_{H_2} u_2), v)_{L^2(D)} \quad \forall v \in H_0^1(D). \quad (9.7)$$

Therefore, Lemma 3 takes the form:

**Lemma 4.** Finding the minimax estimate of  $l(\varphi)$  is equivalent to the problem of optimal control of the system described by equation (9.7) with the cost function (9.6).

Theorems 1 and 2 are transformed into the following assertions.

**Theorem 9.** There exists a unique minimax estimate of  $l(\mathbf{j}, \varphi)$  which has the form

$$\widehat{\widehat{l(\mathbf{j},\varphi)}} = (y_2, \hat{u}_2)_{H_2} + \hat{c}, \qquad (9.8)$$

where

$$\hat{c} = -\int_D \hat{z}_2(x) f_0(x) \, dx, \quad \hat{u}_2 = \tilde{Q}_2 C_2 p_2,$$
(9.9)

and the functions  $\hat{z}_2$  and  $p_2 \in H^1_0(D)$  are determined from solution of the following problem:

$$-\int_{D} (\mathbf{A}^{T}(x) \operatorname{\mathbf{grad}} \hat{z}_{2}(x), \operatorname{\mathbf{grad}} v_{1}(x))_{\mathbb{R}^{n}} dx - \int_{D} c(x) \hat{z}_{2}(x) v_{1}(x) dx$$
$$= \int_{D} (l_{2}(x) - C_{2}^{t} J_{H_{2}} \tilde{Q}_{2} C_{2} p_{2}(x)) v_{1}(x) dx \quad \forall v_{1} \in H_{0}^{1}(D), \quad (9.10)$$

$$-\int_{D} (\mathbf{A}(x) \operatorname{\mathbf{grad}} p_{2}(x), \operatorname{\mathbf{grad}} v_{2}(x))_{\mathbb{R}^{n}} dx - \int_{D} c(x) p_{2}(x) v_{2}(x) dx$$
$$= \int_{D} v_{2}(x) Q^{-1} \hat{z}_{2}(x) dx \quad \forall v_{2} \in H_{0}^{1}(D). \quad (9.11)$$

Problem (9.10)–(9.11) is uniquely solvable. The error of estimation  $\sigma$  is given by the expression

$$\sigma = l(p_2)^{1/2}.$$
 (9.12)

**Theorem 10.** The minimax estimate of  $l(\mathbf{j}, \varphi)$  has the form

$$\widehat{\widehat{l(\mathbf{j},\varphi)}} = l(\hat{\mathbf{j}},\hat{\varphi})),$$

where the function  $\hat{\varphi} \in H_0^1(D)$  is determined from solution of the following problem:

$$-\int_{D} (\mathbf{A}^{T}(x) \operatorname{\mathbf{grad}} \hat{p}_{2}(x), \operatorname{\mathbf{grad}} v_{1}(x))_{\mathbb{R}^{n}} dx - \int_{D} c(x) \hat{p}_{2}(x) v_{1}(x) dx$$
$$= \int_{D} C_{2}^{t} J_{H_{2}} \tilde{Q}_{2}(y_{2} - C_{2} \hat{\varphi})(x) v_{1}(x) dx \quad \forall v_{1} \in H_{0}^{1}(D), \quad (9.13)$$

$$- (\mathbf{A}^{T}(x) \operatorname{\mathbf{grad}} \hat{\varphi}(x), \operatorname{\mathbf{grad}} v_{2}(x))_{\mathbb{R}^{n}} dx - \int_{D} c(x) \hat{\varphi}(x) v_{2}(x) dx$$
$$= \int_{D} v_{2}(x) (Q^{-1} \hat{p}_{2}(x) - f_{0}(x)) dx \quad \forall v_{2} \in H^{1}_{0}(D), \quad (9.14)$$

where equalities (9.13) and (9.14) are fulfilled with probability 1. Problem (9.13), (9.14) is uniquely solvable.

The random fields  $\hat{\varphi}$  and  $\hat{p}_2$ , whose realizations satisfy equations (9.13) and (9.14), belong to the space  $L^2(\Omega, H_0^1(D))$ .

#### REFERENCES

- Lions, J.L.: Optimal Control of Systems Described by Partial Differential Equations, Springer-Verlag Berlin · Heidelberg · New York (1971)
- [2] Vahaniya, N. N., Tarieladze, V. I. and Chobanyan, S. A.: Probability Distributions in Banach Spaces. Kluwer Acad. Publ., Dordrecht (1989)
- [3] Krasovskii, N.N.: Theory of Motion Control. Nauka, Moscow (1968)
- [4] Kurzhanskii, A.B.: Control and Observation under Uncertainties. Nauka, Moscow (1977)
- [5] Nakonechnyi, O.G.: Minimax Estimation of Functionals of Solutions to Variational Equations in Hilbert Spaces. Kiev State University, Kiev (1985)
- [6] Nakonechnyi, O.G.: Optimal Control and Estimation for Partial Differential Equations. Kyiv University, Kyiv (2004)
- [7] Nakonechniy, O. G., Podlipenko, Yu. K., Shestopalov, Yu. V.: Estimation of parameters of boundary value problems for linear ordinary differential equations with uncertain data. arXiv:0912.2872v1, 1-72, (2009)
- [8] Hutson, V., Pym, J., Cloud, M.: Applications of Functional Analysis and Operator Theory: Second Edition. Elsevier, Amsterdam (2005)

- [9] Ernst, O. G., Powell, C. E., Silvester, D. J., and Ullman, E.: Efficient solvers for a linear stochastic Galerkin mixed formulation of diffusion problems with random data. Technical Report 2007.126, Manchester Institue for Mathematical Sciences, School of Mathematics (2007)
- [10] Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods. Springer-Verlag (1991)
- [11] Nedelec, J.C.: Mixed finite element in  $\mathbb{R}^3$ . Numer. Math., 35, pp. 315-341 (1980).
- [12] Raviart, P.A., Thomas, J.M.: A mixed finite element method for second order elliptic problems, Mathematical Aspects of the Finite Element Method, Lecture Notes in Math. 606, Springer-Verlag, New York (1977)
- [13] Gatica G.N.: A Simple Introduction to the Mixed Finite Element Method: Theory and Applications. Springer (2014)
- [14] Schwab, C., Gittelson, C.J.: Sparse tensor discretization of high-dimensional parametric and stochastic PDEs. Acta Numerica. doi: 10.1017/S0962492911000055
- [15] Badriev, I.B., Karchevsky, M.M.: Duality methods in applied problems. Publ. of Kazan State University, Kazan (1987)
- [16] Yosida, K.: Functional analysis, 6 ed. Springer (1980)
- [17] Hoppe, R.H.W., Wohlmuth, B.: Adaptive Multilevel Techniques for Mixed Finite Element Discretizations of Elliptic Boundary Value Problems. SIAM J. Numer. Anal., vol. 34, no 4, (1997)
- [18] Podlipenko, Y., Shestopalov, Y.: Guaranteed estimates of functionals from solutions and data of interior Maxwell problems under uncertainties. In Inverse Problems and Large-Scale Computations, Springer Proc. in Mathematics & Statistics, 2013, Vol. 52. pp. 135-168.