

FRACTIONAL SPHERICAL RANDOM FIELDS

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Abstract

In this paper we study the solutions of different forms of fractional equations on the unit sphere $\mathbf{S}_1^2 \subset \mathbb{R}^3$ possessing the structure of time-dependent random fields. We study the correlation functions of the random fields emerging in the analysis of the solutions of the fractional equations and examine their long-range behaviour.

Keywords: fractional equations, spherical Brownian motion, subordinators, random fields, Laplace-Beltrami operators, spherical harmonics.

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1 Introduction

In this paper we deal with various forms of random fields on the unit sphere \mathbf{S}_1^2 indexed by the spherical Brownian motion. We restrict ourselves to isotropic random fields for which the expansion in terms of spherical harmonics holds (see [14] and the references therein). The explicit law of the Brownian motion on \mathbf{S}_1^2 was first obtained in [18]. For Brownian motion on \mathbf{S}_1^d , see [11, pag. 338]. Time-dependent random fields on the line or on arbitrary Euclidean spaces have been studied by several authors (see, for example, [12, 2, 13] and the references therein). We here study time-dependent random fields on the sphere \mathbf{S}_1^2 , governed by different stochastic differential equations.

We first study random fields emerging from the Cauchy problem

$$\begin{cases} \left(\gamma - \mathbb{D}_M + \frac{\partial^\beta}{\partial t^\beta} \right) X_t(x) = 0, & x \in \mathbf{S}_1^2, t > 0, 0 < \beta < 1, \gamma > 0 \\ X_0(x) = T(x), \end{cases} \quad (1.1)$$

where \mathbb{D}_M is a suitable differential operator defined below, $\frac{\partial^\beta}{\partial t^\beta}$ is the Dzerbayshan-Caputo fractional derivative. By $T(x)$, $x \in \mathbf{S}_1^2$ we denote an isotropic Gaussian field on the unit sphere. We are able to obtain the solution $X_t(x)$ of (1.1) and to show that its covariance function displays a long-memory behaviour.

We then consider the non-homogeneous fractional equation

$$(\gamma - \mathbb{D}_M)^\beta X(x) = T(x), \quad x \in \mathbf{S}_1^2, 0 < \beta < 1 \quad (1.2)$$

of which

$$\left(\gamma - \mathbb{D}_M - \varphi \frac{\partial}{\partial t} \right)^\beta X_t(x) = T_t(x), \quad x \in \mathbf{S}_1^2, t > 0, 0 < \beta < 1, \gamma > 0, \varphi \geq 0 \quad (1.3)$$

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is the time-dependent extension. We obtain a solution to (1.3) which is a random field on the sphere with covariance function with a short-range dependence.

The couple $(B_s(t), T(x + B_s(t)))$ describes a random motion on the unit-radius sphere with dynamics governed by fractional stochastic equations (1.1) and (1.3).

Random fields similar to those examined here are considered in the analysis of the cosmic microwave background radiation (CMB radiation). In this case, the correlation structure turns out to be very important as well as the angular power spectrum. The angular power spectrum plays a key role in the study of the corresponding random field. In particular, the high-frequency behaviour of the angular power spectrum is related to some anisotropies of the CMB radiation (see for example [9, 14]). Such relations have been also investigated in [10] where a coordinates change driven by a fractional equation has been considered.

Diffusions on the sphere arise in several contexts. At the cellular level, diffusion is an important mode of transport of substances. The cell wall is a lipid membrane and biological substances like lipids and proteins diffuse on it. In general biological membranes are curved surfaces. Spherical diffusions also crop up in the swimming of bacteria, surface smoothening in computer graphics [[5]] and global migration patterns of marine mammals [6].

2 Preliminaries

2.1 Isotropic random fields on the unit-radius sphere

We consider the square integrable 2-weakly isotropic Gaussian random field

$$\{T(x); x \in \mathbf{S}_1^2\} \quad (2.1)$$

on the sphere $\mathbf{S}_1^2 = \{x \in \mathbf{R}^3 : |x| = 1\}$ for which

$$\begin{aligned} \mathbb{E}T(gx) &= 0, \\ \mathbb{E}T^2(gx) &= \mathbb{E}T^2(x) \\ \mathbb{E}[T(gx_1)T(gx_2)] &= \mathbb{E}[T(x_1)T(x_2)]. \end{aligned}$$

for all $g \in SO(3)$ where $SO(3)$ is the special group of rotations in \mathbf{R}^3 . We will consider the spectral representation

$$T(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{l,m} \mathcal{Y}_{l,m}(x) = \sum_{l=0}^{\infty} T_l(x) \quad (2.2)$$

where

$$a_{l,m} = \int_{\mathbf{S}_1^2} T(x) \mathcal{Y}_{l,m}^*(x) \lambda(dx), \quad -l \leq m \leq +l, l \geq 0 \quad (2.3)$$

are the Fourier random coefficients of T . The convergence in (2.2) must be meant in the sense that

$$\lim_{L \rightarrow \infty} \mathbb{E} \left[\int_{\mathbf{S}_1^2} \left(T(x) - \sum_{l=0}^L \sum_{m=-l}^{+l} a_{l,m} \mathcal{Y}_{l,m}(x) \right)^2 \lambda(dx) \right] = 0 \quad (2.4)$$

where $\lambda(dx)$ is the Lebesgue measure on the sphere \mathbf{S}_1^2 , $\{\mathcal{Y}_{l,m}(x) : l \geq 0, m = -l, \dots, +l, x \in \mathbf{S}_1^2\}$ is the set of spherical harmonics representing an orthonormal basis for the space $L^2(\mathbf{S}_1^2, \lambda(dx))$. By $\mathcal{Y}_{l,m}^*(x)$ we denote the conjugate of $\mathcal{Y}_{l,m}(x)$. For the sake of clarity we observe that for all $x \in \mathbf{S}_1^2$ and $0 \leq \vartheta \leq \pi$, $0 \leq \varphi \leq 2\pi$:

$$\lambda(dx) = \lambda(d\vartheta, d\varphi) = d\varphi d\vartheta \sin \vartheta$$

and

$$x = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta).$$

We shall write $f(x)$ instead of $f(\vartheta, \varphi)$ when no confusion arises.

The random coefficients (2.3) are zero-mean Gaussian complex random variables such that ([3])

$$\mathbb{E}[a_{l,m} a_{l',m'}^*] = \delta_l^{l'} \delta_m^{m'} \mathbb{E}|a_{l,m}|^2 \quad (2.5)$$

where

$$\mathbb{E}|a_{l,m}|^2 = C_l, \quad l \geq 0 \quad (2.6)$$

is the angular power spectrum of the random field T which under the assumption of Gaussianity fully characterizes the dependence structure of T . Clearly, δ_a^b is the Kronecker symbol.

For a fixed integer l we define $\mu_l = l(l+1)$. The spherical harmonics $Y_{l,m}(\vartheta, \varphi)$ are defined as

$$\mathcal{Y}_{l,m}(\vartheta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} Q_{l,m}(\cos \vartheta) e^{im\varphi}, \quad 0 \leq \vartheta \leq \pi, \quad 0 \leq \varphi \leq 2\pi$$

where

$$Q_{l,m}(z) = (-1)^m (1-z^2)^{\frac{m}{2}} \frac{d^m}{dz^m} Q_l(z), \quad |z| < 1$$

are the associated Legendre functions and Q_l are the the Legendre polynomials with Rodrigues representation

$$Q_l(z) = \frac{1}{2^l l!} \frac{d^l}{dz^l} (z^2 - 1)^l, \quad |z| < 1.$$

We remind that the spherical harmonics solve

$$\Delta_{\mathbb{S}_1^2} \mathcal{Y}_{l,m} = -\mu_l \mathcal{Y}_{l,m}, \quad l \geq 0, \quad |m| \leq l \quad (2.7)$$

where

$$\Delta_{\mathbb{S}_1^2} = \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial}{\partial \vartheta} \right)$$

is the spherical Laplace operator or Laplace-Beltrami operator.

In view of (2.5), the covariance function of $T(x)$ writes

$$\mathbb{E}[T(x)T(y)] = \sum_{lm} C_l \mathcal{Y}_{l,m}(x) \mathcal{Y}_{l,m}^*(y) = \sum_l C_l \frac{2l+1}{4\pi} Q_l(\langle x, y \rangle) \quad (2.8)$$

where in the last step we used the addition formula for spherical harmonics

$$\sum_{m=-l}^{+l} \mathcal{Y}_{l,m}(y) \mathcal{Y}_{l,m}^*(x) = \frac{2l+1}{4\pi} Q_l(\langle x, y \rangle). \quad (2.9)$$

and the inner product

$$\langle x, y \rangle = \cos d(x, y) = \cos \vartheta_x \cos \vartheta_y + \sin \vartheta_x \sin \vartheta_y \cos(\varphi_x - \varphi_y)$$

where $d(\cdot, \cdot)$ is the spherical distance between the points x, y .

For the details on this material we refer to the book by Marinucci and Peccati [14].

2.2 Subordinators and fractional operators

Let $F(t)$, $t \geq 0$ be a Lévy subordinator with characteristic function

$$\mathbb{E} e^{i\xi F(t)} = e^{-t\Phi(\xi)} = e^{-t(ib\xi + \int_0^\infty (e^{i\xi y} - 1)M(dy))}, \quad (2.10)$$

where $b \geq 0$ is the drift and $M(\cdot)$ is the Lévy measure M on $\mathbf{R}_+ \setminus \{0\}$ satisfying the condition:

$$\int_0^\infty (y \wedge 1)M(dy) < \infty.$$

$\int_0^\infty (y \wedge 1)M(dy) < \infty$ and $M(-\infty, 0) = 0$. The Laplace transform of the law of a subordinator $F(t)$, $t > 0$ defined above, can be written as

$$\mathbb{E}e^{-\xi F(t)} = e^{-t\Psi(\xi)} = e^{t\Phi(i\xi)} = e^{-t(b\xi + \int_0^\infty (1 - e^{-\xi y})M(dy))}, \quad (2.11)$$

where $\Psi(\xi)$ is known as Laplace exponent. If $F(t)$, $t \geq 0$, is the β -stable subordinator, then $\Psi(\xi) = \xi^\beta$, $\beta \in (0, 1)$. Hereafter, we assume $b = 0$.

We write the transition density of a Brownian motion on the unit sphere (see [18]) as follows

$$\begin{aligned} Pr\{x + B_t \in dy\}/dy &= Pr\{B_t \in dy | B_0 = x\}/dy \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} e^{-t\mu_l} \mathcal{Y}_{l,m}(y) \mathcal{Y}_{l,m}^*(x) \\ &= \sum_l e^{-t\mu_l} \frac{2l+1}{4\pi} Q_l(\langle x, y \rangle) \end{aligned} \quad (2.12)$$

where we used the addition formula for spherical harmonics (2.9). Furthermore, we shall write

$$P_t f(x) = \mathbb{E}f(x + B_t) = \int_{\mathbf{S}_1^2} f(y) Pr\{x + B_t \in dy\} \quad (2.13)$$

where $P_t f(x)$ is the solution to the initial-value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_{\mathbf{S}_1^2} u, & x \in \mathbf{S}_1^2, t > 0 \\ u(x, 0) = f(x) \end{cases} \quad (2.14)$$

for a measurable function $f(x)$, $x \in \mathbf{S}_1^2$.

Let f be a square integrable function on the unit sphere, that is $f \in L^2(\mathbf{S}_1^2)$. We define the following operator

$$\mathbb{D}_M f(x) = \int_0^\infty (P_s f(x) - f(x)) M(ds) \quad (2.15)$$

where, from (2.12) and (2.13), we have that

$$P_s f(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} e^{-s\mu_l} \mathcal{Y}_{l,m}(x) f_{l,m} \quad (2.16)$$

and $f_{l,m}$ are the Fourier coefficients of f . The operator (2.15) can be rewritten as

$$\mathbb{D}_M f(x) = \int_{\mathbf{S}_1^2} (f(y) - f(x)) \widehat{J}(x, y) \lambda(dy) \quad (2.17)$$

where λ is the Lebesgue measure on \mathbf{S}_1^2 and

$$\widehat{J}(x, y) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} Q_l(\langle y, x \rangle) \widehat{\Psi}(\mu_l)$$

with $\widehat{\Psi}(\mu) = \int_0^\infty e^{-s\mu} M(ds)$ when the integral exists. Indeed we can write

$$\mathbb{D}_M f(x) = \int_0^\infty (P_s f(x) - f(x)) M(ds)$$

$$\begin{aligned}
&= \int_0^\infty \mathbb{E}[(f(x+B_s) - f(x))] M(ds) \\
&= \int_0^\infty \int_{\mathbf{S}_1^2} (f(y) - f(x)) Pr\{x+B_s \in dy\} M(ds) \\
&= \int_{\mathbf{S}_1^2} (f(y) - f(x)) \widehat{J}(x, y) \lambda(dy)
\end{aligned}$$

where

$$\begin{aligned}
\widehat{J}(x, y) \lambda(dy) &= \int_0^\infty Pr\{x+B_s \in dy\} M(ds) \\
&= \lambda(dy) \sum_l \frac{2l+1}{4\pi} Q_l(\langle y, x \rangle) \int_0^\infty e^{-s\mu_l} M(ds) \\
&= \lambda(dy) \sum_l \frac{2l+1}{4\pi} Q_l(\langle y, x \rangle) \widehat{\Psi}(\mu_l).
\end{aligned}$$

Furthermore, from (2.16), the operator (2.15) can be written as follows

$$\begin{aligned}
\mathbb{D}_M f(x) &= \int_0^\infty (P_s f(x) - P_0 f(x)) M(ds) \\
&= \sum_{lm} f_{l,m} \mathcal{Y}_{l,m}(x) \int_0^\infty (e^{-s\mu_l} - 1) M(ds) \\
&= [by (2.11)] = - \sum_{lm} f_{l,m} \mathcal{Y}_{l,m}(x) \Psi(\mu_l) \\
&= - \sum_{lm} \left(\int_{\mathbf{S}_1^2} f(y) Y_{l,m}^*(y) \lambda(dy) \right) \mathcal{Y}_{l,m}(x) \Psi(\mu_l) \\
&= - \int_{\mathbf{S}_1^2} f(y) \left(\sum_{lm} \Psi(\mu_l) \mathcal{Y}_{l,m}(x) \mathcal{Y}_{l,m}^*(y) \right) \lambda(dy) \\
&= - \int_{\mathbf{S}_1^2} f(y) J(x, y) \lambda(dy)
\end{aligned}$$

where

$$J(x, y) = \sum_{l=0}^\infty \sum_{m=-l}^{+l} \Psi(\mu_l) \mathcal{Y}_{l,m}(x) \mathcal{Y}_{l,m}^*(y) = \sum_{l=0}^\infty \Psi(\mu_l) \frac{2l+1}{4\pi} Q_l(\langle x, y \rangle) \quad (2.18)$$

exists (in the last step we have applied the addition formula (2.9)).

We introduce the Sobolev space

$$H^s(\mathbf{S}_1^2) = \left\{ f \in L^2(\mathbf{S}_1^2) : \sum_{l=0}^\infty (2l+1)^{2s} f_l < \infty \right\} \quad (2.19)$$

where

$$f_l = \sum_{|m| \leq l} |f_{l,m}|^2 = \sum_{|m| \leq l} \left| \int_{\mathbf{S}_1^2} f(x) \mathcal{Y}_{l,m}^*(x) \lambda(dx) \right|^2, \quad l = 0, 1, 2, \dots$$

Definition 1. Let Ψ be the symbol of a subordinator. Let $f \in H^s(\mathbf{S}_1^2)$ and $s > 4$. Then,

$$\mathbb{D}_M f(x) = - \sum_{l=0}^\infty \sum_{m=-l}^{+l} f_{l,m} \mathcal{Y}_{l,m}(x) \Psi(\mu_l) \quad (2.20)$$

where

$$f_{l,m} = \int_{\mathbb{S}_1^2} f(x) \mathcal{Y}_{l,m}^*(x) \lambda(dx)$$

are the Fourier coefficients of the initial condition.

The series (2.20) converges absolutely and uniformly. Indeed, $f_l < l^{-2s}$ with $s > 4$ (being $f \in H^s(\mathbb{S}_1^2)$), $\|Y_{l,m}\|_\infty < l^{1/2}$ (see [16]) and $\Psi(\mu_l) \leq l^2$ and thus, by considering that

$$\sum_m |f_{l,m}| \leq \left(\sum_m |f_{l,m}|^2 \right)^{\frac{1}{2}} ((2l+1))^{\frac{1}{2}} = \sqrt{(2l+1) f_l} \leq l^{-s+\frac{1}{2}}$$

we get the claim.

Definition 2. $\mathbb{P}_t = \exp(t\mathbb{D}_M)$ is the semigroup associated with (2.17) with symbol $\widehat{\mathbb{P}}_t = \exp(-t\Psi)$ where $-\Psi$ is the Fourier multiplier of \mathbb{D}_M .

3 Some fractional equations on the sphere

We recall the Dzerbayshan-Caputo fractional derivative

$$\frac{\partial^\beta u}{\partial t^\beta}(x, t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u(x, s)}{\partial s} \frac{ds}{(t-s)^\beta} \quad (3.1)$$

for $0 < \beta < 1$, $x \in \mathbb{R}$, $t > 0$, see, e.g., [15], p. 38.

The inverse \mathfrak{L}_t^β of a β -stable subordinator \mathfrak{H}_t^β can be defined by the following relationship

$$Pr\{\mathfrak{L}_t^\beta < x\} = Pr\{\mathfrak{H}_x^\beta > t\}$$

for $x, t > 0$, see, e.g., [15], p. 101.

The Mittag-Leffler function is defined as

$$E_\beta(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta k + 1)}, \quad x \in \mathbb{R}, \beta > 0, \quad (3.2)$$

see, e.g., [15], p. 35.

We assume also that the random field T introduced in (2.2) is Gaussian and its Fourier coefficients $a_{l,m}$ are independent complex zero-mean Gaussian r.v.'s. We shall use the following notation

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{+l} = \sum_{lm}$$

when no confusion arises.

We pass now to the first theorem. Denote by $F^\Psi(\mathfrak{L}_t^\beta)$ the subordinator with symbol Ψ time-changed by the inverse of a stable subordinator of order $\beta \in (0, 1)$.

Theorem 1. *Let us consider $\gamma \geq 0$ and $\beta \in (0, 1)$. The solution to the fractional equation*

$$\left(\gamma - \mathbb{D}_M + \frac{\partial^\beta}{\partial t^\beta} \right) X_t(x) = 0, \quad x \in \mathbb{S}_1^2, t \geq 0 \quad (3.3)$$

with initial condition $X_0(x) = T(x)$ is a time-dependent random field on the sphere \mathbb{S}_1^2 written as

$$X_t(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{l,m} E_\beta(-t^\beta(\gamma + \Psi(\mu_l))) \mathcal{Y}_{l,m}(x) \quad (3.4)$$

where

$$a_{l,m} = \int_{\mathbf{S}_1^2} X_0(x) \mathcal{Y}_{l,m}^*(x) \lambda(dx). \quad (3.5)$$

Furthermore, the following representation holds

$$X_t(x) = \mathbb{E} \left[T(x + B(\gamma \mathfrak{L}_t^\beta + F^\Psi(\mathfrak{L}_t^\beta))) \middle| \mathfrak{F}_T \right] \quad (3.6)$$

where \mathfrak{F}_T is the σ -field generated by $X_0 = T$.

Proof. First we notice that

$$-\frac{\partial}{\partial t} \mathbb{E} e^{-\xi(\gamma t + F_t)} \Big|_{t=0} = \xi \gamma + \Psi(\xi) \quad (3.7)$$

which coincides with (2.11) for $\gamma = b$. Indeed, we are dealing with the symbol Ψ of the subordinator F_t without drift. Furthermore, it is well-known that the Mittag-Leffler function E_β is an eigenfunction of the Dzerbayshan-Caputo fractional derivative, that is

$$\frac{\partial^\beta}{\partial t^\beta} E_\beta(-t^\beta \mu) = -\mu E_\beta(-t^\beta \mu). \quad (3.8)$$

We assume that (3.4) holds true. From the fact that

$$\mathbb{D}_M \mathcal{Y}_{l,m}(x) = \int_0^\infty (P_s \mathcal{Y}_{l,m}(x) - \mathcal{Y}_{l,m}(x)) M(ds)$$

where $\mathcal{Y}_{l,m}(x) = (-1)^m \mathcal{Y}_{l,-m}^*(x)$ and

$$P_s \mathcal{Y}_{l,m}(x) = e^{-s\mu_l} \mathcal{Y}_{l,m}(x) \quad (3.9)$$

we obtain that

$$\begin{aligned} \mathbb{D}_M \mathcal{Y}_{l,m}(x) &= \int_0^\infty (e^{-s\mu_l} \mathcal{Y}_{l,m}(x) - \mathcal{Y}_{l,m}(x)) M(ds) \\ &= \int_0^\infty (e^{-s\mu_l} - 1) M(ds) \mathcal{Y}_{l,m}(x) \\ &= -\Psi(\mu_l) \mathcal{Y}_{l,m}(x). \end{aligned}$$

Formula (3.9) can be obtained by considering that

$$\begin{aligned} P_s \mathcal{Y}_{l,m}(x) &= \mathbb{E} \mathcal{Y}_{l,m}(x + B_s) \\ &= \sum_{l'm'} e^{-s\mu_{l'}} \mathcal{Y}_{l',m'}^*(x) \int_{\mathbf{S}_1^2} \mathcal{Y}_{l,m}(y) \mathcal{Y}_{l',m'} \lambda(dy) \\ &= \sum_{l'm'} e^{-s\mu_{l'}} \mathcal{Y}_{l',m'}^*(x) (-1)^{m'} \int_{\mathbf{S}_1^2} \mathcal{Y}_{l,m}(y) \mathcal{Y}_{l',-m'}^* \lambda(dy) \\ &= \sum_{l'm'} e^{-s\mu_{l'}} \mathcal{Y}_{l',m'}^*(x) (-1)^{m'} \delta_l^{l'} \delta_m^{-m'} = e^{-s\mu_l} \mathcal{Y}_{l,m}(x). \end{aligned}$$

Thus, we get that

$$(\gamma - \mathbb{D}_M) X_t(x) = \sum_{l=0}^\infty \sum_{m=-l}^{+l} a_{l,m} (\gamma + \Psi(\mu_l)) E_{\beta,1}(-t^\beta \gamma - t^\beta \Psi(\mu_l)) \mathcal{Y}_{l,m}(x)$$

and, from (3.8), we arrive at

$$\left(\frac{\partial^\beta}{\partial t^\beta} + \gamma - \mathbb{D}_M \right) X_t(x) =$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{l,m} \left(\frac{\partial^\beta}{\partial t^\beta} + \gamma + \Psi(\mu_l) \right) E_{\beta,1}(-t^\beta \gamma - t^\beta \Psi(\mu_l)) \mathcal{Y}_{l,m}(x) = 0$$

term by term and therefore equation (3.3) is satisfied. This concludes the proof. \square

Remark 1. *Since*

$$T(x) = \sum_{lm} a_{l,m} \mathcal{Y}_{l,m}(x). \quad (3.10)$$

we have that

$$P_t T(x) = \mathbb{E}[T(x + B_t) | \mathfrak{F}_T] = \sum_{lm} e^{-t\mu_l} a_{l,m} \mathcal{Y}_{l,m}(x) = T_t(x). \quad (3.11)$$

This represents the solution to (3.3) with $\beta = 1$, $\gamma = 0$ and $\mathbb{D}_M = \Delta_{\mathbb{S}_1^2}$. From (3.6), for $\beta = 1$, $\Psi(\xi) = \xi$, that is for the elementary subordinator $F(t) = t$ (and $\mathfrak{L}_t^1 = t$) we have that

$$X_t(x) = \mathbb{E}[T(x + B(\gamma t + t)) | \mathfrak{F}_T] = \sum_{lm} a_{l,m} e^{-t(\gamma+1)\mu_l} \mathcal{Y}_{l,m}(x).$$

Remark 2. *The series (3.4) converges in $L^2(dP \times d\lambda)$ sense for all $t \geq 0$, that is*

$$\lim_{L \rightarrow \infty} \mathbb{E} \left[\int_{\mathbb{S}_1^2} \left(X_t(x) - \sum_{l=0}^L \sum_{m=-l}^{+l} a_{l,m} \mathcal{T}_l(t) \mathcal{Y}_{l,m}(x) \right)^2 \lambda(dx) \right] = 0, \quad \forall t. \quad (3.12)$$

where, in formula (3.4), the time-dependent random coefficients are

$$\mathcal{T}_l(t) = E_\beta(-t^\beta(\gamma + \Psi(\mu_l))), \quad l \geq 0.$$

Throughout the paper the convergence (3.12) in mean square sense on \mathbb{S}_1^2 is considered.

Theorem 2. *Let us consider $\gamma, \varphi \geq 0$ and $\beta \in (0, 1]$. A solution to the fractional equation*

$$\left(\gamma - \mathbb{D}_M - \varphi \frac{\partial}{\partial t} \right)^\beta X_t(x) = T_t(x), \quad x \in \mathbb{S}_1^2, t \geq 0 \quad (3.13)$$

where $T_t(x)$ is given in (3.11), is a time-dependent random field on the sphere \mathbb{S}_1^2 written as

$$X_t(x) = \sum_{lm} a_{l,m} e^{-t\mu_l} (\gamma + \varphi\mu_l + \Psi(\mu_l))^{-\beta} \mathcal{Y}_{l,m}(x), \quad (3.14)$$

where $e^{-t\mu_l} a_{l,m}$ are the Fourier coefficients involved in the representation (3.11) of the innovation process $T_t(x)$ in (3.11) in terms of spherical harmonics.

Proof. We have that

$$\begin{aligned} X_t(x) &= \left(\gamma - \mathbb{D}_M - \varphi \frac{\partial}{\partial t} \right)^{-\beta} T_t(x) \\ &= \int_0^\infty ds \frac{s^{\beta-1}}{\Gamma(\beta)} e^{s\varphi \frac{\partial}{\partial t} - s\gamma + s\mathbb{D}_M} T_t(x) \\ &= \int_0^\infty ds \frac{s^{\beta-1}}{\Gamma(\beta)} e^{s\varphi \frac{\partial}{\partial t} - s\gamma} \mathbb{P}_s T_t(x) \\ &= \int_0^\infty ds \frac{s^{\beta-1}}{\Gamma(\beta)} e^{-s\gamma} \mathbb{P}_s T_{t+\varphi s}(x) \end{aligned}$$

where we used the translation rule

$$e^{a\frac{\partial}{\partial z}} f(z) = f(z + a), \quad a \in \mathbb{R}$$

which holds for bounded continuous functions f on $(0, +\infty)$ (see, for example, formula (3.9) in [8] and the references therein for details). From the fact that

$$\mathbb{P}_s \mathcal{Y}_{l,m}(x) = e^{-s\Psi(\mu_l)} \mathcal{Y}_{l,m}(x) \quad (3.15)$$

where $\mathbb{P}_s = \exp(s\mathbb{D}_M)$ we get that

$$\begin{aligned} X_t(x) &= \sum_{lm} a_{l,m} \left(\int_0^\infty ds \frac{s^{\beta-1}}{\Gamma(\beta)} e^{-s\gamma} e^{-(t+\varphi s)\mu_l} \mathbb{P}_s \mathcal{Y}_{l,m}(x) \right) \\ &= \sum_{lm} a_{l,m} \left(\int_0^\infty ds \frac{s^{\beta-1}}{\Gamma(\beta)} e^{-s\gamma} e^{-(t+\varphi s)\mu_l} e^{-s\Psi(\mu_l)} \right) \mathcal{Y}_{l,m}(x) \\ &= \sum_{lm} a_{l,m} e^{-t\mu_l} \left(\int_0^\infty ds \frac{s^{\beta-1}}{\Gamma(\beta)} e^{-s\gamma - s\varphi\mu_l - s\Psi(\mu_l)} \right) \mathcal{Y}_{l,m}(x) \\ &= \sum_{lm} a_{l,m} e^{-t\mu_l} (\gamma + \varphi\mu_l + \Psi(\mu_l))^{-\beta} \mathcal{Y}_{l,m}(x) \end{aligned}$$

and this concludes the proof. \square

We now examine the special case $\varphi = 0$.

Corollary 1. *Let $\alpha \in (0, 1)$, $\beta \in (0, 1]$. The solution to*

$$(\gamma - \mathbb{D}_M)^\beta X(x) = T(x) \quad (3.16)$$

is written as

$$X(x) = \sum_{lm} a_{l,m} (\gamma + \Psi(\mu_l))^{-\beta} \mathcal{Y}_{l,m}(x). \quad (3.17)$$

Proof. For $\beta \in (0, 1)$ we consider the following relation concerning the fractional power of operators (Bessel potential). For $f \in L^2(\mathbf{S}_1^2)$ we have that

$$\begin{aligned} (\gamma - \mathbb{D}_M)^\beta f(x) &= \frac{\beta}{\Gamma(1-\beta)} \int_0^\infty \frac{ds}{s^{\beta+1}} (1 - e^{-s\gamma + s\mathbb{D}_M}) f(x) \\ &= \frac{\beta}{\Gamma(1-\beta)} \int_0^\infty \frac{ds}{s^{\beta+1}} (f(x) - e^{-s\gamma} \mathbb{P}_s f(x)) \end{aligned}$$

where, we recall that $\mathbb{P}_s f$ is the transition semigroup associated with the operator \mathbb{D}_M and $u(x, t) = \mathbb{P}_t f(x)$ solves the Cauchy problem $(\partial_t - \mathbb{D}_M)u(x, t) = 0$ with $u(x, 0) = f(x)$. Therefore, if we assume that there exists the following spectral representation for the solution X as a random function on \mathbf{S}_1^2 ,

$$X(x) = \sum_{lm} \hat{a}_{l,m} \mathcal{Y}_{l,m}(x) \quad (3.18)$$

then we can immediately write

$$\begin{aligned} (\gamma - \mathbb{D}_M)^\beta X(x) &= \frac{\beta}{\Gamma(1-\beta)} \sum_{lm} \hat{a}_{l,m} \int_0^\infty \frac{ds}{s^{\beta+1}} (\mathcal{Y}_{l,m}(x) - e^{-s\gamma} \mathbb{P}_s \mathcal{Y}_{l,m}(x)) \\ &= \frac{\beta}{\Gamma(1-\beta)} \sum_{lm} \hat{a}_{l,m} \int_0^\infty \frac{ds}{s^{\beta+1}} (1 - e^{-s\gamma} e^{-s\Psi(\mu_l)}) \mathcal{Y}_{l,m}(x) \end{aligned}$$

$$= \sum_{lm} \hat{a}_{l,m} (\gamma + \Psi(\mu_l))^\beta \mathcal{Y}_{l,m}(x).$$

The equation (3.16) turns out to be satisfied only if

$$\hat{a}_{l,m} = a_{l,m} (\gamma + \Psi(\mu_l))^{-\beta}.$$

On the other hand, by repeating the arguments of the proof of Theorem 3 we have that

$$\begin{aligned} X(x) &= (\gamma - \mathbb{D}_M)^{-\beta} T(x) \\ &= \int_0^\infty ds \frac{s^{\beta-1}}{\Gamma(\beta)} e^{-s\gamma + s\mathbb{D}_M} T(x) \\ &= \int_0^\infty ds \frac{s^{\beta-1}}{\Gamma(\beta)} e^{-s\gamma} \mathbb{P}_s T(x) \\ &= \sum_{lm} a_{l,m} \int_0^\infty ds \frac{s^{\beta-1}}{\Gamma(\beta)} e^{-s\gamma} e^{-s\Psi(\mu_l)} \mathcal{Y}_{l,m}(x) \\ &= \sum_{lm} a_{l,m} (\gamma + \Psi(\mu_l))^{-\beta} \mathcal{Y}_{l,m}(x). \end{aligned}$$

This confirms result (3.17). \square

We now study the covariance of the random fields introduced so far. Let us consider the representation

$$X_t(x) = \sum_{lm} a_{l,m} \mathcal{T}_l(t) \mathcal{Y}_{l,m}(x) = \sum_l \mathcal{T}_l(t) T_l(x) \quad (3.19)$$

already introduced in Remark 2. We also recall that, for $x, y \in \mathbb{S}_1^2$,

$$\mathbb{E}[X_0(x) X_0(y)] = \sum_l \frac{2l+1}{4\pi} C_l Q_l(\langle x, y \rangle) = \mathbb{E}[T(x) T(y)]. \quad (3.20)$$

Furthermore,

$$\mathbb{E}[T(x) T(y)] = \sum_l \mathbb{E}[T_l(x) T_l(y)]. \quad (3.21)$$

This is due to the fact that the coefficients $a_{l,m}$ are uncorrelated over l .

Remark 3. *We observe that*

- for $X_t(x)$ as in Theorem 1,

$$\mathcal{T}_l(t) = E_\beta(-t^\beta(\gamma + \Psi(\mu_l))) \geq \frac{1}{1 + \Gamma(1-\beta)t^\beta(\gamma + \Psi(\mu_l))}, \quad t \geq 0, l \geq 0 \quad (3.22)$$

For this inequality, consult [17, Theorem 4].

- for $X_t(x)$ as in Theorem 2,

$$\mathcal{T}_l(t) = e^{-t\mu_l} (\gamma + \varphi\mu_l + \Psi(\mu_l))^{-\beta} \leq e^{-tl^2} (\gamma + \varphi l^2 + \Psi(l^2))^{-\beta}, \quad t \geq 0, l \geq 0 \quad (3.23)$$

- for $X(x)$ as in Corollary 1,

$$\mathcal{T}_l(0) = (\gamma + \Psi(\mu_l))^{-\beta} \leq (\gamma + \Psi(l^2))^{-\beta}, \quad t \geq 0, l \geq 0 \quad (3.24)$$

Remark 4. Let $B(\tau_t)$ be a time-changed Brownian motion on the unit sphere. We refer to it as a coordinates change for the random field on the sphere T . From the previous results, we observe that

$$X_t(x) = \mathbb{E}[T_0(x + B(\tau_t)) | \mathfrak{F}_T] = \mathbb{E}[T_{\tau_t}(x) | \mathfrak{F}_T], \quad x \in \mathbb{S}_1^2, \quad t > 0 \quad (3.25)$$

where

$$T_t(x) = \sum_l e^{-t\mu_l} T_l(x), \quad x \in \mathbb{S}_1^2, \quad t > 0. \quad (3.26)$$

Moreover,

$$\mathcal{T}_l(t) = \mathbb{E} e^{-\mu_l \tau_t}. \quad (3.27)$$

We can state the following result for which the spherical Brownian motions underlying $X_t(x)$ and $X_t(y)$ are assumed independent.

Theorem 3. For $x, y \in \mathbb{S}_1^2$, for all $g \in SO(3)$, we have that

$$\mathbb{E}[X_t(gx) X_s(gy)] = \sum_l \frac{2l+1}{4\pi} C_l \mathcal{T}_l(t) \mathcal{T}_l(s) P_l(\langle x, y \rangle), \quad t, s \geq 0 \quad (3.28)$$

Proof. First we observe that

$$\mathbb{E}[a_{l,m} a_{l',m'}] = (-1)^m \delta_l^{l'} \delta_{-m}^{m'} C_l \quad (3.29)$$

from the property $\mathcal{Y}_{l,m}(x) = (-1)^m \mathcal{Y}_{l,-m}^*(x)$ of the spherical harmonics. From the representation (3.19) we can write

$$\begin{aligned} \mathbb{E}[X_t(x) X_s(y)] &= \sum_{lm} \sum_{l'm'} \mathbb{E}[a_{l,m} a_{l',m'}] \mathcal{T}_l(t) \mathcal{T}_{l'}(s) \mathcal{Y}_{l,m}(x) \mathcal{Y}_{l',m'}(y) \\ &= \sum_{lm} C_l \mathcal{T}_l(t) \mathcal{T}_l(s) \mathcal{Y}_{l,m}(x) \mathcal{Y}_{l,m}^*(y) \\ &= \sum_l \frac{2l+1}{4\pi} C_l \mathcal{T}_l(t) \mathcal{T}_l(s) P_l(\langle x, y \rangle) \end{aligned}$$

where $\mathcal{T}_l(t)$ is given as in (3.27) and we used the addition formula in order to arrive at $P_l(\langle x, y \rangle)$. \square

Remark 5. We can immediately see that the variance becomes

$$\mathbb{E}[X_t(gx)]^2 = \sum_l \frac{2l+1}{4\pi} C_l |\mathcal{T}_l(t)|^2, \quad \forall g \in SO(3). \quad (3.30)$$

We recall that C_l is the angular power spectrum of T and, it is usually assumed to be $C_l \sim l^{-\gamma}$ with $\gamma \geq 2$ for large l to ensure summability (or $C_l \sim L(l)/t^\theta$ where $L(\cdot)$ is slowly varying function as $l \rightarrow \infty$). As Remark 3 shows we have the high-frequency behaviour also for $\mathcal{T}_l(t)$ in both the variable $t > 0$ and the frequency $l > 0$. The convergence of (3.30) therefore entails different correlation structures for the solutions $X_t(x)$ of the equations investigated so far.

We say that the zero mean process $X_t(x)$ exhibits a long range dependence if

$$\sum_{h=1}^{\infty} \mathbb{E}[X_{t+h}(x) X_t(y)] = \infty, \quad x, y, \in \mathbb{S}_1^2. \quad (3.31)$$

Conversely, we say that X exhibits a short range dependence if the series (3.31) converges.

Remark 6. We write

$$\mathcal{K}_t(x, y) = \sum_{h \geq 1} \mathbb{E}[X_{t+h}(x) X_t(y)], \quad t \geq 0$$

for $x, y \in \mathbb{S}_1^2$. From the discussion above, we have that

- for $X_t(x)$ as in Theorem 1,

$$\begin{aligned}
\mathcal{K}_t(x, y) &= \sum_{h \geq 1} \sum_{l \geq 0} \frac{2l+1}{2\pi} C_l E_\beta(-t^\beta(\gamma + \Psi(\mu_l))) E_\beta(-(t+h)^\beta(\gamma + \Psi(\mu_l))) \quad (3.32) \\
&\geq \sum_{h \geq 1} \sum_{l \geq 0} \frac{2l+1}{2\pi} C_l \frac{1}{1 + \Gamma(1-\beta)t^\beta(\gamma + \Psi(\mu_l))} \frac{1}{1 + \Gamma(1-\beta)(t+h)^\beta(\gamma + \Psi(\mu_l))} \\
&\geq \sum_{l \geq 0} \frac{2l+1}{2\pi} C_l \frac{1}{1 + \Gamma(1-\beta)t^\beta(\gamma + \Psi(\mu_l))} \sum_{h \geq 1} \frac{1}{1 + \Gamma(1-\beta)(t+h)^\beta(\gamma + \Psi(\mu_l))} \\
&= \infty,
\end{aligned}$$

that is the random field exhibits a long-range dependence;

- for $X_t(x)$ as in Theorem 2,

$$\begin{aligned}
\mathcal{K}_t(x, y) &\leq \sum_{h \geq 1} \sum_{l \geq 0} \frac{2l+1}{4\pi} C_l e^{-2tl^2 - hl^2} (\gamma + \varphi l^2 + \Psi(l^2))^{-2\beta} P_l(\langle x, y \rangle) \quad (3.33) \\
&\leq \sum_{l \geq 0} \frac{2l+1}{4\pi} \frac{C_l}{e^{l^2} - 1} e^{-2tl^2} (\gamma + \varphi l^2 + \Psi(l^2))^{-2\beta} P_l(\langle x, y \rangle) \\
&< \infty,
\end{aligned}$$

that is the random field has a short range dependence.

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