Nonparametric statistical inference for the context tree of a stationary ergodic process*

Sandro Gallo

Mathematics Departement
Federal University of São Carlos, Brazil
e-mail: sandro.gallo@ufscar.br

Florencia Leonardi†

Institute of Mathematics and Statistics
University of São Paulo, Brazil
e-mail: florencia@usp.br

Abstract: We consider the problem of estimating the context tree of a stationary ergodic process with finite alphabet without imposing additional conditions on the process. As a starting point we introduce a *Hamming* metric in the space of irreducible context trees and we use the properties of the weak topology in the space of ergodic stationary processes to prove that if the Hamming metric is unbounded, there exist no consistent estimators for the context tree. Even in the bounded case we show that there exist no two-sided confidence bounds. However we prove that one-sided inference is possible in this general setting and we construct a consistent estimator that is a lower bound for the context tree of the process with an explicit formula for the coverage probability. We develop an efficient algorithm to compute the lower bound and we apply the method to test a linguistic hypothesis about the context tree of codified written texts in European Portuguese.

Primary 62M09, 62G15, 62G20; secondary 60G10, 60J10.
Keywords and phrases: variable length markov chain, context tree, confidence bounds, consistent estimation, nonparametric inference.

1. Introduction

In this work we address the issue of whether or not there exist consistent estimators (and confidence bounds) for the *context tree* of a discrete time stationary ergodic process with finite alphabet. In words, the context tree of a stochastic process is a set of finite strings or left-infinite sequences that determines the portion of the past the process has to look at in order to decide the distribution

*This article was produced as part of the activities of FAPESP Research, Innovation and Dissemination Center for Neuromathematics, grant 2013/07699-0, São Paulo Research Foundation. It has also received financial support from the projects *Stochastic systems: equilibrium and non-equilibrium, limits in scale and percolation*, grant CNPq 474233/2012-0, and *Stochastic chains of long range*, grant FAPESP 2015/09094-3.

†Partially supported by a CNPq-Brazil fellowship 304836/2012-5 and a L'Oréal Fellowship for Women in Science.

of its next symbol. For example, an i.i.d. process has the empty string as context tree since it has no dependence on the past. A k-steps Markov chain has a context tree containing at least one string of length k, and a non-Markovian chain (sometimes coined infinite memory process) has a context tree having at least one left-infinite sequence.

Finite context trees were introduced by Rissanen (1983) as an efficient tool for data compression. The corresponding processes were originally called *Variable length Markov Chains* (VLMC) and its estimation was first addressed in Bühlmann and Wyner (1999). Recently, they have received increasing attention in the applied statistics literature, being used in a wide range of problems from different areas (Bejerano and Yona, 2001; Dalevi, Dubhashi and Hermansson, 2006; Busch et al., 2009; Galves et al., 2012, for instance). Its success in real word applications seems to stem from its parsimony (including memory only where data needs) and its capacity to capture structural dependencies in the data. The counterpart of the model, when compared to finite step Markov models for instance, is that estimation is a much complicated task. When he introduced the model, Rissanen (1983) also provided an algorithm for recovering the context tree out of a given sample. Since then, a large part of the related statistical literature has focussed on consistent estimation of the context tree in the finite and infinite memory case, an incomplete list includes Bühlmann and Wyner (1999); Galves and Leonardi (2008); Collet, Galves and Leonardi (2008); Csiszár and Talata (2006); Garivier and Leonardi (2011).

Most of the above cited works make some assumptions on the processes, such as lower bounding the transition probabilities or imposing mixing conditions, additionally to ergodicity. In the present paper, we precisely refer to our statistical inference problem as *nonparametric* because we make no further assumptions concerning the distribution of the process, else than ergodicity. In this nonparametric setting, Csiszár and Talata (2006) proved the consistency of the *Bayesian Information Criterion* (BIC) when the context trees are truncated to a given finite length (the truncation being necessary only for infinite context trees). Interestingly, nothing has been done concerning confidence bounds as far as we know.

Given a sample of a stationary ergodic process, it is natural to wonder whether this process has a finite or infinite context tree. This cannot be consistently decided in this general class (Bailey, 1976; Morvai and Weiss, 2005). That is, there exists no two-valued function of the sample which, as the sample increases, stabilizes to the value "yes" for every process having a finite context tree and "no" for every process having an infinite context tree. Thus, when considering the discrete metric in the space of trees, the existence of a universal consistent estimator relies on assumptions that cannot be checked empirically. This situation has its counterpart in nonparametric statistics for i.i.d observations. For instance, Fraiman and Meloche (1999) observed that it is impossible to decide, out of a random sample, whether or not the underlying distribution has a finite number of modes. Assuming a priori that the number of modes is finite, they can be consistently estimated.

In the present work the space of irreducible context trees with finite alphabet is equipped with the Hamming distance. Using only topological arguments we prove that if this metric in the space of trees is unbounded, there exists no consistent estimator of the context tree in the class of stationary ergodic processes. In the bounded metric case, we construct an estimator that is consistent and also a nonparametric lower bound with an explicit coverage probability, based on a result of Garivier and Leonardi (2011). Finally, following Donoho (1988), we also prove that it is not possible to obtain nonparametric upper bounds even in the smaller class of processes having finite context trees. To our knowledge, this is the first work considering the problem of construction of nonparametric confidence bounds for context trees.

Notation, definitions and main results are given in the next section. In Section 3 we show how to compute the lower confidence bound and we present a practical application, testing a linguistic hypothesis about the memory of stressed and non-stressed syllables in European Portuguese written texts. The proofs of the results are given in Section 4.

2. Definitions and results

In this section we present the main definitions and theoretical results of this paper. We begin by describing the notion of irreducible tree and we introduce a *Hamming* distance in the set of all irreducible trees over a finite alphabet. Then we proceed by defining the context tree of a stationary ergodic process and by establishing some topological properties of the set of all stationary ergodic probability measures with respect to the weak topology. The last part of the section is dedicated to the statements of the main results of the paper.

2.1. Metric tree space

Let A be a finite set called alphabet. For any $m \leq n$, we denote by a_m^n the string $a_m \dots a_n$ of symbols in A with length $n - m + 1$. This notation is also valid for $m = -\infty$ in which case we obtain a left-infinite sequence $a_{-\infty}^n$. If $m > n$ we let a_m^n denote the empty string λ. The length of a string w will be denoted by $|w|$. For any $j \in \{0, 1, \dots\}$, we let A^j denote the set of strings in A having length j, in particular $A^0 = \{\lambda\}$. We also let $A^\star = \cup_{j \geq 0} A^j$ denote the set of all finite strings on A and we denote by A^∞ the set of all left-infinite sequences $a_{-\infty}^n$ with symbols in A.

We will need to concatenate strings; for instance, if $v \in A^i$ and $w \in A^j$ are strings of length i and j respectively, then vw denotes the string of length $i + j$ obtained by putting the symbols in w after the ones in v. We also extend concatenation to the case where $v \in A^\infty$ is an infinite string on the left. We say that w is a *suffix* of the sequence s if there exists a sequence v such that $s = vw$. When $|v| \geq 1$ we say that w is a proper suffix of s.

A *tree* τ is any set of strings or perhaps of left-infinite sequences, called *leaves*, such that no $w \in \tau$ is a proper suffix of any other $s \in \tau$. This property

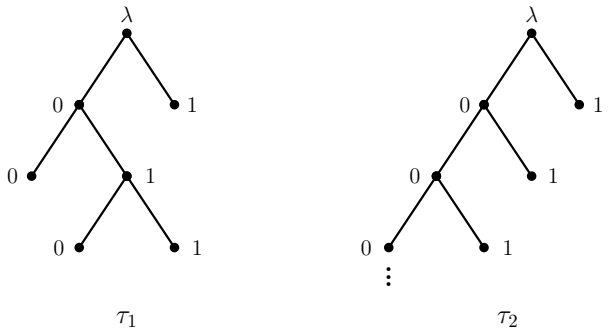


FIG 1. *Graphical representation of the trees* $\tau_1 = \{00, 010, 110, 1\}$ *and* $\tau_2 = \{10_1^i : i = 0, 1, \dots\} \cup \{0^\infty\}$. *In both cases, the contexts corresponds to the sequences obtained by concatenating the symbols from the leaves to the root of the trees. In the case of* τ_2 *we only show the strings of length at most 3.*

enables us to represent the set τ as a graphical rooted tree by identifying the elements in τ with *paths* from the terminal nodes of the tree to the root. As an example of finite tree, consider the set $\tau_1 = \{00, 010, 110, 1\}$ over the alphabet $A = \{0, 1\}$. On the other hand, an example of an infinite tree over A is given by $\tau_2 = \{10_1^i : i = 0, 1, \dots\} \cup \{0^\infty\}$, which has a unique infinite element, the left-infinite sequence 0^∞. The graphical representation of these trees can be found in Fig. 1. Special cases of trees are given by the entire set A^∞ of left-infinite sequences, denoted in this paper by τ^∞, and the tree consisting of the unique empty string λ, denoted by τ^{root}.

We say that the tree τ is *irreducible* if no $w \in \tau$ can be replaced by a proper suffix without violating the tree property. Both trees in Fig. 1 are irreducible, as well as τ^∞ and τ^{root}. An example of a non-irreducible tree is $\tau_3 = \{000, 010, 110, 1\}$, because substituting 000 by 00 leads to τ_1 that satisfies the tree property.

We will call a *node* of τ any finite string that is a suffix of some $s \in \tau$. Sometimes it will be convenient to identify τ with the set of its nodes $\bar{\tau} \subset A^\star$. In fact it is easy to verify that τ uniquely determines $\bar{\tau}$ and *vice versa*. In the case of τ_1 given before, the set $\bar{\tau}_1$ is the set of all strings represented in Fig. 1, that is $\bar{\tau}_1 = \{010, 110, 00, 10, 0, 1, \emptyset\}$. In the case of τ_2 we have $\bar{\tau}_2 = \tau_2 \cup \{0_1^i : i = 0, 1, \dots\}$.

Let $\mathcal{T}$ denote the set of all irreducible trees on A, with the following partial order

$$\tau \prec (\preceq)\, \tau' \quad \text{if and only if} \quad \bar{\tau} \subsetneq (\subseteq)\, \bar{\tau}'.$$

Given a tree $\tau \in \mathcal{T}$ and a constant $k \in \mathbb{N}$, we denote by $\tau|_k$ the truncated tree at level k, defined by the set of its nodes

$$\bar{\tau}|_k = \{v \in \bar{\tau} : |v| \le k\}.$$

Finally, $\mathcal{T}$ is equipped with the *Hamming* distance defined by

$$d_\phi(\tau,\tau') \;=\; \sum_{v\in A^\star} \phi(v)\,\left|\mathbf{1}_{\{v\in\bar\tau\}} - \mathbf{1}_{\{v\in\bar\tau'\}}\right|, \tag{2.1}$$

where $\phi\colon A^\star \to \mathbb{R}^+$. In the summable case $\sum_{v\in A^\star}\phi(v) < +\infty$ we have that $(\mathcal{T}, d_\phi)$ is a bounded metric space.

2.2. Context tree of a stationary ergodic process

Let $\{X_i\colon i\in\mathbb{Z}\}$ be a stationary and ergodic process assuming values in the alphabet A. We denote by $P(a_n^m)$ the stationary probability of the string a_n^m, that is

$$P(a_n^m) \;=\; \mathrm{Prob}\big(X_n^m = a_n^m\big)\,.$$

If $s\in A^\star$ is such that $P(s)>0$ we write

$$P(a|s) \;=\; \mathrm{Prob}\big(X_0 = a \,|\, X_{-|s|}^{-1} = s\big),$$

with the convention that if $s=\emptyset$ then $P(a|s) = \mathrm{Prob}\big(X_0=a\big)$.

A process as above is said to have law, or measure, P.

Definition 2.1. *We say that the string $s\in A^\star$ is a* context *for a process with measure P if it satisfies*

1. *$P(s)>0$ or $s=\emptyset$.*
2. *For all $a\in A$ and all $w\in A^\star$ such that s is suffix of v*

$$\mathit{Prob}\big(X_0 = a \,|\, X_{-|v|}^{-1} = v\big) \;=\; P(a|s)\,. \tag{2.2}$$

3. *No proper suffix of s satisfies 2.*

An infinite context *is a left-infinite sequence $x_{-\infty}^{-1}$ such that its finite suffixes $x_{-n}^{-1}, n=1,2,\dots$ have positive probability but none of them is a context.*

By this definition, the set of contexts of a process with measure P is an irreducible tree, it will be denoted by τ_P.

Example 2.2. Consider the stationary Markov chain of order 3 over the alphabet $A=\{0,1\}$ defined by the transition probabilities

w	$P(0\|w)$	$P(1\|w)$
$ab1$	0.2	0.8
$a00$	0.5	0.5
010	0.3	0.7
110	0.7	0.3

where $a,b\in A$ are arbitrary. This is an example of what is called a *Variable Length Markov Chain* (VLMC). By Definition 2.1, the only contexts of this process are the strings 1, 00, 010 and 110. The context tree τ_P is the tree τ_1 represented in Fig. 1.

Example 2.3. Suppose that the process $\{X_i : i \in \mathbb{Z}\}$ takes values in $\{0,1\}$, and in order to decide the probability distribution of the next symbol based on the past realization, we only need to know the distance to the last occurrence of a 1. Then, for any $k \geq 0$, any $i \geq 1$ and any $v, w \in A^i$

$$P(1|v10^k) = P(1|w10^k).$$

According to Definition 2.1, the strings 10^k, $k \geq 0$, as well as the semi-infinite sequence 0^∞ are context of this process. Therefore, the context tree τ_P is τ_2 shown in Fig. 1.

2.3. The weak topology in the space of stationary ergodic processes

Let Σ be the σ-algebra on $\Omega = A^{\mathbb{Z}}$ obtained as the product of the discrete σ-algebra on A. Let $\mathcal{P}$ denote the set of all stationary ergodic probability measures over (Ω, Σ).

Define the following distance in $\mathcal{P}$

$$D(P,Q) = \sum_{k \in \mathbb{N}} 2^{-k} |P - Q|_k \,,$$

where

$$|P - Q|_k = \sum_{a_1^k \in A^k} |P(a_1^k) - Q(a_1^k)|$$

is the k-th order variational distance. This distance is known in the literature as the *weak distance*, and the topology induced by it is known as the *weak topology* (Shields, 1996, Section I.9).

We now state a basic lemma about the topological properties of the space $\mathcal{P}$ with respect to the weak topology.

Lemma 2.4. *The space $(\mathcal{P}, D)$ is a Baire space.*

2.4. Consistent estimation and confidence bounds

As mentioned in the Introduction, in this paper we are interested in the estimation of properties of the context tree τ_P from samples $X_1, \ldots, X_n$ of size n of the corresponding stationary and ergodic process P. Up to now this problem has been reduced to the consistent identification of the set of contexts (in the finite case) or of a truncated version of the context tree (in the infinite case). The latter corresponds to a special case of our distance d_ϕ; for instance when the interest is in estimating contexts of length at most k we can consider $\phi(v) = 0$ for all $|v| > k$. In the sequel we define the notion of consistency of a sequence of estimators in a general setting.

Let $F : \mathcal{P} \to \mathcal{F}$ be a functional with values in some metric space $(\mathcal{F}, d)$.

Definition 2.5. *We say that F is consistently estimable on $\mathcal{P}$ (in probability) if there exists a sequence $\{F_n\}_{n\in\mathbb{N}}$ of statistics, with $F_n\colon A^n \to \mathcal{F}$, such that for all $P \in \mathcal{P}$*

$$d\left(F_n(X_1,\dots,X_n), F(P)\right) \xrightarrow{P} 0\,.$$

In this case we say that $\{F_n\}_{n\in\mathbb{N}}$ is a consistent estimator for F on $\mathcal{P}$. We say that F is *strongly* consistent on $\mathcal{P}$ if the convergence takes place almost surely with respect to the probability measure P, and in this case we say that $\{F_n\}_{n\in\mathbb{N}}$ is a strongly consistent estimator for F on $\mathcal{P}$.

The following result establishes a necessary condition for the existence of consistent estimators of a bounded real functional defined on $\mathcal{P}$.

Proposition 2.6. *Assume $F\colon \mathcal{P} \to \mathbb{R}$ is bounded (that is there exists $R \in \mathbb{R}$ such that $|F(P)| \le R$ for all $P \in \mathcal{P}$). If F is consistently estimable on $\mathcal{P}$ then F must be continuous on a dense subset of $\mathcal{P}$.*

In this paper we are concerned with the functional $T\colon \mathcal{P} \to \mathcal{T}$ that assigns to any measure $P \in \mathcal{P}$ its associated context tree $\tau_P \in \mathcal{T}$. The first question we address here is if it is possible to decide, out from a finite sample, if the sum of the function ϕ over the nodes of the context tree is finite or not.

Theorem 2.7. *If $\sum_{v\in A^\star} \phi(v) = +\infty$ then the functional*

$$L(P) = \mathbf{1}\{\sum_{v\in\tilde{\tau}_P} \phi(v) < +\infty\}$$

is not consistently estimable on $\mathcal{P}$.

This result states, in particular, that the functional that attributes the value 1 if the measure is Markovian, and 0 otherwise, is not consistently estimable when ϕ is not summable. This is a known result; see Morvai and Weiss (2005) and references therein. However, our proof is completely different from theirs and it is mainly based on topological properties of $\mathcal{P}$.

Our main result about consistent estimation for the context tree on $\mathcal{P}$ is given in the following theorem.

Theorem 2.8. *T is consistently estimable on $\mathcal{P}$ if and only if $\sum_{v\in A^\star} \phi(v)$ is finite.*

The *only if* part of this theorem is a direct consequence of Theorem 2.7. The *if* part is proved constructively later, because the estimator $\{T_n^c\}_{n\in\mathbb{N}}$ defined by (2.3) below will be proved to be consistent when ϕ is summable.

As mentioned before, the present work is also concerned with the obtention of confidence bounds for the context tree of a stationary and ergodic process. We use the following general definition of upper and lower confidence bounds, taken from Donoho (1988). Suppose $\mathcal{F}$ is equipped with a partial order $<$ with supremum and infimum.

Definition 2.9. *Given $n \ge 1$, a statistic $U_n\colon A^n \to \mathcal{F}$ is called a non-trivial upper confidence bound for F on $\mathcal{P}$ with coverage probability at least $1-\alpha$ if*

$$\sup_{P\in\mathcal{P}} P(U_n < \sup_{P'\in\mathcal{P}} F(P')) = 1$$

and

$$\inf_{P\in\mathcal{P}} P(F(P) \le U_n) \;\ge\; 1-\alpha.$$

Analogously we say that L_n is a non-trivial lower confidence bound for F on $\mathcal{P}$ with coverage probability at least $1-\alpha$ if $-L_n$ is a non-trivial upper confidence bound for $-F$ on $\mathcal{P}$ with coverage probability at least $1-\alpha$.

Our first theorem concerning confidence bounds is a negative result stating that the functional T does not admit a non-trivial upper confidence bound neither on $\mathcal{P}$ nor in the class of stationary ergodic measures with finite context tree.

Theorem 2.10. *If U_n is an upper bound that satisfies $\sup_{P\in\mathcal{P}} P(U_n \prec \tau^\infty) = 1$ then the coverage probability $\inf_{P\in\mathcal{P}} P(\tau_P \preceq U_n) = 0$. This is also satisfied even in the smaller class $\mathcal{P}_f \subset \mathcal{P}$ of stationary ergodic measures having finite context tree.*

The functional T does however admit non-trivial lower confidence bounds on $\mathcal{P}$. In what follows, we construct a sequence of statistics which will be proved to be a non-trivial lower confidence bound and a consistent estimator of T on $\mathcal{P}$, when $\sum_{v\in A^\star}\phi(v) < +\infty$.

We will first define a discrepancy measure between a sample $X_1,\dots,X_n$ and a measure $Q\in\mathcal{P}$. To do so, we need to introduce some more notation and definitions. Given a string w, denote by $N_n(w)$ the number of occurrences of w in the sample $X_1,\dots,X_n$; that is

$$N_n(w) = \begin{cases} \sum_{i=0}^{n-|w|} \mathbf{1}\{X_{i+1}^{i+|w|} = w\} & n \ge |w| \\ 0 & n < |w|. \end{cases}$$

If $N_{n-1}(w) > 0$, we define for any $a\in A$ the estimated transition probability

$$\hat{p}_n(a|w) := \frac{N_n(wa)}{N_{n-1}(w)}\,.$$

Denote also by $C_{n-1}(w)$ the set of *children* of w that appear in the sample at least once, that is

$$C_{n-1}(w) = \{bw\colon b\in A \text{ and } N_{n-1}(bw) > 0\}$$

and by S_n the set of all such strings w; that is

$$S_n = \{w\in A^\star\colon N_{n-1}(w) > 0\}\,.$$

Finally, for any context tree τ, let

$$\tau^* := \{u\in A^\star\colon u\notin \bar{\tau}\}.$$

Now, we can define our discrepancy measure as a function $d_n\colon A^n\times\mathcal{P}\to\mathbb{R}$

$$d_n(X_1^n, Q) \;:=\; \max_{w\in S_n\cap\tau_Q^*} \{\, N_{n-1}(w) \max_{a\in A} |\hat{p}_n(a|w) - Q(a|w)| \,\}$$

if $S_n \cap \tau_Q^* \neq \emptyset$. If $S_n \cap \tau_Q^* = \emptyset$ we define $d_n(X_1^n, Q) = 0$.

We are now ready to introduce the lower bound for the functional T. Given a constant $c > 0$, for any $n \in \mathbb{N}$ let $T_n^c \colon A^n \to \mathcal{T}$ be defined by

$$T_n^c(X_1^n) \;=\; \inf\{\, \tau_Q \colon d_n(X_1^n, Q) \leq c\log(n)\,\}\,, \tag{2.3}$$

where the infimum is taken with respect to the order $\prec$ between trees, and the logarithm is taken in base 2. Note that since the tree τ^{root} is the smallest element of $\mathcal{T}$ with respect to $\prec$, this infimum always exists. In Section 3 we show how to practically compute $T_n^c(X_1^n)$.

We now state the main result of this paper.

Theorem 2.11. *Given $0 < \alpha < 1$ and $n > 2$, for any c satisfying*

$$(|A|-1)\Big(\frac{\log((|A|-1)/\alpha)}{\log(n)} + 2\Big) \;\leq\; c \;\leq\; \frac{n-1}{2|A|\log(n)} \tag{2.4}$$

we have that the statistic T_n^c is a non-trivial lower confidence bound for T on $\mathcal{P}$, with nonparametric coverage probability of at least $1-\alpha$. Moreover, if $\sum_{v\in A^\star} \phi(v) < \infty$, for any $c > 2(|A|-1)$ the sequence $\{T_n^c\}_{n\in\mathbb{N}}$ is a consistent estimator of T on $\mathcal{P}$ and if $c > 3(|A|-1)$ then $\{T_n^c\}_{n\in\mathbb{N}}$ is strongly consistent.

3. Computation and application of the lower confidence bound

In this section we show how to compute the confidence bound (2.3) and we present a practical application of Theorem 2.11 to linguistic data.

3.1. Tree lower bound algorithm

Let $X_1, \dots, X_n$ be a given sample and let $c > 0$ be a fixed constant. To compute the tree $T_n^c(X_1^n)$, we will identify its nodes, i.e. the set $\bar{T}_n^c(X_1^n)$. By definition, we know that $w \in \bar{T}_n^c(X_1^n)$ if and only if every process Q satisfying $d_n(X_1^n, Q) \leq c\log(n)$ has context tree with w as a node. The following proposition gives a simple criteria to check whether or not we have to include a string w in the set $\bar{T}_n^c(X_1^n)$. It relies on two quantities, $l_n(w,a)$ and $u_n(w,a)$, which are defined for any $w \in S_n$ and any $a \in A$ by

$$l_n(w,a) = \max_{sw\in S_n} \Big\{\hat{p}(a|sw) - \frac{c\log(n)}{N_{n-1}(sw)}\Big\} \tag{3.1}$$

$$u_n(w,a) = \min_{sw\in S_n} \Big\{\hat{p}(a|sw) + \frac{c\log(n)}{N_{n-1}(sw)}\Big\}\,. \tag{3.2}$$

Proposition 3.1. *Let w be a finite string with $N_{n-1}(w) > 0$. Then there exists a process Q satisfying $d_n(X_1^n, Q) \leq c\log(n)$ and having w as a context if and only if the following conditions hold*

1. *For any $a \in A$, $l_n(w,a) \leq u_n(w,a)$.*

2. $\sum_{a\in A} l_n(w,a) \leq 1 \leq \sum_{a\in A} u_n(w,a)$.

We now give a simple algorithm (see Fig 2) to construct the estimated tree. Let us explain how it works. Since every context tree has the root λ as node, then $\lambda \in \bar{T}_n^c(X_1^n)$, and we can inicialize the algorithm with λ. We then proceed iteratively as follows, until we exhaust the set S_n.

Suppose that a string w has been included in $\bar{T}_n^c(X_1^n)$. If $C_{n-1}(w)\cap S_n \neq \emptyset$ and at least one of the conditions of Proposition 3.1 is not satisfied for w, this means that there do not exist processes Q satisfying $d_n(X_1^n, Q) \leq c\log(n)$ and having w as a context. In other words, all processes such that $d_n(X_1^n, Q) \leq c\log(n)$ has w as a proper suffix of their contexts. Thus the set $C_{n-1}(w)\cap S_n$ must belong to $\bar{T}_n^c(X_1^n)$. On the other hand, if both conditions of Proposition 3.1 are satisfied for w, then there exists at least one process Q such that $d_n(X_1^n, Q) \leq c\log(n)$ and having w as a context. In this case we let w be a context of $\bar{T}_n^c(X_1^n)$ and we stop checking its descendants (strings of the form $sw \in S_n$, with $s \in A^\star$).

Tree lower bound (TLB) algorithm

(1) Initialise with $\bar{T}_n^c(X_1^n) \leftarrow \{\lambda\}$ and $S \leftarrow \{\lambda\}$.

(2) While $S \neq \emptyset$, pick any $w \in S$ and do:

(a) Remove w from S;

(b) For any $a \in A$ compute the values $l_n(w,a)$ and $u_n(w,a)$ (see (3.1) and (3.2)).

(c) If for some $a \in A$, $u_n(w,a) < l_n(w,a)$ or if

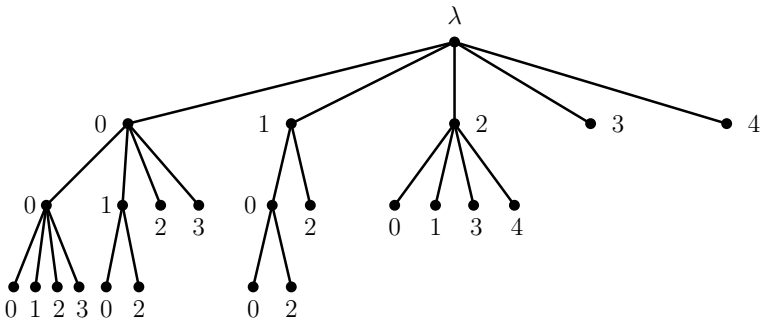
$$1 \notin \Big[\sum_{a\in A} l_n(w,a)\,;\, \sum_{a\in A} u_n(w,a)\Big]$$

then add $C_n(w)$ to $\bar{T}_n^c(X_1^n)$ and to S.

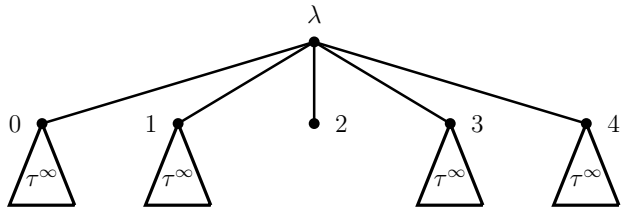
FIG 2. *Algorithmic steps to compute the lower bound in* (2.3).

3.2. One-sided test of hypotheses for context trees

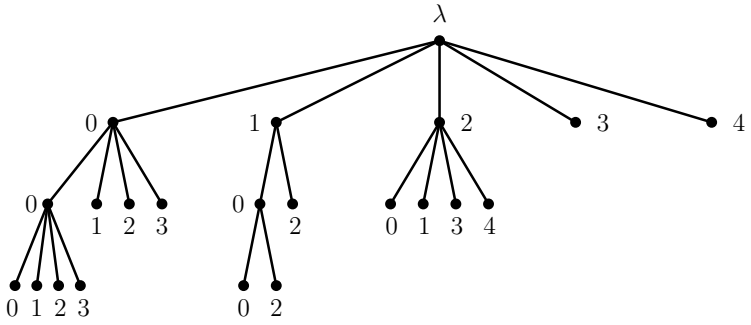
In this subsection we present an application of the lower confidence bound introduced in (2.3) to test a hypothesis about the context tree of codified texts written in European Portuguese. This dataset, that is publicly available, was first analyzed in Galves et al. (2012) where a method to estimate a context tree was proposed and then applied to solve a linguistic conjecture about the rhythmic distinction between European and Brazilian Portuguese. The written texts were codified into the alphabet $A = \{0, 1, 2, 3, 4\}$ taking into account the stressed syllables and the boundaries of words; see Galves et al. (2012) for details. The European Portuguese context tree obtained in the cited work is the one shown in Fig. 3(a). Another analysis of the same dataset with similar results can be found in Belloni and Oliveira (2015).



(a) European Portuguese context tree.



(b) Tree τ_0 for the test of hypotheses.



(c) Estimated lower bound.

FIG 3. *On top we show the European Portuguese context tree over the alphabet* $A = \{0, 1, 2, 3, 4\}$ *estimated from a corpus of codified written texts. On the middle we show a representation of the tree* τ_0 *used for the definition of the test of hypotheses. The triangles with* τ^∞ *written inside represent infinite complete trees that ramify from symbols* $0, 1, 3$ *and* 4*. In other words,* τ_0 *has a unique finite context, which is* 2*. The bottom tree corresponds to the lower bound* $T_n^c(X_1^n)$ *computed on the same sample as the tree on top, with* $\alpha = 0.05$*.*

An interesting difference with a corresponding linguistic interpretation between the two languages observed from the codified data was the ramification of string "2" into the set of contexts "02", "12", "32"and "42" that appears in the European Portuguese context tree in Fig. 3(a) (in the Brazilian Portuguese context tree this ramification did not occur and the string "2" was identified as a context). A natural idea is then to test if there is enough evidence in the data supporting that the European Portuguese context tree ramifies from the sequence "2" or not.

It is well known that tests of hypotheses can be constructed using confidence bounds. Let τ_0 be a tree and suppose we want to test the hypotheses

$$H_0 : \tau_P \preceq \tau_0 \quad vs. \quad H_1 : \tau_P \not\preceq \tau_0 \,.$$

Given n and α, consider the test that rejects H_0 if and only if $T_n^c(X_1^n) \in R = \{\tau \in \mathcal{T} : \tau \not\preceq \tau_0\}$, with $c = (|A| - 1)(\log((|A| - 1)/\alpha)/\log(n) + 2)$. By Theorem 2.11 we have that

$$\sup_{P:\tau_P \preceq \tau_0} P(T_n^c(X_1^n) \not\preceq \tau_0) \;\leq\; \sup_{P \in \mathcal{P}} P(T_n^c(X_1^n) \not\preceq \tau_P) \;\leq\; \alpha \,.$$

Thus, the test defined by the rejection region R has significance level α for the hypotheses $H_0 : \tau_P \preceq \tau_0$ vs. $H_1 : \tau_P \not\preceq \tau_0$.

In our application the null hypothesis is defined by a tree τ_0 having the string "2" as a context. Since we impose no further condition, we let 2 be the unique finite context of τ_0, that is

$$\tau_0 = \{2\} \cup \{wa \colon w \in \tau^\infty, a \in A, a \neq 2\} \,.$$

This tree is represented in Fig. 3(b). We set the significance level $\alpha = 0.05$, and as our sample size is $n = 107.761$ we have $c = 9,513$. The estimated tree with the TLB algorithm of Fig 2 is given in Fig. 3(c). We see that $T_n^c(X_1^n)$ belongs to the rejection region R therefore we reject the null hypothesis at the significance level $\alpha = 0.05$, confirming in this way the results of Galves et al. (2012) about the ramification of sequence "2" in the European Portuguese context tree.

The algorithm described in Fig. 2 was coded in the `R` language and is available upon request.

4. Proofs

Proof of Lemma 2.4. With respect to the weak topology, the set of all stationary probability measures over $(\Omega, \mathcal{F})$ is a compact Hausdorff space (Shields, 1996) and the subspace $\mathcal{P}$ of all stationary and ergodic probability measures over $(\Omega, \mathcal{F})$ is a G_δ set (Parthasarathy, 1961, Theorem 2.1). Therefore, $\mathcal{P}$ is a Baire space with the induced topology. □

Proof of Proposition 2.6. The proof uses the same arguments of Lemma 1.1 in Fraiman and Meloche (1999). The difference is that here we do not have independent random variables and the space $\mathcal{P}$ is not a complete metric space

with respect to D. But the same result can be obtained in our setting, as we show in the sequel. Recall that in the conditions of the proposition, there exists $R \in \mathbb{R}$ such that $|F(P)| \leq R$ for all $P \in \mathcal{P}$, and assume that $\{F_n\}_{n\in\mathbb{N}}$ is a consistent estimator for F. Define

$$S_n = F_n I_{\{|F_n|\leq R\}} + sg(F_n) R I_{\{|F_n|>R\}},$$

where I is the indicator function and sg is the sign of F_n. It is not hard to show that $\{S_n\}_{n\in\mathbb{N}}$ is also a consistent estimator for F, for details see (Fraiman and Meloche, 1999, Lemma 1.1). As for any $n \in \mathbb{N}$ the function S_n is bounded by R we have that the convergence in probability to $F(P)$ implies convergence in mean. Therefore we have that

$$\phi_n(P) := \mathbb{E}_P(S_n) \;\to\; F(P)$$

as $n \to \infty$. Moreover,

$$\begin{aligned} |\phi_n(P) - \phi_n(Q)| \;&=\; \Big| \sum_{x_1^n \in A^n} S_n(x_1^n)\, P(x_1^n) - \sum_{x_1^n \in A^n} S_n(x_1^n)\, Q(x_1^n) \Big| \\ &\leq \sum_{x_1^n \in A^n} \big|S_n(x_1^n)\big|\, \big|\, P(x_1^n) - Q(x_1^n)\big| \\ &\leq\; R\, 2^n D(P,Q)\,. \end{aligned}$$

Therefore, for each n, ϕ_n is uniformly continuous with respect to the weak topology (induced by D) on $\mathcal{P}$. Then, by Lemma 2.4 and the Baire's Cathegory Theorem, the function F must be continuous on a dense subset of $\mathcal{P}$. □

To continue we need two basic lemmas that constitute the core of all our negative results.

Lemma 4.1. *Any measure $P \in \mathcal{P}$ can be approximated with respect to D by a sequence of measures $\{P_n\}_{n\in\mathbb{N}}$ in $\mathcal{P}$ each of which have as context tree a given tree τ, with $\tau_P \preceq \tau$. In particular, τ can be infinite.*

Proof. We proceed in two steps, first we define a sequence of Markov measures $\{P^{[k]}\}_{k\in\mathbb{N}}$ converging to P and then for any $k \in \mathbb{N}$, we construct a sequence of stationary ergodic measures $\{P_i^{[k]}\}_{i\in\mathbb{N}}$ each of which have context tree τ and that converges to $P^{[k]}$. The conclusion of the proof then follows by a diagonal argument, since convergence in D (or in the weak topology) corresponds to convergence of the measure of cylinders (Shields, 1996, Section I.9).

For any $k \in \mathbb{N}$, let $P^{[k]}$ be the k-steps canonical Markov approximation of P, which is a Markov chain of order k with transition probabilities

$$P^{[k]}(a|a_{-k}^{-1}) := P(a|a_{-k}^{-1})\,, \quad a \in A\,,\; a_{-k}^{-1} \in A^k\,. \tag{4.1}$$

An important observation is that $\tau_{P^{[k]}} \preceq \tau_P$, since for any semi-infinite sequence $a_{-\infty}^{-1} \in A^\infty$ the length of the context of $P^{[k]}$ along $a_{-\infty}^{-1}$ is at most the length of the context of P. Moreover, it is well known that the sequence $\{P^{[k]}\}_{k\in\mathbb{N}}$

converges weakly to P (see Rudolph and Schwarz (1977) for instance), then the first step is proven.

To continue, let us introduce the continuity rate of a process $\tilde{P}$ along a given past $a_{-\infty}^{-1}$, which is the non-increasing sequence $\{\beta_l^{\tilde{P}}(a_{-\infty}^{-1})\}_{l\in\mathbb{N}}$ defined as

$$\beta_l^{\tilde{P}}(a_{-\infty}^{-1}) := \sup_{a,b_{-\infty}^{-1},c_{-\infty}^{-1}} |\tilde{P}(a|b_{-\infty}^{-1}a_{-l}^{-1}) - \tilde{P}(a|c_{-\infty}^{-1}a_{-l}^{-1})|\,, \quad l \geq 1.$$

Observe that $\beta_{l-1}^{\tilde{P}}(a_{-\infty}^{-1}) > 0$ means $a_{-l}^{-1} \in \bar{\tau}_{\tilde{P}}$ and therefore $\beta_l^{\tilde{P}}(a_{-l}^{-1}) > 0$ for all l means that the infinite sequence $a_{-\infty}^{-1} \in \tau_{\tilde{P}}$. Let $\tilde{P}$ be a measure in $\mathcal{P}$ satisfying the following three conditions:

(i) $\inf_{a_{-\infty}^0}\{\tilde{P}(a_0|a_{-\infty}^{-1})\} > 0$.

(ii) For any $a_{-\infty}^{-1} \in A^\infty$ and any $l \in \mathbb{N}$, $\beta_l^{\tilde{P}}(a_{-\infty}^{-1}) > 0$.

(iii) $\sum_{l\in\mathbb{N}} \sup_{a_{-\infty}^{-1}} \beta_l^{\tilde{P}}(a_{-\infty}^{-1}) < \infty$.

It should be clear to the reader that such a measure $\tilde{P} \in \mathcal{P}$ can always be selected. An example of this is the observable chain in a Hidden Markov Model, that under simple assumptions satisfy conditions (i)-(iii) above, see for instance Collet and Leonardi (2014).

Now consider any context tree τ such that $\tau_P \preceq \tau$. For all $i \in \mathbb{N}$ define the kernel

$$P_i^{[k]}(a|a_{-\infty}^{-1}) = (1 - 1/i)\, P^{[k]}(a|a_{-k}^{-1}) + 1/i\, \tilde{P}(a|a_{-l}^{-1})\,,\ a \in A\,,\ a_{-l}^{-1} \in \tau\,.$$

We have $\inf_{a_{-\infty}^0}\{P_i^{[k]}(a_0|a_{-\infty}^{-1})\} \geq 1/i \inf_{a_{-\infty}^0}\{\tilde{P}(a_0|a_{-\infty}^{-1})\} > 0$ for any $i \in \mathbb{N}$. Thus, this kernel satisfies (i) and let us show that it also satisfies property (iii). For any $a_{-\infty}^{-1}$ with $a_{-l}^{-1} \in \tau$ we have

$$|P_i^{[k]}(a|b_{-\infty}^{-1}a_{-r}^{-1}) - P_i^{[k]}(a|c_{-\infty}^{-1}a_{-r}^{-1})| = 1/i\, |\tilde{P}(a|b_{-\infty}^{-l-1}a_{-r}^{-1}) - \tilde{P}(a|c_{-\infty}^{-l-1}a_{-r}^{-1})|$$

for all $r < l$ or $|P_i^{[k]}(a|b_{-\infty}^{-1}a_{-r}^{-1}) - P_i^{[k]}(a|c_{-\infty}^{-1}a_{-r}^{-1})| = 0$ if $r \geq l$, for all $a \in A$ and all $b_{-\infty}^{-1}, c_{-\infty}^{-1} \in A^\infty$. Conditions (i) and (iii) ensure that there exists a unique stationary ergodic measure having kernel $P_i^{[k]}$; see for instance Bressaud, Fernández and Galves (1999); Fernández and Galves (2002).

By the above observations, the contexts of $P_i^{[k]}$ are exactly the sequences in τ, since $\tau_{P^{[k]}} \preceq \tau_P \preceq \tau$. Now, since $\{P_i^{[k]}\}_{i\in\mathbb{N}}$ converges uniformly to $P^{[k]}$ as $i \to \infty$ we also have $\{P_k^{[k]}\}_{k\in\mathbb{N}}$ converging in D to P as k diverges. □

Lemma 4.2. *Any measure $P \in \mathcal{P}$ can be approximated with respect to D by a sequence of measures $\{P_n\}_{n\in\mathbb{N}}$ in $\mathcal{P}$ each of which have a finite context tree.*

Proof. We prove this lemma using a similar two-steps argument as in the previous one. First, we use the same sequence of canonical Markov approximations $\{P^{[k]}\}_{k\in\mathbb{N}}$ defined in (4.1) to approximate P. Second, as we do not know whether

these Markov measures are ergodic or not, we construct, for any $k \geq 1$, a sequence $\{P_i^{[k]}\}_{i\in\mathbb{N}}$ of ergodic Markov measures converging to $P^{[k]}$ when $i \to \infty$. The conclusion of the proof also follows from a diagonal argument.

The construction of $P_i^{[k]}$ is also carried as in the previous lemma, by specifying the kernel of transition probabilities

$$P_i^{[k]}(a|a_{-\infty}^{-1}) = (1-1/i)\, P^{[k]}(a|a_{-k}^{-1}) + \frac{1}{i|A|}\,,\ a \in A\,,\ a_{-\infty}^{-1} \in A^{\infty}\,.$$

Is is easy to see that this definition leads to a Markovian (i.e. with finite context tree) ergodic measure, and that the sequence $\{P_i^{[k]}\}_{i\in\mathbb{N}}$ converges to $P^{[k]}$ when $i \to \infty$. As before we have that $\{P_k^{[k]}\}_{k\in\mathbb{N}}$ converges to P when $k \to \infty$ and this concludes the proof. □

We are now ready to prove Theorem 2.7.

Proof of Theorem 2.7. Assume that $\sum_{v\in\bar{\tau}^{\infty}} \phi(v) = +\infty$. Then Lemmas 4.1 and 4.2 imply that any $P \in \mathcal{P}$ having $L(P) = 0$ (respectively $L(P) = 1$) is limit in D of a sequence of measures $\{P_n\}_{n\in\mathbb{N}}$ in $\mathcal{P}$ satisfying $L(P_n) = 1$ (respectively $L(P_n) = 0$) for all $n \in \mathbb{N}$. In other words, the functional L is discontinuous (with respect to the D-distance) at any point of $\mathcal{P}$. Together with Proposition 2.6, this proves that L is not consistently estimable on $\mathcal{P}$ when $\sum_{v\in\bar{\tau}^{\infty}} \phi(v) = +\infty$. □

Proof of Theorem 2.8. As we already mentioned, the proof of the *if* part of the theorem follows from Theorem 2.11 which states that $\{T_n^c\}_{n\in\mathbb{N}}$ is actually a consistent estimator of τ_P. It remains to prove the *only if* part. Assume $\sum_{v\in\bar{\tau}^{\infty}} \phi(v) = +\infty$ and suppose there exists $\{T_n\}_{n\in\mathbb{N}}$, a consistent estimator of T on $\mathcal{P}$. Define $L_n \colon A^n \to \{0,1\}$ by $L_n(x_1^n) = \mathbf{1}\{\sum_{v\in\bar{T}_n(x_1^n)} \phi(v) < +\infty\}$. We will prove that $\{L_n\}_{n\in\mathbb{N}}$ is a consistent estimator of L, which is a contradiction with Theorem 2.7, concluding the proof of the theorem.

Fix $P \in \mathcal{P}$. As $\{T_n\}_{n\in\mathbb{N}}$ is consistent we have that for any $\epsilon > 0$

$$\lim_{n\to\infty} P(d_\phi(T_n(X_1^n), \tau_P) \leq \epsilon) \;=\; 1\,.$$

We will prove that for any $\epsilon > 0$ the ball of center τ_P and radius ϵ contains only trees where L is constant and equal to $L(\tau_P)$. By the definition of d_ϕ, see (2.1), for $\tau' \in \mathcal{T}$,

$$d_\phi(\tau_P, \tau') \;=\; \sum_{v\in\bar{\tau}'} \phi(v) \;+\; \sum_{v\in\bar{\tau}_P} \phi(v) - 2 \sum_{v\in\bar{\tau}'\cap\bar{\tau}_P} \phi(v).$$

Then if $d_\phi(\tau_P, \tau') < \epsilon$ we have $L(\tau_P) = 1$ if and only if $L(\tau') = 1$. Therefore

$$\lim_{n\to\infty} P(\, L_n(X_1^n) = L(P)\,) \;\geq\; \lim_{n\to\infty} P(\, d_\phi(T_n(X_1^n), \tau_P) \leq \epsilon\,) \;=\; 1$$

which proves that L is consistently estimable on $\mathcal{P}$. But by Theorem 2.7, L is not consistently estimable on $\mathcal{P}$, which is a contradiction. □

Proof of Theorem 2.10. Suppose U_n satisfies

$$\sup_{P\in\mathcal{P}} P(U_n \prec \tau^\infty) = 1\,.$$

Given $\delta > 0$ choose a measure $P_\delta \in \mathcal{P}_f$ such that

$$P_\delta(U_n \prec \tau^\infty) \;\geq\; 1 - \frac{\delta}{3}\,. \tag{4.2}$$

This can always be done because the set $\mathcal{P}_f$ is dense in $\mathcal{P}$ (see Lemma 4.2). Let $\{\tau_k\}_{k\in\mathbb{N}}$ be an increasing sequence of finite trees and denote by V_k the event

$$V_k = \{\tau_k \prec U_n\}\,.$$

We have $V_k \supset V_{k+1}$ for all $k \geq 1$ and $\cap_{k\geq 1} V_k = \{U_n = \tau^\infty\}$. Therefore

$$\lim_{k\to\infty} P_\delta(V_k) \;=\; P_\delta(U_n = \tau^\infty) \;\leq\; \frac{1}{3}\delta\,.$$

Now let k^* be such that

$$P_\delta(V_{k^*}) \;<\; \frac{2}{3}\delta\,,$$

and denote by τ the finite tree given by $\bar\tau = \bar\tau_{k^*} \cup \bar\tau_{P_\delta}$. We have

$$P_\delta(\tau \not\preceq U_n) \;\geq\; P_\delta(\tau_{k^*} \not\preceq U_n) \;\geq\; 1 - \frac{2}{3}\delta\,.$$

By Lemma 4.1, there exists a measure $Q_\delta \in \mathcal{P}_f$ with $\tau_{Q_\delta} = \tau$ such that

$$D(P_\delta, Q_\delta) \;<\; 2^{-n}\frac{\delta}{3}\,.$$

Moreover we have

$$\begin{aligned}
&P_\delta(\tau \not\preceq U_n) \;-\; Q_\delta(\tau \not\preceq U_n) \\
&\qquad= \sum_{x_1^n\in A^n} \mathbf{1}_{\{\tau\not\preceq U_n(x_1^n)\}}\, P_\delta(x_1^n) - \sum_{x_1^n\in A^n} \mathbf{1}_{\{\tau\not\preceq U_n(x_1^n)\}}\, Q_\delta(x_1^n) \\
&\qquad\leq \sum_{x_1^n\in A^n} \mathbf{1}_{\{\tau\not\preceq U_n(x_1^n)\}}\, |\, P_\delta(x_1^n) - Q_\delta(x_1^n)\,| \\
&\qquad\leq\; 2^n D(P_\delta, Q_\delta) \;<\; \frac{\delta}{3}\,.
\end{aligned}$$

Therefore

$$Q_\delta(\tau \not\preceq U_n) \;>\; P_\delta(\tau \not\preceq U_n) \;-\; \frac{\delta}{3} \;\geq\; 1 - \delta\,.$$

As δ is arbitrary we have just proved that

$$\inf_{Q\in\mathcal{P}} Q(\tau_Q \preceq U_n) \;=\; \inf_{Q\in\mathcal{P}_f} Q(\tau_Q \preceq U_n) \;=\; 0\,. \qquad \square$$

In order to prove Theorem 2.11 we will need the following lemma.

Lemma 4.3. *Given $P \in \mathcal{P}$, let $X_1, \dots, X_n$ be a sample of size n with law P. Then for any constant $c > 0$ we have*

$$P\big(\, d_n(X_1^n, P) \le c \log(n) \,\big) \;\ge\; 1 - \frac{|A| - 1}{n^{c/(|A|-1)-2}} \,.$$

Proof. First note that if $S_n \cap \tau_P^* = \emptyset$ then the assertion of the lemma is trivial because in this case $d_n(X_1^n, P) = 0$ and then $P(d_n(X_1^n, P) \le c \log n) = 1$. Now suppose $S_n \cap \tau_P^* \neq \emptyset$ and let $w \in S_n \cap \tau_P^*$. We recall the reader that all the logarithms are taken in base 2. This is not in fact a real restriction, as if the base is $r > 1$ we can replace c by $c/\log_2(r)$ and the result holds as well. First note that we can write

$$\max_{a \in A} |\hat{p}_n(a|w) - p(a|w)| \;=\; \sum_{\substack{a \in A \\ p(a|w) > \hat{p}_n(a|w)}} p(a|w) - \hat{p}_n(a|w) \,.$$

For a proof of this equivalence see for instance (Levin, Peres and Wilmer, 2009, Proposition 4.2 and Remark 4.3). Therefore we have that the event

$$B_n(w) = \{N_{n-1}(w) \max_{a \in A} |\hat{p}_n(a|w) - p(a|w)| > c \log(n)\}$$

is included in the event

$$E_n(w) = \bigcup_{a \in A} E_n(w, a) \tag{4.3}$$

where

$$E_n(w,a) = \Big\{ (N_{n-1}(w) p(a|w) - N_n(wa)) \mathbf{1}\{p(a|w) > \hat{p}_n(a|w)\} > \frac{c \log(n)}{|A| - 1} \Big\} \,.$$

and the union (4.3) has at most $|A| - 1$ non empty sets. Now define the random variables

$$W_n(w,a) \;=\; 2^{N_{n-1}(w) \log(1 + p(a|w)) - N_n(wa)} \,, \quad n \ge |w| \,.$$

Then, by the inequality $p(a|w) \le \log_2(1 + p(a|w))$ valid in the interval $(0, 1]$ we have that for $a \in A$ such that $p(a|w) > \hat{p}_n(a|w)$, the event $E_n(w,a)$ is included in the event

$$F_n(w,a) = \Big\{ W_n(w,a) > n^{c/(|A|-1)} \Big\} \,, \tag{4.4}$$

As in (Garivier and Leonardi, 2011, Proposition A.1), we will show that when $w \in \tau_P^*$, the sequence $\{W_n(w,a)\}_{n \in \mathbb{N}}$ is a martingale with respect to the filtration $\{\sigma(X_1, \dots, X_{n-1})\}_{n \in \mathbb{N}}$. In fact, note that by the definition of $N_n(\cdot)$ we have that

$$\begin{aligned} \mathbb{E}(2^{(N_{n+1}(wa) - N_n(wa))} | X_1, \dots, X_n) \;&=\; \mathbb{E}(2^{\mathbf{1}\{X_{n+1-|w|}^{n+1} = wa\}} | X_1, \dots, X_n) \\ &=\; 2^{(\mathbf{1}\{X_{n+1-|w|}^{n} = w\}) \log(1 + p(a|w))} \\ &=\; 2^{(N_n(w) - N_{n-1}(w)) \log(1 + p(a|w))} \end{aligned}$$

which implies that $\mathbb{E}(W_{n+1}(w,a)|X_1,\dots,X_n) = W_n(w,a)$. Thus $\mathbb{E}(W_n(w,a)) = \mathbb{E}(W_{|w|}(w,a)) = 1$ and therefore, by Markov's inequality and a union bound we have that

$$P(E_n(w)) \;\le\; (|A|-1)\sup_{a\in A} P(F_n(w,a)) \;\le\; \frac{|A|-1}{n^{c/(|A|-1)}}\,.$$

One more union bound over $S_n \cap \tau_P^*$ yields

$$\begin{aligned} P\big(\, d_n(X_1^n,P) > \, c\log n\big) \;&=\; P\big(\, \cup_{w\in S_n} B_n(w)\,\big) \\ &\le\; n^2 \sup_{w\in S_n} \{\, P\big(\, E_n(w)\,\big)\,\} \\ &\le\; \frac{|A|-1}{n^{c/(|A|-1)-2}}\,. \end{aligned}$$

□

Proof of Theorem 2.11. Observe that for any $P \in \mathcal{P}$ the event $\{d_n(X_1^n,P) \le c\log(n)\}$ implies $\{T_n^c(X_1^n) \preceq \tau_P\}$. Therefore, by Lemma 4.3 and the condition on c we have

$$\begin{aligned} P(\, T_n^c(X_1^n) \preceq \tau_P\,) \;&\ge\; P(d_n(X_1^n,P) \le c\log(n)) \\ &\ge\; 1-(|A|-1)/n^{c/(|A|-1)-2} \\ &\ge\; 1-\alpha\,. \end{aligned}$$

To prove that $T_n^c(X_1^n)$ is not trivial we have to prove that

$$\sup_{P\in\mathcal{P}} P(\tau^{\text{root}} \prec T_n^c(X_1^n)) = 1\,. \tag{4.5}$$

For that consider the transition matrix Q^ϵ on $A=\{1,\dots,k\}$ defined by

$$\begin{aligned} Q^\epsilon(i,i) \;&=\; 1-Q^\epsilon(i,i+1) \;=\; \epsilon\,, \qquad \text{for } i=1,\dots,k-1, \quad \text{and} \\ Q^\epsilon(k,k) \;&=\; 1-Q^\epsilon(k,1) \;=\; \epsilon\,. \end{aligned}$$

The parameter ϵ will be chosen adequately later. Observe that if $\epsilon > 0$ there exists a unique stationary ergodic Markov chain P specified by Q^ϵ. It is also easy to see, by symmetry, that $P(i) = 1/k$ for any $i = 1,\dots,k$. Now for any $n \ge 3$ denote by $\mathcal{M}_n$ the set of strings $x_1^n \in A^n$ such that $x_{i+1} = x_i + 1$ for $i = 1,\dots,n-1$. Observe that there are exactly k such strings, independently of n, each beginning in a different symbol of A. Moreover, each of these strings have equal measure

$$P(x_1^n) = \frac{1}{k}(1-\epsilon)^{n-1}.$$

On the other hand, thanks to Proposition 3.1 and the TLB algorithm in Fig. 2, the k trees $\{T_n^c(x_1^n) : x_1^n \in \mathcal{M}_n\}$ will be different from τ^{root}, because if $c \le (n-1)/2|A|\log(n)$, as $N_{n-1}(x_1) \ge N_{n-1}(x_2) \ge \lceil \frac{n-1}{|A|} \rceil$ we will have

$$u_n(x_2,x_2) \;\le\; \hat{p}_n(x_2|x_2) + \frac{c\log(n)}{N_{n-1}(x_2)} \;<\; 1/2.$$

and

$$l_n(x_1, x_2) \;\geq\; \hat{p}_n(x_2|x_1) - \frac{c\log(n)}{N_{n-1}(x_1)} \;>\; 1/2\,.$$

In other words,

$$P(\tau^{\mathrm{root}} \prec T_n^c(X_1^n)) \;\geq\; P(X_1^n \in \mathcal{M}_n) = k \times \frac{1}{k}(1-\epsilon)^{n-1}\,.$$

To conclude, observe that for any $\delta > 0$, we can take $\epsilon = 1-(1-\delta)^{1/(n-1)}$, and we get

$$P(\tau^{\mathrm{root}} \prec T_n^c(X_1^n)) \;\geq\; 1-\delta$$

proving that (4.5) holds.

Now we will prove the consistency of the estimator $\{T_n^c(X_1^n)\}_{n\in\mathbb{N}}$ for any $c > 2(|A|-1)$, by showing that for any $P \in \mathcal{P}$ and any $\epsilon > 0$ the event $d_\phi(T_n^c(X_1^n), \tau_P) \leq \epsilon$ occurs with probability converging to 1 as $n \to \infty$. To begin, notice that by Lemma 4.3 we have that

$$P(d_n(X_1^n, P) \leq c\log n) \;\geq\; 1 - \frac{|A|-1}{n^\delta} \quad \text{for all } n \geq 1\,,$$

where $\delta = c/(|A|-1) - 2 > 0$. This implies, by the definition of $T_n^c(X_1^n)$, that

$$P(T_n^c(X_1^n) \preceq \tau_P) \;\geq\; 1 - \frac{|A|-1}{n^\delta} \quad \text{for all } n \geq 1\,. \tag{4.6}$$

Now, recall that in the conditions of the theorem $\sum_{v\in\bar{\tau}^\infty}\phi(v) < \infty$, and take $k \in \mathbb{N}$ such that

$$\sum_{u\in\bar{\tau}_P\colon\,|u|>k} \phi(u) \;<\; \epsilon\,. \tag{4.7}$$

Thus we have with probability at least $1-(|A|-1)/n^\delta$ that

$$\begin{aligned} d_\phi(T_n^c(X_1^n), \tau_P) \;&\leq \sum_{u\in\bar{\tau}_P|_k\setminus\bar{T}_n^c(X_1^n)} \phi(u) + \sum_{u\in\bar{\tau}_P\colon\,|u|>k} \phi(u) \\ &\leq \sum_{u\in\bar{\tau}_P|_k\setminus\bar{T}_n^c(X_1^n)} \phi(u) \;+\; \epsilon\,. \end{aligned} \tag{4.8}$$

Therefore it is enough to prove that $\bar{\tau}_P|_k \setminus \bar{T}_n^c(X_1^n) = \emptyset$ with probability converging to 1 as $n \to \infty$, or what is stronger, with probability equal to 1 for n sufficiently large, a fact that we refer as to occur *eventually almost surely* or *e.a.s.* for short. To prove this last assertion, for any $v \in \bar{\tau}_P|_k$ we will show that the set $\{Q\colon d_n(X_1^n, Q) \leq c\log n\}$ is included in the set $\{Q\colon v \in \bar{\tau}_Q\}$ e.a.s. As the set $\bar{\tau}_P|_k$ is finite we will have $\{Q\colon d_n(X_1^n, Q) \leq c\log n\} \subset \{Q\colon \tau_Q \succeq \tau_P|_k\}$ e.a.s. and therefore $T_n^c(X_1^n) \succeq \tau_P|_k$ e.a.s., which in turns implies that $\bar{\tau}_P|_k \setminus \bar{T}_n^c(X_1^n) = \emptyset$ e.a.s. So, let $v \in \bar{\tau}_P|_k$, $v = v_1^j$, and denote by v' its largest proper suffix, that is $v' = v_2^j$. It can be shown that we can always find a symbol $b \in A$ and another finite string $w \in A^\star$ such that the following conditions hold

(i) $P(a|v_1wv') \neq P(a|bwv')$ for some $a \in A$ and
(ii) v is a suffix (proper or not) of v_1wv'.

If v is a context for P, i.e if $v \in \tau_P \cap \bar{\tau}_P|_k$ then it is enough to take $w = \lambda$ and $b \neq v_1$ satisfying (i) (such b must always exist because v is a context). In this case we have $v = v_1wv'$ and (ii) is also satisfied. On the other hand, if v is not a context then it is an internal node of τ_P and we must have some $w = w'v_1$, with $w' \in A^\star$, and $b \in A$ such that (i)-(ii) are satisfied. If not, this would imply that for any $u \in A^\star$ $P(\cdot|uv) = P(\cdot|v)$, contradicting the fact that v is a proper suffix of a context. Using the triangle inequality we have, for any $Q \in \mathcal{P}$, that

$$\begin{aligned} |Q(a|v_1wv') - Q(a|bwv')| \;\geq\; & |P(a|v_1wv') - P(a|bwv')| \\ & - |P(a|v_1wv') - \hat{p}_n(a|v_1wv')| - |\hat{p}_n(a|v_1wv') - Q(a|v_1wv')| \\ & - |Q(a|bwv') - \hat{p}_n(a|bwv')| - |\hat{p}_n(a|bwv') - P(a|bwv')|\,. \end{aligned} \tag{4.9}$$

Now, by ergodicity we have that P-almost surely, for any finite sequence $s \in A^\star$

$$\left|\frac{N_n(s)}{n} - P(s)\right| \to 0 \tag{4.10}$$

when $n \to \infty$, which in turns implies that for any $s \in A^\star$

$$|\hat{p}_n(a|s) - P(a|s)| \to 0\,. \tag{4.11}$$

In particular by (4.10), for any finite sequence $s \in A^\star$ we will have $N_n(s) \geq nP(s)/2$ e.a.s. This fact, together with (4.11) and the inequality (4.9) implies that for a sufficiently large n, any measure Q such that $d_n(X_1^n, Q) \leq c \log n$ will satisfy

$$|Q(a|v_1wv') - Q(a|bwv')| \;>\; 0 \tag{4.12}$$

and thus $v \in \bar{\tau}_Q$. Therefore $\{Q\colon d_n(X_1^n, Q) \leq c \log n\} \subset \{Q\colon v \in \bar{\tau}_Q\}$ and $\bar{\tau}_P|_k \setminus \bar{T}_n^c(X_1^n) = \emptyset$ e.a.s, as required, showing that

$$P(d_\phi(T_n^c(X_1^n), \tau_P) \leq \epsilon) \;\to\; 1 \quad \text{ when } n \to \infty\,.$$

To finish the proof we only emphasize that if $c > 3(|A| - 1)$ then $\delta > 1$ and therefore the bound in (4.6) is summable. This fact together with the Borel-Cantelli lemma implies that $\mathcal{T}_n^c(X_1^n) \preceq \tau$ e.a.s. Therefore, as the other inclusion $\tau|_k \preceq \mathcal{T}_n^c(X_1^n)$ also holds e.a.s, by (4.8) we will have that for all n sufficiently large

$$d_\phi(T_n^c(X_1^n), \tau_P) \;\leq\; \epsilon$$

with probability one. □

Proof of Proposition 3.1. First suppose there exists a process Q satisfying $d_n(X_1^n, Q) \leq c \log(n)$ and having w as a context. Then for any $s \in A^*$ such that $sw \in S_n \cap \tau_Q^*$ and any $a \in A$ we have that

$$|Q(a|sw) - \hat{p}(a|sw)| \;\leq\; \frac{c \log(n)}{N_{n-1}(sw)}\,.$$

Therefore for any $a \in A$ and any $s \in A^*$ such that $sw \in S_n \cap \tau_Q^*$ we have

$$\hat{p}(a|sw) - \frac{c\log(n)}{N_{n-1}(sw)} \;\le\; Q(a|w) \;\le\; \hat{p}(a|sw) + \frac{c\log(n)}{N_{n-1}(sw)} \tag{4.13}$$

because w is a context for Q which implies

$$l_n(w,a) \;\le\; Q(a|w) \;\le\; u_n(w,a)$$

by maximizing (minimizing) with respect to s the left (right) side of (4.13), and the first condition is proven. Now, by summing this last inequality over $a \in A$ we obtain that $\sum_{a\in A} l_n(w,a)$ and $\sum_{a\in A} u_n(w,a)$ must satisfy the second condition as well.

Now suppose 1. and 2. hold. Then it is easy to see that these conditions allow us to choose some positive values $Q(a|w)$ in the interval $(l_n(w,a); u_n(w,a))$ for any $a \in A$ in such a way that $\sum_{a\in A} Q(a|w) = 1$. Then, it is straightforward to construct a stationary ergodic Variable Length Markov chain having $\{Q(a|w)\}_{a\in A}$ as the conditional distribution of the next symbol in the sequence given the past string w, and such that w is a context for Q. The other contexts $v \in \tau_Q \setminus \{w\}$ can be chosen arbitrarily large in such a way that $N_{n-1}(v) = 0$, implying that $S_n \cap \tau_Q^* = \{sw\colon N_{n-1}(sw) > 0\}$. The conditions on $\{Q(a|w)\}_{a\in A}$ implies that for all $sw \in S_n$ and all $a \in A$ we have

$$\hat{p}_n(a|sw) - \frac{c\log(n)}{N_{n-1}(sw)} \;\le\; Q(a|sw) \;\le\; \hat{p}_n(a|sw) + \frac{c\log(n)}{N_{n-1}(sw)}$$

and therefore

$$d_n(X_1^n, Q) \;=\; \max_{v\in S_n\cap\tau_Q^*} \{N_{n-1}(v) \max_{a\in A} |\hat{p}_n(a|v) - Q(a|v)|_1\} \;\le\; c\log(n)\,. \quad \square$$

Discussion

The main contribution of this work is the introduction of a lower confidence bound for the context tree in the class of stationary ergodic probability measures over $A^{\mathbb{Z}}$, with A a finite alphabet. We derive an explicit formula for the coverage probability of this confidence bound, based on a martingale deviation inequality developed in Garivier and Leonardi (2011), and we show the almost sure convergence of this estimator with respect to the Hamming distance d_ϕ, when ϕ is summable. To our knowledge, this is the first lower confidence bound for context trees and it is also the first strong consistent estimator that do not restrict the length of the estimated contexts, as for example does the BIC context tree estimator in Csiszár and Talata (2006) that only allows candidate contexts of length $o(\log n)$. Using only topological arguments we also prove that if ϕ is not summable then there exists no consistent estimator of the context tree in the class of stationary and ergodic processes. This is not the case in the class of processes having finite context trees because in this case the BIC estimator is

strongly consistent. On the other hand we also prove that it is not possible to obtain nonparametric upper confidence bounds even in the smaller class of processes having finite context trees, because any process can be approximated in D by stationary ergodic processes having arbitrary large context trees, as shown in Lemma 4.1. We also show in this work a practical application of the lower confidence bound to test a hypothesis involving the presence of finite contexts in codified written texts of European Portuguese. We support at the confidence level of 95% the results obtained in Galves et al. (2012) for this dataset, where only point estimation of the context tree was addressed.

Acknowledgements

We are thankful to Antonio Galves, Ricardo Fraiman and Miguel Abadi for interesting discussion on the subject and to the Associate Editor and an anonymous referee for suggestions to improve the exposition of the results.

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