FINE AND COARSE MULTIFRACTAL ZETA-FUNCTIONS: ON THE MULTIFRACTAL FORMALISM FOR MULTIFRACTAL ZETA-FUNCTIONS

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Abstract. Multifractal analysis refers to the study of the local properties of measures and functions, and consists of two parts: the fine multifractal theory and the coarse multifractal theory. The fine and the coarse theory are linked by a web of conjectures known collectively as the Multifractal Formalism. Very roughly speaking the Multifractal Formalism says that the multifractal spectrum from fine theory equals the Legendre transform of the Renyi dimensions from the coarse theory.

Recently *fine* multifractal zeta-functions, i.e. multifractal zeta-functions designed to produce detailed information about the fine multifractal theory, have been introduced and investigated. The purpose of this work is to complement and expand this study by introducing and investigating *coarse* multifractal zeta-functions, i.e. multifractal zeta-functions designed to produce information about the coarse multifractal theory, and, in particular, to establish a *Multifractal Fortmalism for Zeta-Functions* linking fine multifractal zeta-functions and coarse multifractal zeta-functions via the Legendre transform.

Several applications are given, including applications to multifractal analysis of graph-directed self-conformal measures and multifractal analysis of ergodic Birkhoff averages of continuous functions on graph-directed self-conformal sets.

In judiciously chosen examples the *Multifractal Fortmalism for Zeta-Functions* reduces to known results linking different types of classical multifractal spectra to the Legendre transform of certain Renyi dimensions, and in other examples the *Multifractal Fortmalism for Zeta-Functions* provides new results linking multifractal spectra to the Legendre transform of various Renyi dimensions. In particular, our results leads to new representations for many different types of multifractal spectra of ergodic Birkhoff averages of continuous functions.

1. Introduction.

Mutifractal analysis emerged in the 1990's as a powerful tool for analysing the local behaviour of measures with widely varying intensity. Multifractal analysis consists of two distinct parts: the *fine* part and the *coarse* part. the fine multifractal theory and the coarse multifractal theory. The fine and the coarse theory are linked, via the Legendre transform, by a web of conjectures known collectively as the Multifractal Formalism.

Recently *fine* multifractal zeta-functions, i.e. multifractal zeta-functions designed to produce detailed information about the fine multifractal theory, have been introduced and investigated. The purpose of this work is to complement and expand this study by introducing and investigating *coarse* multifractal zeta-functions, i.e. multifractal zeta-functions designed to produce information about the coarse multifractal theory, and, in particular, to establish a *Multifractal Fortmalism for Zeta-Functions* linking fine multifractal zeta-functions and coarse multifractal zeta-functions via the Legendre transform.

Multifractal analysis. Multifractal analysis consists of two distinct parts: the *fine* part and the *coarse* part. These two parts are linked together by the so-called Multifractal Formalism.

Fine multifractal analysis. The first main ingredient in multifractal analysis is the multifractal spectrum. The multifractal spectrum is defined as follows. For a Borel measure μ on $\mathbb{R}^d$ and a

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positive number α, let us consider the set of those points x in $\mathbb{R}^d$ for which the measure $\mu(B(x,r))$ of the ball $B(x,r)$ with center x and radius r behaves like r^α for small r, i.e. the set

$$\left\{ x \in K \,\middle|\, \lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\}. \tag{1.1}$$

If the intensity of the measure μ varies very widely, it may happen that the sets in (1.1) display a fractal-like character for a range of values of α. In this case it is natural to study the Hausdorff dimensions of the sets in (1.1) as α varies. We therefore define the fine multifractal spectrum of μ by

$$f_\mu(\alpha) = \dim_{\mathsf{H}} \left\{ x \in K \,\middle|\, \lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\}, \tag{1.2}$$

where $\dim_{\mathsf{H}}$ denotes the Hausdorff dimension. It is clear that the fine multifractal spectrum is defined using local (i.e. *fine*) properties of the measure, namely, the local dimension.

Coarse multifractal analysis. The second main ingredient in multifractal analysis is the Renyi dimensions. The Renyi dimensions are defined using global (i.e. *coarse*) properties of the measure, namely, they quantify the varying intensity of a measure by analyzing its moments at different scales. Formally, for $q \in \mathbb{R}$, the q'th Renyi dimensions $\tau_\mu(q)$ of μ is defined by

$$\tau_\mu(q) = \lim_{r \searrow 0} \frac{\log \displaystyle\int_K \mu(B(x,r))^{q-1} \, d\mu(x)}{-\log r}, \tag{1.3}$$

provided the limit exists.

The Multifractal Formalism: linking fine and coarse multifractal analysis. One of the main problems in multifractal analysis is to understand the multifractal spectrum and the Renyi dimensions, and their relationship with each other. Indeed, based on a remarkable insight together with a clever heuristic argument, theoretical physicists Halsey et al. [HaJeKaPrSh] suggested in the 1980's that the multifractal spectrum can be computed from the Renyi dimensions using a principle known as the Multifractal Formalism. More precisely, the Multifractal Formalism predicts that the multifractal spectrum equals the Legendre transform of the Renyi dimensions. Before stating this formally, we remind the reader that if X is an inner product space with inner product $\langle \cdot | \cdot \rangle$ and $f : X \to \mathbb{R}$ is a real valued function, then the Legendre transform $f^* : \mathbb{R} \to [-\infty, \infty]$ of f is defined by

$$f^*(x) = \inf_y (\langle x | y \rangle + f(y)). \tag{1.4}$$

We can now state a mathematically precise version of the Multifractal Formalism.

Definition. The Multifractal Formalism for Measures. *Let μ be a Borel measure on $\mathbb{R}^d$ with support K. We will say that μ satisfies the Multifractal Formalism if the following conditions hold:*

(i) *The limit in (1.3) exists for all q.*
(ii) *The function τ_μ is convex and differentiable.*
(iii) $\tau_\mu(0) = \dim_{\mathsf{H}} K$.
(iv) *The multifractal spectrum of μ equals the Legendre transform of the Renyi dimensions, i.e.*

$$f_\mu(\alpha) = \tau_\mu^*(\alpha) \quad \text{for all } \alpha.$$

Conditions (ii) and (iii) are usually not included in the Multifractal Formalism; however, since these properties play an important role later, we have decided to include them. During the past 20 years there has been an enormous interest in verifying the Multifractal Formalism and computing the multifractal spectra of measures in the mathematical literature. In the mid 1990's Cawley & Mauldin [CaMa] and Arbeiter & Patzschke [ArPa] verified the Multifractal Formalism for self-similar measures satisfying the so-called Open Set Condition, and within the last 20 years the multifractal spectra of various classes of measures in Euclidean space $\mathbb{R}^d$ exhibiting some degree of self-similarity have been computed rigorously, cf. the textbooks [Fa2,Pe2] and the references therein. On the other hand, it is also known that most measures (both in the sense of Baire category and in the sense of "shyness") do not satisfy the Multifractal Formalism, see, for example, [Bay1,Bay2,Ol3,Ol4]. While most measures do not satisfy the Multifractal Formalism, it is well-known and not difficult to show that the multifractal spectrum is always bounded above by the Legendre transform of the Renyi dimension. Since this result plays an important role later, we have decided to state it formally.

Proposition A [LauNg,Ol1,Pe1]. *Let μ be a Borel measure on $\mathbb{R}^d$ and assume that the limit in (1.3) exists for all q. Then*

$$f_\mu(\alpha) \le \tau_\mu^*(\alpha) \ \text{ for all } \alpha.$$

More generally, if $K \subseteq \mathbb{R}^d$ and $\Phi : K \to \mathbb{R}$ is a function, then the fine multifractal spectrum of Φ is defined by

$$f_\Phi(\alpha) = \dim_{\mathsf{H}} \left\{ x \in K \,\middle|\, \Phi(x) = \alpha \right\}. \tag{1.5}$$

Of course, if $\Phi(x) = \lim_{r\searrow 0} \frac{\log \mu(B(x,r))}{\log r}$, then the multifractal spectrum of Φ equals the multifractal spectrum of μ from (1.2). Other interesting and important examples are obtained by, for example, letting $\Phi(x)$ equal the ergodic Birkhoff average of a continuous function at x, or by letting $\Phi(x)$ equal the local entropy of a dynamical system at x. In this more general setting it is also natural to attempt to find a "coarse moment scaling" function $\tau_\Phi : \mathbb{R} \to \mathbb{R}$ such that $f_\Phi(\alpha) = \tau_\Phi^*(\alpha)$ for all α, and during the past 10 years an extensive theory analysing this problem have been developed, see, for example, the texts by Pesin [Pe2] and Barrierra [Bar].

Multifractal zeta-functions. Dynamical zeta-functions were introduced by Artin & Mazur in the mid 1960's [ArMa] based on an analogy with the number theoretical zeta-functions associated with a function field over a finite ring. Subsequently Ruelle [Rue1,Rue2] associated zeta-functions to certain statistical mechanical models in one dimensions. During the past 35 years many parallels have been drawn between number theory zeta-functions, dynamical zeta-functions, and statistical mechanics zeta-functions. However, much more recently and motivated by the powerful techniques provided by the use of the Artin-Mazur zeta-functions in number theory and the use of the Ruelle zeta-functions in dynamical systems, Lapidus and collaborators (see the intriguing books by Lapidus & van Frankenhuysen [Lap-vF1,Lap-VF2] and the references therein) have recently introduced and pioneered to use of zeta-functions in fractal geometry. Inspired by this development, within the past 4–5 years several authors have paralleled this development by introducing zeta-functions into multifractal geometry.

Fine multifractal zeta-functions. Indeed, in 2009, Lapidus and collaborators [LapRo,LapLe-VeRo] introduced various intriguing multifractal zeta-functions designed to provide information about the multifractal spectrum $f_\mu(\alpha)$ of self-similar measures μ, and a number of connections with multifractal spectra were suggested and in some cases proved; for example, in simplified cases the multifractal spectrum of a self-similar measure could be recovered from a zeta-function. The key idea in [LapRo,LapLe-VeRo] is both simple and attractive: while traditional zeta-functions are defined by "summing over all data", the multifractal zeta-functions in [LapRo,LapLe-VeRo] are defined by only "summing over data that are multifractally relevant". Ideas similar to those in [LapRo,LapLe-VeRo] have much more recently been revisited and investigated in [Bak,MiOl1,MiOl2,Ol7] where the authors introduce related multifractal zeta-functions tailored to study the multifractal spectra of self-conformal measures and a number of connections with very general types of multifractal spectra were established. Indeed, in [MiOl2,Ol7] we proposed a theory of *fine* multifractal zeta-functions paralleling the existing theory of dynamical zeta-functions introduced and developed by Ruelle [Rue1,Rue2] and others [Bal1,Bal2,ParPo1,ParPo2]. In particular, in the setting of graph-directed self-conformal constructions, we introduced a family of *fine* multifractal zeta-functions designed to provide precise information of very general classes of multifractal spectra, including, for example, the multifractal spectra of self-conformal measures and the multifractal spectra of ergodic Birkhoff averages of continuous functions on self-conformal sets. More precisely, we fix a metric space X and a continuous map $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ where $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ denotes the shift space modelling the underlying graph-directed self-conformal construction G (the precise definition of $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ will be given in Section 2) and $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ denotes the family of probability measures on $\Sigma_{\mathsf{G}}^{\mathbb{N}}$. For each subset C of X and each continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$, we now associated a *fine* dynamical multifractal zeta-function $\zeta_C^{\mathsf{dyn},U}(\varphi;z)$ defined by

$$\zeta_C^{\mathsf{dyn},U}(\varphi;z)=\sum_n\,\frac{z^n}{n}\left(\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n\\ UL_n[\mathbf{i}]\subseteq C}}\,\sup_{\mathbf{u}\in[\mathbf{i}]}\,\exp\sum_{k=0}^{n-1}\varphi(S^k\mathbf{u})\right), \tag{1.6}$$

where z is a complex variable; in (1.6) we have used the following notation, namely, S denotes the shift map, L_n denotes the n'th empirical measure (i.e. $L_n\mathbf{i}=\frac{1}{n}\sum_{k=0}^{n-1}\delta_{S^k\mathbf{i}}$ for $\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$), Σ_{G}^n denotes the family of all permissible words of length n and if $\mathbf{i}\in\Sigma_{\mathsf{G}}^n$, then $[\mathbf{i}]$ denotes the cylinder generated by $\mathbf{i}$; the precise definitions will be given in Section 2. One of main results from [MiOl2,Ol7] shows that if α is a point in X and $\varphi<0$, then there is a unique real number $\not{f}(\alpha)$ such that

$$\lim_{r\searrow 0}\sigma_{\mathsf{rad}}\big(\,\zeta_{B(\alpha,r)}^{\mathsf{dyn},U}(\,\not{f}(\alpha)\,\varphi;\cdot)\,\big)=1\,,$$

where σ_{rad} denotes the radius of convergence; furthermore, $\not{f}(\alpha)$ satisfies a variational principle and for judicious choices of X, U and φ, the number $\not{f}(\alpha)$ equals important fine multifractal spectra, see, [MiOl2,Ol7] for a large number of examples illustrating this.

Coarse multifractal zeta-functions. Because of the importance of the Renyi dimensions, it is natural to construct a theory of *coarse* multifractal zeta-functions paralleling the existing theory of fractal zeta-functions introduced and pioneered by Lapidus & van Frankenhuysen [Lap-vF1,Lap-VF2]. This has recently been done by Levy-Vehel & Mendivil [Le-VeMe] who introduced multifractal zeta-functions tailored to provide information about the Renyi dimensions $\tau_\mu(q)$ of self-similar measures μ. Ideas related to those in [Le-VeMe] have also been investigated in [Ol5,Ol6] where related multifractal zeta-functions designed to study the Renyi dimensions and the (closely related) multifractal Minkowski volume of self-conformal measures are introduced and investigated.

While Renyi dimensions play an important role in multifractal analysis, the Multifractal Formalism underpinnning the link between the Renyi dimensions and the multifractal spectrum is fundamental for a fuller understanding of the Renyi dimensions as well as the genuinely mysterious relationship between these and the multifractal spectrum. In recognition of this viewpoint, and in order to place the somewhat ad-hoc multifractal zeta-functions from [Le-VeMe,Ol5,Ol6] in to a broader context, it is natural to ask if the "fine multifractal spectrum" $\not{f}(\alpha)$ defined using the *fine* multifractal zeta-functions satisfies a multifractal formalism based on a natural family of *coarse* multifractal zeta-functions. More precisely, assuming X is an inner product space (indeed, since the classical Multifractal Formalism involves the Legendre transform, it seems natural to work in a setting which allows Legendre transforms, and the most natural setting for this in within the context of inner product spaces), we pose the following question: is it possible to associated a natural *coarse* multifractal zeta-function $\zeta_q^{\mathsf{co},U}(\varphi;s)$, where s is a complex variable, to each continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ and each point $q\in X$, such that if we let

$$\tau(q)=\sigma_{\mathsf{ab}}\big(\,\zeta_q^{\mathsf{co},U}(\varphi;\cdot)\,\big)$$

where σ_{ab} denotes the abscissa of convergence, then

$$\not{f}(\alpha)=\tau^*(\alpha)\quad\text{for all (or some) }\alpha?$$

Adopting this viewpoint, the purpose of this paper is to introduce and develop a meaningful theory of *coarse* multifractal zeta-functions in the setting of graph-directed self-conformal constructions and establish a multifractal formalism for zeta-functions relating the *fine* multifractal zeta-function in (1.6) and the *fine* multifractal zeta-function in (1.7) below via the Legendre transform. The *coarse* multifractal zeta-function $\zeta_q^{\mathsf{co},U}(\varphi;s)$ is defined by

$$\zeta_q^{\mathsf{co},U}(\varphi;s)=\sum_n\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n}\exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]}\big\langle q\big|UL_n\mathbf{u}\big\rangle+s\right)\sup_{\mathbf{u}\in[\mathbf{i}]}\sum_{k=0}^{n-1}\varphi(S^k\mathbf{u})\right) \tag{1.7}$$

where s is complex variable and where we have used the same notation as in (1.6).

We illustrate the definition of $\zeta_q^{\mathsf{co},U}(\varphi;s)$ by considering the following simple example involving self-similar measures. Self-similar measures form a special case of the family of graph-directed self-conformal measures and are defined as follows. Let $S_1,\dots,S_N:\mathbb{R}^d\to\mathbb{R}^d$ be contraction similarities and let r_i denote the contraction ratio of S_i, i.e. $|S_ix-S_iy|=r_i|x-y|$ for all $x,y\in\mathbb{R}^d$. Also, fix a probability vector $(p_1,\dots,p_N)$. The self-similar set K and the self-similar measure μ associated with the list $(S_1,\dots,S_N,p_1,\dots,p_N)$ is the unique set and the unique probability measure satisfying

$$K=\bigcup_i S_i(K)\,,\quad \mu=\sum_i p_i\mu\circ S_i^{-1}\,. \tag{1.8}$$

In this particular case, the shift space $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ is given by $\Sigma_{\mathsf{G}}^{\mathbb{N}}=\{1,\dots,N\}^{\mathbb{N}}$. We now define maps $\varphi,\psi:\{1,\dots,N\}^{\mathbb{N}}\to\mathbb{R}$ and $U:\mathcal{P}(\{1,\dots,N\}^{\mathbb{N}})\to\mathbb{R}$ by

$$\begin{aligned}\varphi(\mathbf{i})&=\log r_{i_1}\,,\\ \psi(\mathbf{i})&=\log p_{i_1}\end{aligned}$$

for $\mathbf{i}=i_1i_2\ldots\in\{1,\dots,N\}^{\mathbb{N}}$ and

$$U\mu=\frac{\int\psi\,d\mu}{\int\varphi\,d\mu}$$

for $\mu\in\mathcal{P}(\{1,\dots,N\}^{\mathbb{N}})$. For this choice of φ and U, a simple calculation shows that (the reader is referred to Section 5 for details and for several other examples)

$$\begin{aligned}\zeta_q^{\mathsf{co},U}(\varphi;s)&=\sum_n\sum_{\mathbf{i}=i_1\ldots i_n\in\{1,\dots,N\}^n}\exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]}q\,U(L_n\mathbf{u})+s\right)\sup_{\mathbf{u}\in[\mathbf{i}]}\sum_{k=0}^{n-1}\varphi(S^k\mathbf{u})\right)\\
&=\sum_n\sum_{\mathbf{i}=i_1\ldots i_n\in\{1,\dots,N\}^n}\exp\left(\left(q\,\frac{\sum_{k=0}^{n-1}\log p_{i_k}}{\sum_{k=0}^{n-1}\log r_{i_k}}+s\right)\sum_{k=0}^{n-1}\log r_{i_k}\right)\\
&=\sum_n\sum_{\mathbf{i}=i_1\ldots i_n\in\{1,\dots,N\}^n}p_{i_1}^q\cdots p_{i_n}^q\,r_{i_1}^s\cdots r_{i_n}^s\\
&=\sum_n\left(\sum_{i=1}^N p_i^q r_i^s\right)^n\\
&=\frac{\sum_{i=1}^N p_i^q r_i^s}{1-\sum_{i=1}^N p_i^q r_i^s}\end{aligned} \tag{1.9}$$

provided the geometric series $\sum_n(\sum_{i=1}^N p_i^q r_i^s)^n$ converges, i.e. provided $|\sum_{i=1}^N p_i^q r_i^s|<1$. The zeta-function in (1.9), i.e. the zeta-function

$$\zeta_q(s)=\frac{\sum_{i=1}^N p_i^q r_i^s}{1-\sum_{i=1}^N p_i^q r_i^s}\,, \tag{1.10}$$

and similar zeta-functions have, in fact, been investigated intensively during the past 10 years by Lapidus et at [[Lap-vF1,Lap-VF2] (for $q=0$) and others [Le-VeMe,Ol5,Ol6]. For example, it is well-known that if K and μ denote the self-similar set and the self-similar measure satisfying (1.8), then the Renyi dimension $\tau_\mu(q)$ of μ (see (1.3)) equals the abscissa of convergence of the zeta-function in (1.10), see [Le-VeMe,Ol5,Ol6] (and [Lap-vF1,Lap-VF2] for the case $q=0$). In addition, explicit formulas for the Minkowski volume of the fractal K and multifractal μ can be found from the poles and the residues of the zeta-function in (1.10); see Lapidus & van Frankenhuysen's books [Lap-vF1,Lap-VF2] for a detailed discussion of this in the fractal case (i.e. the the case $q=0$) and Olsen [Ol5,Ol6] for a discussion of the multifractal case (i.e. the case $q\in\mathbb{R}$).

The *coarse* multifractal zeta-functions introduced in this paper may therefore be view as a continuation of the work initiated in [Le-VeMe,Ol5,Ol6] placing the zeta-functions from [Le-VeMe,Ol5,Ol6] in to a broader context provider a fuller understanding and allowing further applications (see, in particular, Section 5 where applications to multifractal analysis of graph-directed self-conformal measures and multifractal analysis of various ergodic spectra are considered).

The Multifractal Formalism for zeta-functions: linking fine and coarse multi fractal zeta-functions. For judicious choices of X, U and φ, the abscissa of convergence

$$\tau(q) = \sigma_{\mathsf{ab}}\big(\,\zeta_q^{\mathsf{co},U}(\varphi;\cdot)\,\big)$$

of the coarse multifractal zeta-function $\zeta_q^{\mathsf{co},U}(\varphi;\cdot)$ has many of the properties that any meaningful "coarse moment scaling" function is expected to have, including the following (see the statement of the Multifractal Formalism for Measures and Proposition A):

- the function τ is convex and (under some mild assumptions) differentiable;
- the "fine multifractal spectrum" $f(\alpha)$ is *always* bounded above by the Legendre transform of τ;
- if τ is differentiable, then $f(\alpha)$ equals the Legendre transform of τ, i.e. the Multifractal Formalism is satisfied.

This is the content of the Theorem 1.1 below; we emphasise that Theorem 1.1 is special case of the three main results in this paper, namely, Theorem 4.2, Theorem 4.3 and Theorem 4.4, where more general results are presented.

Theorem 1.1. The multifractal formalism for multifractal zeta-functions; a special case. *Let X be an inner product space and let $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ be continuous with respect to the weak topology. Fix a Hölder continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ with $\varphi<0$. For $C\subseteq X$ and $q\in X$, we let $\zeta_C^{\mathsf{dyn},U}(\varphi;\cdot)$ and $\zeta_q^{\mathsf{co},U}(\varphi;\cdot)$ denote the fine and coarse multifractal zeta-functions in (1.6) and (1.7), respecively, and let*

$$\tau(q) = \sigma_{\mathsf{ab}}\big(\,\zeta_q^{\mathsf{co},U}(\varphi;\cdot)\,\big)\,.$$

(1) *The function τ is convex and if there is a Hölder continuous function $\psi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ such that $U\mu=\dfrac{\int\psi\,d\mu}{\int\varphi\,d\mu}$ for all μ, then τ is real analytic.*

(2) *Fix α be a point in X and let $f(\alpha)$ be the unique real number such that*

$$\limsup_{r\searrow 0}\sigma_{\mathsf{rad}}\big(\,\zeta_{B(\alpha,r)}^{\mathsf{dyn},U}(f(\alpha)\,\varphi;\cdot)\,\big)=1\,.$$

Then

$$f(\alpha)\le\tau^*(\alpha)\,.$$

If, in addition, $X=\mathbb{R}^M$ and there is $q\in\mathbb{R}^M$ such that τ is differentiable at q with $\alpha=-\nabla\tau(q)$, then

$$f(\alpha)=\tau^*(\alpha)\,. \tag{1.11}$$

(Observe that since τ is convex, we conclude that τ is differentiable almost everywhere, and the conclusion in (1.11) is therefore satisfied for "many" points α.)

For judicious choices of X, U and φ equation (1.11) reduces to known Legendre transform representations of different types of classical multifractal spectra, and for other choices of X, U and φ equation (1.11) provides new Legendre transform representations of multifractal spectra; for example, for certain choices of X, U and φ equation (1.11) equals the well-know Legendre transform representation of the multifractal spectrum of a graph directed self-conformal measure (see the examples in Section 5.1), and for other choices of X, U and φ equation (1.11) provides new Legendre transform representations of the multifractal spectrum of different types of multifractal spectra of ergodic averages of continuous functions on graph-directed self-conformal sets (see the examples in Section 5.2).

The remaining part of the paper is organised as follows. In Section 2 we briefly recall the definitions of self-conformal constructions and in Section 3 we recall the definition of the fine dynamical multifractal zeta-functions introduced in [MiOl2,Ol7]. In Section 4 we provide the definition of the coarse multifractal zeta-functions and we state our main results. Section 5 contains a number of examples illustrating the main results. Finally, the proofs are presented in Sections 6–9.

2. The setting: Graph-directed self-conformal sets and graph-directed self-conformal measures.

Notation from symbolic dynamics. We first recall the notation and terminology from symbolic dynamics that will be used in this paper. Fix a finite directed multigraph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ where V denotes the set of vertices of G and E denotes the set of edges of G. For an edge $\mathsf{e} \in \mathsf{E}$, we write $\mathrm{i}(\mathsf{e})$ for the initial vertex of e and we write $\mathrm{t}(\mathsf{e})$ for the terminal vertex of e. For $\mathsf{i}, \mathsf{j} \in \mathsf{V}$, write

$$\begin{aligned} \mathsf{E}_{\mathsf{i}} &= \Big\{ \mathsf{e} \in \mathsf{E} \,\Big|\, \mathrm{i}(\mathsf{e}) = \mathsf{i} \Big\}, \\ \mathsf{E}_{\mathsf{i},\mathsf{j}} &= \Big\{ \mathsf{e} \in \mathsf{E} \,\Big|\, \mathrm{i}(\mathsf{e}) = \mathsf{i} \text{ and } \mathrm{t}(\mathsf{e}) = \mathsf{j} \Big\}; \end{aligned} \tag{2.1}$$

i.e. E_{i} is the family of all edges starting at i; and $\mathsf{E}_{\mathsf{i},\mathsf{j}}$ is the is the family of all edges starting at i and ending at j. Also, for a positive integer n, we write

$$\begin{aligned} \Sigma_{\mathsf{G}}^{n} = \Big\{ \mathsf{e}_1 \ldots \mathsf{e}_n \,\Big|\, & \mathsf{e}_i \in \mathsf{E} \text{ for } 1 \le i \le n, \\ & \mathrm{t}(\mathsf{e}_1) = \mathrm{i}(e_2), \\ & \mathrm{t}(e_{i-1}) = \mathrm{i}(\mathsf{e}_i) \text{ and } \mathrm{t}(\mathsf{e}_i) = \mathrm{i}(e_{i+1}) \text{ for } 1 < i < n, \\ & \mathrm{t}(e_{n-1}) = \mathrm{i}(\mathsf{e}_n) \Big\} \\ \Sigma_{\mathsf{G}}^{*} = \bigcup_m & \Sigma_{\mathsf{G}}^{m}, \\ \Sigma_{\mathsf{G}}^{\mathbb{N}} = \Big\{ \mathsf{e}_1 \mathsf{e}_2 \ldots \,\Big|\, & \mathsf{e}_i \in \mathsf{E} \text{ for } 1 \le i, \\ & \mathrm{t}(\mathsf{e}_1) = \mathrm{i}(e_2), \\ & \mathrm{t}(e_{i-1}) = \mathrm{i}(\mathsf{e}_i) \text{ and } \mathrm{t}(\mathsf{e}_i) = \mathrm{i}(e_{i+1}) \text{ for } 1 < i \Big\}; \end{aligned} \tag{2.2}$$

i.e. Σ_{G}^{n} is the family of all finite strings $\mathbf{i} = \mathsf{e}_1 \ldots \mathsf{e}_n$ consisting of finite paths in G of length n; Σ_{G}^{*} is the family of all finite strings $\mathbf{i} = \mathsf{e}_1 \ldots \mathsf{e}_m$ with $m \in \mathbb{N}$ consisting of finite paths in G; and $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ is the family of all infinite strings $\mathbf{i} = \mathsf{e}_1 \mathsf{e}_2 \ldots$ consisting of infinite paths in G. For a finite string $\mathbf{i} = \mathsf{e}_1 \ldots \mathsf{e}_n \in \Sigma_{\mathsf{G}}^{n}$, we write

$$\mathrm{i}(\mathbf{i}) = \mathrm{i}(\mathsf{e}_1), \quad \mathrm{t}(\mathbf{i}) = \mathrm{t}(\mathsf{e}_n), \tag{2.3}$$

and for an infinite string $\mathbf{i} = \mathsf{e}_1 \mathsf{e}_2 \ldots \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$, we write

$$\mathrm{i}(\mathbf{i}) = \mathrm{i}(\mathsf{e}_1). \tag{2.4}$$

Next, for an infinite string $\mathbf{i} = \mathsf{e}_1 \mathsf{e}_2 \ldots \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$ and a positive integer n, we will write $\mathbf{i}|n = \mathsf{e}_1 \ldots \mathsf{e}_n$. In addition, for a positive integer n and a finite string $\mathbf{i} = \mathsf{e}_1 \ldots \mathsf{e}_n \in \Sigma_{\mathsf{G}}^{n}$ with length equal to n, we will write $|\mathbf{i}| = n$, and we let $[\mathbf{i}]$ denote the cylinder generated by $\mathbf{i}$, i.e.

$$[\mathbf{i}] = \Big\{ \mathbf{j} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\Big|\, \mathbf{j}|n = \mathbf{i} \Big\}. \tag{2.5}$$

Finally, let $S : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \Sigma_{\mathsf{G}}^{\mathbb{N}}$ denote the shift map, i.e.

$$S(\mathsf{e}_1 \mathsf{e}_2 \ldots) = \mathsf{e}_2 \mathsf{e}_3 \ldots .$$

Graph-directed self-conformal sets and graph-directed self-conformal measures. Next, we recall the definition of graph-directed self-conformal sets and measures. A graph-directed conformal iterated function system with probabilities is a list

$$\Big(\mathsf{V},\, \mathsf{E},\, (V_{\mathsf{i}})_{\mathsf{i} \in \mathsf{V}},\, (X_{\mathsf{i}})_{\mathsf{i} \in \mathsf{V}},\, (S_{\mathsf{e}})_{\mathsf{e} \in \mathsf{E}},\, (p_{\mathsf{e}})_{\mathsf{e} \in \mathsf{E}} \Big)$$

where

- For each $\mathsf{i} \in \mathsf{V}$ we have: V_i is an open, connected subset of $\mathbb{R}^d$.
- For each $\mathsf{i} \in \mathsf{V}$ we have: $X_\mathsf{i} \subseteq V_\mathsf{i}$ is a compact set with $X_\mathsf{i}^{\circ\,-} = X_\mathsf{i}$.
- For each $\mathsf{i}, \mathsf{j} \in \mathsf{V}$ and $\mathsf{e} \in E_{\mathsf{i},\mathsf{j}}$ we have: $S_\mathsf{e} : V_\mathsf{j} \to V_\mathsf{i}$ is a contractive $C^{1+\gamma}$ diffeomorphism with $0 < \gamma < 1$ such that $S_\mathsf{e}(X_\mathsf{j}) \subseteq X_\mathsf{i}$.
- The Conformality Condition. For $\mathsf{i}, \mathsf{j} \in \mathsf{V}$, $\mathsf{e} \in E_{\mathsf{i},\mathsf{j}}$ and $x \in V_\mathsf{j}$, let $(DS_\mathsf{e})(x) : \mathbb{R}^d \to \mathbb{R}^d$ denote the derivative of S_e at x. For each $\mathsf{i}, \mathsf{j} \in \mathsf{V}$ and $\mathsf{e} \in E_{\mathsf{i},\mathsf{j}}$ we have: $(DS_\mathsf{e})(x)$ is a contractive similarity map, i.e. there exists $s_\mathsf{e}(x) \in (0, 1)$ such that $|(DS_\mathsf{e})(x)u - (DS_\mathsf{e})(x)v| = s_\mathsf{e}(x)|u - v|$ for all $u, v \in \mathbb{R}^d$.
- For each $\mathsf{i} \in \mathsf{V}$ we have: $(p_\mathsf{e})_{\mathsf{e}\in\mathsf{E}_\mathsf{i}}$ is a probability vector.

It follows from [Hu] that there exists a unique list $(K_\mathsf{i})_{\mathsf{i}\in\mathsf{V}}$ of non-empty compact sets $K_\mathsf{i} \subseteq X_\mathsf{i}$ such that

$$K_\mathsf{i} = \bigcup_{\mathsf{e}\in\mathsf{E}_\mathsf{i}} S_\mathsf{e} K_{\mathsf{t}(\mathsf{e})}\,, \tag{2.6}$$

and a unique list $(\mu_\mathsf{i})_{\mathsf{i}\in\mathsf{V}}$ of probability measures with $\operatorname{supp}\mu_\mathsf{i} = K_\mathsf{i}$ such that

$$\mu_\mathsf{i} = \sum_{\mathsf{e}\in E_\mathsf{i}} p_\mathsf{e}\, \mu_{\mathsf{t}(\mathsf{e})} \circ S_\mathsf{e}^{-1}\,. \tag{2.7}$$

The sets $(K_\mathsf{i})_{\mathsf{i}\in\mathsf{V}}$ and measures $(\mu_\mathsf{i})_{\mathsf{i}\in\mathsf{V}}$ are called the self-conformal sets and self-conformal measures associated with the list (1.21), respectively. We will frequently assume that the so-called Open Set condition (OSC) is satisfied. The OSC is defined as follows:

- The Open Set Condition: There exists a list $(U_\mathsf{i})_{\mathsf{i}\in\mathsf{V}}$ of open non-empty and bounded sets $U_\mathsf{i} \subseteq X_\mathsf{i}$ with $S_\mathsf{e}\big(U_\mathsf{j}\big) \subseteq U_\mathsf{i}$ for all $\mathsf{i}, \mathsf{j} \in \mathsf{V}$ and all $\mathsf{e} \in \mathsf{E}_{\mathsf{i},\mathsf{j}}$ such that $S_{\mathsf{e}_1}\big(U_{\mathsf{t}(\mathsf{e}_1)}\big) \cap S_{\mathsf{e}_2}\big(U_{\mathsf{t}(\mathsf{e}_2)}\big) = \varnothing$ for all $\mathsf{i} \in \mathsf{V}$ and all $\mathsf{e}_1, \mathsf{e}_2 \in \mathsf{E}_\mathsf{i}$ with $\mathsf{e}_1 \neq \mathsf{e}_2$.

For $\mathbf{i} = \mathsf{e}_1 \ldots \mathsf{e}_n \in \Sigma_\mathsf{G}^n$, we write

$$\begin{aligned} S_\mathbf{i} &= S_{\mathsf{e}_1} \cdots S_{\mathsf{e}_n}\,, \\ K_\mathbf{i} &= S_{\mathsf{e}_1} \cdots S_{\mathsf{e}_n}\big(K_{\mathsf{t}(\mathsf{e}_n)}\big)\,, \end{aligned} \tag{2.8}$$

and we define the projection $\pi : \Sigma_\mathsf{G}^\mathbb{N} \to \mathbb{R}^d$ by

$$\{\,\pi(\mathbf{i})\,\} = \bigcap_n K_{\mathbf{i}|n} \tag{2.9}$$

for $\mathbf{i} = \mathsf{e}_1\mathsf{e}_2 \ldots \in \Sigma_\mathsf{G}^\mathbb{N}$. Finally, we define $\Lambda : \Sigma_\mathsf{G}^\mathbb{N} \to \mathbb{R}$ by

$$\Lambda(\mathbf{i}) = \log\big|\,\big(DS_{\mathsf{e}_1}\big)\big(\pi_{\mathsf{t}(\mathsf{e}_1)}(S\mathbf{i})\big)\,\big| \tag{2.10}$$

for $\mathbf{i} = \mathsf{e}_1\mathsf{e}_2 \ldots \in \Sigma_\mathsf{G}^\mathbb{N}$; loosely speaking the map Λ represents the local change of scale as one goes from $\pi_{\mathsf{t}(\mathsf{e}_1)}(S\mathbf{i})$ to $\pi_{\mathsf{i}(\mathsf{e}_1)}(\mathbf{i})$.

3. Fine dynamical multifractal zeta-functions.

The purpose of this section is to recall the fine dynamical multifractal zeta-functions introduced in [MiOl2,Ol7]. Throughout this section, and in the remaining parts of the paper, we will used the following notation. Namely, if $(a_n)_n$ is a sequence of complex numbers and if f is the power series defined by $f(z) = \sum_n a_n z^n$ for $z \in \mathbb{C}$, then we will denote the radius of convergence of f by $\sigma_{\mathsf{rad}}(f)$, i.e. we write

$$\sigma_{\mathsf{rad}}(f) = \text{"the radius of convergence of } f\text{"}\,.$$

The fine dynamical multifractal zeta-functions in [MiOl2,Ol7] are motivated by the notion of pressure from the thermodynamic formalism and the dynamical zeta-functions introduced by Ruelle

[Rue1,Rue2]; see, also [Bal1,Bal2,ParPo1,ParPo2]. Because of this we now briefly recall the definition of pressure and dynamical zeta-function. Let $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ be a continuous function. The pressure of φ is defined by

$$P(\varphi) = \lim_{n} \frac{1}{n} \log \sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^{n}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^{k}\mathbf{u}\,, \tag{3.1}$$

see [Bo] or [ParPo2]; we note that it is well-known that the limit in (3.1) exists. Also, the dynamical zeta-function of φ is defined by

$$\zeta^{\mathsf{dyn}}(\varphi; z) = \sum_{n} \frac{z^{n}}{n} \left(\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^{n}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^{k}\mathbf{u} \right) \tag{3.2}$$

for those complex numbers z for which the series converge, see [ParPo2].

We now define fine dynamical multifractal zeta-functions from [MiOl2,Ol7]. However, we first introduce the following notation. We denote the family of Borel probability measures on $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ and the family of shift invariant Borel probability measures on $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ by $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ and $\mathcal{P}_{S}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$, respectively, i.e. we write

$$\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) = \left\{ \mu \,\middle|\, \mu \text{ is a Borel probability measures on } \Sigma_{\mathsf{G}}^{\mathbb{N}} \right\}, \tag{3.3}$$

$$\mathcal{P}_{S}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) = \left\{ \mu \,\middle|\, \mu \text{ is a shift invariant Borel probability measures on } \Sigma_{\mathsf{G}}^{\mathbb{N}} \right\}; \tag{3.4}$$

we will always equip $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ and $\mathcal{P}_{S}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ with the weak topologies. We now fix a metric space X and a continuous map $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$. The multifractal zeta-function framework developed in [MiOl2,Ol7] depends on the space X and the map U; judicious choices of X and U will provide important examples, including, multifractal spectra of graph-directed self-conformal measures (see [MiOl2,Ol7] and Section 5.1) and a variety of multifractal spectra of ergodic averages of continuous functions on graph-directed self-conformal sets (see [MiOl2,Ol7] and Section 5.2). Next, for a positive integer n, let $L_{n} : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ be defined by

$$L_{n}\mathbf{i} = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{S^{k}\mathbf{i}}\,; \tag{3.5}$$

recall, that $S : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \Sigma_{\mathsf{G}}^{\mathbb{N}}$ denotes the shift map. We can now define the multifractal pressure and zeta-function associated with the space X and the map U.

Definition. The multifractal pressure $\underline{P}_{C}^{U}(\varphi)$ and $\overline{P}_{C}^{U}(\varphi)$ associated with the space X and the map U. *Let X be a metric space and let $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Fix a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$. For $C \subseteq X$, we define the lower and upper mutifractal pressure of φ associated with the space X and the map U by*

$$\begin{aligned} \underline{P}_{C}^{U}(\varphi) &= \liminf_{n} \frac{1}{n} \log \sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^{n} \\ UL_{n}[\mathbf{i}]\subseteq C}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^{k}\mathbf{u}\,, \\ \overline{P}_{C}^{U}(\varphi) &= \limsup_{n} \frac{1}{n} \log \sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^{n} \\ UL_{n}[\mathbf{i}]\subseteq C}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^{k}\mathbf{u}\,. \end{aligned} \tag{3.6}$$

Definition. The dynamical multifractal zeta-function $\zeta_C^{\mathsf{dyn},U}(\varphi;\cdot)$ associated with the space X and the map U. *Let X be a metric space and let $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ be continuous with respect to the weak topology. Fix a continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$. For $C\subseteq X$, we define the dynamical multifractal zeta-function $\zeta_C^{\mathsf{dyn},U}(\varphi;\cdot)$ associated with the space X and the map U by*

$$\zeta_C^{\mathsf{dyn},U}(\varphi;z)=\sum_n\frac{z^n}{n}\left(\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n\\ UL_n[\mathbf{i}]\subseteq C}}\ \sup_{\mathbf{u}\in[\mathbf{i}]}\ \exp\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}\right) \tag{3.7}$$

for those complex numbers z for which the series converges.

Remark. Comparing the definition of the pressure (3.1) (the dynamical zeta-function (3.2)) and the definition of the multifractal pressure (3.6) (the dynamical multifractal zeta-function (3.7)), it is clear that the multifractal pressure (the dynamical multifractal zeta-function) is obtained by only summing over those strings $\mathbf{i}\in\Sigma_{\mathsf{G}}^n$ that are multifractally relevant, i.e. those strings $\mathbf{i}\in\Sigma_{\mathsf{G}}^n$ for which $UL_n[\mathbf{i}]\subseteq C$.

Remark. It is clear that if $C=X$, then the multifractal "constraint" $UL_n[\mathbf{i}]\subseteq C$ is vacuously satisfied, and that, in this case, the multifractal pressure and dynamical multifractal zeta-function reduce to the usual pressure and the usual dynamical zeta-function, i.e. $\underline{P}_X^U(\varphi)=\overline{P}_X^U(\varphi)=P(\varphi)$ and $\zeta_X^{\mathsf{dyn},U}(\varphi;\cdot)=\zeta^{\mathsf{dyn}}(\varphi;\cdot)$.

Recall, the following classical and beautiful result known. If $(K_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ denotes the the graph-directed self-conformal sets (2.6) and the OSC is satisfied, then $\dim_{\mathsf{H}}K_{\mathsf{i}}=s$ for all i where s is the unique real number such that $P(s\Lambda)=0$ (alternatively, s is the unique real number such that $\sigma_{\mathsf{rad}}\big(\,\zeta^{\mathsf{dyn}}(s\Lambda;\cdot)\,\big)=1$); here Λ is the scaling function defined by (2.10). The formula $P(s\Lambda)=0$ is known as Bowen's formula. The next proposition shows that if $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ is a continuous function with $\varphi<0$, then there is are unique real numbers $\underline{f}(C)$ and $\overline{f}(C)$ solving the natural multifractal versions of Bowen's equations.

Proposition B [MiOl2,Ol7]. Solutions to mutifractal Bowen equations. *Let X be a metric space and let $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ be continuous with respect to the weak topology. Let $C\subseteq X$ be a subset of X. Fix a continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ with $\varphi<0$. Then there are unique real numbers $\underline{f}(C)$ and $\overline{f}(C)$ such that*

$$\lim_{r\searrow0}\overline{P}_{B(C,r)}^U(\,\underline{f}(C)\,\varphi)=0\,, \tag{3.8}$$

$$\overline{P}_C^U(\,\overline{f}(C)\,\varphi)=0\,; \tag{3.9}$$

alternatively, $\underline{f}(C)$ and $\overline{f}(C)$ are the unique real numbers such that

$$\lim_{r\searrow0}\sigma_{\mathsf{rad}}\big(\,\zeta_{B(C,r)}^{\mathsf{dyn},U}(\,\underline{f}(C)\,\varphi;\cdot)\,\big)=1\,,$$

$$\sigma_{\mathsf{rad}}\big(\,\zeta_C^{\mathsf{dyn},U}(\,\overline{f}(C)\,\varphi;\cdot)\,\big)=1\,.$$

If $C=\{\alpha\}$, then we will write $\underline{f}(\alpha)$ and $\overline{f}(\alpha)$ for $\underline{f}(C)$ and $\overline{f}(C)$, respectively.

The main results in [MiOl2,Ol7] relate the solutions $\underline{f}(C)$ and $\overline{f}(C)$ of the multifractal Bowen equations (3.8) and (3.9) to various multifractal spectra. However, it is clear that if $C=\{\alpha\}$, then the sum in (3.7) may be empty, and the solution $\overline{f}(\alpha)$ to the equation $\overline{P}_C^U(\,\overline{f}(\alpha)\,\varphi)=0$ is therefore $-\infty$, i.e. $\overline{f}(\alpha)=-\infty$; in fact, examples in [MiOl2,Ol7] shows that this may happen for all but countably many α. Hence, if C is "too small", then it may happen that the pressure (3.6) and the zeta-function (3.7) do not encode sufficient information allowing us to recover any meaningful dynamical or geometric characteristics associated with U, including, for example, multifractal spectra. Because of this problem, [MiOl2,Ol7] proceed in the following two equally naturally ways. Either, we can consider a family of enlarged "target" sets shrinking to the original main "target" $\{\alpha\}$; this approach will be referred to as the shrinking target approach. Or, alternatively, we can consider a fixed enlarge "target" set and regard this as our original main "target"; this approach will be referred to as the fixed target approach.

The shrinking target setting. In the shrinking target setting, [MiOl2,Ol7] provide a variational principle for the solution $f(C)$ to the multifractal Bowen equation (3.8); thisn is Theorem C below. Below we denote the entropy of a measure $\mu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ by $h(\mu)$.

Theorem C [MiOl2,Ol7]. The shrinking target multifractal Bowen equation. *Let X be a metric space and let $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Let $C \subseteq X$ be a subset of X. Fix a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ with $\varphi < 0$ and let $f(C)$ be the unique real number such that*

$$\lim_{r \searrow 0} \sigma_{\mathsf{rad}}\big(\zeta_{B(C,r)}^{\mathsf{dyn},U}(f(C)\,\varphi;\cdot)\big) = 1\,.$$

Then

$$f(C) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ U\mu \in \overline{C}}} -\frac{h(\mu)}{\int \varphi\, d\mu}\,.$$

The fixed target setting. If the set C is "too small", then it follows from the above discussion that the solution $f(C)$ to the equation $\overline{P}_C^U(f(C)\,\varphi) = 0$ does not encode any meaningful dynamical or geometric information about U. However, if the set C satisfies a non-degeneracy condition guaranteeing that it is not "too small" (namely condition (3.10) below), then results can be obtained. Indeed, [MiOl2,Ol7] provides a variational principles for the solution $f(C)$ to the multifractal Bowen equation (3.9) in the fixed target setting.; this is Theorem D below.

Theorem D [MiOl2,Ol7]. The fixed target multifractal Bowen equation. *Let X be a normed vector space. Let $\Gamma : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous and affine and let $\Delta : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to \mathbb{R}$ be continuous and affine with $\Delta(\mu) \neq 0$ for all $\mu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. Define $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ by $U = \frac{\Gamma}{\Delta}$. Let C be a closed and convex subset of X and assume that*

$$\mathring{C} \cap U\big(\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big) \neq \varnothing\,. \tag{3.10}$$

Let $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ be continuous with $\varphi < 0$. Let $f(C)$ be the unique real number such that

$$\sigma_{\mathsf{rad}}\big(\zeta_C^{\mathsf{dyn},U}(\,f(C)\,\varphi;\cdot)\big) = 1\,.$$

Then

$$f(C) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ U\mu \in C}} -\frac{h(\mu)}{\int \varphi\, d\mu}\,.$$

4. Coarse multifractal zeta-functions.

This is the main sections in the paper. In this section we introduce the the coarse multifractal zeta-functions and show that the fine dynamical multifractal zeta-functions defined in Section 3 and the coarse multifractal zeta-functions defined below are linked by a Multifractal Formalism. Finally, in Section 5 we provide a number of examples, including multifractal spectra of graph-directed self-conformal measures and multifractal spectra of ergodic averages of continuous functions on graph-directed self-conformal sets.

Throughout this section, and in the remaining parts of the paper, we will use the following notation. If $(\lambda_n)_n$ and $(a_n)_n$ are sequences of real numbers and if f is the "zeta"-function defined by $f(s) = \sum_n a_n e^{\lambda_n s}$ for $s \in \mathbb{C}$, then we will denote the abscissa of convergence of f by $\sigma_{\mathsf{ab}}(f)$, i.e.

$$\sigma_{\mathsf{ab}}(f) = \text{"the abscissa of convergence of } f\text{"} \ .$$

We can give define the coarse multifractal zeta-function and the corresponding Renyi dimension associated with the space X and the map U.

Definition. The coarse multifractal zeta-function $\zeta_C^{\mathsf{co},U}(\varphi;\cdot)$ associated with the space X and the map U. *Let X be an inner product space with inner product $\langle\cdot|\cdot\rangle$ and let $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Fix a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$. For $q \in X$, we define the coarse multifractal zeta-function $\zeta_q^{\mathsf{co},U}(\varphi;\cdot)$ associated with the space X and the map U by*

$$\zeta_q^{\mathsf{co},U}(\varphi;s) = \sum_{\mathbf{i}} \exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]} \langle q \big| U L_{|\mathbf{i}|}\mathbf{u}\,\rangle + s\right) \sup_{\mathbf{u}\in[\mathbf{i}]} \sum_{k=0}^{|\mathbf{i}|-1} \varphi(S^k\mathbf{u})\right)$$

$$= \sum_n \sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} \exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]} \langle q \big| U L_n\mathbf{u}\,\rangle + s\right) \sup_{\mathbf{u}\in[\mathbf{i}]} \sum_{k=0}^{n-1} \varphi(S^k\mathbf{u})\right)$$

for those complex numbers s for which the series converges.

We define the zeta-function Renyi dimension $\tau(q)$ associated with the space X and the map U by

$$\tau(q) = \sigma_{\mathsf{ab}}\big(\,\zeta_q^{\mathsf{co},U}(\varphi;\cdot)\,\big)\,. \tag{4.1}$$

Properties of τ. It turns out that the zeta-function Renyi dimension satisfies a variational principle. This is the statement of Theorem 4.1 below.

Theorem 4.1. The variational principle for τ. *Let X be an inner product space with inner product $\langle\cdot|\cdot\rangle$ and let $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Let $q \in X$. Fix a Hölder continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ with $\varphi < 0$, and let the function τ be defined by (4.1).*

(1) *We have*

$$\tau(q) = \sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})} \left(-\frac{h(\mu)}{\int\varphi\,d\mu} - \langle q \big| U\mu\rangle \right).$$

(2) *We have*

$$\sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})} \left(h(\mu) + \big(\langle q \big| U\mu\rangle + \tau(q)\big) \int\varphi\,d\mu \right) = 0\,.$$

Theorem 4.1 is proved in Section 6 and Section 7 using techniques from large deviation theory

Remark. Recall that if $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ is a continuous function, then $P(\varphi)$ denotes the pressure of φ (see (3.1)). Also, recall that it follows from the variational principle that

$$P(\varphi) = \sup_{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})} \left(h(\mu) + \int \varphi \, d\mu \right).$$

We conclude from this and Theorem 4.1 that $\tau(0)$ is the solution to the following pressure equation, namely,

$$P(\,\tau(0)\varphi\,) = \sup_{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})} \left(h(\mu) + \tau(0) \int \varphi \, d\mu \right) = 0\,.$$

This viewpoint suggests that $\tau(q)$ may be viewed as the solution to the "non-linear" pressure equation given by

$$\sup_{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})} \left(h(\mu) + \big(\langle q | U\mu \rangle + \tau(q)\big) \int \varphi \, d\mu \right) = 0\,; \tag{4.1}$$

observe that (4.1) is a "non-linear" pressure equation since it (in addition to the usual "linear" term $\tau(q) \int \varphi \, d\mu$ depending linearly on μ) contains the "non-linear" term $\langle q | U\mu \rangle \int \varphi \, d\mu$ depending non-linearly on μ.

The next result shows that τ has the expected properties, including, (1) convexity. (2) differentiability (under certain additional conditions), and (3) $\tau(0)$ equals the Hausdorff dimension of the invariant set of the associated dynamical system.

Theorem 4.2. Properties of τ. *Let X be an inner product space and let $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Fix a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$, and let the function τ be defined by (4.1).*

(1) *The function τ is convex.*
(2) *If φ is Hölder continuous with $\varphi < 0$ and there is a Hölder continuous function $\psi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}^M$ such that*
$$U\mu = \frac{\int \psi \, d\mu}{\int \varphi \, d\mu}$$
for all $\mu \in \mathcal{P}\big(\Sigma_{\mathsf{G}}^{\mathbb{N}}\big)$, then the function τ is real analytic.
(3) *Let $(K_{\mathsf{i}})_{\mathsf{i} \in \mathsf{V}}$ be the graph-directed self-conformal sets defined by (2.6) and let $\varphi = \Lambda$ where $\Lambda : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ is the scaling function defined by (2.10). If the OSC is satisfied and $\mathsf{i} \in \mathsf{V}$, then*
$$\tau(0) = \dim_{\mathsf{H}} K_{\mathsf{i}}$$
for all inner product spaces X and all continuous maps $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$.

The multifractal formalism and τ. The next result shows that the fine dynamical multifractal zeta-functions defined in Section 3 and the coarse multifractal zeta-functions are linked by a Multifractal Formalism. Recall, that if X is an inner product space with inner product $\langle \cdot | \cdot \rangle$ and $f : X \to \mathbb{R}$ is a function, then the Legendre transform $f^* : X \to [-\infty, \infty]$ of f is defined by

$$f^*(x) = \inf_y (\langle x | y \rangle + f(y))\,.$$

Our main results are divided into two parts. In analogy with the classical Multifractal Formalism for measures (see Section 1) the first part provides a natural zeta-function analogue of Proposition A, namely, the "zeta-function multifractal spectrum" $\mathit{f}(C)$ is *always* majorized by the Legendre transform of τ; more precisely $\mathit{f}(C) \le \sup_{\alpha \in \overline{C}} \tau^*(\alpha)$. This is the statement of Theorem 4.3.

The second part shows that if τ is differentiable, then the "zeta-function multifractal spectrum" $\mathit{f}(C)$ equals the Legendre transform of τ; more precisely $\mathit{f}(C) = \sup_{\alpha \in \overline{C}} \tau^*(\alpha)$. In other words, if τ is differentiable, then the Multifractal Formalism is satisfied. This is the statement of Theorem 4.4. We use the following notation is the statement of Theorem 4.4 below, namely, if $f : \mathbb{R}^M \to \mathbb{R}$ is a convex function, then we write $D_{\mathsf{sub}} f(x)$ for the subdifferential of f at x.

Theorem 4.3. The majorant theorem. *Let X be an inner product space and let $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ be continuous with respect to the weak topology. Fix a continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ with $\varphi<0$, and let the function τ be defined by (4.1). Let $C\subseteq X$ be a subset of X and let $f(C)$ be the unique real number such that*

$$\limsup_{r\searrow 0}\sigma_{\mathsf{rad}}\big(\zeta_{B(C,r)}^{\mathsf{dyn},U}(f(C)\,\varphi;\cdot)\big)=1\,.$$

Then

$$f(C)\leq\sup_{\alpha\in\overline{C}}\tau^{*}(\alpha)\,.$$

Theorem 4.4. The Multifractal Formalism for multifractal zeta-functions. *Let $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to\mathbb{R}^{M}$ be continuous with respect to the weak topology. Fix a Hölder continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ with $\varphi<0$, and let the function τ be defined by (4.1).*

(1) *Let $C\subseteq\mathbb{R}^{M}$ and let $f(C)$ be the unique real number such that*

$$\limsup_{r\searrow 0}\sigma_{\mathsf{rad}}\big(\zeta_{B(C,r)}^{\mathsf{dyn},U}(f(C)\,\varphi;\cdot)\big)=1\,.$$

Assume that there is a subset Q of $\mathbb{R}^{M}$ such that τ is differentiable at all $q\in Q$ and if $\alpha\in\overline{C}\cap U\big(\mathcal{P}_{S}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big)$ and $\tau^{*}(\alpha)>-\infty$, then there is $q\in Q$ with $\alpha\in-D_{\mathsf{sub}}\tau(q)$. Then

$$f(C)=\sup_{\alpha\in\overline{C}}\tau^{*}(\alpha)\,.$$

(2) *Let $\alpha\in\mathbb{R}^{M}$ and let $f(\alpha)$ be the unique real number such that*

$$\limsup_{r\searrow 0}\sigma_{\mathsf{rad}}\big(\zeta_{B(C,r)}^{\mathsf{dyn},U}(f(\alpha)\,\varphi;\cdot)\big)=1\,.$$

Assume that there is a point q in $\mathbb{R}^{M}$ such that τ is differentiable at q and if $\alpha\in\overline{C}\cap U\big(\mathcal{P}_{S}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big)$ and $\tau^{*}(\alpha)>-\infty$, then $\alpha\in-D_{\mathsf{sub}}\tau(q)$. Then

$$f(\alpha)=\tau^{*}(\alpha)\,.$$

(Observe that since τ is convex, we conclude that τ is differentiable almost everywhere, and the conclusion in Theorem 4.4 is therefore satisfied for "many" points α.)

Theorem 4.3 is proved in Section 8 and Theorem 4.4 is proved in Section 9. Theorem 4.4 shows that if τ is differentiable, then the Multifractal Formalism for multifractal zeta-functions is satisfied. It is therefore of interest to find conditions guaranteeing the differentiability of τ. Corollary 4.5 below provides such a condition. In Corollary 4.5 we will use the following notation. If $\mathbf{f}=(f_1,\ldots,f_M):\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}^{M}$ is a continuous function taking values in $\mathbb{R}^{M}$ and $\mu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$, then we will write

$$\int\mathbf{f}\,d\mu=\left(\int f_1\,d\mu,\ldots,\int f_M\,d\mu\right).$$

We can now state Corollary 4.5.

Corollary 4.5. *Fix a Hölder continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ with $\varphi<0$, and let the function τ be defined by (4.1). Let $\psi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}^{M}$ be a Hölder continuous function and define $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to\mathbb{R}^{M}$ by*

$$U\mu=\frac{\int\psi\,d\mu}{\int\varphi\,d\mu}\,.$$

Let $C\subseteq\mathbb{R}^{M}$ be a subset of $\mathbb{R}^{M}$. Let $f(C)$ be the unique real number such that

$$\limsup_{r\searrow 0}\sigma_{\mathsf{rad}}\big(\zeta_{B(C,r)}^{\mathsf{dyn},U}(f(C)\,\varphi;\cdot)\big)=1\,.$$

Then

$$f(C)=\sup_{\alpha\in\overline{C}}\tau^{*}(\alpha)\,.$$

Proof
We conclude from Theorem 4.2 that τ is real analytic, and the desired result therefore follows from Theorem 4.4. □

5. Applications: multifractal spectra of measures and multifractal spectra of ergodic Birkhoff averages

We will now consider several of applications of Theorems 4.1–4.3 to multifractal spectra of measures and ergodic averages. In particular, we consider the following examples:

- Section 5.1: Multifractal spectra of graph-directed self-conformal measures.
- Section 5.2: Multifractal spectra of ergodic Birkhoff averages of continuous functions on graph-directed self-conformal sets.

5.1. Multifractal spectra of graph directed self-conformal measures. Let $(\mathsf{V},\mathsf{E},(V_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}},(X_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}},(S_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}},(p_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}})$ be a graph-directed conformal iterated function system with probabilities (see Section 2) and let $(K_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ and $(\mu_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ be the list of graph-directed self-conformal sets and the list of graph-direted self-conformal measures associated with the list $(\mathsf{V},\mathsf{E},(V_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}},(X_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}},(S_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}},(p_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}})$, respectively, i.e. the sets in the list $(K_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ are the unique non-empty compact sets satisfying (2.6) and the measures in the list $(\mu_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ are the unique probability measures satisfying (2.7). Recall that the Hausdorff multifractal spectrum $f_{\mu_{\mathsf{i}}}$ of μ_{i} is defined by

$$f_{\mu_{\mathsf{i}}}(\alpha)=\dim_{\mathsf{H}}\left\{x\in K_{\mathsf{i}}\,\middle|\,\lim_{r\searrow 0}\frac{\log\mu_{\mathsf{i}}(B(x,r))}{\log r}=\alpha\right\},$$

for $\alpha\in\mathbb{R}$, see Section 1. If the OSC is satisfied, then the multifractal spectrum $f_{\mu_{\mathsf{i}}}(\alpha)$ can be computed as follows. Define $\Phi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ by

$$\Phi(\mathbf{i})=\log p_{\mathsf{i}(\mathbf{i})}\tag{5.1}$$

for $\mathbf{i}=\mathsf{e}_1\mathsf{e}_2\ldots\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$ and recall that the map $\Lambda:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ is defined in (2.10). Next, we define the function $\beta:\mathbb{R}\to\mathbb{R}$ by

$$P\big(\,q\Phi+\beta(q)\Lambda\,\big)=0\,.\tag{5.2}$$

If the OSC is satisfied, then it follows from [EdMa,Col1,Col2,Pa] that

$$f_{\mu_{\mathsf{i}}}(\alpha)=\beta^*(\alpha)\,.\tag{5.3}$$

Of course, in general, the limit $\lim_{r\searrow 0}\frac{\log\mu_{\mathsf{i}}(B(x,r))}{\log r}$ may not exist. Indeed, recently Barreira & Schmeling [BaSc] (see also Olsen & Winter [OlWi1,OlWi2], Xiao, Wu & Gao [XiWuGa] and Moran [Mo]) have shown that in many cases the set D_{i} of divergence points of μ_{i}, i.e. the set of points x for which the limit $\lim_{r\searrow 0}\frac{\log\mu_{\mathsf{i}}(B(x,r))}{\log r}$ does not exist, is highly "visible", namely it has full Hausdorff dimension, i.e. $\dim_{\mathsf{H}}D_{\mathsf{i}}=\dim_{\mathsf{H}}K_{\mathsf{i}}$. This suggests that the set of divergence points has a surprising rich fractal structure. In order to explore this more carefully, Olsen & Winter [OlWi1,OlWi2] introduced various generalised multifractal spectra functions designed to "see" different sets of divergence points. To define these spectra we introduce the following notation. If M is a metric space and $f:(0,\infty)\to M$ is a function, then we write $\operatorname{acc}_{r\searrow 0}f(r)$ for the set of accumulation points of f as $r\searrow 0$, i.e.

$$\operatorname*{acc}_{r\searrow 0}f(r)=\Big\{x\in M\,\Big|\,x\text{ is an accumulation point of }f\text{ as }r\searrow 0\Big\}.$$

Olsen & Winter [OlWi1] introduced and investigated the generalised Hausdorff multifractal spectrum $F_{\mu_{\mathsf{i}}}$ of μ_{i} defined by

$$F_{\mu_{\mathsf{i}}}(C)=\dim_{\mathsf{H}}\left\{x\in K_{\mathsf{i}}\,\middle|\,\operatorname*{acc}_{r\searrow 0}\frac{\log\mu_{\mathsf{i}}(B(x,r))}{\log r}\subseteq C\right\}$$

for $C\subseteq\mathbb{R}$. There is a natural divergence point analogue of (5.3). Namely if the OSC is satisfied, then

$$F_{\mu_{\mathsf{i}}}(C)=\sup_{\alpha\in C}\beta^*(\alpha)\,.$$

for all $C\subseteq\mathbb{R}$, see [Mo,OlWi1,LiWuXi] (see also [Ca,Vo] for earlier but related results in a slightly different setting).

As a first application of Corollary 4.5 we obtain a fine dynamical multifractal zeta-function with an associated Bowen equation whose solution $\mathscr{f}(C)$ equals the generalised multifractal spectrum of the graph-directed self-conformal measure μ_{i} and a family of coarse multifractal zeta-functions with abscissae of convergence whose Legendre transform equals $\mathscr{f}(C)$. This is the content of Theorem 5.1 below. Theorem 5.1 follows by applying Corollary 4.5 to $X=\mathbb{R}$ and the function $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to\mathbb{R}$ defined by $U\mu=\frac{\int\Phi\,d\mu}{\int\Lambda\,d\mu}$ where Φ and Λ are defined in (5.1) and (2.10), respectively. For this choice of X and U, the zeta-functions $\zeta_C^{\mathsf{dyn},U}(\varphi;z)$ and $\zeta_C^{\mathsf{co},U}(\varphi;z)$ are clearly given by

$$\zeta_C^{\mathsf{dyn},U}(\varphi;z)=\sum_n\frac{z^n}{n}\left(\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n\\ \forall\mathbf{v}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\text{ with }\mathsf{t}(\mathbf{i})=\mathsf{i}(\mathbf{v})\,:\,\frac{\log p_{\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{v})|}\in C}}\sup_{\mathbf{u}\in[\mathbf{i}]}\exp\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}\right)\tag{5.5}$$

and

$$\zeta_q^{\mathsf{co},U}(\varphi;s)=\sum_n\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n}\exp\left(\left(\sup_{\mathbf{v}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\text{ with }\mathsf{t}(\mathbf{i})=\mathsf{i}(\mathbf{v})}q\,\frac{\log p_{\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{v})|}+s\right)\sup_{\mathbf{u}\in[\mathbf{i}]}\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}\right).\tag{5.6}$$

It follows from the Principle of Bounded Variation that $\log|DS_{\mathbf{i}}(\pi\mathbf{v})|$ behaves asymptotically as $\operatorname{diam}K_{\mathbf{i}}$ as n tends to infinity for all $\mathbf{i}\in\Sigma_{\mathsf{G}}^n$ and $\mathbf{v}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathsf{t}(\mathbf{i})=\mathsf{i}(\mathbf{v})$; this, in turn, implies that the radius of convergence and the abscissa of convergence of the zeta-functions in (5.5) and (5.6) behave asymptotically as the the radius of convergence and the abscissa of convergence of the zeta-functions

$$\zeta_C^{\mathsf{dyn\text{-}con}}(\varphi;z)=\sum_n\frac{z^n}{n}\left(\sum_{\frac{\log p_{\mathbf{i}}}{\log\operatorname{diam}_{\mathsf{N}}K_{\mathbf{i}}|}\in C}\sup_{\mathbf{u}\in[\mathbf{i}]}\exp\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}\right)\tag{5.7}$$

and

$$\zeta_q^{\mathsf{co\text{-}con}}(\varphi;z)=\sum_n\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n}\exp\left(\left(q\,\frac{\log p_{\mathbf{i}}}{\log\operatorname{diam}_{\mathsf{N}}K_{\mathbf{i}}}+s\right)\sup_{\mathbf{u}\in[\mathbf{i}]}\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}\right),\tag{5.8}$$

obtained from (5.5) and (5.6) by replacing $|DS_{\mathbf{i}}(\pi\mathbf{v})|$ by the normalised diameter $\operatorname{diam}_{\mathsf{N}}K_{\mathbf{i}}$ of $K_{\mathbf{i}}$ defined by

$$\operatorname{diam}_{\mathsf{N}}K_{\mathbf{i}}=\frac{\operatorname{diam}K_{\mathbf{i}}}{\operatorname{diam}K_{\mathsf{i}}}$$

for each finite string $\mathbf{i}\in\Sigma_{\mathsf{G}}^*$ with initial vertex equal to i; Proposition 5.3 provides a precise statement of this. For this reason we have decided to formulate Theorem 5.1 using the more natural zeta-functions (5.7) and (5.8) (instead of (5.5) and (5.6)).

Theorem 5.1. Fine dynamical multifractal zeta-functions and coarse multifractal zeta-functions for multifractal spectra of graph-directed self-conformal measures. *Let $(\mu_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ be the list of graph-directed self-conformal measures associated with the list $\big(\mathsf{V},\,\mathsf{E},\,(V_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}},\,(X_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}},\,(S_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}},\,(p_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}\big)$, i.e. μ_{i} is the unique probability measure such that $\mu_{\mathsf{i}}=\sum_{\mathsf{e}\in E_{\mathsf{i}}}p_{\mathsf{e}}\,\mu_{\mathsf{t}(\mathsf{e})}\circ S_{\mathsf{e}}^{-1}$.*

For $C\subseteq\mathbb{R}$ and a continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$, we define the fine dynamical graph-directed self-conformal multifractal zeta-function $\zeta_C^{\mathsf{dyn\text{-}con}}(\varphi;z)$ by (5.7).

For $q\in\mathbb{R}$ and a continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$, we define the coarse graph-directed self-conformal multifractal zeta-function $\zeta_q^{\mathsf{co\text{-}con}}(\varphi;s)$ by (5.8).

Let Λ be defined by (2.10) and let β be defined by (5.2), and write

$$\tau(q)=\sigma_{\mathsf{ab}}\big(\zeta_q^{\mathsf{co\text{-}con}}(\Lambda;s)\big).\tag{5.9}$$

(1) *For all q, we have*

$$\tau(q)=\beta(q).$$

(2) *Assume that $C \subseteq \mathbb{R}$ is closed.*

(2.1) *There is a unique real number $\mathscr{f}(C)$ such that*

$$\lim_{r \searrow 0} \sigma_{\mathsf{rad}}\big(\,\zeta_{B(C,r)}^{\mathsf{dyn\text{-}con}}(\,\mathscr{f}(C)\,\Lambda;\cdot)\,\big) = 1\,.$$

It $\alpha \in \mathbb{R}$ and $C = \{\alpha\}$, then we will write $\mathscr{f}(\alpha) = \mathscr{f}(C)$.

(2.2) *We have*

$$\mathscr{f}(C) = \sup_{\alpha \in C} \tau^*(\alpha)\,.$$

(2.3) *If the OSC is satisfied, then we have*

$$\mathscr{f}(C) = F_{\mu_{\mathsf{i}}}(C) = \dim_{\mathsf{H}}\left\{ x \in K_{\mathsf{i}} \;\middle|\; \operatorname*{acc}_{r \searrow 0} \frac{\log \mu_{\mathsf{i}}(B(x,r))}{\log r} \subseteq C \right\} = \sup_{\alpha \in C} \tau^*(\alpha)\,.$$

In particular, if the OSC is satisfied and $\alpha \in \mathbb{R}$, then we have

$$\mathscr{f}(\alpha) = f_{\mu_{\mathsf{i}}}(\alpha) = \dim_{\mathsf{H}}\left\{ x \in K_{\mathsf{i}} \;\middle|\; \lim_{r \searrow 0} \frac{\log \mu_{\mathsf{i}}(B(x,r))}{\log r} = \alpha \right\} = \tau^*(\alpha)\,.$$

(3) *Assume that $C \subseteq \mathbb{R}$ is a closed interval with $\overset{\circ}{C} \cap \big(-\beta'(\mathbb{R})\big) \neq \varnothing$.*

(3.1) *There is a unique real number $\mathscr{F}(C)$ such that*

$$\sigma_{\mathsf{rad}}\big(\,\zeta_{C}^{\mathsf{dyn\text{-}con}}(\,\mathscr{F}(C)\,\Lambda;\cdot)\,\big) = 1\,.$$

(3.2) *We have*

$$\mathscr{F}(C) = \sup_{\alpha \in C} \tau^*(\alpha)\,.$$

(3.3) *If the OSC is satisfied then*

$$\mathscr{F}(C) = F_{\mu_{\mathsf{i}}}(C) = \dim_{\mathsf{H}}\left\{ x \in K_{\mathsf{i}} \;\middle|\; \operatorname*{acc}_{r \searrow 0} \frac{\log \mu_{\mathsf{i}}(B(x,r))}{\log r} \subseteq C \right\} = \sup_{\alpha \in C} \tau^*(\alpha)\,.$$

We will prove Theorem 5.1 below. However, we first note that if all the maps S_{e} are similarities, then the coarse multifractal zeta-functions in (5.6) and (5.8) can be computed explicitly; this is the content of Theorem 5.2 below. In order to state this result, we introduce the following notation. Assuming that the maps S_{e} are similarities, i.e. if for each $\mathsf{e} \in \mathsf{E}$ there is a number $r_{\mathsf{e}} \in (0,1)$ such that

$$|S_{\mathsf{e}}(x) - S_{\mathsf{e}}(y)| = r_{\mathsf{e}}|x-y|$$

for all $x, y \in X_{\mathsf{t(e)}}$, then we define the matrix $A(q,s)$ for $q \in \mathbb{R}$ and $s \in \mathbb{C}$ by

$$A(q,s) = \big(\,a_{\mathsf{i,j}}(q,s)\,\big)_{\mathsf{i,j} \in \mathsf{V}} \tag{5.10}$$

where

$$a_{\mathsf{i,j}}(q,s) = \sum_{\mathsf{e} \in \mathsf{E}_{\mathsf{i,j}}} p_{\mathsf{e}}^q r_{\mathsf{e}}^s\,.$$

We can now state Theorem 5.2.

Theorem 5.2. *Let $(\mu_{\mathsf{i}})_{\mathsf{i} \in \mathsf{V}}$ be the list of graph-directed self-conformal measures associated with the list $\big(\mathsf{V},\, \mathsf{E},\, (V_{\mathsf{i}})_{\mathsf{i} \in \mathsf{V}},\, (X_{\mathsf{i}})_{\mathsf{i} \in \mathsf{V}},\, (S_{\mathsf{e}})_{\mathsf{e} \in \mathsf{E}},\, (p_{\mathsf{e}})_{\mathsf{e} \in \mathsf{E}}\,\big)$, i.e. μ_{i} is the unique probability measure such that $\mu_{\mathsf{i}} = \sum_{\mathsf{e} \in E_{\mathsf{i}}} p_{\mathsf{e}}\, \mu_{\mathsf{t(e)}} \circ S_{\mathsf{e}}^{-1}$. Assume that all the maps S_{e} are similarities and let the matrix $A(q,s)$ be defined my (5.10). Let Λ be defined by (2.10) and let the zeta-functions $\zeta_q^{\mathsf{co},U}(\Lambda;s)$ and $\zeta_q^{\mathsf{co\text{-}con}}(\Lambda;s)$ be defined by (5.6) and (5.8), respectively. Finally, let $\tau(q)$ be defined by (5.9).*

If $s \in \mathbb{C}$ with $\operatorname{Re} s > \tau(q)$, then $I - A(q,s)$ is invertible and we have

$$\zeta_q^{\mathsf{co},U}(\Lambda;s) = \zeta_q^{\mathsf{co\text{-}con}}(\Lambda;s) = \mathbf{1}^\top \left(I - A(q,s)\right)^{-1} A(q,s)\,\mathbf{1}$$

where $\mathbf{1} = (1)_{\mathsf{i}\in\mathsf{V}}$ is the column vector in $\mathbb{R}^{\mathsf{V}}$ consisting of 1's and $\mathbf{1}^\top$ denotes the transpose of $\mathbf{1}$.

In particular, if the graph G has only one vertex and N edges labelled $1, 2, \ldots, N$, then we have

$$\zeta_q^{\mathsf{co},U}(\Lambda;s) = \zeta_q^{\mathsf{co\text{-}con}}(\Lambda;s) = \frac{\sum_{i=1}^N p_i^q r_i^s}{1 - \sum_{i=1}^N p_i^q r_i^s}\,. \tag{5.11}$$

Proof

We fist note that, in this particular case, the zeta-functions in (5.6) and (5.8) coincide, i.e. $\zeta_q^{\mathsf{co},U}(\Lambda;s) = \zeta_q^{\mathsf{co\text{-}con}}(\Lambda;s)$.

Next, fix $s \in \mathbb{C}$ with $\operatorname{Re} s > \tau(q)$ and write $A = A(q,s)$ for brevity. Since $\sum_{k=0}^{n-1} \Lambda S^k \mathbf{u} = \log r_{\mathbf{i}}$ for all $\mathbf{i} \in \Sigma_{\mathsf{G}}^*$ and all $\mathbf{u} \in [\mathbf{i}]$, we conclude that

$$\begin{aligned}
\zeta_q^{\mathsf{co},U}(\Lambda;s) &= \sum_n \sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} \exp\left(\left(\sup_{\mathbf{v}\in\Sigma_{\mathsf{G}}^{\mathbb{N}} \text{ with } \mathsf{t}(\mathbf{i})=\mathsf{i}(\mathbf{v})} q\,\frac{\log p_{\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{v})|} + s \right) \sup_{\mathbf{u}\in[\mathbf{i}]} \sum_{k=0}^{n-1} \Lambda S^k \mathbf{u} \right)\\
&= \sum_n \sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} \exp\left(\left(\sup_{\mathbf{v}\in\Sigma_{\mathsf{G}}^{\mathbb{N}} \text{ with } \mathsf{t}(\mathbf{i})=\mathsf{i}(\mathbf{v})} q\,\frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} + s \right) \log r_{\mathbf{i}} \right)\\
&= \sum_n \sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} p_{\mathbf{i}}^q r_{\mathbf{i}}^s\,.
\end{aligned}$$

Noticing that $\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} p_{\mathbf{i}}^q r_{\mathbf{i}}^s = \mathbf{1}^\top A^n\,\mathbf{1}$, it therefore follows that

$$\zeta_q^{\mathsf{co},U}(\Lambda;s) = \sum_n \mathbf{1}^\top A^n\,\mathbf{1}\,. \tag{5.12}$$

Let $M_{\mathsf{V}}(\mathbb{C})$ denote the vector space of $\mathsf{V}\times\mathsf{V}$ square matrices with complex entries and let $\|\cdot\|$ be a norm on $M_{\mathsf{V}}(\mathbb{C})$ (since $M_{\mathsf{V}}(\mathbb{C})$ is finite dimensional, all norms on $M_{\mathsf{V}}(\mathbb{C})$ are equivalent and it is therefore not important which norm we use). We now claim that (1) the matrix $I - A$ is invertible, and (2) the series $\sum_n A^n$ converges with respect to the norm $\|\cdot\|$ and

$$\sum_n A^n = (I-A)^{-1} A\,. \tag{5.13}$$

We will now prove (5.13). It follows from Theorem 5.1 that $\sigma_{\mathsf{ab}}\big(\zeta_q^{\mathsf{co},U}(\Lambda;\cdot)\big) = \sigma_{\mathsf{ab}}\big(\zeta_q^{\mathsf{co\text{-}con}}(\Lambda;\cdot)\big) = \beta(q)$. It also follows from [EdMa] that $\beta(q)$ is the unique real number such that $\rho_{\mathsf{spec\text{-}rad}}\,A(q,\beta(q)) = 1$ and the function $t \to \rho_{\mathsf{spec\text{-}rad}}\,A(q,t)$ is strictly decreasing; here and below we use the following notation, namely, if M is a square matrix, then we will write $\rho_{\mathsf{spec\text{-}rad}}\,M$ for the spectral radius of M. Consequently, since $\operatorname{Re} s > \sigma_{\mathsf{ab}}\big(\zeta_q^{\mathsf{co},U}(\Lambda;\cdot)\big) = \beta(q)$, we conclude that $\rho_{\mathsf{spec\text{-}rad}} A = \rho_{\mathsf{spec\text{-}rad}}\,A(q,s) < 1$. It also follows from the spectral formula that $\lim_n \|A^n\|^{\frac{1}{n}} = \rho_{\mathsf{spec\text{-}rad}} A$, and we therefore deduce that $\lim_n \|A^n\|^{\frac{1}{n}} = \rho_{\mathsf{spec\text{-}rad}} A < 1$. This implies that we can find a positive constant c_0 with $0 < c_0 < 1$ and an integer N_0 such that $\|A^n\|^{\frac{1}{n}} \le c_0$ for $n \ge N_0$, i.e. $\|A^n\| \le c_0^n$ for $n \ge N_0$, whence $\sum_{n\ge 0}\|A^n\| \le \sum_{n=0}^{N_0}\|A^n\| + \sum_{n>N_0} c_0^n < \infty$. We conclude from this and the completeness of $M_{\mathsf{V}}(\mathbb{C})$ that the series $\sum_{n\ge0} A^n$ converges with respect to $\|\cdot\|$. Writing $B = \sum_{n\ge0} A^n$, we clearly have $(I-A)B = I$ and $B(I-A) = I$. This shows that $I - A$ is invertible with $(I-A)^{-1} = B = \sum_{n\ge0} A^n$. It is not difficult to see that this equality implies (5.13). This completes the proof of (5.13).

Next, we note that it follows easily from (5.13) that

$$\sum_n \mathbf{1}^\top A^n\,\mathbf{1} = \mathbf{1}^\top \left(\sum_n A^n\right)\mathbf{1} = \mathbf{1}^\top (I-A)^{-1} A\,\mathbf{1}\,. \tag{5.14}$$

The desired result now follows immediately from (5.12) and (5.14). □

The zeta-function in (5.11) and its connection to the Renyi dimensions $\tau_\mu(q)$ of self-similar measures μ has recently been investigated in [Le-VeMe,Ol5,Ol6].

We will now prove Theorem 5.1. We note that Part (2) and Part (3) follow from Part (1) and [MiOl2,Ol7]; hence it suffices to prove Part (1). Recall that the functions Φ : and Λ are defined in (5.1) and (2.10), respectively. The proof of Theorem 5.1 now follows by applying Corollary 4.5 to the function $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to\mathbb{R}$ defined by

$$U\mu=\frac{\int\Phi\,d\mu}{\int\Lambda\,d\mu}\,, \tag{5.15}$$

noticing that in this particular case the zeta-function $\zeta_q^{\mathsf{co},U}(\varphi;s)$ is given by (5.6). We first prove the following auxiliary result showing that the abscissa of convergence of the zeta-functions $\zeta_q^{\mathsf{co},U}(\varphi;s)$ and $\zeta_q^{\mathsf{co\text{-}con}}(\varphi;s)$ are equal.

Proposition 5.3. *Let U be defined by (5.15). Fix a continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ with $\varphi<0$.*

(1) *There is a sequence $(\Delta_n)_n$ with $\Delta_n>0$ and $\Delta_n\to 0$ such that for all $n\in\mathbb{N}$, $\mathbf{i}\in\Sigma_{\mathsf{G}}^n$ and $\mathbf{u}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathrm{t}(\mathbf{i})=\mathrm{i}(\mathbf{u})$, we have $|\,\frac{\log p_{\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|}-\frac{\log p_{\mathbf{i}}}{\log\operatorname{diam}_{\mathsf{N}}K_{\mathbf{i}}}\,|\le\Delta_n$.*

(2) *For all $q\in X$, we have $\sigma_{\mathsf{ab}}\big(\zeta_q^{\mathsf{co},U}(\varphi;\cdot)\big)=\sigma_{\mathsf{ab}}\big(\zeta_q^{\mathsf{co\text{-}con}}(\varphi;\cdot)\big)$.*

Proof

(1) It is well-known and follows from the Principle of Bounded Distortion (see, for example, [Bar,Fa]) that there is a constant $c>0$ such that for all integers n and all $\mathbf{i}\in\Sigma_{\mathsf{G}}^n$ and all $\mathbf{u},\mathbf{v}\in[\mathbf{i}]$, we have $\frac{1}{c}\le\frac{|DS_{\mathbf{i}}(\pi S^n\mathbf{u})|}{\operatorname{diam}_{\mathsf{N}}K_{\mathbf{i}}}\le c$ and $\frac{1}{c}\le\frac{|DS_{\mathbf{i}}(\pi S^n\mathbf{u})|}{|DS_{\mathbf{i}}(\pi S^n\mathbf{v})|}\le c$. It is not difficult to see that the desired result follows from this.

(2) It is not difficult to see that this statement follows from Part 1 and we have therefore decided to omit the details. ☐

We can now prove Theorem 5.1.

Proof of Theorem 5.1

(1) Let the map U be defined by (5.15). Proposition 5.3 and Theorem 4.1 clearly imply that $0=\sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(h(\mu)+(qU\mu+\tau(q))\int\Lambda\,d\mu)=\sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(h(\mu)+\int(q\Phi+\tau(q)\Lambda)\,d\mu$. It follows from this and the variational principle that $P(q\Phi+\tau(q)\Lambda)=0$, and (5.2) therefore implies that $\tau(q)=\beta(q)$.

(2)–(3) These statements follow from (1) and [MiOl2,Ol7]. ☐

5.2. Multifractal spectra of ergodic Birkhoff averages. We first fix $\gamma\in(0,1)$ and define the metric d_γ on $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ as follows. For $\mathbf{i},\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathbf{i}\neq\mathbf{j}$, we will write $\mathbf{i}\wedge\mathbf{j}$ for the longest common prefix of $\mathbf{i}$ and $\mathbf{j}$. The metric d_γ is now defined by $\mathsf{d}_\gamma(\mathbf{i},\mathbf{j})=\gamma^{|\mathbf{i}\wedge\mathbf{j}|}$ if $\mathbf{i}\neq\mathbf{j}$, and $\mathsf{d}_\gamma(\mathbf{i},\mathbf{j})=0$ if $\mathbf{i}=\mathbf{j}$. for $\mathbf{i},\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$; throughout this section, we equip $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ with the metric d_γ, and continuity and Lipschitz properties of functions $f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ from $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ to $\mathbb{R}$ will always refer to the metric d_γ. Multifractal analysis of Birkhoff averages has received significant interest during the past 10 years, see, for example, [BaMe,FaFe,FaFeWu,FeLaWu,Oli,Ol2,OlWi1,OlWi2]. The multifractal spectrum F_f^{erg} of ergodic Birkhoff averages of a continuous function $f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ is defined by

$$F_f^{\mathsf{erg}}(\alpha)=\dim_{\mathsf{H}}\pi\left\{\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,\middle|\,\lim_n\frac{1}{n}\sum_{k=0}^{n-1}f(S^k\mathbf{i})=\alpha\right\}$$

for $\alpha\in\mathbb{R}$; recall, that the map π is defined in Section 2. One of the main problems in multifractal analysis of Birkhoff averages is the detailed study of the multifractal spectrum F_f^{erg}. For example, it is proved (in different settings and at various levels of generality) in [FaFe,FaFeWu,FeLaWu,Oli,Ol2,OlWi1] that if $f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ is continuous and α is a real number, then

$$F_f^{\mathsf{erg}}(\alpha)=\dim_{\mathsf{H}}\pi\left\{\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,\middle|\,\lim_n\frac{1}{n}\sum_{k=0}^{n-1}f(S^k\mathbf{i})=\alpha\right\}=\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ \int f\,d\mu=\alpha}}-\frac{h(\mu)}{\int\Lambda\,d\mu}\,.$$

where Λ be defined by (2.10).

As a second application of Theorems 4.1–4.4 we will now obtain a fine dynamical multifractal zeta-function with an associated Bowen equation whose solution $\mathscr{f}(C)$ equals the multifractal spectrum of ergodic Birkhoff averages and a family of coarse multifractal zeta-functions with abscissae of convergence whose Legendre transform equals $\mathscr{f}(C)$. We first state and prove a rather general result, namely Theorem 5.4 below. Using Theorem 5.4 we then deduce analogous results for a number of different multifractal spectra of ergodic Birkhoff averages, including $F_f^{\mathsf{erg}}\alpha)$; see Theorem 5.5 and Theorem 5.6. Recall that if $\mathbf{f}:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}^M$ is a continuous function with $\mathbf{f}=(f_1,\dots,f_M)$, then we will write $\int\mathbf{f}\,d\mu=(\int f_1\,d\mu,\dots,\int f_M\,d\mu)$ for $\mu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. Also, if $(x_n)_n$ is a sequence of points in a metric space M, then we write $\operatorname{acc}_n x_n$ for the set of accumulation points of the sequence $(x_n)_n$, i.e.

$$\operatorname*{acc}_n x_n=\left\{x\in M\,\middle|\,x\text{ is an accumulation point of }(x_n)_n\right\}.$$

Theorem 5.4. Multifractal zeta-functions for abstract multifractal spectra of ergodic Birkhoff averages. *Fix $\gamma\in(0,1)$ and $W\subseteq\mathbb{R}^I$ and let $\mathbf{\Phi}:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}^I$ be a Lipschitz function with respect to the metric d_γ such that $\{\int\mathbf{\Phi}\,d\mu\,|\,\mu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\}\subseteq W$. Let $Q:W\to\mathbb{R}^M$ be a continuous function.*

For $C\subseteq\mathbb{R}^M$ and a continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$, we define the abstract fine dynamical ergodic multifractal zeta-function associated with Q by

$$\zeta_C^{\mathsf{dyn\text{-}erg}}(\varphi;z)=\sum_n\frac{z^n}{n}\left(\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n\\ \forall\mathbf{v}\in[\mathbf{i}]\,:\,Q\left(\frac{1}{n}\sum_{k=0}^{n-1}\mathbf{\Phi}(S^k\mathbf{v})\right)\in C}}\sup_{\mathbf{u}\in[\mathbf{i}]}\exp\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}\right).$$

For $\mathbf{q}\in\mathbb{R}^M$ and a continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$, we define the abstract coarse ergodic multifractal zeta-function associated with Q by

$$\zeta_{\mathbf{q}}^{\mathsf{co\text{-}erg}}(\varphi;s)=\sum_n\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n}\exp\left(\left(\sup_{\mathbf{v}\in[\mathbf{i}]}\left\langle\mathbf{q}\,\middle|\,Q\left(\frac{1}{n}\sum_{k=0}^{n-1}\mathbf{\Phi}(S^k\mathbf{v})\right)\right\rangle+s\right)\sup_{\mathbf{u}\in[\mathbf{i}]}\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}\right).$$

Let $\Lambda:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ be defined by (2.10) and write

$$\tau(\mathbf{q})=\sigma_{\mathsf{ab}}\left(\zeta_{\mathbf{q}}^{\mathsf{co\text{-}erg}}(\Lambda;\cdot)\right).$$

(1) *For all $\mathbf{q}$, we have*

$$\tau(\mathbf{q})=\sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}\left(-\frac{h(\mu)}{\int\Lambda\,d\mu}-\left\langle\mathbf{q}\,\middle|\,Q\left(\int\mathbf{\Phi}\,d\mu\right)\right\rangle\right).$$

(2) *Assume that $C\subseteq\mathbb{R}^M$ is closed.*

(2.1) *There is a unique real number $\mathscr{f}(C)$ such that*

$$\lim_{r\searrow0}\sigma_{\mathsf{rad}}\left(\zeta_{B(C,r)}^{\mathsf{dyn\text{-}erg}}(\mathscr{f}(C)\,\Lambda;\cdot)\right)=1.$$

(2.2) *If for each $\boldsymbol{\alpha}\in C$, there is $\mathbf{q}\in\mathbb{R}^M$ such that $\boldsymbol{\alpha}=-\nabla\tau(\mathbf{q})$, then we have*

$$\mathscr{f}(C)=\sup_{\boldsymbol{\alpha}\in C}\tau^*(\boldsymbol{\alpha}).$$

(2.3) *If the OSC is satisfied and if for each $\boldsymbol{\alpha}\in C$, there is $\mathbf{q}\in\mathbb{R}^M$ such that $\boldsymbol{\alpha}=-\nabla\tau(\mathbf{q})$, then we have*

$$\mathscr{f}(C)=\dim_{\mathsf{H}}\pi\left\{\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,\middle|\,\operatorname*{acc}_n Q\left(\frac{1}{n}\sum_{k=0}^{n-1}\mathbf{\Phi}(S^k\mathbf{i})\right)\subseteq C\right\}=\sup_{\boldsymbol{\alpha}\in C}\tau^*(\boldsymbol{\alpha}).$$

Proof
(1) This follows by applying Theorem 4.1 to the map $U:\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to\mathbb{R}^M$ defined by $U\mu=Q(\int\mathbf{\Phi}\,d\mu)$.
(2) This follows from Part (1), Theorem 4.4 and [MiOl2,Ol7]. □

Below we present two corollaries of Theorem 5.4. Theorem 5.5 studies the multifractal spectrum $F_f^{\mathsf{erg}}(\alpha)$ of continuos functions f. The Legendre transform representation of the spectrum $F_f^{\mathsf{erg}}(\alpha)$ in Part (2.3) of Theorem 5.4 seems to be new.

Theorem 5.5. Multifractal zeta-functinons for multifractal spectra of vector valued ergodic Birkhoff averages. *Fix $\gamma \in (0,1)$ and let $\mathbf{f}:\Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}^M$ be a Lipschitz function with respect to the metric d_γ. For $C \subseteq \mathbb{R}^M$ and a continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$, we define the dynamical ergodic multifractal zeta-function by*

$$\zeta_C^{\mathsf{dyn\text{-}vec\text{-}erg}}(\varphi;z) = \sum_n \frac{z^n}{n} \left(\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n \\ \forall \mathbf{v}\in[\mathbf{i}]\,:\, \frac{1}{n}\sum_{k=0}^{n-1}\mathbf{f}(S^k\mathbf{v})\in C}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u} \right).$$

For $\mathbf{q}\in\mathbb{R}^M$ and a continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$, we define the coarse ergodic multifractal zeta-function by

$$\zeta_{\mathbf{q}}^{\mathsf{co\text{-}vec\text{-}erg}}(\varphi;s) = \sum_n \sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} \exp\left(\left(\sup_{\mathbf{v}\in[\mathbf{i}]} \left\langle \mathbf{q} \middle| \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{f}(S^k\mathbf{v}) \right\rangle + s \right) \sup_{\mathbf{u}\in[\mathbf{i}]} \sum_{k=0}^{n-1} \varphi S^k\mathbf{u} \right).$$

Let $\Lambda:\Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ be defined by (2.10) and write

$$\tau(\mathbf{q}) = \sigma_{\mathsf{ab}}\big(\zeta_{\mathbf{q}}^{\mathsf{co\text{-}vec\text{-}erg}}(\Lambda;s) \big).$$

(1) *For all $\mathbf{q}$, we have*

$$\tau(\mathbf{q}) = \sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})} \left(-\frac{h(\mu)}{\int \Lambda\, d\mu} - \left\langle \mathbf{q} \middle| \int \mathbf{f}\, d\mu \right\rangle \right).$$

(2) *Assume that $C \subseteq \mathbb{R}^M$ is closed.*

(2.1) *There is a unique real number $\not{f}(C)$ such that*

$$\lim_{r\searrow 0} \sigma_{\mathsf{rad}}\big(\zeta_{B(C,r)}^{\mathsf{dyn\text{-}vec\text{-}erg}}(\not{f}(C)\,\Lambda;\cdot) \big) = 1.$$

If $\boldsymbol{\alpha}\in\mathbb{R}^M$ and $C=\{\boldsymbol{\alpha}\}$, then we will write $\not{f}(\boldsymbol{\alpha}) = \not{f}(C)$.

(2.2) *If for each $\boldsymbol{\alpha}\in C$, there is $\mathbf{q}\in\mathbb{R}^M$ such that $\boldsymbol{\alpha} = -\nabla\tau(\mathbf{q})$, then we have*

$$\not{f}(C) = \sup_{\boldsymbol{\alpha}\in C} \tau^*(\boldsymbol{\alpha}).$$

(2.3) *If the OSC is satisfied and if for each $\boldsymbol{\alpha}\in C$, there is $\mathbf{q}\in\mathbb{R}^M$ such that $\boldsymbol{\alpha} = -\nabla\tau(\mathbf{q})$, then we have*

$$\not{f}(C) = \dim_{\mathsf{H}} \pi \left\{ \mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, \operatorname*{acc}_n \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{f}(S^k\mathbf{i}) \subseteq C \right\} = \sup_{\boldsymbol{\alpha}\in C} \tau^*(\boldsymbol{\alpha}).$$

In particular, if the OSC is satisfied and $\boldsymbol{\alpha}\in\mathbb{R}^M$ and there is $\mathbf{q}\in\mathbb{R}^M$ such that $\boldsymbol{\alpha} = -\nabla\tau(\mathbf{q})$, then we have

$$\not{f}(\boldsymbol{\alpha}) = \dim_{\mathsf{H}} \pi \left\{ \mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, \lim_n \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{f}(S^k\mathbf{i}) = \boldsymbol{\alpha} \right\} = \tau^*(\boldsymbol{\alpha}).$$

(Observe that since τ is convex, we conclude that τ is differentiable almost everywhere, and the conclusion in Part (2.3) is therefore satisfied for "many" points $\boldsymbol{\alpha}$.)

Proof
This follows immediately by applying Theorem 5.4 to $W=\mathbb{R}^M$ and the maps $\mathbf{\Phi}:\Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}^M$ and $Q:W\to\mathbb{R}^M$ defined by $\mathbf{\Phi}=\mathbf{f}$ and $Q(\mathbf{x})=\mathbf{x}$. □

As a final application of Theorem 5.4 we consider a type of relative ergodic multifractal spectra involving quantities similar to those appearing in Hölder's inequality; for this reason we have decided to refer to these multifractal spectra as "Hölder-like relative ergodic Birkhoff averages".

Theorem 5.6. Multifractal zeta-functinons for multifractal spectra of Hölder-like relative ergodic Birkhoff averages. *Fix $\gamma \in (0,1)$ and let $f_1,\dots,f_M,g_1,\dots,g_M : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ be Lipschitz functions with respect to the metric d_γ and assume that $f_i(\mathbf{i}) > 0$ for all i and all $\mathbf{i}$, and that $g_i(\mathbf{i}) > 0$ for all i and all $\mathbf{i}$. Fix $s_1,\dots,s_M,t_1,\dots,t_M > 0$. For $C \subseteq \mathbb{R}$ and a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$, we define the dynamical Hölder-like relative ergodic multifractal zeta-function by*

$$\zeta_C^{\text{dyn-Höl-erg}}(\varphi;z) = \sum_n \frac{z^n}{n} \left(\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n \\ \forall \mathbf{v}\in[\mathbf{i}]\,:\, \frac{\prod_{i=1}^M \left(\frac{1}{n}\sum_{k=0}^{n-1} f_i(S^k\mathbf{v})\right)^{s_i}}{\prod_{i=1}^M \left(\frac{1}{n}\sum_{k=0}^{n-1} g_i(S^k\mathbf{v})\right)^{t_i}} \in C}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u} \right).$$

For $q \in \mathbb{R}$ and a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$, we define the coarse Hölder-like ergodic multifractal zeta-function by

$$\zeta_q^{\text{co-Höl-erg}}(\varphi;s) = \sum_n \sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} \exp\left(\left(\sup_{\mathbf{v}\in[\mathbf{i}]} q\, \frac{\prod_{i=1}^M \left(\frac{1}{n}\sum_{k=0}^{n-1} f_i(S^k\mathbf{v})\right)^{s_i}}{\prod_{i=1}^M \left(\frac{1}{n}\sum_{k=0}^{n-1} g_i(S^k\mathbf{v})\right)^{t_i}} + s \right) \sup_{\mathbf{u}\in[\mathbf{i}]} \sum_{k=0}^{n-1} \varphi S^k \mathbf{u} \right).$$

Let $\Lambda : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ be defined by (2.10) and write

$$\tau(q) = \sigma_{\mathsf{ab}}\big(\, \zeta_q^{\text{co-Höl-erg}}(\Lambda;s)\,\big)\,.$$

(1) *For all q, we have*

$$\tau(q) = \sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})} \left(-\frac{h(\mu)}{\int \Lambda\,d\mu} - q\,\frac{\prod_{i=1}^M \left(\int f_i\,d\mu\right)^{s_i}}{\prod_{i=1}^M \left(\int g_i\,d\mu\right)^{t_i}} \right).$$

(2) *Assume that $C \subseteq \mathbb{R}^M$ is closed.*

(2.1) *There is a unique real number $\mathscr{f}(C)$ such that*

$$\lim_{r\searrow 0} \sigma_{\mathsf{rad}}\big(\, \zeta_{B(C,r)}^{\text{dyn-Höl-erg}}(\mathscr{f}(C)\,\Lambda;\cdot)\,\big) = 1\,.$$

If $\alpha \in \mathbb{R}$ and $C = \{\alpha\}$, then we will write $\mathscr{f}(\alpha) = \mathscr{f}(C)$.

(2.2) *If for each $\alpha \in C$, there is $q \in \mathbb{R}$ such that $\alpha = -\tau'(q)$, then we have*

$$\mathscr{f}(C) = \sup_{\alpha\in C} \tau^*(\alpha)\,.$$

(2.3) *If the OSC is satisfied and if for each $\alpha \in C$, there is $q \in \mathbb{R}$ such that $\alpha = -\tau'(q)$, then we have*

$$\mathscr{f}(C) = \dim_{\mathsf{H}} \pi \left\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, \operatorname*{acc}_n \frac{\prod_{i=1}^M \left(\frac{1}{n}\sum_{k=0}^{n-1} f_i(S^k\mathbf{i})\right)^{s_i}}{\prod_{i=1}^M \left(\frac{1}{n}\sum_{k=0}^{n-1} g_i(S^k\mathbf{i})\right)^{t_i}} \subseteq C \right\} = \sup_{\alpha\in C} \tau^*(\alpha)\,.$$

In particular, if the OSC is satisfied and $\alpha \in \mathbb{R}$ and there is $q \in \mathbb{R}$ such that $\alpha = -\tau'(q)$, , then we have

$$\mathscr{f}(\alpha) = \dim_{\mathsf{H}} \pi \left\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, \lim_n \frac{\prod_{i=1}^M \left(\frac{1}{n}\sum_{k=0}^{n-1} f_i(S^k\mathbf{i})\right)^{s_i}}{\prod_{i=1}^M \left(\frac{1}{n}\sum_{k=0}^{n-1} g_i(S^k\mathbf{i})\right)^{t_i}} = \alpha \right\} = \tau^*(\alpha)\,.$$

(Observe that since τ is convex, we conclude that τ is differentiable almost everywhere, and the conclusion in Part (2.3) is therefore satisfied for "many" points α.)

Proof
This follows immediately by applying Theorem 5.4 to $W = \mathbb{R}^M \times (\mathbb{R}\setminus\{0\})^M$ and the maps $\mathbf{\Phi} : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}^{2M}$ and $Q : W \to \mathbb{R}$ defined by $\mathbf{\Phi} = (f_1,\dots,f_M,g_1,\dots,g_M)$ and $Q(x_1,\dots,x_M,y_1,\dots,y_M) = \frac{\prod_{i=1}^M x_i^{s_i}}{\prod_{i=1}^M y_i^{t_i}}$. □

6. Proof of Theorem 4.1 and Theorem 4.2, Part 1: The map M_n.

The map M_n. Since the graph $\mathsf{G}=(\mathsf{V},\mathsf{E})$ is strongly connected, it follows that for each $\mathbf{i}\in\Sigma_{\mathsf{G}}^*$, we can choose $\widehat{\mathbf{i}}\in\Sigma_{\mathsf{G}}^*$ with $|\widehat{\mathbf{i}}|\le|\mathsf{V}|$ such that $\mathrm{t}(\mathbf{i})=\mathrm{i}(\widehat{\mathbf{i}})$ and $\mathrm{t}(\widehat{\mathbf{i}})=\mathrm{i}(\mathbf{i})$, and we now define $\bar{\mathbf{i}}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$ by

$$\bar{\mathbf{i}}=\mathbf{i}\,\widehat{\mathbf{i}}\,\mathbf{i}\,\widehat{\mathbf{i}}\,\mathbf{i}\,\widehat{\mathbf{i}}\dots.$$

For a positive integer n, we define $M_n:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ by

$$\begin{aligned}M_n\mathbf{i}&=L_{n+|\,\widehat{\mathbf{i}|n}\,|}\left(\overline{\mathbf{i}|n}\right)\\&=\frac{1}{n+|\,\widehat{\mathbf{i}|n}\,|}\sum_{k=0}^{n+|\,\widehat{\mathbf{i}|n}\,|-1}\delta_{S^k(\,\overline{\mathbf{i}|n}\,)}\end{aligned}\tag{6.1}$$

for $\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$; recall, that the map $L_n:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ is defined in (3.5).

Why the map M_n? The main reason for introduction the map M_n is the following. In order to prove Theorem 4.1 we will use results from large deviation theory. In particular, we will use Varadhan's integral lemma which says that if X is a complete separable metric space and $(P_n)_n$ is a sequence of probability measures on X satisfying the large deviation property with rate constants $a_n\in(0,\infty)$ for $n\in\mathbb{N}$ and rate function $I:\mathbb{R}\to[-\infty,\infty]$ (this terminology will be explained in Section 7), then

$$\lim_n\frac{1}{a_n}\log\int\exp(a_nF)\,dP_n=-\inf_{x\in X}(\,I(x)-F(x))$$

for any bounded continuous function $F:X\to\mathbb{R}$ (see Section 7 for more a detailed and precise discussion of this result).

More precisely, in Section 7 we intend to use Varadhan's integral lemma to analyse the asymptotic behaviour of the integral

$$\frac{1}{n}\log\int\exp(nF(L_n(\overline{\mathbf{i}|n})))\,d\Pi(\mathbf{i})\tag{6.2}$$

as $n\to\infty$ where Π is the Parry measure on $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ (the Parry measure will be defined in Section 7) and the function $F:\mathcal{P}\big(\Sigma_{\mathsf{G}}^{\mathbb{N}}\big)\to\mathbb{R}$ is defined by $F(\mu)=(\langle q|U\mu\rangle+s)\int\varphi\,d\mu$. Defining $\Lambda_n:\mathcal{P}\big(\Sigma_{\mathsf{G}}^{\mathbb{N}}\big)\to\mathbb{R}$ by $\Lambda_n(\mathbf{i})=L_n(\,\overline{\mathbf{i}|n}\,)$, then (6.2) can be written as

$$\frac{1}{n}\log\int\exp(nF)\,d(\Pi\circ\Lambda_n^{-1})\,.\tag{6.3}$$

Consequently, if the sequence $(\Pi\circ\Lambda_n^{-1})_n$ satisfied the large deviation property with rate constants $a_n=n$, then Varadhan's integral lemma could be applied to analyse the asymptotic behaviour of the sequence of integrals in (6.3). However, it follows from results by Orey & Pelikan [OrPe1,OrPe2] that the sequence $(\Pi\circ L_n)_n$ satisfies the large deviation property with rate constants $a_n=n$ and Varadhan's integral lemma can therefore be applied to provide information about the asymptotic behaviour of the sequence of integral defined by

$$\frac{1}{n}\log\int\exp(nF)\,d(\Pi\circ L_n^{-1})\,.\tag{6.4}$$

In order to utilise the knowledge of the asymptotic behaviour of (6.4) for analysing the asymptotic behaviour of (6.3), we must therefore show that the measures

$$\Pi\circ\Lambda_n^{-1}$$

and

$$\Pi\circ L_n^{-1}$$

are "close". In fact, for technical reasons we will prove and use a similar statement involving the measures

$$\Pi\circ M_n^{-1}$$

and

$$\Pi\circ L_n^{-1}\,.$$

Indeed, below we prove a number of results showing that the maps M_n and L_n (and therefore also the measures $\Pi\circ M_n^{-1}$ and $\Pi\circ L_n^{-1}$) are "close". These results play an important role in Section 7. In particular, they allow us to: (1) use Orey & Pelikan's result from [OrPe1,OrPe2] saying that the sequence $(\Pi\circ L_n^{-1})_n$ satisfies the large deviation property to prove that the sequence $(\Pi\circ M_n^{-1})_n$ also satisfies the large deviation property (see Theorem 7.2), and (2) replace all occurencies of $L_n(\overline{\mathbf{i}|n})$ in the formulas in Section 7 by $M_n\mathbf{i}$ allowing us to use the large deviation property of the sequence $(\Pi\circ M_n^{-1})_n$. The above discussion explains the mean reason for introducing the map M_n and the associated measure $\Pi\circ M_n^{-1}$.

Comparing M_n and L_n. We now prove various continuity statements saying that the maps M_n and L_n are "close". These statements play an important role in Section 7 where we apply Varadhan's integral lemma to prove Theorem 4.1. We first introduce the metric L on $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. Fix $\gamma\in(0,1)$ and let d_γ denote the metric on $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ introduced in Section 5.2. For a function $f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$, we let $\operatorname{Lip}_\gamma(f)$ denote the Lipschitz constant of f with respect to the metric d_γ, i.e. $\operatorname{Lip}_\gamma(f)=\sup_{\mathbf{i},\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}},\mathbf{i}\neq\mathbf{j}}\frac{|f(\mathbf{i})-f(\mathbf{j})|}{\mathsf{d}_\gamma(\mathbf{i},\mathbf{j})}$ and we define the metric L in $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ by

$$\mathsf{L}(\mu,\nu)=\sup_{\substack{f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \operatorname{Lip}_\gamma(f)\le1}}\left|\int f\,d\mu-\int f\,d\nu\right|; \tag{6.5}$$

we note that it is well-known that L is a metric and that L induces the weak topology. Below we will always equip the space $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ with the metric L. We can now state and prove the main results in this section.

Lemma 6.1. *Let (X,d) be a metric space and let $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ be continuous with respect to the weak topology.*

(1) *We have*

$$\sup_{\mathbf{u}\in\Sigma_{\mathsf{G}}^n}\ \sup_{\substack{\mathbf{k},\mathbf{l}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{k})\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{l})}}\mathsf{d}\big(UL_n(\mathbf{uk})\,,\,UM_n(\mathbf{ul})\big)\to0\quad\textit{as } n\to\infty.$$

(2) *If $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ is a Hölder continuous function, then we have*

$$\sup_{\mathbf{u}\in\Sigma_{\mathsf{G}}^n}\ \sup_{\substack{\mathbf{k},\mathbf{l}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{k})\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{l})}}\left|\int\varphi\,d(L_n(\mathbf{uk}))-\int\varphi\,d(M_n(\mathbf{ul}))\right|\to0\quad\textit{as } n\to\infty.$$

Proof
We first note that if $\mathbf{u}\in\Sigma_{\mathsf{G}}^n$ and $\mathbf{k}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{k})$, and $f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ is a continuous function,

then

$$
\begin{aligned}
\left|\int f\,d(L_n(\mathbf{uk})) - \int f\,d(L_{n+|\,\widehat{\mathbf{u}}\,|}\overline{\mathbf{u}})\right|
&= \left|\frac{1}{n}\sum_{i=0}^{n-1} f(S^i(\mathbf{uk})) - \frac{1}{n+|\,\widehat{\mathbf{u}}\,|}\sum_{i=0}^{n+|\,\widehat{\mathbf{u}}\,|-1} f(S^i\overline{\mathbf{u}})\right|\\
&\le \left|\frac{1}{n}\sum_{i=0}^{n-1} f(S^i(\mathbf{uk})) - \frac{1}{n+|\,\widehat{\mathbf{u}}\,|}\sum_{i=0}^{n-1} f(S^i\overline{\mathbf{u}})\right|\\
&\qquad + \left|\frac{1}{n+|\,\widehat{\mathbf{u}}\,|}\sum_{i=n}^{n+|\,\widehat{\mathbf{u}}\,|-1} f(S^i\overline{\mathbf{u}})\right|\\
&\le \left|\frac{1}{n}\sum_{i=0}^{n-1} f(S^i(\mathbf{uk})) - \frac{1}{n}\sum_{i=0}^{n-1} f(S^i\overline{\mathbf{u}})\right|\\
&\qquad + \left|\frac{1}{n}\sum_{i=0}^{n-1} f(S^i\overline{\mathbf{u}}) - \frac{1}{n+|\,\widehat{\mathbf{u}}\,|}\sum_{i=0}^{n-1} f(S^i\overline{\mathbf{u}})\right|\\
&\qquad + \left|\frac{1}{n+|\,\widehat{\mathbf{u}}\,|}\sum_{i=n}^{n+|\,\widehat{\mathbf{u}}\,|-1} f(S^i\overline{\mathbf{u}})\right|\\
&\le \frac{1}{n}\sum_{i=0}^{n-1} |f(S^i(\mathbf{uk})) - f(S^i\overline{\mathbf{u}})|\\
&\qquad + \frac{|\,\widehat{\mathbf{u}}\,|}{n+|\,\widehat{\mathbf{u}}\,|}\frac{1}{n}\sum_{i=0}^{n-1}\|f\|_\infty\\
&\qquad + \frac{1}{n+|\,\widehat{\mathbf{u}}\,|}\sum_{i=n}^{n+|\,\widehat{\mathbf{u}}\,|-1}\|f\|_\infty\\
&= \frac{1}{n}\sum_{i=0}^{n-1} |f(S^i(\mathbf{uk})) - f(S^i\overline{\mathbf{u}})| + 2\frac{|\,\widehat{\mathbf{u}}\,|}{n+|\,\widehat{\mathbf{u}}\,|}\|f\|_\infty\\
&\le \frac{1}{n}\sum_{i=0}^{n-1} |f(S^i(\mathbf{uk})) - f(S^i\overline{\mathbf{u}})| + 2\frac{|\mathsf{V}|}{n}\|f\|_\infty\,. \quad [\text{since } |\,\widehat{\mathbf{u}}\,| \le |\mathsf{V}|]
\end{aligned}
\tag{6.6}
$$

(1). Let $\varepsilon > 0$. Fix $\gamma \in (0,1)$ and let L be the metric on $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ defined in (6.5). Since $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ is continuous and $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ is compact, we conclude that $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ is uniformly continuous. This implies that we can choose $\delta > 0$ such that all measures $\mu, \nu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ satisfy the following implication:

$$
\mathsf{L}(\mu,\nu) < \delta \quad\Rightarrow\quad \mathsf{d}(U\mu, U\nu) < \frac{\varepsilon}{2}\,. \tag{6.7}
$$

Next, choose a positive integer N_0 such that $\frac{1}{N_0}(2|\mathsf{V}| + \frac{1}{1-\gamma}) < \delta$.

Now, fix $n \ge N_0$, $\mathbf{u} \in \Sigma_{\mathsf{G}}^n$ and $\mathbf{k}, \mathbf{l} \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathrm{t}(\mathbf{u}) = \mathrm{i}(\mathbf{k})$ and $\mathrm{t}(\mathbf{u}) = \mathrm{i}(\mathbf{l})$. It follows that

$$
\begin{aligned}
\mathsf{L}\big(\,L_n(\mathbf{uk})\,,\, M_n(\mathbf{ul})\,\big) &\le \mathsf{L}\big(\,L_n(\mathbf{uk})\,,\, L_n(\mathbf{ul})\,\big) + \mathsf{L}\big(\,L_n(\mathbf{ul})\,,\, M_n(\mathbf{ul})\,\big)\\
&\le \mathsf{L}\big(\,L_n(\mathbf{uk})\,,\, L_n(\mathbf{ul})\,\big) + \mathsf{L}\big(\,L_n(\mathbf{ul})\,,\, L_{n+|\,\widehat{\mathbf{u}}\,|}\overline{\mathbf{u}}\,\big)
\end{aligned}
$$

We first estimate the distance $\mathsf{L}\big(\,L_n(\mathbf{uk})\,,\, L_n(\mathbf{ul})\,\big)$. Indeed, since $\frac{1}{N_r(1-\gamma)} \le \frac{1}{N_0}(2|\mathsf{V}| + \frac{1}{1-\gamma}) < \delta$, it follows that

$$\begin{aligned}
\mathsf{L}\big(\, L_n(\mathbf{uk})\,,\, L_n(\mathbf{ul})\,\big) &= \sup_{\substack{f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \operatorname{Lip}_\gamma(f)\le 1}} \left| \int f\, d(L_n(\mathbf{uk})) - \int f\, d(L_n(\mathbf{ul}))\right| \\
&= \sup_{\substack{f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \operatorname{Lip}_\gamma(f)\le 1}} \left| \frac{1}{n}\sum_{i=0}^{n-1} f(S^i(\mathbf{uk})) - \frac{1}{n}\sum_{i=0}^{n-1} f(S^i(\mathbf{ul}))\right| \\
&\le \sup_{\substack{f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \operatorname{Lip}_\gamma(f)\le 1}} \frac{1}{n}\sum_{i=0}^{n-1} |f(S^i(\mathbf{uk})) - f(S^i(\mathbf{ul}))| \\
&\le \frac{1}{n}\sum_{i=0}^{n-1} \mathsf{d}_\gamma\big(\, S^i(\mathbf{uk})\,,\, S^i(\mathbf{ul})\,\big) \\
&= \frac{1}{n}\sum_{i=0}^{n-1} \gamma^{|S^i(\mathbf{uk})\wedge S^i(\mathbf{ul})|} \\
&\le \frac{1}{N_0}\sum_{i=0}^{n-1} \gamma^{n-i} \\
&\le \frac{1}{N_0(1-\gamma)} \\
&< \delta\,,
\end{aligned}$$

and we therefore conclude from (6.7) that

$$\mathsf{d}\big(\, UL_n(\mathbf{uk})\,,\, UL_n(\mathbf{ul})\,\big) < \frac{\varepsilon}{2}\,. \tag{6.8}$$

Next, we estimate the distance $\mathsf{L}\big(\, L_n(\mathbf{ul})\,,\, L_{n+|\widehat{\mathbf{u}}|}\overline{\mathbf{u}}\,\big)$. We start by observing that if we fix $\mathbf{i}_0\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$, then

$$\begin{aligned}
\mathsf{L}(\mu,\nu) &= \sup_{\substack{f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \operatorname{Lip}_\gamma(f)\le 1}} \left| \int f\, d\mu - \int f\, d\nu\right| \\
&= \sup_{\substack{f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \operatorname{Lip}_\gamma(f)\le 1}} \left| \int (f - f(\mathbf{i}_0))\, d\mu - \int (f - f(\mathbf{i}_0))\, d\nu\right| \\
&= \sup_{\substack{g:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \operatorname{Lip}_\gamma(g)\le 1\\ g(\mathbf{i}_0)=0}} \left| \int g\, d\mu - \int g\, d\nu\right|
\end{aligned} \tag{6.9}$$

for all $\mu,\nu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. It follows from (6.9) and (6.6) that

$$\begin{aligned}
\mathsf{L}\big(\, L_n(\mathbf{ul})\,,\, L_{n+|\widehat{\mathbf{u}}|}\overline{\mathbf{u}}\,\big) &= \sup_{\substack{g:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \operatorname{Lip}_\gamma(g)\le 1\\ g(\mathbf{i}_0)=0}} \left| \int g\, d(L_n(\mathbf{ul})) - \int g\, d(L_{n+|\widehat{\mathbf{u}}|}\overline{\mathbf{u}})\right| \\
&\le \sup_{\substack{g:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \operatorname{Lip}_\gamma(g)\le 1\\ g(\mathbf{i}_0)=0}} \left(\frac{1}{n}\sum_{i=0}^{n-1} |g(S^i(\mathbf{ul})) - g(S^i\overline{\mathbf{u}})| + 2\frac{|\mathsf{V}|}{n}\|g\|_\infty\right)
\end{aligned}$$

$$\leq \sup_{\substack{g:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(g)\leq 1\\ g(\mathbf{i}_0)=0}} \left(\frac{1}{n}\sum_{i=0}^{n-1}\mathrm{Lip}_\gamma(g)\ \mathsf{d}_\gamma\big(\,S^i(\mathbf{ul})\big)\,,\,S^i\overline{\mathbf{u}}\,\big) + 2\frac{|\mathsf{V}|}{n}\|g\|_\infty\right)$$

$$\leq \sup_{\substack{g:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(g)\leq 1\\ g(\mathbf{i}_0)=0}} \left(\frac{1}{n}\sum_{i=0}^{n-1}\gamma^{|S^i(\mathbf{ul})\wedge S^i\overline{\mathbf{u}}|} + 2\frac{|\mathsf{V}|}{n}\|g\|_\infty\right)$$

$$\leq \sup_{\substack{g:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(g)\leq 1\\ g(\mathbf{i}_0)=0}} \left(\frac{1}{n}\sum_{i=0}^{n-1}\gamma^{n-i} + 2\frac{|\mathsf{V}|}{n}\|g\|_\infty\right)$$

$$\leq \sup_{\substack{g:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(g)\leq 1\\ g(\mathbf{i}_0)=0}} \left(\frac{1}{n(1-\gamma)} + 2\frac{|\mathsf{V}|}{n}\|g\|_\infty\right). \tag{6.10}$$

However, if $g:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ satisfies $\mathrm{Lip}_\gamma(g)\leq 1$ and $g(\mathbf{i}_0)=0$, then $|g(\mathbf{i})| = |g(\mathbf{i})-g(\mathbf{i}_0)| \leq \mathsf{d}_\gamma(\mathbf{i},\mathbf{i}_0)\leq 1$ for all $\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$, whence $\|g\|_\infty\leq 1$. It therefore follows from (6.10) that if $n\geq N_0$, $\mathbf{u}\in\Sigma_{\mathsf{G}}^n$ and $\mathbf{l}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{l})$, then $\mathsf{L}\big(\,L_n(\mathbf{ul})\,,\,L_{n+|\widehat{\mathbf{u}}|}\overline{\mathbf{u}}\,\big)\leq \frac{1}{n}(\frac{1}{1-\gamma}+2|\mathsf{V}|)\leq\frac{1}{N_0}(\frac{1}{1-\gamma}+2|\mathsf{V}|)<\delta$, and we therefore conclude from (6.7) that

$$\mathsf{d}\big(\,UL_n(\mathbf{ul})\,,\,L_{n+|\widehat{\mathbf{u}}|}\overline{\mathbf{u}}\,\big)<\frac{\varepsilon}{2}\,. \tag{6.11}$$

Finally, combining (6.8) and (6.11) shows that $\mathsf{d}\big(\,UL_n(\mathbf{uk})\,,\,UM_n(\mathbf{ul})\,\big)\leq\mathsf{d}\big(\,UL_n(\mathbf{uk})\,,\,UL_n(\mathbf{ul})\,\big)+\mathsf{d}\big(\,UL_n(\mathbf{ul})\,,\,UL_{n+|\widehat{\mathbf{u}}|}\overline{\mathbf{u}}\,\big)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$.

(2) Since φ is Hölder continuous there are constant c and a with $c,a>0$ such that $|\varphi(\mathbf{i})-\varphi(\mathbf{j})|\leq c\,\mathsf{d}_\gamma(\mathbf{i},\mathbf{j})^a$ for all $\mathbf{i},\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$. It follows from this and (6.6) that

$$\sup_{\mathbf{u}\in\Sigma_{\mathsf{G}}^n}\ \sup_{\substack{\mathbf{k},\mathbf{l}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{k})\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{l})}}\left|\int\varphi\,d(L_n(\mathbf{uk}))-\int\varphi\,d(M_n(\mathbf{ul}))\right|$$

$$=\sup_{\mathbf{u}\in\Sigma_{\mathsf{G}}^n}\ \sup_{\substack{\mathbf{k},\mathbf{l}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{k})\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{l})}}\left|\int\varphi\,d(L_n(\mathbf{uk}))-\int\varphi\,d(L_{n+|\widehat{\mathbf{u}}|}\overline{\mathbf{u}})\right|$$

$$\leq\sup_{\mathbf{u}\in\Sigma_{\mathsf{G}}^n}\ \sup_{\substack{\mathbf{k},\mathbf{l}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{k})\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{l})}}\left(\frac{1}{n}\sum_{i=0}^{n-1}|\varphi(S^i(\mathbf{uk}))-\varphi(S^i\overline{\mathbf{u}})|+2\frac{|\mathsf{V}|}{n}\|\varphi\|_\infty\right)$$

$$\leq\sup_{\mathbf{u}\in\Sigma_{\mathsf{G}}^n}\ \sup_{\substack{\mathbf{k},\mathbf{l}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{k})\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{l})}}\left(\frac{1}{n}\sum_{i=0}^{n-1}c\,\mathsf{d}_\gamma\big(\,S^i(\mathbf{uk}))\,,\,S^i\overline{\mathbf{u}}\,\big)^a+2\frac{|\mathsf{V}|}{n}\|\varphi\|_\infty\right)$$

$$\leq\sup_{\mathbf{u}\in\Sigma_{\mathsf{G}}^n}\ \sup_{\substack{\mathbf{k},\mathbf{l}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{k})\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{l})}}\left(\frac{1}{n}\sum_{i=0}^{n-1}c\,\gamma^{a|S^i(\mathbf{ul})\wedge S^i\overline{\mathbf{u}}|}+2\frac{|\mathsf{V}|}{n}\|\varphi\|_\infty\right)$$

$$\leq\sup_{\mathbf{u}\in\Sigma_{\mathsf{G}}^n}\ \sup_{\substack{\mathbf{k},\mathbf{l}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{k})\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{l})}}\left(\frac{1}{n}\sum_{i=0}^{n-1}c\,\gamma^{a(n-i)}+2\frac{|\mathsf{V}|}{n}\|\varphi\|_\infty\right)$$

$$\leq\frac{c}{n}\,\frac{1}{1-\gamma^a}+2\frac{|\mathsf{V}|}{n}\|\varphi\|_\infty\to 0\ \text{ as } n\to\infty.$$

This completes the proof. $\square$

Lemma 6.2. *Let X be an inner product space with inner product $\langle\cdot|\cdot\rangle$ and let $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ be continuous with respect to the weak topology. Fix a Hölder continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ and let s be a real number and $q\in X$. We have*

$$\sup_{\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}}\left|\left(\sup_{\mathbf{u}\in[\overline{\mathbf{i}|n}]}\langle q|UL_n\mathbf{u}\rangle+s\right)\int\varphi\,d\big(L_n\big(\overline{\mathbf{i}|n}\big)\big)-\big(\langle q|UM_n\mathbf{i}\rangle+s\big)\int\varphi\,d(M_n\mathbf{i})\right|\to 0\quad\text{as } n\to\infty.$$

Proof
We note that if $\mathbf{u}\in\Sigma_{\mathsf{G}}^{n}$ and $\mathbf{k},\mathbf{l},\mathbf{m}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{k})$, $\mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{l})$ and $\mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{m})$ then Cauchy–Schwarz inequality implies that

$$\begin{aligned}
&\left|\big(\langle q|UL_n(\mathbf{um})\rangle+s\big)\int\varphi\,d(L_n(\mathbf{uk}))-\big(\langle q|UM_n(\mathbf{ul})\rangle+s\big)\int\varphi\,d(M_n(\mathbf{ul}))\right|\\
&\qquad\le\left|\big(\langle q|UL_n(\mathbf{um})\rangle+s\big)\int\varphi\,d(L_n(\mathbf{uk}))-\big(\langle q|UM_n(\mathbf{ul})\rangle+s\big)\int\varphi\,d(L_n(\mathbf{uk}))\right|\\
&\qquad\quad+\left|\big(\langle q|UM_n(\mathbf{ul})\rangle+s\big)\int\varphi\,d(L_n(\mathbf{uk}))-\big(\langle q|UM_n(\mathbf{ul})\rangle+s\big)\int\varphi\,d(M_n(\mathbf{ul}))\right|\\
&\qquad\le\big(|\langle q|UL_n(\mathbf{um})-UM_n(\mathbf{ul})\rangle|+|s|\big)\|\varphi\|_\infty\\
&\qquad\quad+\big(|\langle q|UM_n(\mathbf{ul})\rangle|+|s|\big)\left|\int\varphi\,d(L_n(\mathbf{uk}))-\int\varphi\,d(M_n(\mathbf{ul}))\right|\\
&\qquad\le\big(\|q\|\,\|UL_n(\mathbf{um})-UM_n(\mathbf{ul})\|+|s|\big)\|\varphi\|_\infty\\
&\qquad\quad+\big(\|q\|\,\|U\|_\infty+|s|\big)\left|\int\varphi\,d(L_n(\mathbf{uk}))-\int\varphi\,d(M_n(\mathbf{ul}))\right|,
\end{aligned}\tag{6.12}$$

where $\|U\|_\infty<\infty$ since $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ is continuous and $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ is compact with respect to the weak topology. The desired result follows immediately from Lemma 6.1 and (6.12) . $\square$

Lemma 6.3. *Let X be an inner product space with inner product $\langle\cdot|\cdot\rangle$ and let $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ be continuous with respect to the weak topology. Let $q\in X$. We have*

$$\sup_{\mathbf{u}\in\Sigma_{\mathsf{G}}^{n}}\ \sup_{\substack{\mathbf{k},\mathbf{l}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{k})\\ \mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{l})}}\big|\langle q|UL_n(\mathbf{uk})\rangle-\langle q|UL_n(\mathbf{ul})\rangle\big|\to 0\quad\text{as } n\to\infty.$$

Proof
We note that if $\mathbf{u}\in\Sigma_{\mathsf{G}}^{n}$ and $\mathbf{k},\mathbf{l}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{k})$ and $\mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{l})$, then $M_n(\mathbf{uk})=M_n(\mathbf{ul})$, and Cauchy–Schwarz inequality therefore implies that

$$\begin{aligned}
\big|\langle q|UL_n(\mathbf{uk})\rangle-\langle q|UL_n(\mathbf{ul})\rangle\big|&\le\|q\|\,\|UL_n(\mathbf{uk})-UL_n(\mathbf{ul})\|\\
&\le\|q\|\Big(\|UL_n(\mathbf{uk})-UM_n(\mathbf{uk})\|+\|UM_n(\mathbf{uk})-UM_n(\mathbf{ul})\|\\
&\qquad\qquad+\|UM_n(\mathbf{ul})-UL_n(\mathbf{ul})\|\Big)\\
&\le\|q\|\Big(\|UL_n(\mathbf{uk})-UM_n(\mathbf{uk})\|+\|UM_n(\mathbf{ul})-UL_n(\mathbf{ul})\|\Big).
\end{aligned}$$

The desired result follows immediately from this and Lemma 6.1. $\square$

7. Proof of Theorems 4.1 and Theorem 4.2, Part 2: The proofs.

The purpose of this section is to use the continuity results in Section 6 and Varadhan's integral lemma from large deviation theory to prove Theorem 4.1. We first introduce the sequence of measures (see (7.3) below) that we will apply Varadhan's integral lemma to.

The measure Π. Let $B = (b_{\mathsf{i},\mathsf{j}})_{\mathsf{i},\mathsf{j}\in\mathsf{V}}$ denote the matrix defined by

$$b_{\mathsf{i},\mathsf{j}} = |\mathsf{E}_{\mathsf{i},\mathsf{j}}|\,;$$

recall, that $\mathsf{E}_{\mathsf{i},\mathsf{j}}$ denotes the set of edges from i to j. We denote the spectral radius of B by λ. Since $\mathsf{G} = (\mathsf{V},\mathsf{E})$ is strongly connected, we conclude that the matrix B is irreducible, and it therefore follows from the Perron-Frobenius theorem that there is a unique right eigen-vector $\mathbf{u} = (u_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ of B with eigen-value λ and a unique left eigen-vector $\mathbf{v} = (v_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ of B with eigen-value λ, i.e.

$$\begin{aligned} \mathbf{u}B &= \lambda\mathbf{u}\,,\\ B\mathbf{v} &= \lambda\mathbf{v}\,, \end{aligned} \tag{7.1}$$

such that $u_{\mathsf{i}}, v_{\mathsf{i}} > 0$ for all i, $\sum_{\mathsf{i}} u_{\mathsf{i}}v_{\mathsf{i}} = 1$ and $\sum_{\mathsf{i}} u_{\mathsf{i}} = 1$. For $\mathsf{e}\in\mathsf{V}$, write $p_{\mathsf{e}} = v_{\mathsf{i}(\mathsf{e})}^{-1}\,v_{\mathsf{t}(\mathsf{e})}\,\lambda^{-1}$. A simple calculation shows that $\sum_{\mathsf{e}\in\mathsf{E}_{\mathsf{i}}} p_{\mathsf{e}} = 1$ for all i and that $\sum_{\mathsf{i}}\sum_{\mathsf{e}\in\mathsf{E}_{\mathsf{i},\mathsf{j}}} u_{\mathsf{i}}\,v_{\mathsf{i}}\,p_{\mathsf{e}} = u_{\mathsf{j}}\,v_{\mathsf{j}}$ for all j. It follows from this that there is a unique Borel probability measure $\Pi\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ such that

$$\begin{aligned} \Pi[\mathbf{i}] &= u_{\mathsf{i}(\mathsf{e}_1)}\,v_{\mathsf{i}(\mathsf{e}_1)}\,p_{\mathsf{e}_1}\,\cdots\,p_{\mathsf{e}_n}\\ &= u_{\mathsf{i}(\mathsf{e}_1)}\,v_{\mathsf{t}(\mathsf{e}_n)}\,\lambda^{-n}\\ &= u_{\mathsf{i}(\mathbf{i})}\,v_{\mathsf{t}(\mathbf{i})}\,\lambda^{-n} \end{aligned} \tag{7.2}$$

for all $\mathbf{i} = \mathsf{e}_1\ldots\mathsf{e}_n\in\Sigma_{\mathsf{G}}^{*}$.

The measure Π_n. Finally, for a positive integer n, we define the probability measures $\Pi_n\in\mathcal{P}(\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}))$ by

$$\Pi_n = \Pi\circ M_n^{-1}\,; \tag{7.3}$$

recall, that the map $M_n:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ is defined in (6.1).

We now turn towards the proof of the main result in this section, namely, Theorem 4.1. The proof of Theorem 4.1 uses Varadhan's integral lemma from large deviation theory and the fact that the sequence $(\Pi_n)_n$ has the large deviation property. We now state the definition of the large deviation property and Varadhan's integral lemma.

Definition. *Let X be a complete separable metric space and let $(P_n)_n$ be a sequence of probability measures on X. Let $(a_n)_n$ be a sequence of positive numbers with $a_n\to\infty$ and let $I: X\to[0,\infty]$ be a lower semi-continuous function with compact level sets. The sequence $(P_n)_n$ is said to have the large deviation property with rate constants $(a_n)_n$ and rate function I if the following two condistions hold:*

(i) *For each closed subset K of X, we have*

$$\limsup_n \frac{1}{a_n}\log P_n(K) \le -\inf_{x\in K} I(x)\,;$$

(ii) *For each open subset G of X, we have*

$$\liminf_n \frac{1}{a_n}\log P_n(G) \ge -\inf_{x\in G} I(x)\,.$$

Theorem 7.1. Varadhan's integral lemma. *Let X be a complete separable metric space and let $(P_n)_n$ be a sequence of probability measures on X. Assume that the sequence $(P_n)_n$ has the large deviation property with rate constants $(a_n)_n$ and rate function I. Let $F : X \to \mathbb{R}$ be a continuous function satisfying the following two conditions:*

(i) *For all n, we have*

$$\int \exp(a_n F)\, dP_n < \infty\,.$$

(ii) *We have*

$$\lim_{M\to\infty} \limsup_n \frac{1}{a_n} \log \int_{\{M\le F\}} \exp(a_n F)\, dP_n = -\infty\,.$$

(Observe that the Conditions (i)–(ii) are satisfied if F is bounded.) Then we have

$$\lim_n \frac{1}{a_n} \log \int \exp(a_n F)\, dP_n = -\inf_{x\in X} (I(x) - F(x))\,.$$

Proof
This follows from [El, Theorem II.7.1] or [DeZe, Theorem 4.3.1]. □

The next result says that the sequence $(\Pi_n)_n$ has the large deviation property.

Theorem 7.2. *Define $I : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to [0,\infty]$ by*

$$I(\mu) = \begin{cases} \log\lambda - h(\mu) & \text{for } \mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}});\\ \infty & \text{for } \mu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \setminus \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}). \end{cases}$$

Then the sequence $(\Pi_n)_n$ has the large deviation deviation property with respect to the sequence $(n)_n$ and rate function I.

Proof
Let

$$\Gamma_n = \Pi \circ L_n^{-1}\,.$$

It follows from Orey & Pelikan [OrPe1,OrPe2] (see also [JiQiQi, Remark 7.2.2]) that the sequence $(\Gamma_n)_n$ has the large deviation property with respect to the sequence $(n)_n$ and rate function I. Using the results from Section 6 (saying that the measures $\Pi_n = \Pi \circ M_n^{-1}$ and $\Gamma_n = \Pi \circ L_n^{-1}$ are "close") it can be shown that that this implies that the sequence $(\Pi_n)_n$ also has the large deviation property with respect to the sequence $(n)_n$ and rate function I; the reader is referred to [MiOl2] for a detailled proof of this. □

We now turn towards the proof of Theorem 4.1.

Theorem 7.3. *Let X be an inner product space with inner product $\langle\cdot|\cdot\rangle$ and let $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Fix a Hölder continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ and let s be a real number and $q \in X$. For each positive integer n and each $\mathbf{i} \in \Sigma_{\mathsf{G}}^n$, we write*

$$u_{\mathbf{i}} = \sup_{\mathbf{u}\in[\mathbf{i}]} \langle q|UL_n\mathbf{u}\,\rangle\,,$$

$$s_{\mathbf{i}} = \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^k\mathbf{u}\,.$$

Then

$$\liminf_n \frac{1}{n} \log \sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} \exp((u_{\mathbf{i}} + s)s_{\mathbf{i}})k = \sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})} \left(h(\mu) + (\langle q|U\mu\rangle + s) \int \varphi\, d\mu \right),$$

$$\limsup_n \frac{1}{n} \log \sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} \exp((u_{\mathbf{i}} + s)s_{\mathbf{i}}) = \sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})} \left(h(\mu) + (\langle q|U\mu\rangle + s) \int \varphi\, d\mu \right).$$

Proof
For brevity write $t_{\mathbf{i}} = \exp((u_i + s)s_{\mathbf{i}})$. We also define $F : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to \mathbb{R}$ by

$$F(\mu) = (\langle q|U\mu\rangle + s)\int \varphi\, d\mu\,.$$

Observe that since φ and U are bounded, i.e. $\|\varphi\|_\infty < \infty$ and $\|U\|_\infty < \infty$, we conclude that $\|F\|_\infty \leq (\|q\|\,\|U\|_\infty + |s|)\|\varphi\|_\infty < \infty$.

Next, we prove the following claim.

Claim 1. Then there is a constant c such that the following holds: for all $\varepsilon > 0$, there is a positive integer N_ε, such that if $n \geq N_\varepsilon$, then

$$\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} t_{\mathbf{i}} \leq c\,\lambda^n \quad e^{n\varepsilon} \quad \int \exp(nF)\, d\Pi_n\,, \tag{7.4}$$

$$\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} t_{\mathbf{i}} \geq \frac{1}{c}\,\lambda^n\; e^{-n\varepsilon} \quad \int \exp(nF)\, d\Pi_n\,. \tag{7.5}$$

Proof of Claim 1. For each positive integer n, we clearly have

$$\begin{aligned}\int t_{\mathbf{i}|n}\, d\Pi(\mathbf{i}) &= \sum_{\mathbf{k}\in\Sigma_{\mathsf{G}}^n} \int_{[\mathbf{k}]} t_{\mathbf{i}|n}\, d\Pi(\mathbf{i})\\ &= \sum_{\mathbf{k}\in\Sigma_{\mathsf{G}}^n} t_{\mathbf{k}}\;\Pi\big(\,[\mathbf{k}]\,\big)\\ &= \sum_{\mathbf{k}\in\Sigma_{\mathsf{G}}^n} t_{\mathbf{k}}\, u_{\mathsf{i}(\mathbf{k})}\, v_{\mathsf{t}(\mathbf{k})}\,\lambda^{-n}\,.\end{aligned} \tag{7.6}$$

Next, it follows from the Principle of Bounded Distortion (see, for example, [Bar,Fa]) that there is a constant $C > 0$ such that if $n \in \mathbb{N}$, $\mathbf{i} \in \Sigma_{\mathsf{G}}^n$ and $\mathbf{u}, \mathbf{v} \in [\mathbf{i}]$, then $|\sum_{k=0}^{n-1}\varphi S^k\mathbf{u} - \sum_{k=0}^{n-1}\varphi S^k\mathbf{v}| \leq C$. In particular, this implies that for all positive integers n and all $\mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$, we have $|(u_{\mathbf{i}} + s)s_{\mathbf{i}} - (u_{\mathbf{i}} + s)\sum_{k=0}^{n-1}\varphi S^k\bar{\mathbf{i}}| \leq (|u_{\mathbf{i}}| + |s|)C = (|\sup_{\mathbf{u}\in[\mathbf{i}]}\langle q|UL_n\mathbf{u}\,\rangle| + |s|)c_0 \leq (\|q\|\,\|U\|_\infty + |s|)C$, whence

$$\begin{aligned}&\frac{1}{c_0}\exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]}\langle q|UL_n\mathbf{u}\,\rangle + s\right)\sum_{k=0}^{n-1}\varphi S^k\bar{\mathbf{i}}\right) \leq t_{\mathbf{i}}\,,\\ &t_{\mathbf{i}} \leq c_0\exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]}\langle q|UL_n\mathbf{u}\,\rangle + s\right)\sum_{k=0}^{n-1}\varphi S^k\bar{\mathbf{i}}\right),\end{aligned} \tag{7.7}$$

for all positive integers n and all $\mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$ where $c_0 = \exp(\|q\|\,\|U\|_\infty + |s|)C$.

We also note that it follows from Lemma 6.2 that there is a positive integer N_ε such that

$$\begin{aligned}&\left(\sup_{\mathbf{u}\in[\overline{\mathbf{i}|n}]}\langle q|UL_n\mathbf{u}\rangle + s\right)\int\varphi\, d\big(L_n\big(\overline{\mathbf{i}|n}\big)\big) \leq \varepsilon + (\langle q|UM_n\mathbf{i}\rangle + s)\int\varphi\, d(M_n\mathbf{i})\,,\\ &(\langle q|UM_n\mathbf{i}\rangle + s)\int\varphi\, d(M_n\mathbf{i}) \leq \varepsilon + \left(\sup_{\mathbf{u}\in[\overline{\mathbf{i}|n}]}\langle q|UL_n\mathbf{u}\rangle + s\right)\int\varphi\, d\big(L_n\big(\overline{\mathbf{i}|n}\big)\big)\,,\end{aligned} \tag{7.8}$$

for all integers n with $n \geq N_\varepsilon$ and all $\mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$.

Finally, we can find a constant $w > 0$ such that $\frac{1}{w} \leq u_{\mathsf{i}}v_{\mathsf{i}} \leq w$ for all i. Now put $c = c_0 w$.

It follows from (7.6), (7.7) and (7.8) that if n is a positive integer with $n \geq N_\varepsilon$, then we have

$$\begin{aligned}\sum_{\mathbf{k}\in\Sigma_{\mathsf{G}}^n} t_{\mathbf{k}} &\leq w\,\lambda^n \int t_{\mathbf{i}|n}\,d\Pi(\mathbf{i})\\
&\leq c\,\lambda^n \int \exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}|n]}\langle q|UL_n\mathbf{u}\rangle + s\right)\sum_{k=0}^{n-1}\varphi S^k\big(\overline{\mathbf{i}|n}\big)\right)d\Pi(\mathbf{i})\\
&= c\,\lambda^n \int \exp\left(n\left(\sup_{\mathbf{u}\in[\overline{\mathbf{i}|n}]}\langle q|UL_n\mathbf{u}\rangle + s\right)\int \varphi\,d\big(L_n\big(\overline{\mathbf{i}|n}\big)\big)\right)d\Pi(\mathbf{i})\\
&\leq c\,\lambda^n \int \exp\left(n\left(\varepsilon + (\langle q|UM_n\mathbf{i}\rangle + s)\int \varphi\,d(M_n\mathbf{i})\right)\right)d\Pi(\mathbf{i})\\
&= c\,\lambda^n\,e^{n\varepsilon}\int \exp\left(nF(M_n\mathbf{i})\right)\,d\Pi(\mathbf{i})\,.\end{aligned}$$

This proves inequality (7.4). Inequality (7.5) is proved similarly. This completes the proof of Claim 1.

We now turn towards the proof the statement in the theorem. Let $\varepsilon > 0$. We first observe that it follows immediately from Claim 1 that

$$\begin{aligned}\liminf_n \frac{1}{n}\log\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} t_{\mathbf{i}} &\geq \log\lambda - \varepsilon + \liminf_n \frac{1}{n}\log\int \exp\left(nF\right)\,d\Pi_n\,,\\
\limsup_n \frac{1}{n}\log\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} t_{\mathbf{i}} &\leq \log\lambda + \varepsilon + \limsup_n \frac{1}{n}\log\int \exp\left(nF\right)\,d\Pi_n\,.\end{aligned}\tag{7.9}$$

Since inequalities (7.9) hold for all $\varepsilon > 0$, we now conclude that

$$\begin{aligned}\liminf_n \frac{1}{n}\log\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} t_{\mathbf{i}} &\geq \log\lambda + \liminf_n \frac{1}{n}\log\int \exp\left(nF\right)\,d\Pi_n\,,\\
\limsup_n \frac{1}{n}\log\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} t_{\mathbf{i}} &\leq \log\lambda + \limsup_n \frac{1}{n}\log\int \exp\left(nF\right)\,d\Pi_n\,.\end{aligned}\tag{7.10}$$

Next, define $I : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to [0,\infty]$ by

$$I(\mu) = \begin{cases}\log\lambda - h(\mu) & \text{for } \mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}});\\ \infty & \text{for } \mu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\setminus\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}).\end{cases}$$

It follows from Theorem 7.2 that the sequence $(\Pi_n)_n \subseteq \mathcal{P}\big(\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big)$ has the large deviation property with respect to the sequence $(n)_n$ and rate function I, and we therefore conclude from Theorem 7.1 that

$$\begin{aligned}\lim_n \frac{1}{n}\log\int \exp\left(nF\right)\,d\Pi_n &= -\inf_{\nu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})}\left(I(\nu) - F(\nu)\right)\\
&= -\inf_{\nu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}\left(I(\nu) - F(\nu)\right).\end{aligned}\tag{7.11}$$

Finally, we introduce the following notation and prove the two claims below. Write

$$\begin{aligned}\underline{Q}_q^U(\varphi) &= \liminf_n \frac{1}{n}\log\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n}\exp((u_{\mathbf{i}}+s)s_{\mathbf{i}})\,,\\
\overline{Q}_q^U(\varphi) &= \limsup_n \frac{1}{n}\log\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n}\exp((u_{\mathbf{i}}+s)s_{\mathbf{i}})\,.\end{aligned}$$

Claim 2. We have $\underline{Q}_q^U(\varphi) \geq \sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(h(\mu) + (\langle q|U\mu\rangle + s)\int\varphi\,d\mu)$.

Proof of Claim 2. Combining (7.10) and (7.11) now yields

$$\begin{aligned}\underline{Q}_q^U(\varphi) &= \liminf_n \frac{1}{n}\log\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} t_{\mathbf{i}}\\ &\geq \log\lambda + \liminf_n \frac{1}{n}\log\int\exp\left(nF\right)\,d\Pi_n\\ &\geq \log\lambda - \inf_{\nu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(I(\nu) - F(\nu))\\ &= \log\lambda + \sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(F(\mu) - I(\mu))\\ &= \sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}\left(h(\mu) + (\langle q|U\mu\rangle + s)\int\varphi\,d\mu\right).\end{aligned}$$

This completes the proof of Claim 2.

Claim 3. We have $\overline{Q}_q^U(\varphi) \leq \sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(h(\mu) + (\langle q|U\mu\rangle + s)\int\varphi\,d\mu)$.

Proof of Claim 3. The proof of Claim 3 is similar to the proof of Claim 2 and is therefore omitted. This completes the proof of Claim 3.

The statement in Theorem 7.3 now follows immediately by combining Claim 2 and Claim 3. □

We can now prove Theorem 4.1 and Theorem 4.2. We first prove Theorem 4.1; for the benefit of the reader we have decided to repeat the statement of Theorem 4.1.

Theorem 4.1. The variational principle for τ. *Let X be an inner product space with inner product $\langle\cdot|\cdot\rangle$ and let $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Fix a Hölder continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ with $\varphi < 0$ and let $q \in X$.*

(1) *We have*

$$\tau(q) = \sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}\left(-\frac{h(\mu)}{\int\varphi\,d\mu} - \langle q\big|U\mu\rangle\right).$$

(2) *We have*

$$\sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}\left(h(\mu) + \left(\langle q\big|U\mu\rangle + \tau(q)\right)\int\varphi\,d\mu\right) = 0\,.$$

Proof

(1) For brevity write

$$u = \sup_{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})}\left(-\frac{h(\mu)}{\int\varphi\,d\mu} - \langle q|U\mu\rangle\right).$$

We will also use the same notation as in Theorem 7.3. Namely, for each positive integer n and each $\mathbf{i} \in \Sigma_{\mathsf{G}}^n$, we write $u_{\mathbf{i}} = \sup_{\mathbf{u}\in[\mathbf{i}]}\langle q|UL_n\mathbf{u}\rangle$ and $s_{\mathbf{i}} = \sup_{\mathbf{u}\in[\mathbf{i}]}\exp\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}$. Since $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{N}$ is continuous with $\varphi < 0$, we also conclude that there is a constant $c > 0$, such that

$$\varphi \leq -c\,.$$

We must now prove the following two inequalities

$$\tau(q) \leq u\,, \tag{7.12}$$

$$u \leq \tau(q)\,. \tag{7.13}$$

Proof of (7.12). We must prove that if $s > u$, then

$$\sum_{\mathbf{i}} \exp((u_{\mathbf{i}} + s)s_{\mathbf{i}}) < \infty\,.$$

Let $s > u$ and write $\varepsilon = \frac{s-u}{3} > 0$. It follows from the definition of u that if $\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}})$, then we have $-\frac{h(\mu)}{\int \varphi\, d\mu} - \langle q|U\mu\rangle < u + \varepsilon$, whence $h(\mu) + \langle q|U\mu\rangle \int \varphi\, d\mu < -(u+\varepsilon)\int \varphi\, d\mu$ where we have used the fact that $\int \varphi\, d\mu < 0$ because $\varphi < 0$. This implies that if $\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}})$, then

$$\begin{aligned} h(\mu) + (\langle q|U\mu\rangle + s)\int \varphi\, d\mu &= h(\mu) + (\langle q|U\mu\rangle + (u + 3\varepsilon))\int \varphi\, d\mu \\ &\le 2\varepsilon \int \varphi\, d\mu \\ &\le -2\varepsilon c\,. \end{aligned}$$

We deduce from this inequality and Theorem 7.3 that

$$\begin{aligned} &\limsup_n \frac{1}{n} \log \sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} \exp((u_{\mathbf{i}} + s)s_{\mathbf{i}}) \\ &\qquad = \sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})} \left(h(\mu) + (\langle q|U\mu\rangle + s)\int \varphi\, d\mu \right) \qquad \text{[by Theorem 7.3]} \\ &\qquad \le -2\varepsilon c \\ &\qquad < -\varepsilon c\,. \end{aligned} \tag{7.14}$$

Inequality (7.14) shows that there is an integer N_0 such that $\frac{1}{n}\log\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} \exp((u_{\mathbf{i}} + s)s_{\mathbf{i}}) \le -\varepsilon c$ for all $n \ge N_0$, whence

$$\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} \exp((u_{\mathbf{i}} + s)s_{\mathbf{i}}) \le e^{-\varepsilon c n} \tag{7.15}$$

for all $n \ge N_0$. Using (7.15) we now conclude that

$$\begin{aligned} \sum_{\mathbf{i}} \exp((u_{\mathbf{i}} + s)s_{\mathbf{i}}) &= \sum_{n<N_0} \sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} \exp((u_{\mathbf{i}} + s)s_{\mathbf{i}}) \;+\; \sum_{n\ge N_0} \sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} \exp((u_{\mathbf{i}} + s)s_{\mathbf{i}}) \\ &\le \sum_{n<N_0} \sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n} \exp((u_{\mathbf{i}} + s)s_{\mathbf{i}}) \;+\; \sum_{n\ge N_0} e^{-\varepsilon c n} \\ &< \infty\,. \end{aligned}$$

This completes the proof of (7.12).

Proof of 7.13). We must prove that if $s < u$, then

$$\sum_{\mathbf{i}} \exp((u_{\mathbf{i}} + s)s_{\mathbf{i}}) = \infty\,.$$

Let $s < u$ and write $\varepsilon = \frac{u-s}{3} > 0$. It follows from the definition of u that there is a measure $\mu_\varepsilon \in \mathcal{P}_S(\Sigma^{\mathbb{N}})$, such that $u - \varepsilon \le -\frac{h(\mu_\varepsilon)}{\int \varphi\, d\mu_\varepsilon} - \langle q|U\mu_\varepsilon\rangle$, whence $h(\mu_\varepsilon) + \langle q|U\mu_\varepsilon\rangle \int \varphi\, d\mu_\varepsilon \ge -(u-\varepsilon)\int \varphi\, d\mu_\varepsilon$ where we have used the fact that $\int \varphi\, d\mu_\varepsilon < 0$ because $\varphi < 0$. This implies that

$$\begin{aligned} h(\mu_\varepsilon) + (\langle q|U\mu_\varepsilon\rangle + s)\int \varphi\, d\mu_\varepsilon &= h(\mu_\varepsilon) + (\langle q|U\mu_\varepsilon\rangle + (u - 3\varepsilon))\int \varphi\, d\mu_\varepsilon \\ &\ge -2\varepsilon \int \varphi\, d\mu_\varepsilon \\ &\ge 2\varepsilon c\,. \end{aligned}$$

We deduce from this inequality and Theorem 7.3 that

$$\begin{aligned}\limsup_n \frac{1}{n}\log\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n}\exp((u_{\mathbf{i}}+s)s_{\mathbf{i}})\\ &= \sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}\left(h(\mu)+(\langle q|U\mu\rangle+s)\int\varphi\,d\mu\right) \qquad \text{[by Theorem 7.3]}\\ &\geq h(\mu_\varepsilon)+(\langle q|U\mu_\varepsilon\rangle+s)\int\varphi\,d\mu_\varepsilon\\ &\geq 2\varepsilon c\\ &> \varepsilon c\,. \end{aligned} \tag{7.16}$$

Inequality (7.16) shows that there is an integer N_0 such that $\frac{1}{n}\log\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n}\exp((u_{\mathbf{i}}+s)s_{\mathbf{i}})\geq\varepsilon c$ for all $n\geq N_0$, whence

$$\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n}\exp((u_{\mathbf{i}}+s)s_{\mathbf{i}})\geq e^{\varepsilon cn} \tag{7.17}$$

for all $n\geq N_0$. Using (7.17) we now conclude that

$$\begin{aligned}\sum_{\mathbf{i}}\exp((u_{\mathbf{i}}+s)s_{\mathbf{i}}) &= \sum_{n<N_0}\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n}\exp((u_{\mathbf{i}}+s)s_{\mathbf{i}}) \;+\; \sum_{n\geq N_0}\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n}\exp((u_{\mathbf{i}}+s)s_{\mathbf{i}})\\ &\geq \sum_{n<N_0}\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n}\exp((u_{\mathbf{i}}+s)s_{\mathbf{i}}) \;+\; \sum_{n\geq N_0}e^{\varepsilon cn}\\ &= \infty\,.\end{aligned}$$

This completes the proof of (7.13).
(2) This statement follows easily from (1). □

Finally, we prove Theorem 4.2.

Proof of theorem 4.2
We denote the inner product in X by $\langle\cdot|\cdot\rangle$.
(1) Fix $q,p\in X$ and $s,t\in[0,1]$ with $s+t=1$ and let $\varepsilon>0$. Note that it follows from Lemma 6.3 that we can find a positive integer N_ε such that if $n\geq N_\varepsilon$ and $\mathbf{i}\in\Sigma_{\mathsf{G}}^n$, then $\langle q|UL_n\mathbf{u}\rangle\geq\langle q|UL_n\mathbf{v}\rangle-\frac{\varepsilon}{2}$ and $\langle p|UL_n\mathbf{u}\rangle\geq\langle p|UL_n\mathbf{w}\rangle-\frac{\varepsilon}{2}$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in[\mathbf{i}]$. Hence, if $n\geq N_\varepsilon$ and $\mathbf{i}\in\Sigma_{\mathsf{G}}^n$, then $\langle sq+tp|UL_n\mathbf{u}\rangle= s\langle q|UL_n\mathbf{u}\rangle+t\langle p|UL_n\mathbf{u}\rangle\geq s(\langle q|UL_n\mathbf{v}\rangle-\frac{\varepsilon}{2})+t(\langle p|UL_n\mathbf{w}\rangle-\frac{\varepsilon}{2})=s\langle q|UL_n\mathbf{v}\rangle+t\langle p|UL_n\mathbf{w}\rangle-\frac{\varepsilon}{2}$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in[\mathbf{i}]$. This implies that if $n\geq N_\varepsilon$ and $\mathbf{i}\in\Sigma_{\mathsf{G}}^n$, then

$$\sup_{\mathbf{u}\in[\mathbf{i}]}\langle sq+tp|UL_{|\mathbf{i}|}\mathbf{u}\rangle\geq s\sup_{\mathbf{u}\in[\mathbf{i}]}\langle q|UL_{|\mathbf{i}|}\mathbf{u}\rangle+t\sup_{\mathbf{u}\in[\mathbf{i}]}\langle p|UL_{|\mathbf{i}|}\mathbf{u}\rangle-\frac{\varepsilon}{2}$$

Writing

$$M_\varepsilon=\sum_{n<N_\varepsilon}\sum_{\mathbf{i}\in\Sigma_{\mathsf{G}}^n}\exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]}\langle sq+tp|UL_{|\mathbf{i}|}\mathbf{u}\rangle+s\tau(q)+t\tau(p)+\varepsilon\right)\sup_{\mathbf{u}\in[i]}\sum_{k=0}^{n-1}\varphi(S^k\mathbf{u})\right)$$

and using Hölder's inequality together with the fact that $\varphi < 0$, we therefore conclude that

$$\begin{aligned}
&\zeta^{\mathsf{co},U}_{sq+tp}(\varphi; s\tau(q)+t\tau(p)+\varepsilon)\\
&\quad= \sum_{\mathbf{i}} \exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]}\langle sq+tp|UL_{|\mathbf{i}|}\mathbf{u}\rangle + s\tau(q)+t\tau(p)+\varepsilon\right)\sup_{\mathbf{u}\in[i]}\sum_{k=0}^{|\mathbf{i}|-1}\varphi(S^k\mathbf{u})\right)\\
&\quad= M_\varepsilon + \sum_{n\geq N_\varepsilon}\sum_{\mathbf{i}\in\Sigma^n_{\mathsf{G}}} \exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]}\langle sq+tp|UL_{|\mathbf{i}|}\mathbf{u}\rangle + s\tau(q)+t\tau(p)+\varepsilon\right)\sup_{\mathbf{u}\in[i]}\sum_{k=0}^{n-1}\varphi(S^k\mathbf{u})\right)\\
&\quad\leq M_\varepsilon + \sum_{n\geq N_\varepsilon}\sum_{\mathbf{i}\in\Sigma^n_{\mathsf{G}}} \exp\Bigg(\Bigg(s\sup_{\mathbf{u}\in[\mathbf{i}]}\langle q|UL_{|\mathbf{i}|}\mathbf{u}\rangle + t\sup_{\mathbf{u}\in[\mathbf{i}]}\langle p|UL_{|\mathbf{i}|}\mathbf{u}\rangle\\
&\qquad\qquad\qquad + s\tau(q)+t\tau(p)+\tfrac{\varepsilon}{2}\Bigg)\sup_{\mathbf{u}\in[i]}\sum_{k=0}^{n-1}\varphi(S^k\mathbf{u})\Bigg)\\
&\quad= M_\varepsilon + \sum_{n\geq N_\varepsilon}\sum_{\mathbf{i}\in\Sigma^n_{\mathsf{G}}}\Bigg(\exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]}\langle q|UL_{|\mathbf{i}|}\mathbf{u}\rangle + \tau(q)+\tfrac{\varepsilon}{2}\right)\sup_{\mathbf{u}\in[i]}\sum_{k=0}^{|\mathbf{i}|-1}\varphi(S^k\mathbf{u})\right)^s\\
&\qquad\qquad\qquad \times\exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]}\langle p|UL_{|\mathbf{i}|}\mathbf{u}\rangle + \tau(p)+\tfrac{\varepsilon}{2}\right)\sup_{\mathbf{u}\in[i]}\sum_{k=0}^{|\mathbf{i}|-1}\varphi(S^k\mathbf{u})\right)^t\Bigg)\\
&\quad\leq M_\varepsilon + \left(\sum_{n\geq N_\varepsilon}\sum_{\mathbf{i}\in\Sigma^n_{\mathsf{G}}}\exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]}\langle q|UL_{|\mathbf{i}|}\mathbf{u}\rangle + \tau(q)+\tfrac{\varepsilon}{2}\right)\sup_{\mathbf{u}\in[i]}\sum_{k=0}^{|\mathbf{i}|-1}\varphi(S^k\mathbf{u})\right)\right)^s\\
&\qquad\qquad \times\left(\sum_{n\geq N_\varepsilon}\sum_{\mathbf{i}\in\Sigma^n_{\mathsf{G}}}\exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]}\langle p|UL_{|\mathbf{i}|}\mathbf{u}\rangle + \tau(p)+\tfrac{\varepsilon}{2}\right)\sup_{\mathbf{u}\in[i]}\sum_{k=0}^{|\mathbf{i}|-1}\varphi(S^k\mathbf{u})\right)\right)^t\\
&\quad\leq M_\varepsilon + \zeta^{\mathsf{co},U}_q(\varphi;\tau(q)+\tfrac{\varepsilon}{2})^s\ \zeta^{\mathsf{co},U}_p(\varphi;\tau(p)+\tfrac{\varepsilon}{2})^t
\end{aligned} \tag{7.18}$$

Since $\varepsilon > 0$, it follows from the definition of $\tau(q)$ and $\tau(p)$ that $\zeta^{\mathsf{co},U}_q(\varphi;\tau(q)+\frac{\varepsilon}{2}) < \infty$ and $\zeta^{\mathsf{co},U}_p(\varphi;\tau(p)+\frac{\varepsilon}{2}) < \infty$, and we therefore conclude from (7.18) that $\zeta^{\mathsf{co},U}_{sq+tp}(\varphi;s\tau(q)+t\tau(p)+\varepsilon) \leq M_\varepsilon + \zeta^{\mathsf{co},U}_q(\varphi;\tau(q)+\frac{\varepsilon}{2})^s\ \zeta^{\mathsf{co},U}_p(\varphi;\tau(p)+\frac{\varepsilon}{2})^t < \infty$. This inequality shows that $\tau(sq+tp) \leq s\tau(q)+t\tau(p)+\varepsilon$ for all $\varepsilon > 0$. Finally, letting ε tend to 0 gives $\tau(sq+tp) \leq s\tau(q)+t\tau(p)$.
(2) For a Hölder continuous function $f : \Sigma^{\mathbb{N}}_{\mathsf{G}} \to \mathbb{R}$, we write $P(f)$ for the pressure of f. Next, define $\Phi : \mathbb{R}^{M+1} \to \mathbb{R}$ by

$$\Phi(q,s) = \sup_{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}}_{\mathsf{G}})}\left(h(\mu) + (\langle q|U\mu\rangle + s)\int\varphi\,d\mu\right)$$

for $q \in \mathbb{R}^M$ and $s \in \mathbb{R}$. Observe, that since $U\mu = \frac{\int\psi\,d\mu}{\int\varphi\,d\mu}$ for $\mu \in \mathcal{P}_S(\Sigma^{\mathbb{N}}_{\mathsf{G}})$, we conclude that

$$\begin{aligned}
\Phi(q,s) &= \sup_{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}}_{\mathsf{G}})}\left(h(\mu) + \left(\left\langle q\left|\frac{\int\psi\,d\mu}{\int\varphi\,d\mu}\right.\right\rangle + s\right)\int\varphi\,d\mu\right)\\
&= \sup_{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}}_{\mathsf{G}})}\left(h(\mu) + \int(\langle q|\psi\rangle + s\varphi)\,d\mu\right).
\end{aligned} \tag{7.19}$$

It follows from (7.19) and the Variational Principle (see [Wa]) that $\Phi(q,s)$ equals the pressure of $\langle q|\psi\rangle + s\varphi$, i.e. $\Phi(q,s) = P(\langle q|\psi\rangle + s\varphi)$, and we therefore conclude from [Rue1] that Φ is real analytic. Since also Theorem 4.1 implies that $\Phi(q,\tau(q)) = 0$ for all q, we now conclude from the implicit function theorem that τ is real analytic.
(3) It follows from Theorem 4.1 that $\tau(0) = \sup_{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}}_{\mathsf{G}})} -\frac{h(\mu)}{\int\Lambda\,d\mu}$. The desired result follows from this since it is well-known that if the OSC is satisfied, then $\dim_{\mathsf{H}} K_{\mathsf{i}} = \sup_{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}}_{\mathsf{G}})} -\frac{h(\mu)}{\int\Lambda\,d\mu}$ for all

$\mathsf{i} \in \mathsf{V}$ (indeed, if we write $t = \dim_{\mathsf{H}} K_{\mathsf{i}}$, then it follows from Bowen's formula (see [Bar,Fa]) that $0 = P(t\Lambda) = \sup_{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(h(\mu) + t\int \Lambda\, d\mu)$, whence $t = \sup_{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})} -\frac{h(\mu)}{\int \Lambda\, d\mu}$). □

8. Proof of Theorem 4.3

The purpose of this section is to prove Theorem 4.3. We start by proving the following auxiliary result.

Proposition 8.1. *Let X be an inner product space and let $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Fix a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ with $\varphi < 0$. Let $\alpha \in X$ and let $f(\alpha)$ be the unique real number such that*

$$\limsup_{r \searrow 0} \sigma_{\mathsf{rad}}\big(\, \zeta_{B(\alpha,r)}^{\mathsf{dyn},U}(f(\alpha)\,\varphi;\cdot)\,\big) = 1\,.$$

Then

$$f(\alpha) \le \tau^*(\alpha)\,.$$

Proof
Let $\langle\cdot|\cdot\rangle$ denote the inner product in X. Fix $q \in X$. Next, let $\varepsilon > 0$ and fix $r > 0$ with $0 < r < \frac{\varepsilon}{\|q\|}$ where we write $\frac{\varepsilon}{\|q\|} = \infty$ if $q = 0$. We now have

$$\begin{aligned}
&\zeta_q^{\mathsf{co},U}(\varphi; -\langle q|\alpha\rangle + f(\alpha) - \varepsilon)\\
&\qquad = \sum_{\mathbf{i}} \exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]}\langle q|UL_{|\mathbf{i}|}\mathbf{u}\rangle - \langle q|\alpha\rangle + f(\alpha) - \varepsilon\right)\sup_{\mathbf{u}\in[i]}\sum_{k=0}^{|\mathbf{i}|-1}\varphi(S^k\mathbf{u})\right)\\
&\qquad \ge \sum_{\substack{\mathbf{i}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(\alpha,r)}} \exp\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]}\langle q|UL_{|\mathbf{i}|}\mathbf{u}\rangle - \langle q|\alpha\rangle + f(\alpha) - \varepsilon\right)\sup_{\mathbf{u}\in[\mathbf{i}]}\sum_{k=0}^{|\mathbf{i}|-1}\varphi(S^k\mathbf{u})\right).
\end{aligned} \tag{8.1}$$

Next, note that if $\mathbf{i} \in \Sigma_{\mathsf{G}}^*$ with $UL_{|\mathbf{i}|}[\mathbf{i}] \subseteq B(\alpha,r)$ and $\mathbf{u} \in [\mathbf{i}]$, then $UL_{|\mathbf{i}|}\mathbf{u} \in UL_{|\mathbf{i}|}[\mathbf{i}] \subseteq B(\alpha,r)$, whence $\|UL_{|\mathbf{i}|}\mathbf{u} - \alpha\| \le r$, and so $|\langle q|UL_{|\mathbf{i}|}\mathbf{u}\rangle - \langle q|\alpha\rangle| = |\langle q|UL_{|\mathbf{i}|}\mathbf{u} - \alpha\rangle| \le \|q\|\,\|UL_{|\mathbf{i}|}\mathbf{u} - \alpha\| \le \|q\|\,r$. Hence, if $\mathbf{i} \in \Sigma_{\mathsf{G}}^*$ with $UL_{|\mathbf{i}|}[\mathbf{i}] \subseteq B(\alpha,r)$, then (using the fact that $\varphi < 0$)

$$\begin{aligned}
\exp&\left(\left(\sup_{\mathbf{u}\in[\mathbf{i}]}\langle q|UL_{|\mathbf{i}|}\mathbf{u}\rangle - \langle q|\alpha\rangle + f(\alpha) - \varepsilon\right)\sup_{\mathbf{u}\in[\mathbf{i}]}\sum_{k=0}^{|\mathbf{i}|-1}\varphi(S^k\mathbf{u})\right)\\
&\qquad\qquad \ge \exp\left(\big(\|q\|r + f(\alpha) - \varepsilon\big)\sup_{\mathbf{u}\in[\mathbf{i}]}\sum_{k=0}^{|\mathbf{i}|-1}\varphi(S^k\mathbf{u})\right)\\
&\qquad\qquad = \exp\left(\big(f(\alpha) - (\varepsilon - \|q\|r)\big)\sup_{\mathbf{u}\in[\mathbf{i}]}\sum_{k=0}^{|\mathbf{i}|-1}\varphi(S^k\mathbf{u})\right).
\end{aligned} \tag{8.2}$$

Combining (8.1) and (8.2) now shows that

$$\zeta_q^{\mathsf{co},U}(\varphi; -\langle q|\alpha\rangle + f(\alpha) - \varepsilon) \ge \sum_{\substack{\mathbf{i}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(\alpha,r)}} \exp\left(\big(f(\alpha) - (\varepsilon - \|q\|r)\big)\sup_{\mathbf{u}\in[\mathbf{i}]}\sum_{k=0}^{|\mathbf{i}|-1}\varphi(S^k\mathbf{u})\right). \tag{8.3}$$

Next, we observe that if $\eta > 0$, then it follows immediately from the definition of $f(\alpha)$ that $\limsup_{r\searrow 0}\sigma_{\mathsf{rad}}\big(\,\zeta_{B(\alpha,r)}^{\mathsf{dyn},U}((f(\alpha) - \eta)\,\varphi;\cdot)\,\big) < 1$, and we can therefore find positive real numbers δ_η and

ρ_η such that if $0<\rho<\rho_\eta$, then $\sigma_{\mathsf{rad}}\big(\zeta^{\mathsf{dyn},U}_{B(\alpha,\rho)}((f(\alpha)-\eta)\,\varphi;\cdot)\big)<1-\delta_\eta$. In particular, this shows that if $0<\rho<\rho_\eta$ and $t>1-\delta_\eta$, then

$$\sum_n \frac{t^n}{n!}\sum_{\substack{\mathbf{i}\in\Sigma^n_{\mathsf{G}}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(\alpha,\rho)}}\exp\left(\big(f(\alpha)-\eta\big)\sup_{\mathbf{u}\in[\mathbf{i}]}\sum_{k=0}^{|\mathbf{i}|-1}\Phi(S^k\mathbf{u})\right)=\infty\,. \tag{8.4}$$

Since $\varepsilon-\|q\|r>0$, we can choset a positive real number ρ with $0<\rho<\min(\rho_{\varepsilon-\|q\|r},r)$. It now follows from (8.3) and (8.4) that

$$\begin{aligned}
&\zeta^{\mathsf{co},U}_q(\varphi;-\langle q|\alpha\rangle+f(\alpha)-\varepsilon)\\
&\qquad\ge\sum_{\substack{\mathbf{i}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(\alpha,r)}}\exp\left(\big(f(\alpha)-(\varepsilon-\|q\|r)\big)\sup_{\mathbf{u}\in[\mathbf{i}]}\sum_{k=0}^{|\mathbf{i}|-1}\varphi(S^k\mathbf{u})\right)\\
&\qquad\ge\sum_{\substack{\mathbf{i}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(\alpha,\rho)}}\exp\left(\big(f(\alpha)-(\varepsilon-\|q\|r)\big)\sup_{\mathbf{u}\in[\mathbf{i}]}\sum_{k=0}^{|\mathbf{i}|-1}\varphi(S^k\mathbf{u})\right)\\
&\qquad\qquad\qquad\qquad\text{[since } B(\alpha,\rho)\subseteq B(\alpha,r)\text{ because }\rho<r]\\
&\qquad=\sum_n\sum_{\substack{\mathbf{i}\in\Sigma^n_{\mathsf{G}}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(\alpha,\rho)}}\exp\left(\big(f(\alpha)-(\varepsilon-\|q\|r)\big)\sup_{\mathbf{u}\in[\mathbf{i}]}\sum_{k=0}^{n-1}\varphi(S^k\mathbf{u})\right)\\
&\qquad\ge\sum_n\frac{1}{n!}\sum_{\substack{\mathbf{i}\in\Sigma^n_{\mathsf{G}}\\ UL_{|\mathbf{i}|}[\mathbf{i}]\subseteq B(\alpha,\rho)}}\exp\left(\big(f(\alpha)-(\varepsilon-\|q\|r)\big)\sup_{\mathbf{u}\in[\mathbf{i}]}\sum_{k=0}^{n-1}\varphi(S^k\mathbf{u})\right)\\
&\qquad=\infty\,.\qquad\qquad\text{[by (8.4) because }\rho<\rho_{\varepsilon-\|q\|r}]
\end{aligned}\tag{8.5}$$

It follows from (8.5) that $-\langle q|\alpha\rangle+f(\alpha)-\varepsilon\le\sigma_{\mathsf{ab}}\big(\zeta^{\mathsf{co},U}_q(\varphi;\cdot)\big)=\tau(q)$, and so $f(\alpha)\le\langle q|\alpha\rangle+\tau(q)+\varepsilon$. Letting ε tend to 0 and taking infimum over all q, now shows that $f(\alpha)\le\inf_q(\langle q|\alpha\rangle+\tau(q))=\tau^*(\alpha)$. This completes the proof. □

We can now prove Theorem 4.3.

Proof of Theorem 4.3
Using Theorem C and Proposition 8.1 we conclude that

$$\begin{aligned}
f(C)&=\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu\in\overline{C}}}-\frac{h(\mu)}{\int\varphi\,d\mu} &&\text{[by Theorem C]}\\
&=\sup_{\alpha\in\overline{C}}\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma^{\mathbb{N}})\\ U\mu=\alpha}}-\frac{h(\mu)}{\int\varphi\,d\mu}\\
&=\sup_{\alpha\in\overline{C}}\ f(\alpha) &&\text{[by Theorem C]}\\
&\le\sup_{\alpha\in\overline{C}}\ \tau^*(\alpha)\,. &&\text{[by Proposition 8.1]}
\end{aligned}$$

This completes the proof. □

9. Proof of Theorem 4.4

The purpose of this section is to prove Theorem 4.4. We start by proving a simple , but useful, auxiliary result, namely, Lemma 9.1 below. Lemma 9.1 says that if $C \subset X$, then f and τ satisfy the multifractal formalism at C, i.e. $f(C) = \sup_{\alpha \in \overline{C}} \tau^*(\alpha)$, provided each $\alpha \in \overline{C} \cap U\big(\,\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big)$ with $\tau^*(\alpha) > -\infty$ is the image of a "Gibbs like" measure, i.e. there is a measure μ_α with a certain "Gibbs like" property (namely, property (ii) in Lemma 9.1) such that $\alpha = U\mu_\alpha$. This lemma plays a key role in the proof of Theorem 4.4.

Lemma 9.1. *Let X be an inner product space with inner product $\langle\cdot|\cdot\rangle$ and let $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Fix a Hölder continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ with $\varphi < 0$. Let $C \subseteq X$ be a subset of X and let $f(C)$ be the unique real number such that*

$$\limsup_{r \searrow 0} \sigma_{\mathsf{rad}}\big(\,\zeta_{B(C,r)}^{\mathsf{dyn},U}(f(C)\,\varphi;\cdot)\,\big) = 1\,.$$

Assume that for each $\alpha \in \overline{C} \cap U\big(\,\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big)$ with $\tau^*(\alpha) > -\infty$ there is a point $q_\alpha \in X$ and a measure $\mu_\alpha \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ such that

(i) $\alpha = U\mu_\alpha$;

(ii) $$\sup_{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})} \left(-\frac{h(\mu)}{\int \varphi\, d\mu} - \langle q_\alpha | U\mu \rangle \right) = -\frac{h(\mu_\alpha)}{\int \varphi\, d\mu_\alpha} - \langle q_\alpha | U\mu_\alpha \rangle.$$

Then

$$f(C) = \sup_{\alpha \in \overline{C}} \tau^*(\alpha)\,.$$

Proof

For $\alpha \in \overline{C} \cap U\big(\,\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big)$ with $\tau^*(\alpha) > -\infty$ we have (using Theorem 4.1 and Theorem C)

$$\begin{aligned}
\tau^*(\alpha) &= \inf_q (\langle q|\alpha\rangle + \tau(q)) \\
&\le \langle q_\alpha|\alpha\rangle + \tau(q_\alpha) \\
&= \langle q_\alpha|\alpha\rangle + \sup_{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})} \left(-\frac{h(\mu)}{\int \varphi\, d\mu} - \langle q_\alpha | U\mu \rangle \right) && \text{[by Theorem 4.1]} \\
&= \langle q_\alpha|U\mu_\alpha\rangle - \frac{h(\mu_\alpha)}{\int \varphi\, d\mu_\alpha} - \langle q_\alpha | U\mu_\alpha \rangle && \text{[by (i) and (ii)]} \\
&\le -\frac{h(\mu_\alpha)}{\int \varphi\, d\mu_\alpha} \\
&\le \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ U\mu \in \overline{C}}} -\frac{h(\mu)}{\int \varphi\, d\mu} \\
&= f(C)\,. && \text{[by Theorem C]} \qquad (9.1)
\end{aligned}$$

Of course, if $\alpha \in \overline{C} \cap U\big(\,\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big)$ with $\tau^*(\alpha) = -\infty$, then it is clear that $\tau^*(\alpha) \le f(C)$. It follows immediately from this and (9.1) that

$$\sup_{\alpha \in \overline{C} \cap U(\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}))} \tau^*(\alpha) \le f(C)\,.$$

Since it also follows from Theorem 4.3 that $f(C) \le \sup_{\alpha \in \overline{C}} \tau^*(\alpha) = \sup_{\alpha \in \overline{C} \cap U(\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}))} \tau^*(\alpha)$, we therefore conclude that $f(C) = \sup_{\alpha \in \overline{C}} \tau^*(\alpha)$. $\square$

We can now prove Theorem 4.4.

Proof of Theorem 4.4
(1) By Lemma 9.1 it suffices to show that for each $\alpha\in\overline{C}\cap U\big(\,\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big)$ with $\tau^*(\alpha)>-\infty$ there is a point $q_\alpha\in\mathbb{R}^M$ and a measure $\mu_\alpha\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ such that

(i) $\alpha=U\mu_\alpha$;

(ii) $\displaystyle\sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}\left(-\frac{h(\mu)}{\int\varphi\,d\mu}-\langle q_\alpha|U\mu\rangle\right)=-\frac{h(\mu_\alpha)}{\int\varphi\,d\mu_\alpha}-\langle q_\alpha|U\mu_\alpha\rangle.$

Define $\Phi:\mathbb{R}^{M+1}\to\mathbb{R}$ by

$$\Phi(q,s)=\sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}\left(h(\mu)+(\langle q|U\mu\rangle+s)\int\varphi\,d\mu\right)$$

for $q\in\mathbb{R}^M$ and $s\in\mathbb{R}$. Also, for $q\in\mathbb{R}^M$ and $s\in\mathbb{R}$, we write

$$\mathcal{E}_{q,s}=\left\{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\,\middle|\,\Phi(q,s)=h(\mu)+(\langle q|U\mu\rangle+s)\int\varphi\,d\mu\right\}.$$

We first show that $\mathcal{E}_{q,s}\neq\varnothing$. This is the statement of Claim 1 below.

Claim 1. For all $q\in\mathbb{R}^M$ and $s\in\mathbb{R}$, we have $\mathcal{E}_{q,s}\neq\varnothing$.

Proof of Claim 1. Fix $q\in\mathbb{R}^M$ and $s\in\mathbb{R}$ and define the map $H:\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to\mathbb{R}$ by $H(\mu)=h(\mu)+(\langle q|U\mu\rangle+s)\int\varphi\,d\mu$ for $\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. Since the entropy map $\mu\to h(\mu)$ is upper semi-continuous (see,for example, [Wa, Theorem 8.2]) and U is continuous, we conclude that H is upper semi-continuous. We therefore deduce from the compactness of $\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ with respect to the weak topology that the map H has a maximum, i.e. $\mathcal{E}_{q,s}\neq\varnothing$. This completes the proof of Claim 1.

Since $\mathcal{E}_{q,s}\neq\varnothing$, we can choose $\mu_{q,s}\in\mathcal{E}_{q,s}$.

Claim 2. For all q, we have

$$\sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}\left(-\frac{h(\mu)}{\int\varphi\,d\mu}-\langle q|U\mu\rangle\right)=-\frac{h(\mu_{q,\tau(q)})}{\int\varphi\,d\mu_{q,\tau(q)}}-\langle q|U\mu_{q,\tau(q)}\rangle\,.$$

Proof of Claim 2. Since $\Phi(q,\tau(q))=0$ (by Theorem 4.1), we deduce that if $\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$, then $h(\mu)+(\langle q|U\mu\rangle+\tau(q))\int\varphi\,d\mu\le\sup_{\nu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(h(\nu)+(\langle q|U\nu\rangle+\tau(q))\int\varphi\,d\nu)=\Phi(q,\tau(q))=0$. Dividing this inequality by $-\int\varphi\,d\mu$, clearly implies that

$$-\frac{h(\mu)}{\int\varphi\,d\mu}-\langle q|U\mu\rangle-\tau(q)\le 0\,. \tag{9.2}$$

In addition, since $\mu_{q,\tau(q)}\in\mathcal{E}_{q,\tau(q)}$ and $\Phi(q,\tau(q))=0$, we conclude that $0=\Phi(q,\tau(q))=h(\mu_{q,\tau(q)})+(\langle q|U\mu_{q,\tau(q)}\rangle+\tau(q))\int\varphi\,d\mu_{q,\tau(q)}$. Dividing this equality by $-\int\varphi\,d\mu_{q,\tau(q)}$, shows that

$$0=-\frac{h(\mu_{q,\tau(q)})}{\int\varphi\,d\mu_{q,\tau(q)}}-\langle q|U\mu_{q,\tau(q)}\rangle-\tau(q)\,. \tag{9.3}$$

Finally, combining (9.2) and (9.3), we deduce that if $\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$, then

$$-\frac{h(\mu)}{\int\varphi\,d\mu}-\langle q|U\mu\rangle\le-\frac{h(\mu_{q,\tau(q)})}{\int\varphi\,d\mu_{q,\tau(q)}}-\langle q|U\mu_{q,\tau(q)}\rangle\,.$$

The desired conclusion follows immediately from this inequality. This completes the proof of Claim 2.

Next, we compute the gradient $\nabla\tau(q)$ of τ at points q at which τ is differentiable. This is the statement of Claim 3. In Claim 3 we use the following notation, namely, for $i = 1, \ldots, M$, we write $D_i^-\tau(q)$ and $D_i^+\tau(q)$ for the left and right partial derivative of τ with respect to the i'th variable at the point q, respectively; note that since τ is convex (by Theorem 4.2), we conclude that the left and right partial derivatives of τ with respect to the i'th variable exist at all points. Also, if τ is partially differentiable with respect to the i'th variable at q, then we write $D_i\tau(q)$ for the partial derivative of τ with respect to the i'th variable at the point q.

Claim 3. For $i = 1, \ldots, M$ and all $q \in \mathbb{R}^M$, we have

$$D_i^-\tau(q) \leq -\langle e_i | U\mu_{q,\tau(q)}\rangle \leq D_i^+\tau(q) \tag{9.4}$$

where $e_i = (0, \ldots, 0, 1, 0 \ldots, 0)$ is the canonical basis vector in $\mathbb{R}^M$ with 1 in the i'th coordinate and 0's elsewhere. In particular, if τ is partially differentiable at q, then (9.4) implies that $D_i\tau(q) = -\langle e_i | U\mu_{q,\tau(q)}\rangle$, and so

$$\nabla\tau(q) = -U\mu_{q,\tau(q)}\,.$$

Proof of Claim 3. It follows Theorem 4.1 and Claim 2 that if $h \in \mathbb{R}$, then we have

$$\begin{aligned}\tau(q + he_i) - \tau(q) &= \sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}\left(-\frac{h(\mu)}{\int\varphi\,d\mu} - \langle q + he_i | U\mu\rangle\right) - \sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}\left(-\frac{h(\mu)}{\int\varphi\,d\mu} - \langle q | U\mu\rangle\right)\\ &\geq \left(-\frac{h(\mu_{q,\tau(q)})}{\int\varphi\,d\mu_{q,\tau(q)}} - \langle q + he_i | U\mu_{q,\tau(q)}\rangle\right) - \left(-\frac{h(\mu_{q,\tau(q)})}{\int\varphi\,d\mu_{q,\tau(q)}} - \langle q | U\mu_{q,\tau(q)}\rangle\right)\\ &= -h\langle e_i | U\mu_{q,\tau(q)}\rangle\,.\end{aligned} \tag{9.5}$$

For $h < 0$, (9.5) implies that $\frac{\tau(q+he_i)-\tau(q)}{h} \leq -\langle e_i | U\mu_{q,\tau(q)}\rangle$ and for $h > 0$, (9.5) implies that $-\langle e_i | U\mu_{q,\tau(q)}\rangle \leq \frac{\tau(q+he_i)-\tau(q)}{h}$. The result follows immediately from these inequalities. This completes the proof of Claim 3.

Let $\alpha \in \overline{C} \cap U\big(\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big)$ with $\tau^*(\alpha) > -\infty$. Since $\tau^*(\alpha) > -\infty$, it follows from the convexity of τ that there is a point $q_\alpha \in \mathbb{R}^M \cap Q$ such that $\alpha = -\nabla\tau(q_\alpha)$, see [Ro]. Now put $\mu_\alpha = \mu_{q_\alpha,\tau(q_\alpha)}$. We claim that μ_α satisfies (i) and (ii). Indeed, it follows from Claim 2 and Claim 3 that

$$\alpha = -\nabla\tau(q_\alpha) = U\mu_{q_\alpha,\tau(q_\alpha)} = U\mu_\alpha$$

and

$$\sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}\left(-\frac{h(\mu)}{\int\varphi\,d\mu} - \langle q_\alpha | U\mu\rangle\right) = -\frac{h(\mu_{q_\alpha,\tau(q_\alpha)})}{\int\varphi\,d\mu_{q_\alpha,\tau(q_\alpha)}} - \langle q_\alpha | U\mu_{q_\alpha,\tau(q_\alpha)}\rangle = -\frac{h(\mu_\alpha)}{\int\varphi\,d\mu_\alpha} - \langle q_\alpha | U\mu_\alpha\rangle\,.$$

This completes the proof of (1).
(2) This follows immediately by applying the statement in Part (1) to the set $C = \{\alpha\}$. □

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