

STOCHASTIC SHADOWING AND STOCHASTIC STABILITY

DMITRY TODOROV

ABSTRACT. The notion of stochastic shadowing property is introduced. Relations to stochastic stability and standard shadowing are studied. Using tent map as an example it is proved that, in contrast to what happens for standard shadowing, there are significantly non-uniformly hyperbolic systems that satisfy stochastic shadowing property.

1. INTRODUCTION

In this paper I address two problems. The first one is getting information about a chaotic dynamical system without having access to its exact trajectories – i.e. when one has only trajectories with errors (pseudotrajectories). The second problem is the lack of examples of non-uniformly hyperbolic systems having good shadowing properties.

The most standard naive way to approximate an (SRB) invariant measure of a chaotic system is to take a point of a phase space at random, iterate it (numerically) for a long time and then build a histogram. However an issue appears here – when one does numerics, one always gets trajectories with errors that appear at each iteration. It is not clear is there a real trajectory close to each erroneous one and if there is one, does it reflect well the statistics of the invariant measure one was that trying to approximate initially. The same applies of course to experimental observations of chaotic physical systems.

There are two well-known important notions in this context: stochastic stability and shadowing. Both notions state a sort of stability of the system with respect to small per-iteration perturbations. So far no direct relations between them were established (it was stated as a problem in [6]).

Both notions have certain drawbacks when one tries to apply them to a practical situation. Namely classical shadowing notions (see [19, 17]) do not concern statistical information questions at all, and stochastic stability works only on a level of measures and does not take into account the fact that ergodic averages for the unperturbed system can have fluctuations with different properties than ergodic averages for the perturbed one. That is that usually it is desirable that one has closeness of statistical properties (of samples given by erroneous and by exact trajectories) not only in the limit but also for a (big) finite number of iterations for a large set of erroneous trajectories.

It is well known that smooth uniformly hyperbolic systems have shadowing [19]. In fact, they have even stronger property of quantitative (Lipschitz) shadowing. Moreover this property is uniform with respect to perturbations in C^1 topology. It was proved that in fact uniformly hyperbolic systems are the only ones to have Lipschitz and Hölder shadowing (see [21, 26]).

Key words and phrases. shadowing, stochastic stability, ergodicity, hyperbolicity.

However even for simplest systems with singularities nothing like this is known. For piecewise expanding maps of the interval there is a simple observation by Blank [3], Lipschitz shadowing away from singularities by Kifer [14] and a shadowing result for a large set of parameters of tent maps by Coven, Kan and Yorke [8].

A term “ergodic shadowing” has been introduced by Fakhari and Ghane in [9]. The word “ergodic” there was used not because of relation to ergodic properties but because of special notion of closeness between sequences of points. I use term “stochastic shadowing” for a different notion which is more “ergodic” in some sense. It is somewhat closer to a notion of average shadowing introduced by Blank in [3].

Average shadowing itself will be too strong for our purposes – it is difficult to satisfy it for a systems that do not have very strong hyperbolicity (with the ambient manifold being a hyperbolic set).

We will study relations between stochastic shadowing, stochastic stability and standard shadowing. To show usefulness of the new notion we examine family of tent maps as an example. We will show that there is no hope to have uniform Lipschitz shadowing for it but nevertheless it is possible to show the presence of Lipschitz stochastic shadowing.

2. DEFINITIONS

Let M be a compact manifold (possibly with boundary) with a Riemmanian metric dist . Let f be a mapping of M to itself.

For $x \in M$ let δ_x be a δ -measure concentrated at point x . For a sequence of points $\xi = \{x_k\}_{k \in J} \subset M$, where J is either $\mathbb{N} \cup \{0\}$ or $\{0, 1, \dots, N\}$ with number of elements greater than n denote

$$S_n(\xi) = \frac{1}{n+1} \sum_{k=0}^n \delta_{x_k}.$$

For $x \in M$ denote $S_n(x) = S_n(\{f^k(x)\}_{k=0}^\infty)$.

Let μ be a Borel invariant probability measure on M . Invariant means that $\mu(f^{-1}(A)) = \mu(A)$ for every Borel set A .

Definition 1. The mesure μ is *ergodic* if for μ -a.e. point $x \in M$

$$(2.1) \quad S_n(x) \rightarrow \mu, \quad n \rightarrow \infty$$

where the convergence holds in the weak* topology on the space of probability measures on M .

A point $x \in M$ for which (2.1) is satisfied is called *typical* for μ or just *μ -typical*.

Definition 2. The measure μ is called *physical* if the set of μ -typical points has full (normalized) Lebesgue measure on M .

Assumption 1. Assume μ is physical for f .

Definition 3. A sequence of points $\bar{x} = \{x_k\}_{k=0}^\infty$ is called an ε -pseudotrajectory (of f) if

$$\text{dist}(f(x_k), x_{k+1}) \leq \varepsilon, \quad k \geq 0.$$

Let be $\{X_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)}$ a family of Markov chains on M , where $0 < \varepsilon_0 < \text{diam } M/2$.

Definition 4. We call such a family *admissible* if every realization of X_ε is an ε -pseudotrajectory.

Assumption 2. Assume $\{X_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)}$ is admissible.

For $x \in M$ and a Borel set E let $P_\varepsilon(x, E)$ be a transition probability for the chain X_ε i.e.

$$P_\varepsilon(x, E) = P(X_\varepsilon^{n+1} \in E | X_\varepsilon^n = x), \quad n \geq 0.$$

Denote the normalized Lebesgue measure on M by Leb . We will sometimes use specific type of perturbations:

Definition 5. We say that the family $\{X_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)}$ is a *family of uniform perturbations* if for every $0 < \varepsilon < \varepsilon_0$ the transition probability $P_\varepsilon(x, dy)$ for X_ε is given by a uniform distribution in ε -ball around $f(x)$.

I.e. $P_\varepsilon(x, dy)$ has the following density for every $x \in M$:

$$\left(\frac{d}{\text{Leb}} P_\varepsilon(x, \cdot) \right) (y) = \frac{1}{\text{Leb}(B_\varepsilon(f(x)))} \chi_{B_\varepsilon(f(x))}(y),$$

where χ_A is the indicator function for a set A .

Definition 6. A Borel probability measure on M is said to be *stationary for the Markov chain X_ε* if for every Borel set E the following identity holds

$$\mu_\varepsilon(E) = \int P_\varepsilon(x, E) d\mu_\varepsilon(x).$$

Definition 7. A measure μ_ε stationary for the Markov chain X_ε is said to be *ergodic* if for μ_ε -a.e. realization $\bar{x} = \{x_k\}_{0 \leq k \leq \infty}$ of the Markov chain X_ε

$$S_n(\bar{x}) \rightarrow \mu_\varepsilon, \quad n \rightarrow \infty$$

where the convergence holds in the weak* topology on the space of probability measures on M .

Assumption 3. Assume for every $0 < \varepsilon < \varepsilon_0$ there is a unique ergodic stationary measure μ_ε for X_ε .

Denote the space of continuous functions from M to \mathbb{R} with sup-norm by $(C(M), \|\cdot\|_\infty)$ and the standard L^1 space with standard L^1 norm for Borel sigma-algebra by $(L^1(M), \|\cdot\|_{L^1})$. Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be some Banach space of functions from M to \mathbb{R} .

Assumption 4. Assume that $\mathcal{B} \subset (L^1(M), \|\cdot\|_{L^1})$ and $\mathcal{B}_c = C(M) \cap \mathcal{B} \neq \emptyset$.

Let P_ε^n be the n -step transition probability for the Markov chain X_ε .

Definition 8. We say that the stationary measure μ_ε *has exponential decay of correlations for observables from \mathcal{B}* if there exists $\tau > 0$ such that for every $\tau' > \tau$ for every $\phi, \psi \in \mathcal{B}$ there exists a constant $C = C(\tau, \phi, \psi)$ such that

$$\left| \int \left(\int \phi(y) dP_\varepsilon^n(x, y) \right) \psi(x) d\mu(x) - \int \phi d\mu_\varepsilon \int \psi d\mu_\varepsilon \right| \leq C(\tau')^n, \quad n \in \mathbb{N}.$$

Definition 9. We say that the stationary measure μ_ε *has property A for observables from \mathcal{B}* if for any $\delta > 0$, $N \in \mathbb{N}$ and any $\phi \in \mathcal{B}$ there exists a set $A_{\varepsilon, \delta, N}^\phi$ of realizations of X_ε such that for any $\bar{x} \in A_{\varepsilon, \delta, N}^\phi$ for every $n > N$ the following holds:

$$\left| \int \phi dS_n(\bar{x}) - \int \phi d\mu_\varepsilon \right| < \delta$$

and

$$\mu_\varepsilon(A_{\varepsilon, \delta, N}^\phi) \rightarrow 1, \quad N \rightarrow \infty.$$

Remark 2.1. If μ_ε has exponential decay of correlations for observables from \mathcal{B} , then it has property A for observables from \mathcal{B} since for every $\phi \in \mathcal{B}$ we can write the following estimate for some constant $C(\phi) > 0$ and some function $I(\phi, \delta) > 0$:

$$\mu_\varepsilon \left(\left\{ \bar{x} \mid \left| \int \phi dS_n(\bar{x}) - \int \phi d\mu_\varepsilon \right| > \delta \right\} \right) < C(\phi) e^{-I(\phi, \delta)n}.$$

It is left to notice that the sum of the left parts for $n = N \dots \infty$ is less than $C(\phi, \delta) \exp(-I(\phi, \delta)N)$ for some $C(\phi, \delta) > 0$.

Consider $\sigma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that $\sigma(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Definition 10. The map f is said to have *classical shadowing with accuracy σ with respect to $\{X_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)}$* if for any $0 < \varepsilon < \varepsilon_0$ for every realization of X_ε there exists a point $p \in M$ such that

$$\text{dist}(x_k, f^k(p)) < \sigma(\varepsilon), \quad k \geq 0.$$

We call σ a *classical shadowing accuracy function*.

Definition 11. The map f is said to have *standard shadowing with accuracy σ* if it has classical shadowing with accuracy σ with respect to any admissible family of Markov chains.

Remark 2.2. If we do not specify accuracy, this definition coincides with the definition of (one-sided) shadowing property in [19]. If we ask for $\sigma(\varepsilon) = L\varepsilon$ for some $L > 0$ then the definition coincides with the definition of (one-sided) Lipschitz shadowing property from [19].

Definition 12. The map f has *strong stochastic stability with speed σ with respect to $\{X_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)}$* if both μ and μ_ε have densities (with respect to Leb) ρ and ρ_ε respectively and for every $0 < \varepsilon < \varepsilon_0$ the following estimate holds:

$$\|\rho - \rho_\varepsilon\|_{L^1} \leq \sigma(\varepsilon).$$

We call σ a *speed of strong stochastic stability*.

Remark 2.3. Note that here when we do not require an accuracy of shadowing to be a function uniquely defined by the system and its perturbation. For example if a system has classical shadowing with accuracy $\gamma(\varepsilon) = L\varepsilon$ for some $L > 0$ then it has classical shadowing with accuracy $\gamma(\varepsilon) = L\varepsilon^\alpha$ for every $0 < \alpha < 1$.

The same applies for a speed of stochastic stability all similar functions (speeds and accuracies) we will consider later.

Definition 13. The map f has *stochastic stability with speed σ for observables from \mathcal{B} with respect to $\{X_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)}$* if for every $\phi \in \mathcal{B}$ there exists a constant $C(\phi)$ such that we have

$$\left| \int \phi d\mu - \int \phi d\mu_\varepsilon \right| \leq C(\phi)\sigma(\varepsilon).$$

We call σ a *speed of stochastic stability*.

Remark 2.4. If the system has this property it means that in the limit a typical pseudotrajectory approximates invariant measure of the initial system.

Remark 2.5. If μ and μ_ε have densities ρ and ρ_ε respectively with respect to Lebesgue measure on M and the system has strong stochastic stability with speed σ than it also has stochastic stability with speed σ for observables from \mathcal{B}_c .

Fix $\phi \in C(M)$ then

$$\begin{aligned} \left| \int \phi d\mu - \int \phi d\mu_\varepsilon \right| &= \left| \int \phi(\rho - \rho_\varepsilon) d\text{Leb} \right| \leq \\ &\leq \int |\phi| |\rho - \rho_\varepsilon| d\text{Leb} \leq \left(\sup_{x \in M} |\phi(x)| \right) \sigma(\varepsilon). \end{aligned}$$

Remark 2.6. If the Markov chains $\{X_\varepsilon\}$ are generated by random maps (see [6, 11] for details), then the definition resembles the definition of inverse shadowing in [20]. However for this classical inverse shadowing it is shown that it can be quantitatively good only for hyperbolic systems (see [18]).

Definition 14. The map f has *stochastic shadowing with accuracy σ for observables from \mathcal{B} with respect to $\{X_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)}$* for every $0 < \varepsilon < \varepsilon_0$ for every $\phi \in \mathcal{B}$ for every $N \in \mathbb{N}$ there exists a constant $C(\phi)$ and a set $B_{\varepsilon, N}^\phi$ of realizations of X_ε such that for every $\bar{x} = \{x_k\}_{k=0}^\infty \in B_{\varepsilon, N}^\phi$ there exists a point $p \in M$ such that the following holds for $n > N$:

$$\begin{aligned} \left| \int \phi dS_n(\bar{x}) - \int \phi d\mu_\varepsilon \right| &\leq C(\phi)\sigma(\varepsilon), \\ \left| \int \phi dS_n(p) - \int \phi d\mu \right| &\leq C(\phi)\sigma(\varepsilon), \\ \left| \int \phi dS_n(\bar{x}) - \int \phi dS_n(p) \right| &\leq C(\phi)\sigma(\varepsilon), \end{aligned}$$

and $\mu_\varepsilon(B_{\varepsilon, N}^\phi) \rightarrow 1$ as $N \rightarrow \infty$.

We call σ a *stochastic shadowing accuracy function*.

Remark 2.7. If a point x shadows a pseudotrajectory $\bar{x} = \{x_k\}$ in a usual sense, that is

$$\text{dist}(x_n, f^n(x)) \leq \delta, \quad n \in \mathbb{N},$$

then for every C -Lipschitz $\phi : M \rightarrow \mathbb{R}$ we have for every natural n that

$$\left| \int \phi dS_n(\bar{x}) - \int \phi dS_n(x) \right| \leq C\delta.$$

Here is an explanation why does one need another notion similar to stochastic stability.

Remark 2.8. It is easy to see that for every $\phi \in C(M)$ and $n \in \mathbb{N}$ we can write the following representations:

$$\begin{aligned} &\left(\int \phi dS_n(x) - \int \phi dS_n(\bar{x}) \right) = \\ &= \left(\int \phi d\mu - \int \phi d\mu_\varepsilon \right) + \\ (2.2) \quad &\left(\int \phi dS_n(x) - \int \phi d\mu \right) + \left(\int \phi d\mu_\varepsilon - \int \phi dS_n(\bar{x}) \right) \end{aligned}$$

and

$$\begin{aligned} & \left(\int \phi dS_n(x) - \int \phi d\mu \right) = \\ & = \left(\int \phi dS_n(x) - \int \phi dS_n(\bar{x}) \right) + \left(\int \phi dS_n(\bar{x}) - \int \phi d\mu_\varepsilon \right) + \left(\int \phi d\mu_\varepsilon - \int \phi d\mu \right). \end{aligned}$$

Then it is easy to deduce from stochastic stability with speed σ that *in the limit* μ_ε -almost every erroneous trajectory has the same statistics as μ -almost every exact one:

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \left| \int \phi dS_n(x) - \int \phi dS_n(\bar{x}) \right| &\leq C(\phi)\sigma(\varepsilon), \\ \overline{\lim}_{n \rightarrow \infty} \left| \int \phi dS_n(x) - \int \phi d\mu \right| &\leq C(\phi)\sigma(\varepsilon). \end{aligned}$$

A similar observation can be found in [3]. However what one really wants to have is that for many erroneous trajectories one can find an exact trajectory such that for *the same number of iterations* the ergodic average for the exact one is close to μ and also close to the ergodic average for the erroneous one.

Moreover since one usually pick erroneous trajectory at random it is highly desirable so that one could choose a lower bound for the necessary number of iterations uniformly for those many erroneous trajectories.

And as there is no kind of “speed of ergodic theorem” statement in general, terms (2.2) for a fixed n can fluctuate a lot, depending on \bar{x} and x .

We also define one classical notion we will use later.

Definition 15. Let string $[a, b]$ be a finite set of consecutive integers $\{a, a + 1, \dots, b\}$.

Definition 16. f has specification property if for every $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that for every choice of points $x_1, x_2 \in M$ and strings $A_1 = [a_1, b_1]$ and $A_2 = [a_2, b_2]$ with $a_2 - b_1 > N(\varepsilon)$ and every integer $p > b_2 - a_1 + N(\varepsilon)$ there exists a periodic point $x \in M$ with period p such that

$$\begin{aligned} \text{dist}(f^i(x), f^i(x_1)), \quad i \in A_1; \\ \text{dist}(f^i(x), f^i(x_2)), \quad i \in A_2. \end{aligned}$$

2.1. Uniform stability and shadowing. Let $\{f_\alpha\}_{\alpha \in \mathcal{A}}$, where \mathcal{A} is a compact metric space, be a family of maps from M to itself.

Assumption 5. Assume each f_α has a unique physical measure $\mu^{(\alpha)}$.

Let be $\{X_\varepsilon^{(\alpha)}\}_{\alpha \in \mathcal{A}, \varepsilon \in (0, \varepsilon_0)}$ a family of Markov chains on M such that every realisation of $X_{\alpha, \varepsilon}$ is an ε -pseudotrajectory of f_α .

Assumption 6. Assume for every (α, ε) there is an ergodic stationary measure $\mu_\alpha^{(\varepsilon)}$ for $X_\varepsilon^{(\alpha)}$.

We define here uniform versions of the shadowing and stability notions, requiring one accuracy function to suit all the maps of the family.

Definition 17. The family $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ has *uniform* strong stochastic stability with speed σ with respect to $\{X_\varepsilon^{(\alpha)}\}_{\varepsilon \in (0, \varepsilon_0), \alpha \in \mathcal{A}}$ if for every $\alpha \in \mathcal{A}$ the map f_α has strong stochastic stability with speed σ with respect to $\{X_\varepsilon^{(\alpha)}\}_{\varepsilon \in (0, \varepsilon_0)}$.

We call σ a *speed of uniform strong stochastic stability*.

Definition 18. The family $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ has *uniform stochastic stability* with speed σ for observables from \mathcal{B} . with respect to $\{X_\varepsilon^{(\alpha)}\}_{\varepsilon \in (0, \varepsilon_0), \alpha \in \mathcal{A}}$ if for every $\alpha \in \mathcal{A}$ the map f_α has stochastic stability with speed σ with for observables from \mathcal{B} respect to $\{X_\varepsilon^{(\alpha)}\}_{\varepsilon \in (0, \varepsilon_0)}$ with the same $C(\phi)$ for every $\phi \in \mathcal{B}$.

We call σ a *speed of uniform stochastic stability*.

Definition 19. The family $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ is said to have *uniform classical shadowing with accuracy σ with respect to $\{X_\varepsilon^{(\alpha)}\}_{\varepsilon \in (0, \varepsilon_0), \alpha \in \mathcal{A}}$* if for every $\alpha \in \mathcal{A}$ the map f_α has classical shadowing with accuracy σ with respect to $\{X_\varepsilon^{(\alpha)}\}_{\varepsilon \in (0, \varepsilon_0)}$

We call σ a *uniform classical shadowing accuracy function*.

Definition 20. The family $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ has *uniform stochastic shadowing property with accuracy σ for observables from \mathcal{B} with respect to $\{X_\varepsilon^{(\alpha)}\}_{\varepsilon \in (0, \varepsilon_0), \alpha \in \mathcal{A}}$* if for every $\alpha \in \mathcal{A}$ the map f_α has stochastic shadowing property for observables from \mathcal{B} with accuracy σ with respect to $\{X_\varepsilon^{(\alpha)}\}_{\varepsilon \in (0, \varepsilon_0)}$ with the same $C(\phi)$ for every $\phi \in \mathcal{B}$.

We call σ a *uniform stochastic shadowing accuracy function*.

Here are several definitions we will use in Section 5.

Put $c = 1/2$. For $s \in [\sqrt{2}, 2]$ denote by f_s a tent map with slope s :

$$f_s(x) = \begin{cases} sx & x < c, \\ s - sx & c < x < 1. \end{cases}$$

Fix an $s_0 \in [\sqrt{2}, 2]$ and denote $g = f_{s_0}$.

Definition 21. We call a continuous map $\hat{g} : [0, 1] \rightarrow [0, 1]$ *piecewise expanding C^r unimodal* for $r \geq 1$ if there exists $a \in (0, 1)$ such that $g|_{[0, a]}$ is strictly increasing and extends to a C^r map in the neighborhood of $[0, a]$ and $g|_{[a, 1]}$ is strictly decreasing and extends to a C^r map in the neighborhood of $[a, 1]$.

Definition 22. Let $r \geq r_0 \geq 2$ be integers. A $C^{r_0, r}$ -*perturbation* of g is a family of piecewise expanding C^r unimodal maps $g_t : [0, 1] \rightarrow [0, 1]$, $t \in [-1, 1]$ with $f_0 = f$ and satisfying the following properties: there exists neighborhoods I_1, I_2 of $[0, c]$ and $[c, 1]$ respectively so that the C^r norm of the extension of $g_t|_{I_i}$, $i = 1, 2$ is uniformly bounded for small $|t|$ and so that

$$\|(g - g_t)|_{I_i}\|_{C^{r-1}} \leq Ct, \quad i = 1, 2$$

for some $C > 0$. The map $(x, t) \rightarrow g_t(x)$ extends to a C^{r_0} function on a neighbourhood of $(I_1 \cup I_2) \times \{0\}$.

Definition 23. Let $r \geq r_0 \geq 2$ be integers. A $C^{r_0, r}$ -*perturbation* of g is *tangent to the topological class of g* if there exists a $C^{2, 2}$ -perturbation \tilde{g}_t of f such that

$$\sup_{x \in M} |g_t(x) - \tilde{g}_t(x)| = \mathcal{O}(t^2)$$

and homeomorphisms h_t with $h(c) = c$ such that $\tilde{g}_t = h_t \circ g_t \circ h_t^{-1}$.

3. MAIN RESULTS

For the sake of brevity we will not mention the perturbation with respect to which stochastic shadowing or stochastic stability holds, always meaning $\{X_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)}$ that we fixed before.

First we state some general theorems.

Theorem 1. *If the map f has stochastic shadowing with accuracy σ then for observables from \mathcal{B} it has stochastic stability with speed σ for observables from \mathcal{B} .*

Denote by \mathcal{B}_{Lip} the set of Lipschitz functions from M to \mathbb{R} .

Theorem 2. *If the map f has stochastic stability with speed γ for observables from \mathcal{B}_{Lip} and classical shadowing with accuracy σ and exponential decay of correlations for observables from \mathcal{B}_{Lip} then it has stochastic shadowing with accuracy $\max(\sigma, \gamma)$ for observables from \mathcal{B}_{Lip} .*

Theorem 3. *If f is continuous and has stochastic stability with speed σ for observables from \mathcal{B}_c and the following conditions are satisfied:*

- f has specification property,
- μ is not a finite sum of δ measures,
- the stationary measure μ_ε has exponential decay of correlations for observables from \mathcal{B}_c

then it has stochastic shadowing with accuracy σ for observables from \mathcal{B}_c .

As specification property is known to hold for certain systems (see [25]) and for interval maps topological mixing implies specification (see [5]) and strong stochastic stability is proved for many systems (see [27, 4]) we can state the following result:

Corollary 1. *The following systems have stochastic shadowing for observables from \mathcal{B}_c , with accuracy equal to the speed of stochastic stability:*

- uniformly hyperbolic attractors,
- expanding maps of closed manifolds,
- topologically mixing piecewise expanding maps of the interval,
- topologically mixing smooth unimodal maps of the interval with exponential decay of correlations (see [27] for exact conditions to get exponential decay).

Remark 3.1. Classes of perturbations in the previous theorem for which stochastic stability is proved are somewhat different.

We also state uniform versions of some theorems above:

Theorem 4. *Let family $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ have uniform stochastic shadowing with accuracy σ for observables from \mathcal{B} then it has uniform stochastic stability with speed σ for observables from \mathcal{B} .*

Theorem 5. *Let family $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ have uniform stochastic stability with speed γ for observables from \mathcal{B}_{Lip} , uniform classical shadowing with accuracy σ and for each $\alpha \in \mathcal{A}$ the map f_α has exponential decay of correlations for observables from \mathcal{B}_{Lip} then the family $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ has uniform stochastic shadowing with accuracy $\max(\sigma, \gamma)$ for observables from \mathcal{B}_{Lip} .*

Theorem 6. *Let family $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ have uniform stochastic stability with speed σ for observables from \mathcal{B}_c and for every $\alpha \in \mathcal{A}$ the following conditions are satisfied:*

- f_α is continuous,

- f_α has specification property,
- $\mu^{(\alpha)}$ is not a finite sum of δ measures,
- the stationary measure $\mu_\varepsilon^{(\alpha)}$ has exponential decay of correlations for observables from \mathcal{B}_c

then $\{f_\alpha\}_{\alpha \in A}$ has uniform stochastic shadowing with accuracy σ for observables from \mathcal{B}_c .

Finally we state results concerning the example.

Theorem 7. For a full measure (in $[\sqrt{2}, 2]$) set of parameters s for any $\alpha, L > 0$ the map f_s does not have standard shadowing with accuracy $\sigma(\varepsilon) = L\varepsilon^\alpha$.

Nevertheless for every $s \in [\sqrt{2}, 2]$ the map f_s has stochastic shadowing with speed $\sigma(\varepsilon) = \varepsilon$ for observables from $C(M)$ for the family of uniform perturbations.

Remark 3.2. However despite the same accuracy function suits all the f_s the stochastic shadowing may be nonuniform since we can not guarantee uniformity of constants $C(\phi)$.

Theorem 8. Let $s \in [\sqrt{2}, 2]$ be a nonperiodic parameter. If $\{g_t\}$ is a $C^{2,2}$ -perturbation of f_s , tangent to its topological class, then there exists $t_0 > 0$ such that the family $\{g_t\}_{|t| < t_0}$ has uniform stochastic shadowing with speed $\sigma(\varepsilon) = \varepsilon$ with respect to $\{X_\varepsilon\}$ for observables from $C(M)$ for the family of uniform perturbations.

4. PROOFS

Theorem 4.1 (Theorem 1). If the map f has stochastic shadowing with accuracy σ for observables from \mathcal{B} then it has stochastic stability with speed σ for observables from \mathcal{B} .

Proof. Let σ be an stochastic shadowing accuracy function for f . Let ε_0 be a number from the definition of stochastic shadowing. Fix $\varepsilon < \varepsilon_0$ and $\phi \in \mathcal{B}$. There exists a set $B_{\varepsilon, n}^\phi$ from the definition of stochastic shadowing and a natural number N such that $B_{\varepsilon, N}^\phi \neq \emptyset$.

Fix a natural $n > N$. Fix an ε -pseudotrajectory $\bar{x} \in B_{\varepsilon, N}^\phi$. Then there exists a point p such that

$$\begin{aligned} \left| \int \phi d\mu - \int \phi d\mu_\varepsilon \right| &\leq \left| \int \phi d\mu - \int \phi dS_n(x) \right| + \left| \int \phi dS_n(x) - \int \phi dS_n(\bar{x}) \right| + \\ &\quad + \left| \int \phi dS_n(\bar{x}) - \int \phi d\mu_\varepsilon \right| \leq 3C(\phi)\sigma(\varepsilon). \end{aligned}$$

Thus we have stochastic stability with the desired speed. \square

Corollary 4.1. Lower bounds on the speed of stochastic stability would imply lower bounds on the accuracy of stochastic shadowing.

Theorem 4.2. If the map f has stochastic stability with speed γ for observables from \mathcal{B}_{Lip} and classical shadowing with accuracy σ and property A for observables from \mathcal{B}_{Lip} then it has stochastic shadowing with accuracy $\max(\sigma, \gamma)$ for observables from \mathcal{B}_{Lip} .

Proof. Fix $\varepsilon < \varepsilon_0$ and a C -Lipschitz function $\phi : M \rightarrow \mathbb{R}$. Set $B_{\varepsilon, N}^\phi = A_{\varepsilon, \sigma(\varepsilon), N}^\phi$ and fix N large enough so that $B_{\varepsilon, N}^\phi \neq \emptyset$.

Fix $\bar{x} \in B_{\varepsilon, N}^\phi$. By Remark 2.7 there is a point x such that

$$\left| \int \phi dS_n(\bar{x}) - \int \phi dS_n(x) \right| < C\sigma(\varepsilon), \quad n \in \mathbb{N}.$$

We have the following estimate for $n > N$

$$\begin{aligned} \left| \int \phi d\mu - \int \phi dS_n(x) \right| &\leq \left| \int \phi d\mu - \int \phi d\mu_\varepsilon \right| + \left| \int \phi d\mu_\varepsilon - \int \phi dS_n(\bar{x}) \right| + \\ &+ \left| \int \phi dS_n(\bar{x}) - \int \phi dS_n(x) \right| \leq \gamma(\varepsilon) + (1 + C)\sigma(\varepsilon). \end{aligned}$$

□

The following readily follows from Remark 2.1.

Corollary 4.2 (Theorem 2). *If the map f has stochastic stability with speed γ for observables from \mathcal{B}_{Lip} and classical shadowing with accuracy σ and exponential decay of correlations for observables from \mathcal{B}_{Lip} then it has stochastic shadowing with accuracy $\max(\sigma, \gamma)$ for observables from \mathcal{B}_{Lip} .*

Remark 4.3. It should be noticed that a conceptually similar statement was proposed in [3].

Remark 4.4. The condition of presence of classical shadowing is not necessary. If we take $M = S^1$ and f to be an irrational rotation then it is easy to see that the system has stochastic shadowing with respect to family of uniform perturbations while it does not have classical shadowing with respect to the same family of perturbations.

Remark 4.5. One can try to go further in quantifying the property of stochastic shadowing, estimating dependence of N and $\mu_\varepsilon(B_{\varepsilon, N}^\phi)$ on ε to distinguish between rotations and hyperbolic systems.

Remark 4.6. A simple note to make is that if we drop the requirement that the measure μ is ergodic and physical and take $f = Id$ then it is easy to see that the system does not have stochastic shadowing while it has stochastic stability (with any speed for a wide choice of perturbations).

Theorem 4.3 (Theorem 3). *If f is continuous and has stochastic stability with speed σ for observables from \mathcal{B}_c and the following conditions are satisfied:*

- *f has specification property,*
- *μ is not a finite sum of δ measures,*
- *the stationary measure μ_ε has exponential decay of correlations for observables from \mathcal{B}_c*

then it has stochastic shadowing with accuracy σ for observables from \mathcal{B}_c .

Proof. Fix $\phi \in \mathcal{B}_c$ and $0 < \varepsilon < \varepsilon_0$.

By Remark 2.1 the system has property A. Set $B_{\varepsilon, N}^\phi = A_{\varepsilon, \sigma(\varepsilon), N}^\phi$ and fix N large enough so that $B_{\varepsilon, N}^\phi \neq \emptyset$ and $2/N < \sigma(\varepsilon)$. Fix $\bar{x} \in B_{\varepsilon, N}^\phi$.

Presence of specification implies that periodic measures (measures, supported on periodic orbits of f) are weak* dense among all invariant probability measures for f (see [25]).

Since μ is not a finite sum of δ -measures, to approximate it by a periodic measure sufficiently good, the period should be large enough. Thus as ϕ is continuous, there exists a periodic point p_n of period $n > N$ such that

$$\left| \int \phi dS_n(p_n) - \int \phi d\mu \right| < \sigma(\varepsilon).$$

Consider any integer $r = nk + m > N$ where $0 < m < n$. Note that

$$\begin{aligned} Y &= \int \phi dS_r(p_n) - \int \phi dS_{nk}(p_n) = \\ &= \sum_{i=0}^{nk-1} \left(\frac{1}{nk} + \frac{1}{nk+m} \right) \phi(f^i(x)) - \sum_{i=0}^m \frac{1}{nk+m} \phi(f^i(x)) = \\ &= \frac{1}{nk+m} \left(\sum_{j=0}^m \sum_{i=0}^{nk-1} \frac{\phi(f^i(x))}{nk} - \sum_{j=0}^m \sum_{i=0}^{nk-1} \frac{\phi(f^j(x))}{nk} \right). \end{aligned}$$

Then

$$|Y| \leq \frac{2(m+1)}{nk+m} \sup_{x \in M} |\phi(x)| \leq \frac{2}{k} < \frac{2}{N} < \sigma(\varepsilon).$$

Therefore we have

$$\left| \int \phi dS_r(p_n) - \int \phi d\mu \right| < \sigma(\varepsilon).$$

This implies that

$$\begin{aligned} &\left| \int \phi dS_r(\bar{x}) - \int \phi dS_r(p_n) \right| \leq \\ &\leq \left| \int \phi dS_r(\bar{x}) - \int \phi d\mu_\varepsilon \right| + \left| \int \phi d\mu_\varepsilon - \int \phi d\mu \right| + \\ &\quad + \left| \int \phi d\mu - \int \phi dS_r(p_n) \right| \leq 3\sigma(\varepsilon). \end{aligned}$$

□

Remark 4.7. In fact, a summable decay of correlations is probably enough to get the same conclusion.

Remark 4.8. There are properties weaker than the specification property that can guarantee density of periodic measures among all invariant ones (see [10, 15]).

The proofs of uniform versions of the above theorems are just repetitions of proofs of the non-uniform versions.

Despite there is a vast literature on stochastic stability (see references in [27] and more recent in [24, 23]), there are not many results giving exact form of its speed. See [1] for a result about expanding maps of a circle. There exist some numerical studies of speed of stochastic stability in different situations in [16].

However these speeds can sometimes be obtained rather easily by direct applications of results of Keller-Liverani [13]. We give an example when it happens in Section 5.

5. EXAMPLE

It was shown in [8] that for almost all parameters s tent maps f_s have shadowing property. However we are going to show that it is by no means controllable, i.e. not Lipschitz and not uniform. Afterwards we show that nevertheless tent maps have stochastic shadowing with linear accuracy function.

Let $M = [0, 1]$. Set

$$E_{m,\delta} = \{f^m(y) \mid |f^i(y) - f^i(c)| \leq \delta, \quad 0 \leq i \leq m\},$$

$$m(\delta) = \inf \{m \in \mathbb{N} \mid c \in E_{m,\delta}\}.$$

Here we mean that $\inf(\emptyset) = \infty$.

We say that a parameter $s \in [\sqrt{2}, 2]$ is N -periodic if the critical point c is periodic for f_s with (minimal) period equal to N .

For $a, b \in M$ denote

$$\langle a, b \rangle = \begin{cases} [a, b], & a < b; \\ [b, a], & a > b. \end{cases}$$

For an N -periodic s denote

$$\xi_s = \min_{0 < k < N} |c - f_s^k(c)|$$

and for $\varepsilon, \delta > 0$ set

$$n_s(\delta, \varepsilon) = \min \{n \in \mathbb{N} \mid c \in \langle f^n(f(c) - \delta), f^n(f(c) + \varepsilon) \rangle\}.$$

Lemma 5.1. *If s is N -periodic, then for every $\delta, \varepsilon < \xi_s s^{-N}/2$ we have*

$$n_s(\delta, \varepsilon) = N - 1,$$

$$m(\delta) = N - 1.$$

Proof. Denote

$$I_k = \langle f^k(f(c) - \delta), f^k(f(c) + \varepsilon) \rangle.$$

We have $c_{k+1} \in I_k$ while $c \notin I_k$. Due to the assumptions on δ and ε we have $\text{dist}(I_k, c) > 0$ as $k < N$. It means that $c \notin I_k$ until $k = N$. \square

The following theorem is proved in [8].

Theorem 5.1. *Let \mathcal{P} be a set of parameters from $[\sqrt{2}, 2]$ such that for every $\delta > 0$ the number $m(\delta)$ is finite.*

Then \mathcal{P} has full measure in $[\sqrt{2}, 2]$ and for every $s \in \mathcal{P}$ for

$$\varepsilon < \delta(s-1)(m(\delta)+1)^{-1}s^{-m(\delta)-1}$$

every ε -pseudotrajectory can be

$$((s-1)^{-1} + s^4)(m(\delta)+1)s^{m(\delta)+1}\varepsilon$$

-shadowed.

In particular for every N -periodic parameter s_N the map f_{s_N} has standard shadowing with accuracy

$$\sigma(\varepsilon) = ((s-1)^{-1} + s^4)(N+1)s^{N+1}\varepsilon.$$

Denote

$$c_k = f^k(c).$$

We need a following classical result in one-dimensional dynamics (for the proof see, for example [8]):

Lemma 5.2. *The set of parameters s such that there exists N such that s is N -periodic, is dense in $[\sqrt{2}, 2]$ but has zero measure.*

Lemma 5.3 (part of Theorem 7). *For a full measure (in $[\sqrt{2}, 2]$) set of parameters s for any $\alpha, L > 0$ the map f_s does not have standard shadowing with accuracy $\sigma(\varepsilon) = L\varepsilon^\alpha$.*

Proof. Consider

$$\begin{aligned} x_0 &= c, \\ x_1 &= f(c) + \varepsilon, \\ x_k &= f^{k-1}(x_1), k \geq 2. \end{aligned}$$

Suppose \bar{x} can be ω -shadowed by a point y . Then consider $x_\varepsilon = f^2(y)$. We know that x_ε is not equal to x_2 , otherwise $f(y) > f(c)$ which can not hold for an exact trajectory.

Sublemma 5.4. *We prove that*

$$\sigma(\varepsilon) \geq s^{n(\varepsilon)-1}\varepsilon,$$

where

$$n(\varepsilon) = \min \{n \in \mathbb{N} \mid c \in \text{Int}(\langle f^n(x_\varepsilon), f^n(x_2) \rangle)\}.$$

Proof of sublemma. It is easy to see that the distance between $f^k(x_\varepsilon)$ and $f^k(x_2)$ grows exponentially while $k < n(\varepsilon)$.

Thus we can estimate from below the accuracy of shadowing:

$$\sigma(\varepsilon) \geq \left| f^{n(\varepsilon)-1}(x_\varepsilon) - f^{n(\varepsilon)-1}(x_2) \right| = s^{n(\varepsilon)-1} |x_\varepsilon - x_2| \geq s^{n(\varepsilon)-1}\varepsilon.$$

□

If $\sigma(\varepsilon) = \varepsilon^\alpha$ for $0 < \alpha < 1$ then

$$\varepsilon \leq s^{\frac{1-n(\varepsilon)}{1-\alpha}}$$

which implies

$$\sigma(\varepsilon) \leq s^\alpha \frac{1-n(\varepsilon)}{1-\alpha}$$

If c is not periodic then it is easy to see that n goes to infinity as ε goes to 0. By [7] (or, more generally [22]) the set of parameters with non-periodic c has full measure in $[\sqrt{2}, 2]$.

□

Lemma 5.5. *Let U be an open subset of $[\sqrt{2}, 2]$. Then*

$$\inf_{s \in U} \sup_{\delta, \varepsilon > 0} n_s(\delta, \varepsilon) = \infty.$$

Proof. Fix open $U \subset [\sqrt{2}, 2]$. Then for every $N \in \mathbb{N}$ there exists an $n > N$ -periodic parameter $s \in U$. Since we know that

$$\sup_{\delta, \varepsilon > 0} n_s(\delta, \varepsilon) = n,$$

the statement of the lemma easily follows.

□

For an open set $U \subset [\sqrt{2}, 2]$ denote U_{shad} a subset parameters s of U such f_s has standard shadowing (for some accuracy function).

Corollary 5.6. *For any open $U \subset [\sqrt{2}, 2]$ we cannot choose a single constant L such that for every parameter $s \in U_{\text{shad}}$ the map f_s has standard shadowing with accuracy $\sigma(\varepsilon) = L\varepsilon$.*

If there is some regularity of dependence of transfer operator spectral properties on the parameter (usually it is studied in the context “statistical stability”, i.e. continuous dependence of the physical measure on f , or its quantitative versions like “linear response”) then it is possible to get uniform stochastic stability. We illustrate it by an example.

Fix a nonperiodic parameter $s \in [\sqrt{2}, 2]$. Denote $g = f_s$.

Let $\{g_t\}$ be a $C^{2,2}$ -perturbation of g , tangent to its topological class.

For $\phi : [0, 1] \rightarrow \mathbb{R}$ define variation of ϕ by

$$\text{var}_{[0,1]}(\phi) = \sup \left\{ \left| \sum \phi(x_{k+1}) - \phi(x_k) \right| \mid n \geq 1, 0 \leq x_0 < \dots < x_n \leq b \right\}.$$

We consider Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}}) = (BV, \|\cdot\|_{BV})$ where

$$\|\phi\|_{BV} = \text{var}_{[0,1]}(\phi) + \|\phi\|_{L^1}$$

and BV is a set of functions ϕ from L^1 such that $\|\phi\|_{BV} < \infty$.

For $\phi \in BV$ define

$$L_{t,0}\phi(x) = \sum_{g_t(y)=x} \frac{\phi(y)}{|g_t'(y)|}.$$

Let $\{X_\varepsilon^{(t)}\}_{t \in [-1,1], \varepsilon \in (0, \varepsilon_0)}$ be a family of uniform perturbations of g_s .

Note that in our case of $M = [0, 1]$ as $g_s([0, 1])$ is strictly inside of $(0, 1)$, we can write for every natural n

$$\left(\frac{d}{\text{Leb}} P_\varepsilon^n(x, \cdot) \right) (y) = \theta_\varepsilon(y) = \begin{cases} \frac{1}{\text{Leb}(B_\varepsilon(0))}, & |y| \leq \varepsilon; \\ 0, & \text{otherwise.} \end{cases}$$

Define for $0 < \varepsilon < \varepsilon_0$

$$L_{t,\varepsilon}\phi(x) = \int (L_{t,0}\phi)(x - \omega) \theta_\varepsilon(\omega) d\omega.$$

Here is a particular case of the abstract setting introduced in [13]:

For a bounded linear operator $Q : \mathcal{B} \rightarrow \mathcal{B}$ denote

$$\|Q\|_{\text{KL}} = \sup \{ \|Qf\|_{L^1} \mid f \in \mathcal{B}, \|f\|_{\mathcal{B}} \leq 1 \}.$$

Consider a family $\{P_\varepsilon\}_{\varepsilon \geq 0}$ of bounded linear operators on $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ with the following properties:

- (1) there are constants C_1, L such that for all $\varepsilon \geq 0$

$$\|P_\varepsilon^n\|_{L^1} \leq C_1 L^n, \quad \forall n \in \mathbb{N};$$

- (2) there are constants C_2, C_3 and $\alpha \in (0, 1)$, $\alpha < L$ such that for all $\varepsilon \geq 0$

$$\|P_\varepsilon^n \phi\|_{\mathcal{B}} \leq C_2 \alpha^n \|\phi\|_{\mathcal{B}} + C_3 L^n \|\phi\|_{L^1}, \quad \forall n \in \mathbb{N}, \forall \phi \in \mathcal{B};$$

- (3) if $z \in \sigma(P_\varepsilon)$ and $|z| > \alpha$ then z is not in the residual spectrum of P_ε ;

- (4) there is monotone upper-semicontinuous function $\tau : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that $\tau(\varepsilon) > 0$ if $\varepsilon > 0$ and $\tau(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and

$$\|P_0 - P_\varepsilon\|_{\text{KL}} \leq \tau(\varepsilon).$$

It follows that P_ε has 1 as simple eigenvalue for every $\varepsilon \geq 0$. Denote the corresponding eigenfunction by χ_ε .

The next statement follows from [13]:

Theorem 5.2. *If a family of operators $\{P_\varepsilon\}_{\varepsilon \geq 0}$ satisfies conditions 1-4 then there exist constants $\eta = \eta(\alpha, L) > 0$ and ε_0 such that for every $\varepsilon < \varepsilon_0$*

$$\|\chi_0 - \chi_\varepsilon\|_{L^1} \leq (\tau(\varepsilon))^\eta.$$

It is proved in [2] that the family $\{L_{t,0}\}$ satisfies conditions 1-4.

Theorem 5.2 can also be applied to random perturbations. To do this we need another important statement (Corollary, p. 327 from [12]):

Theorem 5.3. *The family L_ε satisfies conditions 1-4 for $\tau(\varepsilon) = L\varepsilon$ for some $L > 0$.*

Corollary 5.7. *There exists $t_0 > 0$ such that the family $\{g_t\}_{|t| < t_0}$ has uniform strong stochastic stability with speed $\sigma(\varepsilon) = L\varepsilon$ with respect to the family of uniform perturbations.*

Corollary 5.8 (Theorem 8). *There exists $t_0 > 0$ such that the family $\{g_t\}_{|t| < t_0}$ has uniform stochastic shadowing with speed $\sigma(\varepsilon) = \varepsilon$ for observables from $C(M)$ with respect to the family of uniform perturbations.*

If we do not care about uniformity, it is possible to prove an analog of Theorem 5.3 (see [4]) even for $g = f_s$ with a periodic parameter s . Therefore we have the following theorem:

Theorem 5.4. *For every $s \in [0, 1]$ the map f_s has strong stochastic stability with speed $\sigma(\varepsilon) = L\varepsilon$ for some $L > 0$ with respect to the family of uniform perturbations.*

Corollary 5.9 (part of Theorem 7). *For every $s \in [\sqrt{2}, 2]$ the map f_s has stochastic shadowing with speed $\sigma(\varepsilon) = \varepsilon$ for observables from $C(M)$ with respect to the family of uniform perturbations.*

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DMA, ÉCOLE NORMALE SUPÉRIEURE, 75005 PARIS, FRANCE, AND CHEBYSHEV LABORATORY, SAINT PETERSBURG STATE UNIVERSITY, 14TH LINE OF VASILJEVSKY ISLAND, 29B, 199178, SAINT-PETERSBURG, RUSSIA