

Pseudo real closed fields, pseudo p -adically closed fields and NTP_2

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Abstract

The main result of this paper is a positive answer to the Conjecture 5.1 of [15] by A. Chernikov, I. Kaplan and P. Simon: If M is a PRC field, then $\text{Th}(M)$ is NTP_2 if and only if M is bounded. In the case of PpC fields, we prove that if M is a bounded PpC field, then $\text{Th}(M)$ is NTP_2 . We also generalize this result to obtain that, if M is a bounded PRC or PpC field with exactly n orders or p -adic valuations respectively, then $\text{Th}(M)$ is strong of burden n . This also allows us to explicitly compute the burden of types, and to describe forking.

Keywords: Model theory, ordered fields, p -adic valuation, real closed fields, p -adically closed fields, PRC, PpC , NIP, NTP_2 .

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1 Introduction

A *pseudo algebraically closed field* (PAC field) is a field M such that every absolutely irreducible affine variety defined over M has an M -rational point. The concept of a PAC field was introduced by J. Ax in [2] and has been extensively studied. The above definition of PAC field has an equivalent model-theoretic version: M is existentially closed (in the language of rings) in each regular field extension of M .

The notion of PAC field has been generalized by S. Basarab in [3] and then by A. Prestel in [35] to ordered fields. Prestel calls a field M *pseudo real closed* (PRC) if M is existentially closed (in the language of rings) in each regular field extension N to which all orderings of M extend. PRC fields were extensively studied by L. van den Dries in [43], Prestel in [35], M. Jarden in [26], [27] and [28], Basarab in [5] and [4], and others.

In analogy to PRC fields, C. Grob [20], Jarden and D. Haran [25] studied the class of pseudo p -adically closed fields. A field M is called a *pseudo p -adically closed* (PpC) if M is existentially closed (in the language of rings) in each regular field extension N to which all the p -adic valuations of M can be extended by p -adic valuations on N . PpC fields have also been studied by I. Efrat and Jarden in [19], Jarden in [29] and others.

The class of PRC fields contains strictly the classes of PAC fields of characteristic 0 and real closed fields (RCF) and the class of PpC fields contains the p -adically closed fields (p CF). It is known that the theories RCF and p CF are NIP. Duret showed in [17] that the complete theory of a PAC field which is not separably closed is not NIP. In [8] Z. Chatzidakis and A. Pillay proved that if M is a bounded (i.e. for any $n \in \mathbb{N}$, M has only finitely many extensions of degree n) PAC field, then $Th(M)$ is simple. In [6] Chatzidakis proved that if M is a PAC field and $Th(M)$ is simple, then M is bounded. PRC and PpC fields were extensively studied, but mainly from the perspective of algebra (description of absolute Galois group, etc), elementary equivalence, decidability, etc. Their stability theoretic properties had not been studied. In this paper we study the stability theoretic properties of the classes of PRC and PpC fields.

In Theorem 4.10 we generalize the result of Duret and we show that the complete theory of a PRC field which is neither algebraically closed nor real closed is not NIP. In Corollary 7.4 we show that the complete theory of a bounded PpC which is not p -adically closed is not NIP. The general case for PpC is still in progress, the main obstacle is that the p -adic valuations are not necessarily definable, and that algebraic extensions are not necessarily PpC.

The class of NTP₂ theories (theories without the tree property of the second kind, see Definition 4.1) was defined by Shelah in [40] in the 1980's and contains strictly the classes of simple and NIP theories. Recently the class of NTP₂ theories has been particularly studied and contains new important examples, A. Chernikov showed in [11] that any ultra-product of p -adics is NTP₂. A. Chernikov and M. Hils showed in [13] that a σ -Henselian valued difference field of equicharacteristic 0 is NTP₂, provided both the residue difference field and the value group (as an ordered difference group) are NTP₂. There are not many more examples of strictly NTP₂ theories. A. Chernikov, I. Kaplan and P. Simon conjectured in [15, Conjecture 5.1] that if M is a PRC field then $Th(M)$ is NTP₂ if and only if M is bounded. Similarly if M is a PpC field.

The main result of this paper is a positive answer to the conjecture by Chernikov, Kaplan and Simon for the case of PRC fields (Theorem 4.23). In fact for bounded PRC fields we obtain a stronger result: In Theorem 4.22 we show that if M is a bounded PRC field with exactly n orders, then $Th(M)$ is *strong* of burden n . We also show that $Th(M)$ is not rosy (Corollary 4.20) and resilient (Theorem 4.30). The class of resilient theories contains the class of NIP theories and is contained in the class of NTP₂ theories. It is an open question to know whether NTP₂ implies resilience.

The case of PpC fields is more delicate, and we obtain only one direction of the conjecture. In Theorem 8.5 we show that the theory of a bounded PpC field with exactly n p -adic valuations is strong of burden n and in Theorem 8.5 we show that this theory is also resilient. That unbounded PpC fields have TP₂ will be discussed in another paper. The problem arises again from the fact that an algebraic extension of a PpC field is not necessarily PpC.

Independently, W. Johnson [30] has shown that the model companion of the theory of fields with several independent orderings has NTP₂, as well as characterized forking, extension bases, the burden, and several other results. He also obtains similar results for the class of existentially closed fields with several valuations, or with several p -adic valuations. Some of his results follow from ours, since his fields are bounded, and PRC or PpC in case of several orderings or p -adic

valuations. His results on fields with several valuations however cannot be obtained by our methods.

The organization of the paper is as follows: In section 2 we give the required preliminaries on ordered fields and pseudo real closed fields. In section 3 we study the theory of bounded PRC fields from a model theoretic point of view. We work in a fixed complete theory of a bounded PRC field, and we enrich the language adding constants for an elementary submodel. In section 3.1.1 we study the one variable definable sets; to do that we work with a notion of interval for multi-ordered fields. We define multi-intervals, and a notion of multi-density in multi-intervals (Definition 3.10). Using that we find a useful description of one variable definable sets in terms of multi-intervals and multi-density. In section 3.1.2 we show that the results obtained in 3.1.1 can be easily generalized to several variables. We define a notion of multi-cell (Definition 3.16), and we find (Theorem 3.17) a description of definable sets in terms of multi-cells and multi-density. This “density theorem” is extremely important and plays a role in most proofs. In 3.2 we show some theorems about amalgamation of types for bounded PRC fields. The main difficulty is caused by the orderings. In section 4 we study the stability theoretic properties of PRC fields and we show the main theorem on NTP_2 and strongness for bounded PRC fields. In section 4.2 we calculate explicitly the burden of complete types, in 4.3 we show that the theory of a bounded PRC field is resilient, and in section 4.4 we give a description of forking. In 4.5 we study the Lascar strong types, in Theorem 4.42 we show that having the same Lascar type equals having the same types. We also show that if a and b have the same Lascar type, then the Lascar distance between a and b is less or equal to two. We see in Remark 4.43 that in the case of PRC bounded fields the Amalgamation Theorem (Theorem 3.21) implies the independence theorem for NTP_2 theories showed by Chernikov and Ben Yaacov in [10].

In section 5 are the preliminaries on p -adically closed fields, and pseudo p -adically closed fields. In sections 6, 7 and 8 we show that some results obtained in section 3 and 4, can be generalized easily to bounded PpC fields.

2 Preliminaries on pseudo real closed fields

In this section we will give all the preliminaries that are required throughout the paper. We assume that the reader is familiar with basic concepts in model theory and algebra.

2.1. Notations and Conventions. Let T be a theory in a language \mathcal{L} , M a model of T and $\phi(\bar{x}, \bar{y})$ an \mathcal{L} -formula. For $A \subseteq M$, $\mathcal{L}(A)$ denotes the set of \mathcal{L} -formulas with parameters in A . If \bar{a} is a tuple in M , we denote by $\phi(M, \bar{a}) := \{\bar{b} \in M^{|\bar{x}|} : M \models \phi(\bar{b}, \bar{a})\}$ and by $\text{tp}_{\mathcal{L}}^M(a/A)$ ($\text{qftp}_{\mathcal{L}}^M(a/A)$) the set of $\mathcal{L}(A)$ -formulas (quantifier-free $\mathcal{L}(A)$ -formulas) φ , such that $M \models \varphi(a)$. Suppose M and N are \mathcal{L} -structures and $A \subseteq M, N$. We denote by $M \equiv_A N$ if M is $\mathcal{L}(A)$ -elementarily equivalent to N . Denote by $\text{acl}_{\mathcal{L}}^M$ and $\text{dcl}_{\mathcal{L}}^M$ the model theoretic algebraic and definable closures in M . We omit M or \mathcal{L} when the structure or the language is clear.

We denote by $\mathcal{L}_{\mathcal{R}} := \{+, -, \cdot, 0, 1\}$ the language of rings. All fields considered will have characteristic zero. If M is a field, we denote by M^{alg} its algebraic closure, by $\text{G}(M) := \text{Gal}(M^{\text{alg}}/M)$ the absolute Galois group of M .

2.1 Ordered fields

A field M is called *formally real* or just *real* if M can be ordered. An ordered field $(M, <)$ is *real closed* if it has no proper ordered algebraic extension. Every ordered field $(M, <)$ has an algebraic extension $(\overline{M}^r, \overline{<}^r)$ which is real closed, and unique up to isomorphism over M . We call this extension the *real closure* of $(M, <)$. The absolute Galois group of \overline{M}^r , $G(\overline{M}^r)$ is cyclic of order 2. Conversely, given an involution σ in $G(M)$, its fixed field $Fix_{M^{alg}}(\sigma)$ is a real closed field and it has a unique ordering $<$ for which the positive elements are exactly the non-zero squares. We refer to the restriction of this ordering to M as: *the ordering of M induced by σ* . If τ and σ in $G(M)$ induce the same ordering on M , then σ and τ are conjugates in $G(M)$ and $Fix_{M^{alg}}(\sigma)$ and $Fix_{M^{alg}}(\tau)$ are isomorphic over M .

A field extension N/M is called *totally real* if each order on M extends to some order on N .

2.2. Amalgamation theorem for ordered fields: If $(M_1, <_1)$ and $(M_2, <_2)$ are extensions of an ordered field $(M, <)$, and if M_1, M_2 are linearly disjoint over M , then M_1M_2 has an ordering $<_3$ that extends both $<_1$ and $<_2$. Moreover, if M_1 and M_2 are algebraic over M , then the extension $<_3$ is unique. A proof of the existence part is given by van den Dries in [43] (Lemma 2.5) and the uniqueness part is shown in section 1 of [26].

2.2 Pseudo real closed fields

Regular extensions: Let N/M be a field extension. We say that N is a *regular extension* of M if N/M is separable and the restriction map: $G(N) \rightarrow G(M)$ is onto. In characteristic 0 this is equivalent to $N \cap M^{alg} = M$.

Fact 2.3. [35, Theorem 1.2] *For a field M the following are equivalent:*

- (1) M is existentially closed (relative to $\mathcal{L}_{\mathcal{R}}$) in every totally real regular extension N of M .
- (2) For every absolutely irreducible variety V defined over M , if V has a simple \overline{M}^r -rational point for every real closure \overline{M}^r of M , then V has an M -rational point.

Prestel showed in Theorem 1.7 of [35] that if M admits only a finite number of orderings, (1) of Fact 2.3 implies that M is existentially closed in N in the language $\mathcal{L}_{\mathcal{R}}$ augmented by predicates for each order $<$ of M .

Definition 2.4. A field M of characteristic 0 satisfying the conditions of Fact 2.3 is called *pseudo real closed* (PRC). By Theorem 4.1 of [35] we can axiomatize the class of PRC fields in $\mathcal{L}_{\mathcal{R}}$. By Theorem 11.5.1 of [19] PAC fields cannot be ordered. So in particular PAC fields of characteristic 0 are PRC fields. Observe also that the class of PRC fields contains the class of real closed fields.

Fact 2.5. *Let M be a PRC field.*

- (1) *If $<$ is an order on M , then M is dense in $(\overline{M}^r, \overline{<}^r)$ ([35, Proposition 1.4]).*

(2) If $<_i$ and $<_j$ are different orders on M , then $<_i$ and $<_j$ induce different topologies ([35, Proposition 1.6]).

(3) If L is an algebraic extension of M , then L is PRC ([35, Theorem 3.1]).

Lemma 2.6. *Let M be a PRC field and $A \subset M$. Then $A^{alg} \cap M = \text{acl}^M(A)$. If in addition M has an $\mathcal{L}_{\mathcal{R}}$ -definable order, then $\text{acl}^M(A) = \text{dcl}^M(A)$.*

Proof. It is clear that if M has a definable order, then $\text{acl}^M(A) = \text{dcl}^M(A)$. Obviously $A^{alg} \cap M \subseteq \text{acl}^M(A)$. Let $A_0 = A^{alg} \cap M$ and let \widetilde{M} be a copy of M by an $\mathcal{L}_{\mathcal{R}}(A_0)$ -isomorphism f , such that M is linearly disjoint from \widetilde{M} over A_0 . As M/A_0 and \widetilde{M}/A_0 are regular and M is linearly disjoint from \widetilde{M} over A_0 , we obtain that $M\widetilde{M}/M$ and $M\widetilde{M}/\widetilde{M}$ are regular. By 2.2 $M\widetilde{M}$ is a totally real regular extension of M and of \widetilde{M} . Then M and \widetilde{M} are existentially closed in $M\widetilde{M}$, so there is an elementary extension M^* of M and of \widetilde{M} such that $M\widetilde{M} \subseteq M^*$.

Let $\alpha \in \text{acl}^M(A)$ and $\tilde{\alpha} = f(\alpha)$. Since $M, \widetilde{M} \prec M^*$, we deduce that $\text{tp}^{M^*}(\alpha/A_0) = \text{tp}^{M^*}(\tilde{\alpha}/A_0)$ and $\text{acl}^{M^*}(A) \subseteq M$. Then $\tilde{\alpha} \in \text{acl}^{M^*}(A) \subseteq M$, since $\alpha \in \text{acl}^M(A)$. It follows that $\tilde{\alpha} \in A_0$, then $\alpha = \tilde{\alpha}$, so $\alpha \in A^{alg} \cap M$. Therefore $\text{acl}^M(A) \subseteq A^{alg} \cap M$. \square

Thanks to the last lemma we get easily the exchange principle and we have a good notion of dimension given by the algebraic closure.

2.3 The theory of PRC fields with n orderings

Definition 2.7. Let M be a field, $n \geq 1$, and $<_1, \dots, <_n$ be n orderings on M . We call the structure $(M, <_1, \dots, <_n)$ an n -fold ordered field.

An n -fold ordered field $(M, <_1, \dots, <_n)$ is n -pseudo real closed (n -PRC) if:

- (1) M is a PRC field,
- (2) if $i \neq j$, then $<_i$ and $<_j$ are different orders on M ,
- (3) $<_1, \dots, <_n$ are the only orderings on M .

Observe that an n -PRC field with $n = 0$ is a PAC field.

Fact 2.8. [27, Proposition 1.4] *Let $(M, <_1, \dots, <_n)$ be an n -PRC field and let V be an absolutely irreducible variety defined over M . Denote by $M^{(i)}$ a fixed real closure of M with respect to $<_i$. For every $1 \leq i \leq n$ take $q_i \in V(M^{(i)})$ a simple point. For each $i \in \{1, \dots, n\}$, let U_i be an $<_i$ -open set such that $q_i \in U_i$. Then V has an M -rational point $q \in \bigcap_{i=1}^n U_i$.*

Fact 2.9. [26, Theorem 3.2] *Let $(M, <_1, \dots, <_n)$ and $(N, <'_1, \dots, <'_n)$ be two n -PRC fields. Let ξ_i and σ_i be involutions in $\text{G}(M)$ and $\text{G}(N)$ that induce $<_i$ and $<'_i$ on M and N respectively. Let L be a common subfield of M and N . Suppose further that there exists an isomorphism $\varphi : \text{G}(N) \rightarrow \text{G}(M)$ such that:*

(a) $\varphi(\sigma)|_{L^{alg}} = \sigma|_{L^{alg}}$ for every $\sigma \in G(N)$.

(b) $\varphi(\sigma_i) = \xi_i$ for $i \in \{1, \dots, n\}$.

Then $(M, <_1, \dots, <_n) \equiv_L (N, <'_1, \dots, <'_n)$.

Corollary 2.10. [26, Corollary 3.3] Let $(M, <_1, \dots, <_n) \subseteq (N, <'_1, \dots, <'_n)$ two n -PRC fields. If the restriction map $G(N) \rightarrow G(M)$ is an isomorphism, then $(M, <_1, \dots, <_n) \prec (N, <'_1, \dots, <'_n)$.

3 Bounded pseudo real closed fields

In this section we study some model theoretic properties of bounded PRC fields. In section 3.1 we give a useful description of definable sets, this description is in some way a generalization to multi-ordered fields of cellular decomposition for real closed fields. In 3.2 we show some results about amalgamation of types.

3.1. Approximation Theorem [37, Theorem 4.1] Let F be a field and τ_1, \dots, τ_n be different topologies on F which are induced by orders or non trivial valuations. For each $i \in \{1, \dots, n\}$, let U_i be a non-empty τ_i -open subset of F . Then $\bigcap_{i=1}^n U_i \neq \emptyset$.

Remark 3.2. If M is a bounded PRC field, then M has only finitely many orders.

Proof. Let $m \in \mathbb{N}$ be such that M has exactly m extensions of degree 2. Suppose by contradiction that there exists $k > m$ and $\{<_i\}_{1 \leq i \leq k}$, distinct orderings on M . For each $1 \leq i \leq k$ choose a_i such that $a_i >_i 0$, $a_i <_j 0$ for all $j \neq i$ (they exist by 3.1 and Fact 2.5). Then the extensions $M(\sqrt{a_i})$, $1 \leq i \leq k$, are proper and linearly disjoint over M . Indeed, $\sqrt{a_i}$ belongs to the real closure $M^{(i)}$ of M with respect to $<_i$, but does not belong to $M^{(j)}$ for $j \neq i$. This contradicts the fact that M has exactly m extensions of degree 2. \square

3.3. Notation & Setting. In this section we fix a bounded PRC field K , which is not real closed and a countable elementary substructure K_0 of K . By Lemma 1.22 of [6] the restriction map: $G(K) \rightarrow G(K_0)$ is an isomorphism, and $K_0^{alg}K = K^{alg}$.

By Remark 3.2 there exists $n \in \mathbb{N}$ such that K has exactly n distinct orders. So K is an n -PRC field. In this section we will work over K_0 , thus we denote by \mathcal{L} the language of rings with constant symbols for the elements of K_0 , $\mathcal{L}^{(i)} := \mathcal{L} \cup \{<_i\}$ and $\mathcal{L}_n := \mathcal{L} \cup \{<_1, \dots, <_n\}$. We let $T := Th_{\mathcal{L}_n}(K)$. If M is a model of T , we denote by $M^{(i)}$ the real closure of M with respect to $<_i$.

Observe that if $n = 0$, K is PAC. In this case the properties of T are well known: by Corollary 4.8 of [8] T is simple and by Corollary 3.1 of [23] T has elimination of imaginaries. **Therefore we will suppose always that $n \geq 1$.**

3.4. Observe that T is model complete: Let $M_1, M_2 \models T$ such that $M_1 \subseteq M_2$. As the restriction maps $G(M_1) \rightarrow G(K_0)$ and $G(M_2) \rightarrow G(K_0)$ are isomorphisms, it follows that the restriction map $G(M_2) \rightarrow G(M_1)$ is also an isomorphism. Then by Corollary 2.10 $M_1 \prec M_2$.

Lemma 3.5. *Let $(M, <_1, \dots, <_n)$ be a model of T . Then for all $i \in \{1, \dots, n\}$ we can define the order $<_i$ by an existential \mathcal{L} -formula.*

Proof. Let $i \in \{1, \dots, n\}$. Then $M \models a >_i 0$ if and only if $M^{(i)} \models \exists \alpha (\alpha^2 = a \wedge \alpha \neq 0)$. Let $\sigma_i \in \text{Gal}(M^{\text{alg}}/M^{(i)})$, $\sigma_i \neq \text{id}$. Define M_2 as the composite field of all the extensions of M of degree 2 and let $\tilde{\sigma}_i = \sigma_i|_{M_2}$. In M we can interpret without quantifiers in the language \mathcal{L} the structure $(M_2, +, \cdot, \tilde{\sigma}_1, \dots, \tilde{\sigma}_n)$, with the action of the automorphism $\tilde{\sigma}_i$, for all $1 \leq i \leq n$. The reader can refer to Appendix 1 of [7] for more details. Therefore we can define the formula “ $a >_i 0$ ” as follows:

$$M \models a >_i 0 \text{ if and only if } M_2 \models \exists \alpha (\alpha \neq 0 \wedge \tilde{\sigma}_i(\alpha) = \alpha \wedge \alpha^2 = a).$$

□

Observe that Lemma 3.5 implies that if $A \subseteq M$ and a is a tuple in M , then $\text{tp}_{\mathcal{L}}(a/A) \vdash \text{tp}_{\mathcal{L}_n}(a/A)$ and $\text{acl}_{\mathcal{L}}(A) = \text{acl}_{\mathcal{L}_n}(A) = \text{dcl}_{\mathcal{L}_n}(A) = \text{dcl}_{\mathcal{L}}(A)$.

Corollary 3.6. *$\text{Th}_{\mathcal{L}}(K)$ is model complete.*

Proof. Use 3.4 and observe that: $x <_i y \leftrightarrow (y - x) >_i 0$, and $\neg(x <_i y) \leftrightarrow y <_i x \vee y = x$.

□

3.7. Types. By Fact 2.9 we can describe the types in T in a simple form: Let M be a model of T , A a subfield of M (containing K_0) and a and b tuples from M . As $K_0^{\text{alg}}M = M^{\text{alg}}$ and $K_0 \subseteq A$, we obtain that $(A(a))^{\text{alg}} = A^{\text{alg}}\text{acl}(A(a))$ and $(A(b))^{\text{alg}} = A^{\text{alg}}\text{acl}(A(b))$. It follows that $\text{tp}^M(a/A) = \text{tp}^M(b/A)$ if and only if there is an \mathcal{L} -isomorphism φ between $\text{acl}(A(a))$ and $\text{acl}(A(b))$, which sends a to b and is the identity on A .

3.1 Density theorem for PRC bounded fields

Lemma 3.8. *(Folklore) Let F be a large algebraically closed field, let $d \in \mathbb{N}$, denote by \mathbb{A}^d the d -dimensional affine space. Let M be a small subfield of F , let $W \subseteq \mathbb{A}^d$ be a Zariski closed set defined over M . Then there exist $m \in \mathbb{N}$ and absolutely irreducible varieties W_1, \dots, W_m defined over M such that:*

$$\bar{x} \in W(M) \text{ if and only if } \bar{x} \in \bigcup_{j \in \{1, \dots, m\}} W_j^{\text{sim}}(M),$$

where $W_j^{\text{sim}}(M) := \{\bar{x} \in W(M) : \bar{x} \text{ is a simple point of } W\}$.

Proof. We can suppose that $W(M)$ is Zariski dense in W since if $V = \overline{W(M)}^z$ is the Zariski closure of $W(M)$, then V is defined over M and $V(M) = W(M)$. Let W_1, \dots, W_m be the absolutely irreducible components of W . Then $W(M) = \bigcup_{j=1}^m W_j(M)$, and $W_j = \overline{W_j(M)}^z$.

Any M -automorphism of F will fix $W_j(M)$ pointwise for all $j \in \{1, \dots, m\}$, hence fix $\overline{W_j(M)}^z$ setwise. Therefore for all $\sigma \in \text{Aut}(F/M)$, $\sigma(W_j) = \sigma(\overline{W_j(M)}^z) = \overline{W_j(M)}^z = W_j$, so W_j is defined over M for all $j \in \{1, \dots, m\}$.

Let $W_j^{\text{sing}}(M) = \{\bar{x} \in W_j(M) : \bar{x} \text{ is a singular point of } W_j\}$. Fix $j \in \{1, \dots, m\}$, we have that $W_j(M) = W_j^{\text{sim}}(M) \cup W_j^{\text{sing}}(M)$, $W_j^{\text{sing}}(M)$ is a closed set in the Zariski topology and its dimension is less than the dimension of $W_j(M)$.

The result follows by induction on the dimension. □

Remark 3.9. Note that if W is defined over $A = A^{\text{alg}} \cap M$, then the W_j are defined over A for all $j \in \{1, \dots, m\}$: The absolutely irreducible components W_j of W are defined over A^{alg} , and by Lemma 3.8 are also defined over M , hence the W_j are defined over $A^{\text{alg}} \cap M = A$.

3.1.1 Density theorem for one variable definable sets

Definition 3.10. Let $(M, <_1, \dots, <_n)$ be a model of T (see 3.3).

(1) A subset of M of the form $I = \bigcap_{i=1}^n (I^i \cap M)$ with I^i a non-empty $<_i$ -open interval in $M^{(i)}$ is called a *multi-interval*. Observe that by 3.1 (Approximation Theorem) and Fact 2.5 every multi-interval is non empty.

(2) A definable subset S of a multi-interval $I = \bigcap_{i=1}^n (I^i \cap M)$ is called *multi-dense* in I if for any multi-interval $J \subseteq I$, $J \cap S \neq \emptyset$. Note that multi-density implies $<_i$ -density in I^i , for all $i \in \{1, \dots, n\}$.

Remark 3.11. Let $(M, <_1, \dots, <_n)$ be a model of T . Let $i \in \{1, \dots, n\}$ and $a \in M^{(i)} \setminus M$ such that $a \in \text{acl}^{M^{(i)}}(c)$, with c a tuple in M . Then $A = \{x \in M : x <_i a\}$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}(c)$ -formula.

Proof. By quantifier elimination of the theory of real closed fields (RCF) and the fact that $\text{acl}^{M^{(i)}}(c) = \text{dcl}^{M^{(i)}}(c)$, we can find a quantifier-free $\mathcal{L}^{(i)}$ -formula $\phi(x, c)$, such that $M^{(i)} \models \forall x(x <_i a \leftrightarrow \phi(x, c))$. Then $x \in A$ if and only if $M \models \phi(x, c)$. □

Proposition 3.12. Let $(M, <_1, \dots, <_n)$ be a model of T . Let $\phi(x, \bar{y})$ be an \mathcal{L}_n -formula, \bar{a} a tuple in M and $b \in M$ such that $M \models \phi(b, \bar{a})$ and $b \notin \text{acl}(\bar{a})$. Then there is a multi-interval

$I = \bigcap_{i=1}^n (I^i \cap M)$ such that:

- (1) $b \in I$,
- (2) $\{x \in I : M \models \phi(x, \bar{a})\}$ is multi-dense in I ,
- (3) $I^i \subseteq M^{(i)}$ has its extremities in $\text{dcl}_{\mathcal{L}^{(i)}}^{M^{(i)}}(\bar{a}) \cup \{\pm\infty\}$, for all $1 \leq i \leq n$,
- (4) the set $I^i \cap M$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}(\bar{a})$ -formula, for all $1 \leq i \leq n$.

Proof. By Corollary 3.6 there exists a quantifier-free $\mathcal{L}(\bar{a})$ -formula $\psi(x, \bar{y})$ such that $M \models \forall x(\phi(x, \bar{a}) \leftrightarrow \exists \bar{y}\psi(x, \bar{y}))$.

As we can define the relation \neq with an existential formula, we can suppose that $\psi(x, \bar{y})$ is a positive formula, i.e defines an algebraic set W defined over $\text{acl}(\bar{a})$.

Then $M \models \phi(x, \bar{a})$ is equivalent to $\exists \bar{y} (x, \bar{y}) \in W(M)$.

Let d be the arity of \bar{y} . As $M \models \phi(b, \bar{a})$ we can find $\bar{y}_0 \in M^d$ such that $(b, \bar{y}_0) \in W(M)$. By Lemma 3.8 and Remark 3.9, there exists an absolutely irreducible variety V defined over $\text{acl}(\bar{a})$ such that (b, \bar{y}_0) is a simple point of V and $V(M) \subseteq W(M)$.

For each $i \in \{1, \dots, n\}$ we define:

$$A_i := \{x \in M^{(i)} : \exists (y_1, \dots, y_d) \in (M^{(i)})^d (x, y_1, \dots, y_d) \text{ is a simple point of } V\}.$$

Observe that $b \in A_i$ and that A_i is $\mathcal{L}^{(i)}$ -definable in $M^{(i)}$ with parameters in $\text{acl}(\bar{a})$. By o-minimality of $M^{(i)}$ there is an $<_i$ -interval $I^i \subseteq M^{(i)}$, with extremities in $\text{dcl}_{\mathcal{L}^{(i)}}^{M^{(i)}}(\bar{a}) \cup \{\pm\infty\}$, such that $b \in I^i$ and $I^i \subseteq A_i$. By Remark 3.11, $I^i \cap M$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}$ -formula.

$$\text{Define } I := \bigcap_{i=1}^n (I^i \cap M) \text{ and } S := \{x \in I : M \models \phi(x, \bar{a})\}.$$

Claim. S is multi-dense in I :

Proof. Let $J \subseteq I$ be a non-empty multi-interval; we need to show that $J \cap S \neq \emptyset$. Let $z \in J$; since $z \in A_i$ for all $i \in \{1, \dots, n\}$, there are $y^{(i)} \in (M^{(i)})^d$ such that each $q_i := (z, y^{(i)})$ is a simple point of V . By Fact 2.8 we can find $q_0 := (z_0, y_0) \in V(M)$ such that q_0 is arbitrary $<_i$ -close to q_i for all $i \in \{1, \dots, n\}$. In particular we can find $z_0 \in J$. Then we obtain that $\exists \bar{y} (z_0, \bar{y}) \in V(M)$, and then $M \models \phi(z_0, \bar{a})$. \square

\square

Theorem 3.13. *Let $(M, <_1, \dots, <_n)$ be a model of T , let $\phi(x, \bar{y})$ be an \mathcal{L}_n -formula and let \bar{a} be a tuple in M . Then there are a finite set $A \subseteq \phi(M, \bar{a})$, $m \in \mathbb{N}$ and I_1, \dots, I_m , with*

$$I_j = \bigcap_{i=1}^n (I_j^i \cap M) \text{ a multi-interval such that:}$$

- (1) $A \subseteq \text{acl}(\bar{a})$,
- (2) $\phi(M, \bar{a}) \subseteq \bigcup_{j=1}^m I_j \cup A$,

- (3) $\{x \in I_j : M \models \phi(x, \bar{a})\}$ is multi-dense in I_j , for all $1 \leq j \leq m$,
- (4) $I_j^i \subseteq M^{(i)}$ has its extremities in $\text{dcl}_{\mathcal{L}^{(i)}}^{M^{(i)}}(\bar{a}) \cup \{\pm\infty\}$, for all $1 \leq j \leq m$ and $1 \leq i \leq n$,
- (5) the set $I_j^i \cap M$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}(\bar{a})$ -formula, for all $1 \leq j \leq m$ and $1 \leq i \leq n$.

Proof. As in Proposition 3.12 using Corollary 3.6 and Lemma 3.8, there are $r \in \mathbb{N}$ and absolutely irreducible varieties W_1, \dots, W_r defined over $\text{acl}(\bar{a})$ such that:

$$M \models \forall x(\phi(x, \bar{a}) \leftrightarrow \exists \bar{y} (x, \bar{y}) \in \bigcup_{j \in \{1, \dots, r\}} W_j^{\text{sim}}(M)).$$

Working with each W_j separately, we can suppose that there is an absolutely irreducible variety W defined over $\text{acl}(\bar{a})$ such that:

$$M \models \forall x(\phi(x, \bar{a}) \longleftrightarrow \exists \bar{y}(x, \bar{y}) \in W^{\text{sim}}(M)).$$

Let $d = |\bar{y}|$, for each $i \in \{1, \dots, n\}$ define:

$$A_i := \{x \in M^{(i)} : \exists \bar{y} \in (M^{(i)})^d(x, \bar{y}) \text{ is a simple point of } W\}.$$

By o-minimality of $M^{(i)}$ there are $r_i \in \mathbb{N}$, $I_1^i, \dots, I_{r_i}^i <_i$ -open intervals in $M^{(i)}$ with extremities in $\text{dcl}_{\mathcal{L}^{(i)}}^{M^{(i)}}(\bar{a}) \cup \{\pm\infty\}$, and a finite set C_i such that $A_i = \bigcup_{j=1}^{r_i} I_j^i \cup C_i$ and $C_i \subseteq \text{acl}(\bar{a})$. By

Remark 3.11, the set $I_j^i \cap M$, with $1 \leq i \leq n$, $1 \leq j \leq r_i$, is definable in M by a quantifier-free $\mathcal{L}^{(i)}(\bar{a})$ -formula.

By the proof of Proposition 3.12, $\phi(M, \bar{a}) \subseteq \bigcup_{\sigma \in J} \bigcap_{i=1}^n (I_{\sigma(i)}^i \cap M) \cup \bigcup_{i=1}^n (C_i \cap M)$, where $J = \{\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, \max\{r_1, \dots, r_n\}\}, \sigma(i) \leq r_i\}$.

For each $\sigma \in J$, define $I_\sigma := \bigcap_{i=1}^n (I_{\sigma(i)}^i \cap M)$ and $S_\sigma := \{x \in I_\sigma : M \models \phi(x, \bar{a})\}$.

Claim. For all $\sigma \in J$, S_σ is multi-dense in I_σ :

Proof. Fix $\sigma \in J$. Let U_σ be a multi-interval such that $U_\sigma \subseteq I_\sigma$, we need to show that $U_\sigma \cap S_\sigma \neq \emptyset$. Let $z \in U_\sigma$. Then $z \in A_i$ for all $i \in \{1, \dots, n\}$. So there is $y^{(i)} \in (M^{(i)})^d$, such that $q_i := (z, y^{(i)})$ is a simple point of W . By Fact 2.8 we can find $q_0 := (z_0, \bar{y}_0) \in W(M)$ such that q_0 is arbitrary $<_i$ -close to q_i for all $i \in \{1, \dots, n\}$, in particular we can find $z_0 \in U_\sigma$ which satisfy ϕ . □

□

3.1.2 Density theorem for several variable definable sets.

The proof of Proposition 3.12 and Theorem 3.17 can be easily generalized to several variables. We assume the reader is familiar with the concept of cells and cell decomposition in o-minimal theories. See chapter 3 of [45] for more details.

Definition 3.14. If $(M, <)$ is an ordered field, and $r \in \mathbb{N}$, then a *box in M^r* is a set of the form $I_1 \times \dots \times I_r$, where I_j is a non-empty $<$ -open interval, for all $j \in \{1, \dots, r\}$.

Remark 3.15. Let F be a field and τ_1, \dots, τ_n distinct topologies on F induced by orders or valuations. Let $r \in \mathbb{N}$. For each $i \in \{1, \dots, n\}$, let U^i be a non-empty τ_i -open set in F^r (endowed with the product topology). Then $\bigcap_{i=1}^n U^i \neq \emptyset$.

Proof. Let $i \in \{1, \dots, n\}$. As U^i is τ_i -open in F^r there exist I_1^i, \dots, I_r^i non-empty τ_i -open subsets of F such that $I_1^i \times \dots \times I_r^i \subseteq U^i$. By the Approximation Theorem (3.1) for all $t \in \{1, \dots, r\}$, $\bigcap_{i=1}^n I_t^i \neq \emptyset$. If $V_t = \bigcap_{i=1}^n I_t^i$, then $\emptyset \neq V_1 \times \dots \times V_r \subseteq \bigcap_{i=1}^n U^i$. \square

Definition 3.16. Let $(M, <_1, \dots, <_n)$ be a model of T and let $r \in \mathbb{N}$.

- (1) A subset of M^r of the form $C = \bigcap_{i=1}^n (C^i \cap M^r)$ with C^i a non-empty $<_i$ -open cell in $(M^{(i)})^r$ is called a *multi-cell in M^r* . Observe that by Remark 3.15 every multi-cell is not empty.
- (2) If $C = \bigcap_{i=1}^n (C^i \cap M^r)$ is a multi-cell in M^r , then C is called a *multi-box in M^r* if C^i is a $<_i$ -box in $(M^{(i)})^r$, for all $i \in \{1, \dots, n\}$.
- (3) A definable subset S of a multi-cell $C = \bigcap_{i=1}^n (C^i \cap M^r)$ in M^r is called *multi-dense in C* if for any multi-box $J \subseteq C$ in M^r , $J \cap S \neq \emptyset$. Note that multi-density in C implies $<_i$ -density in C^i , for all $i \in \{1, \dots, n\}$.

Theorem 3.17. Let $(M, <_1, \dots, <_n)$ be a model of T and let $r \in \mathbb{N}$. Let $\phi(x_1, \dots, x_r, \bar{a})$ be an \mathcal{L}_n -formula. Then there are a set V , $m \in \mathbb{N}$, and C_1, \dots, C_m with $C_j = \bigcap_{i=1}^n (C_j^i \cap M^r)$ a multi-cell in M^r such that:

- (1) $\phi(M, \bar{a}) \subseteq \bigcup_{j=1}^m C_j \cup V$,
- (2) the set V is contained in some proper Zariski closed subset of M^r which is definable over $\text{acl}(\bar{a})$,

- (3) $\{(x_1, \dots, x_r) \in C_j : M \models \phi(x_1, \dots, x_r, \bar{a})\}$ is multi-dense in C_j , for all $1 \leq j \leq m$,
- (4) the set $C_j^i \cap M^r$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}(\bar{a})$ -formula, for all $1 \leq j \leq m$, $1 \leq i \leq n$.

Proof. As in the proof of Theorem 3.13 we can suppose that there is an absolutely irreducible variety W defined over $\text{acl}(\bar{a})$ such that:

$$M \models \forall(x_1, \dots, x_r)(\phi(x_1, \dots, x_r, \bar{a}) \longleftrightarrow \exists \bar{y}(x_1, \dots, x_r, \bar{y}) \in W^{\text{sim}}(M)).$$

Let $d = |\bar{y}|$, for each $i \in \{1, \dots, n\}$ we define

$$A_i := \{(x_1, \dots, x_r) \in (M^{(i)})^r : \exists(\bar{y}) \in (M^{(i)})^d (x_1, \dots, x_r, \bar{y}) \text{ is a simple point of } W\}.$$

Observe that A_i is $\mathcal{L}^{(i)}(\bar{a})$ -definable. By cell decomposition in $M^{(i)}$, there are $r_i \in \mathbb{N}$, pairwise disjoint $<_i$ -open cells $C_1^i, \dots, C_{r_i}^i$, and proper Zariski closed subsets V^i of $(M^{(i)})^r$ such that:

$$(1) A_i = \bigcup_{j=1}^{r_i} C_j^i \cup V^i,$$

$$(2) \bigcup_{j=1}^{r_i} C_j^i \cap V^i = \emptyset,$$

(3) the sets $V^i, C_1^i, \dots, C_{r_i}^i$ are quantifier-free $\mathcal{L}^{(i)}(\bar{a})$ -definable.

Let $V := \bigcup_{i=1}^n (V^i \cap M^r)$. Then V is $\mathcal{L}_n(\bar{a})$ -definable in M and $\dim(V) < r$.

Let $J = \{\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, \max\{r_1, \dots, r_n\}\}, \sigma(i) \leq r_i\}$. For each $\sigma \in J$ define $C_\sigma := \bigcap_{i=1}^n (C_{\sigma(i)}^i \cap M^r)$. Then $\phi(M, \bar{a}) \subseteq \bigcup_{\sigma \in J} C_\sigma \cup V$.

Exactly as the proof in Theorem 3.13 we have that:

$\{(x_1, \dots, x_r) \in C_\sigma : M \models \phi(x_1, \dots, x_r, \bar{a})\}$ is multi-dense in C_σ for all $\sigma \in J$. \square

Lemma 3.18. *Let $(M, <_1, \dots, <_n)$ be a model of T . Let $A \subseteq M$ and let \bar{a} be a tuple of M such that $\text{trdeg}(A(\bar{a})/A) = |\bar{a}|$. For all $i \in \{1, \dots, n\}$, let $\bar{b}_i \in M^{|\bar{a}|}$ be such that $\text{qftp}_{\mathcal{L}^{(i)}}(\bar{b}_i/A) = \text{qftp}_{\mathcal{L}^{(i)}}(\bar{a}/A)$, and let U^i be a non-empty $<_i$ -open set in $(M^{(i)})^{|\bar{a}|}$ such that $\bar{b}_i \in U^i$. Then the type $p(\bar{x}) := \{\bar{x} \in \bigcap_{i=1}^n U^i\} \cup \text{tp}_{\mathcal{L}_n}(\bar{a}/A) \cup \{\bar{x} \neq \bar{a}\}$ is consistent.*

Proof. Let $U := \bigcap_{i=1}^n U^i$. By compactness it is enough to show that if $\psi(\bar{x}) \in \text{tp}_{\mathcal{L}_n}(\bar{a}/A)$, then $M \models \exists \bar{x}(\bar{x} \in U \wedge \psi(\bar{x}) \wedge \bar{x} \neq \bar{a})$. As $\text{trdeg}(A(\bar{a})/A) = |\bar{a}|$, by Proposition 3.17 there exists a multi-cell $C := \bigcap_{i=1}^n (C^i \cap M^{|\bar{a}|})$ in $M^{|\bar{a}|}$ such that:

- (1) $\bar{a} \in C$,
- (2) $\{\bar{x} \in C : M \models \psi(\bar{x})\}$ is multi-dense in C ,
- (3) $C^i \cap M^{|\bar{a}|}$ is definable in M by a quantifier free $\mathcal{L}^{(i)}(A)$ -formula, for all $i \in \{1, \dots, n\}$.

Let $i \in \{1, \dots, n\}$. Since $\bar{a} \in C^i \cap M^{|\bar{a}|}$, $C^i \cap M^{|\bar{a}|}$ is quantifier free $\mathcal{L}^{(i)}(A)$ -definable and $\text{qftp}_{\mathcal{L}^{(i)}}(\bar{b}_i/A) = \text{qftp}_{\mathcal{L}^{(i)}}(\bar{a}/A)$, we deduce that $\bar{b}_i \in C^i \cap M^{|\bar{a}|}$. Let $V^i := U^i \cap C^i \cap M^{|\bar{a}|}$. Observe that $\bar{b}_i \in V^i$ and that V^i is an $<_i$ -open set in $M^{|\bar{a}|}$. Let $V := \bigcap_{i=1}^n V^i$, by Remark 3.15 $V \neq \emptyset$. Since $V \subseteq C$ and every V^i is $<_i$ -open in $M^{|\bar{a}|}$, by multi-density of $\psi(\bar{x})$ in C there exists $\bar{c} \in V$, $\bar{c} \neq \bar{a}$ such that $M \models \psi(\bar{c})$. Then $M \models \bar{c} \in U \wedge \psi(\bar{c}) \wedge \bar{c} \neq \bar{a}$, since $V \subseteq U$. \square

3.2 Amalgamation theorems for PRC bounded fields

Lemma 3.19. *Let F_1 and F_2 be regular extensions of a field k . Let $H \leq G(F_1F_2)$ be a closed subgroup, let $\pi : G(F_1F_2) \rightarrow G(F_1)$ be the restriction map, and suppose that $\pi|_H$ is an isomorphism. Let $\rho \in G(F_1F_2)$ be such that ρ fixes $F_1^{\text{alg}} \cap \text{Fix}(H)$ and assume that for every $\sigma \in H$, $\pi(\sigma) = \pi(\rho^{-1}\sigma\rho)$. Then there is $\tilde{\rho} \in G(F_1^{\text{alg}}F_2)$ such that $\tilde{\rho}^{-1}\sigma\tilde{\rho} = \rho^{-1}\sigma\rho$ for all $\sigma \in H$.*

Proof. The condition $\pi(\sigma) = \pi(\rho^{-1}\sigma\rho)$ for all $\sigma \in H$ implies that $\pi(\rho)$ centralizes $\pi(H)$. We have that $\text{Fix}(\pi(H)) = \text{Fix}(H) \cap F_1^{\text{alg}}$ and that $\text{Fix}(\rho) \cap F_1^{\text{alg}} = \text{Fix}(\pi(\rho))$. By hypothesis $\text{Fix}(H) \cap F_1^{\text{alg}} \subseteq \text{Fix}(\rho) \cap F_1^{\text{alg}}$. So $\pi(\rho) \in \pi(H)$. Let $\tau \in H$ be such that $\pi(\rho) = \pi(\tau)$, and consider $\tilde{\rho} = \tau^{-1}\rho$. Since $\pi(\tau)$ centralizes $\pi(H)$ and π is an isomorphism, $\tau\sigma\tau^{-1} = \sigma$ for all $\sigma \in H$, and therefore $\tilde{\rho}^{-1}\sigma\tilde{\rho} = \rho^{-1}\sigma\rho$, for all $\sigma \in H$ and clearly $\pi(\tilde{\rho}) = 1$. \square

Proposition 3.20. *Let $(M, <_1, \dots, <_n)$ be a model of T and $E = \text{acl}(E) \subseteq M$. Let a_1, a_2, c_1, c_2 be tuples of M such that $E(a_1)^{\text{alg}} \cap E(a_2)^{\text{alg}} = E^{\text{alg}}$ and $\text{tp}_{\mathcal{L}_n}(c_1/E) = \text{tp}_{\mathcal{L}_n}(c_2/E)$. Assume that there is c realizing $\text{qftp}_{\mathcal{L}_n}(c_1/E(a_1)) \cup \text{qftp}_{\mathcal{L}_n}(c_2/E(a_2)) \cup \text{tp}_{\mathcal{L}_n}(c_1/E)$, such that c is ACF-independent from $\{a_1, a_2\}$ over E . Then $\text{tp}_{\mathcal{L}_n}(c_1/E(a_1)) \cup \text{tp}_{\mathcal{L}_n}(c/E(a_2)) \cup \text{qftp}_{\mathcal{L}_n}(c/E(a_1, a_2))$ is consistent.*

Proof. Take c in some elementary extension N of M such that c is ACF-independent from $\{a_1, a_2\}$ over E and realizes $\text{qftp}_{\mathcal{L}_n}^N(c_1/E(a_1)) \cup \text{qftp}_{\mathcal{L}_n}^N(c_2/E(a_2)) \cup \text{tp}_{\mathcal{L}_n}^N(c_1/E)$. For all $1 \leq i \leq n$, fix a real closure $N^{(i)}$ of N for $<_i$. If $A \subset N^{(i)}$, then we set $A^{(i)} = A^{\text{alg}} \cap N^{(i)}$.

As $\text{qftp}_{\mathcal{L}_n}(c/E(a_1)) = \text{qftp}_{\mathcal{L}_n}(c_1/E(a_1))$ there exists an \mathcal{L}_n -isomorphism $\Phi : E(a_1, c_1) \rightarrow E(a_1, c)$, which fixes $E(a_1)$ and sends c_1 to c .

We denote $C := \text{acl}(E(c))$, $A_1 := \text{acl}(E(a_1))$, $A_2 := \text{acl}(E(a_2))$ and $C_1 := \text{acl}(E(c_1))$.

Claim 1. We can extend Φ to an $\mathcal{L}(A_1^{\text{alg}})$ -isomorphism $\tilde{\Phi} : (A_1C_1)^{\text{alg}} \rightarrow (A_1C)^{\text{alg}}$ such that $\tilde{\Phi}|_{A_1C_1}$ is an \mathcal{L}_n -isomorphism and $\tilde{\Phi}(C_1) = C$.

Proof. In $N^{(i)}$ $\text{acl}_{\mathcal{L}^{(i)}} = \text{dcl}_{\mathcal{L}^{(i)}}$, so we can extend Φ uniquely to an $\mathcal{L}^{(i)}$ -isomorphism $\Phi^{(i)} : E(a_1, c_1)^{(i)} \rightarrow E(a_1, c)^{(i)}$. Note that for all $1 \leq i \leq n$, $\Phi^{(i)}$ is the identity on $A_1^{(i)}$.

Since $\text{tp}_{\mathcal{L}_n}(c_1/E) = \text{tp}_{\mathcal{L}_n}(c/E)$, there is an elementary \mathcal{L}_n -isomorphism $\xi : C_1 \rightarrow C$ which fixes E and sends c_1 to c . As above we can extend ξ to an $\mathcal{L}^{(i)}$ -isomorphism $\xi^{(i)} : C_1^{(i)} \rightarrow C^{(i)}$.

As in $N^{(i)}$ $\text{acl}_{\mathcal{L}^{(i)}} = \text{dcl}_{\mathcal{L}^{(i)}}$, we get that $\Phi^{(i)}|_{C_1^{(i)}} = \xi^{(i)}$ for all $1 \leq i \leq n$. This implies that for all $1 \leq i \leq n$, $\Phi^{(i)}|_{C_1} = \xi$, it follows that $\Phi^{(i)}|_{A_1 C_1} = \Phi^{(j)}|_{A_1 C_1}$ for all $i, j \in \{1, \dots, n\}$. Therefore $\Phi^{(i)}|_{A_1 C_1} : A_1 C_1 \rightarrow A_1 C$ is an \mathcal{L}_n -isomorphism and $\Phi^{(i)}(C_1) = C$.

Since C_1/E and C/E are regular extensions and A_1 is ACF-independent from C_1 and from C over E , we have that $C_1 A_1$ and $C A_1$ are linearly disjoint from A_1^{alg} over A_1 . So we can extend $\Phi^{(i)}|_{A_1 C_1}$ to an A_1^{alg} -isomorphism $\tilde{\Phi} : (A_1 C_1)^{\text{alg}} \rightarrow (A_1 C)^{\text{alg}}$. \square

Let $D := \tilde{\Phi}(\text{acl}(A_1 C_1)) \subseteq (A_1 C)^{\text{alg}}$. The map $\Psi : \text{G}(\text{acl}(A_1 C_1)) \rightarrow \text{G}(D)$ defined by $\Psi(\tau) = \tilde{\Phi}\tau\tilde{\Phi}^{-1}$, is an isomorphism inducing the identity on $\text{G}(A_1)$.

Let $\pi : \text{G}(\text{acl}(A_1 C_1)) \rightarrow \text{G}(E)$ the restriction map. As $\text{G}(D) \cong \text{G}(\text{acl}(A_1 C_1)) \cong \text{G}(E)$, there exists $\theta : \text{G}(E) \rightarrow \text{G}(D)$ such that $\Psi = \theta \circ \pi$.

Let $K := A_1 C$, $F := \text{acl}(A_1 A_2)\text{acl}(A_2 C)$, $\tilde{K} := A_1^{\text{alg}} C$, $\tilde{F} := (A_1 A_2)^{\text{alg}}(A_2 C)^{\text{alg}}$ and $\tilde{D} := (A_1 C)^{\text{alg}}$.

Define $S := \{(\sigma, \theta(\sigma|_{E^{\text{alg}}})) : \sigma \in \text{Gal}(\tilde{F}/F)\}$.

Claim 2. S is isomorphic to a subgroup S' of $\text{Gal}(\tilde{F}\tilde{D}/FD)$ which projects onto $\text{Gal}(\tilde{F}/F)$ and onto $\text{G}(D)$.

Proof. By Lemma 2.5 (2) of [7] $(A_1 A_2)^{\text{alg}}(A_2 C)^{\text{alg}} \cap (A_1 C)^{\text{alg}} = (C^{\text{alg}}(A_1^{\text{alg}} \cap A_2^{\text{alg}}))A_1^{\text{alg}} = A_1^{\text{alg}} C^{\text{alg}}$. Since $E^{\text{alg}} M = M^{\text{alg}}$ it follows that $E^{\text{alg}} C = C^{\text{alg}}$. Therefore $\tilde{F} \cap \tilde{D} = A_1^{\text{alg}} C = \tilde{K}$. As \tilde{D}/\tilde{K} is a Galois extension, \tilde{F} and \tilde{D} are linearly disjoint over \tilde{K} .

Since F/A_1 is a regular extension, F is linearly disjoint from A_1^{alg} over A_1 . As $A_1 \subseteq K \subseteq F$, F is linearly disjoint from $K A_1^{\text{alg}}$ over K . Since $K A_1^{\text{alg}} = A_1^{\text{alg}} C = \tilde{K}$, F is linearly disjoint from \tilde{K} over K . Similarly, as D/A_1 is a regular extension and $A_1 \subseteq K \subseteq D$, we get that D is linearly disjoint from \tilde{K} over K .

By Lemma 2.6 of [8] applied to $(K, F, D, \tilde{K}, \tilde{F}, \tilde{D})$ in place of (K, L, M, K_1, L_1, M_1) we obtain that: $\text{Gal}(\tilde{F}\tilde{D}/FD) \simeq \{(\sigma, \tau) \in \text{Gal}(\tilde{F}/F) \times \text{Gal}(\tilde{D}/D) : \sigma|_{\tilde{K}} = \tau|_{\tilde{K}}\}$.

Let $\sigma \in \text{Gal}(\tilde{F}/F)$. Since $\tilde{\Phi}(x) = x$ for all $x \in A_1^{\text{alg}}$, and σ and $\theta(\sigma|_{E^{\text{alg}}})$ are the identity on C , we deduce that σ and $\theta(\sigma|_{E^{\text{alg}}})$ agree on K_1 . Then S is isomorphic to a subgroup S' of $\text{Gal}(\tilde{F}\tilde{D}/FD)$. \square

By claim 2, $L = \text{Fix}(S')$ is a regular extension of D and F . As $N/\text{acl}(A_1 A_2)$ and $N/\text{acl}(A_2 C)$ are regular extensions and contain F , it follows that $L/\text{acl}(A_1 A_2)$ and $L/\text{acl}(A_2 C)$ are regular extensions.

Claim 3. Each of the n orders on F extends to an order on L .

Proof. Fix $1 \leq i \leq n$ and let $H := \text{G}(N^{(i)}) \cong \text{G}(F^{(i)})$. It suffices to show that L is contained in $\text{Fix}(\tilde{H})$ with \tilde{H} a conjugate of H . Let π_1 and π_2 be the restriction maps of $\text{G}(N)$ to $\text{G}(A_1 C_1)$ and $\text{G}(A_1 C)$ respectively.

Denote by $H_1 := \{\pi_1(\sigma) : \sigma \in H\}$ and $H_2 := \{\pi_2(\sigma) : \sigma \in H\}$. Since $\tilde{\Phi}|_{A_1C_1}$ is an $\mathcal{L}_n(A_1)$ -isomorphism and $\tilde{\Phi}(C_1) = C$, it follows that $\Psi(H_1)$ and H_2 are conjugate in $G(A_1C)$. In fact, $\tilde{\Phi}(A_1^{(i)}C_1) = A_1^{(i)}C$, since $\tilde{\Phi}|_{A_1^{alg}} = id$. So we can find $\rho \in G(A_1^{(i)}C)$ such that $\rho^{-1}\pi_2(\sigma)\rho = \Psi(\pi_1(\sigma)) = \theta(\sigma|_{E^{alg}})$, for all $\sigma \in H$. Then $(\rho^{-1}\sigma\rho)|_{A_1^{alg}} = \sigma|_{A_1^{alg}}$, for all $\sigma \in H$.

Applying Lemma 3.19 with $F_1 = A_1$ and $F_2 = C$, we can suppose that $\rho \in G(A_1^{alg}C)$. By Lemma 2.5 (2) of [7] we obtain that $(A_1A_2)^{alg}(A_2C)^{alg}$ and $(A_1C)^{alg}$ are linearly disjoint over $A_1^{alg}C$, therefore we may extend ρ to an element $\tilde{\rho} \in G((A_1A_2)^{alg}(A_2C)^{alg})$ (recall that $A_1^{alg}C \supset C^{alg}$). Let $\tilde{H} := \tilde{\rho}^{-1}H\tilde{\rho}$.

Then we have for all $\sigma \in H$ we have that:

$$\pi_1(\tilde{\rho}^{-1}\sigma\tilde{\rho}) = \rho^{-1}\pi_1(\sigma)\rho = \Psi(\pi_2(\sigma)) = \theta(\sigma|_{E^{alg}}).$$

The image of \tilde{H} by the restriction map inside $\text{Gal}((A_1A_2)^{alg}(A_2C)^{alg}(A_1C)^{alg}/A_1A_2C)$ is contained in S . Hence $\text{Fix}(\tilde{H}) \supseteq \text{Fix}(S) = L$. \square

Let $\tilde{<}_1, \dots, \tilde{<}_n$ be n orders on L extending the n orders on F . Since $L/\text{acl}(A_1A_2)$ and $M/\text{acl}(A_1A_2)$ are regular, we can suppose that they are linearly disjoint, so by Fact 2.2, for all $1 \leq i \leq n$, $<_i$ and $\tilde{<}_i$ have a common extension to an ordering on LM . Hence LM is a totally real extension of M .

Since LM/M is regular, by Fact 2.3, M is existentially closed in LM , so there is an elementary extension M^* of M such that $M \subseteq LM \subseteq M^*$.

Claim 4. (1) $D = (A_1C)^{alg} \cap M^* = \text{acl}^{M^*}(A_1C)$

$$(2) \text{acl}^N(A_1C_1) = (A_1C_1)^{alg} \cap N = (A_1C_1)^{alg} \cap M^* = \text{acl}^{M^*}(A_1C_1)$$

$$(3) \text{acl}^N(A_2C) = (A_2C)^{alg} \cap N = (A_2C)^{alg} \cap M^* = \text{acl}^{M^*}(A_2C)$$

Proof. (1): We have that $D \subseteq D^{alg} \cap M^*$. Suppose that $D \neq D^{alg} \cap M^*$, and let $\alpha \in D^{alg} \cap M^*$, $\alpha \notin D$. Since $D^{alg} = E^{alg}D$, there exists $\beta \in E^{alg} \setminus E$ such that $D(\beta) = D(\alpha) \subset M^*$, so $\beta \in M^*$ and this is a contradiction. Hence $D = D^{alg} \cap M^*$. As $D \subseteq (A_1C)^{alg}$ the claim follows.

(2) and (3): Clear because $M \prec N, M^*$ and by claim 2 \square

Claim 5. We have that c realizes $\text{tp}_{\mathcal{L}_n}^{M^*}(c_1/E(a_1)) \cup \text{tp}_{\mathcal{L}_n}^{M^*}(c/E(a_2)) \cup \text{qftp}_{\mathcal{L}_n}^{M^*}(c/E(a_1, a_2))$.

Proof. Define $\Phi' := \tilde{\Phi}|_{\text{acl}^N(A_1C_1)}$. Then $\Phi' : \text{acl}^N(A_1C_1) \rightarrow D$ satisfies that $\Phi'(c_1) = c$ and $\Phi'|_{A_1} = id$. By Claim 4 (1) and (2) $\Phi' : \text{acl}^{M^*}(A_1C_1) \rightarrow \text{acl}^{M^*}(A_1C)$, therefore by 3.7 $\text{tp}^{M^*}(a_1/E(a_1)) = \text{tp}^{M^*}(c/E(a_1))$. Then c realizes $\text{tp}^{M^*}(c_1/E(a_2))$. \square

\square

\square

Theorem 3.21. Let $(M, <_1, \dots, <_n)$ be a model of T . Let $E = \text{acl}(E) \subseteq M$. Let a_1, a_2, c_1, c_2 be tuples of M such that $E(a_1)^{alg} \cap E(a_2)^{alg} = E^{alg}$ and $\text{tp}_{\mathcal{L}_n}(c_1/E) = \text{tp}_{\mathcal{L}_n}(c_2/E)$. Assume that there is c ACF-independent of $\{a_1, a_2\}$ over E realizing $\text{qftp}_{\mathcal{L}_n}(c_1/E(a_1)) \cup \text{qftp}_{\mathcal{L}_n}(c_2/E(a_2))$. Then $\text{tp}_{\mathcal{L}_n}(c_1/Ea_1) \cup \text{tp}_{\mathcal{L}_n}(c_2/Ea_2) \cup \text{qftp}_{\mathcal{L}_n}(c/E(a_1, a_2))$ is consistent.

Proof. Observe that $\text{trdeg}(E(c_j)/E) = \text{trdeg}(E(c)/E) = \text{trdeg}(E(c, a_j)/E(a_j))$, for $j = 1, 2$. Remark that if $d \in E(c)^{\text{alg}} \cap M = \text{acl}(Ec) = \text{dcl}(Ec)$, then $\text{tp}(d/Ec)$ is isolated by a quantifier-free $\mathcal{L}^{(i)}$ -formula. Then we can suppose that $\text{trdeg}(E(c)/E) = |c| = |c_j|$.

Claim. We can suppose that $\text{tp}_{\mathcal{L}_n}(c/E) = \text{tp}_{\mathcal{L}_n}(c_1/E)$.

Proof. Suppose that M is sufficiently saturated. We need to show that $\text{qftp}_{\mathcal{L}_n}(c/E(a_1)) \cup \text{qftp}_{\mathcal{L}_n}(c/E(a_2)) \cup \text{tp}_{\mathcal{L}_n}(c_1/E)$ is realized by some c^* ACF-independent of $\{a_1, a_2\}$ over E . By compactness it is enough to show that if $\psi_j(x, a_j) \in \text{qftp}_{\mathcal{L}_n}(c/E(a_j))$, for $j = 1, 2$ and $\phi(x) \in \text{tp}_{\mathcal{L}_n}(c_1/E)$, then $\psi_1(x, a_1) \wedge \psi_2(x, a_2) \wedge \phi(x)$ is realized for some c^* which is ACF-independent from $\{a_1, a_2\}$ over E .

Since $\phi(x) \in \text{tp}_{\mathcal{L}_n}(c_1/E)$, there is a multi cell $C := \bigcap_{i=1}^n (C^i \cap M^{|c|})$ in $M^{|c|}$ such that $c_1 \in C$, $\{x \in C : M \models \phi(x)\}$ is multi-dense in C and the set $C \cap M^{|c|}$ is definable in M by a quantifier-free $\mathcal{L}_n(E)$ -formula. Then $c \in C$, since $\text{qftp}_{\mathcal{L}_n}(c/E) = \text{qftp}_{\mathcal{L}_n}(c_1/E)$.

For $j = 1, 2$, as $\psi_j(x, a_j) \in \text{qftp}_{\mathcal{L}_n}(c/E(a_j))$, there is a multi-cell $U_j := \bigcap_{i=1}^n (U_j^i \cap M^{|c|})$ in $M^{|c|}$ such that $c \in U_j$ and $U_j \subseteq \psi_j(M, a_j)$. Then for all $i \in \{1, \dots, n\}$, $U_1^i \cap U_2^i \cap C^i \neq \emptyset$. By saturation we can find for all $i \in \{1, \dots, n\}$ an $<_i$ -cell D^i in $(M^{(i)})^{|c|}$ such that $D^i \subseteq U_1^i \cap U_2^i \cap C^i$.

Let $D := \bigcap_{i=1}^n (D^i \cap M^{|c|})$. By multi-density there is $c^* \in D$, ACF-independent of $\{a_1, a_2\}$ over E , and such that $M \models \phi(c^*)$. Then $M \models \phi(c^*) \wedge \psi(c^*, a_1) \wedge \psi(c^*, a_2)$. □

Since c realizes $\text{qftp}_{\mathcal{L}_n}(c_1/E(a_1)) \cup \text{qftp}_{\mathcal{L}_n}(c_2/E(a_2)) \cup \text{tp}_{\mathcal{L}_n}(c_1/E)$, and is ACF-independent of $\{a_1, a_2\}$ over E , by Proposition 3.20 there is c' realizing $\text{tp}_{\mathcal{L}_n}(c_1/E(a_1)) \cup \text{tp}_{\mathcal{L}_n}(c/E(a_2)) \cup \text{qftp}_{\mathcal{L}_n}(c/E(a_1, a_2))$.

In particular c' realizes $\text{qftp}_{\mathcal{L}_n}(c_2/E(a_2)) \cup \text{qftp}_{\mathcal{L}_n}(c_1/E(a_1)) \cup \text{tp}(c_2/E)$. By Proposition 3.20 again there is c'' ACF-independent of $\{a_1, a_2\}$ over E , realizing $\text{tp}_{\mathcal{L}_n}(c_2/E(a_2)) \cup \text{tp}_{\mathcal{L}_n}(c'/E(a_1)) \cup \text{qftp}(c'/E(a_1, a_2))$. Therefore c'' realizes $\text{tp}_{\mathcal{L}_n}(c_2/E(a_2)) \cup \text{tp}_{\mathcal{L}_n}(c_1/E(a_1)) \cup \text{qftp}_{\mathcal{L}_n}(c/E(a_1, a_2))$. □

Corollary 3.22. *Let $(M, <_1, \dots, <_n)$ be a model of T . Let $E = \text{acl}(E) \subseteq M$ and a_1, a_2, c tuples of M such that: $\text{tp}_{\mathcal{L}_n}(a_1/E) = \text{tp}_{\mathcal{L}_n}(a_2/E)$, c is ACF-independent of $\{a_1, a_2\}$ over E , and $\text{qftp}_{\mathcal{L}_n}(c, a_1/E) = \text{qftp}_{\mathcal{L}_n}(c, a_2/E)$. Suppose that $E(a_1)^{\text{alg}} \cap E(a_2)^{\text{alg}} = E^{\text{alg}}$. Then there exists a tuple c^* in some elementary extension M^* of M such that:*

- (1) $\text{qftp}_{\mathcal{L}_n}(c^*/E(a_1, a_2)) = \text{qftp}_{\mathcal{L}_n}(c/E(a_1, a_2))$,
- (2) $\text{tp}_{\mathcal{L}_n}(c^*, a_1/E) = \text{tp}_{\mathcal{L}_n}(c^*, a_2/E)$,
- (3) $\text{tp}_{\mathcal{L}_n}(c^*, a_1/E) = \text{tp}_{\mathcal{L}_n}(c, a_1/E)$.

Proof. As $\text{tp}_{\mathcal{L}_n}(a_1/E) = \text{tp}_{\mathcal{L}_n}(a_2/E)$, we can find c_2 such that $\text{tp}_{\mathcal{L}_n}(c_2, a_2/E) = \text{tp}_{\mathcal{L}_n}(c, a_1/E)$. Since c realizes $\text{qftp}_{\mathcal{L}_n}(c/E(a_1)) \cup \text{qftp}_{\mathcal{L}_n}(c_2/E(a_2))$, by Theorem 3.21 there is some c^* realizing $\text{tp}_{\mathcal{L}_n}(c/E(a_1)) \cup \text{tp}_{\mathcal{L}_n}(c_2/E(a_2)) \cup \text{qftp}_{\mathcal{L}_n}(c/E(a_1, a_2))$. This implies that $\text{tp}_{\mathcal{L}_n}(c^*, a_1/E) = \text{tp}_{\mathcal{L}_n}(c^*, a_2/E)$, and $\text{tp}_{\mathcal{L}_n}(c^*, a_1/E) = \text{tp}_{\mathcal{L}_n}(c, a_1/E)$. \square

4 PRC fields and their stability theoretic properties

We give all necessary preliminaries about NIP, NTP_2 and strong theories and also some useful lemmas about indiscernible sequences. For more details on NTP_2 and strong theories see [11] and [14].

Fix \mathcal{L} a language and T a complete \mathcal{L} -theory. We work inside a monster model \mathbb{M} of T .

Definition 4.1. Let $\phi(\bar{x}, \bar{y})$ be an \mathcal{L} -formula.

(1) We say that $\phi(\bar{x}, \bar{y})$ has the *independence property (IP)* if for any $m \in \mathbb{N}$ there is a family of tuples $\{b_l : l < m\}$ in $\mathbb{M}^{|\bar{y}|}$ such that for each $A \in \mathcal{P}(m)$ there is a tuple $a_A \in \mathbb{M}^{|\bar{x}|}$, such that $\mathbb{M} \models \phi(a_A, b_l)$ if and only if $l \in A$. A formula $\phi(\bar{x}, \bar{y})$ is *NIP* if it does not have the IP. A theory is called *NIP* if no formula has IP.

(2) We say that $\phi(\bar{x}, \bar{y})$ has TP_2 if there are $(a_{l,j})_{l,j < \omega}$ and $k \in \omega$ such that:

- (a) $\{\phi(\bar{x}, a_{l,j})_{j \in \omega}\}$ is k -inconsistent for all $l < \omega$.
- (b) For all $f : \omega \rightarrow \omega$, $\{\phi(\bar{x}, a_{l,f(l)}) : l \in \omega\}$ is consistent.

A formula $\phi(\bar{x}, \bar{y})$ is *NTP₂* if it does not have TP_2 . A theory is called *NTP₂* if no formula has TP_2 .

By Proposition 5.31 of [42] if T is NIP, then it is NTP_2 .

Definition 4.2. Let $p(x)$ be a (partial) type. An *inp-pattern of depth λ in $p(x)$* , where λ is a finite or infinite cardinal, consists of $(\bar{a}_l, \phi_l(x, y_l), k_l)_{l < \lambda}$ with $\bar{a}_l = (a_{l,j})_{j \in \omega}$ and $k_l \in \omega$ such that:

- (1) $\{\phi_l(x, a_{l,j})\}_{j < \omega}$ is k_l -inconsistent, for each $l < \lambda$.
- (2) $\{\phi_l(x, a_{l,f(l)})\}_{l < \lambda} \cup p(x)$ is consistent, for any $f : \lambda \rightarrow \omega$.

The *burden* of a partial type $p(x)$ is the supremum of the depths of inp-patterns in it. We denote the burden of p by $\text{bdn}(p)$ and by $\text{bdn}(\bar{a}/A)$ the burden of $\text{tp}(\bar{a}/A)$.

4.3. By Theorem 2.5 of [11] if $\text{bdn}(b/A) < \kappa$ and $\text{bdn}(a/Ab) < \lambda$, with κ and λ finite or infinite cardinals, then $\text{bdn}(a, b/A) < \lambda \times \kappa$.

Definition 4.4. Consider a set of sequences $(a_l)_{l \in \omega}$, with $a_l = (a_{l,j})_{j < \kappa}$. We say that they are *mutually indiscernible* over a set C if a_l is indiscernible over $\{C(a_{l'})\}_{l' \neq l}$.

Fact 4.5. [11, Lemma 2.2] For a (partial) type $p(x)$ over a small set E , the following are equivalent:

- (1) There is an inp-pattern of depth λ in $p(x)$.
- (2) There is an array $(\bar{a}_l)_{l < \lambda}$ with rows mutually indiscernible over E and $\phi_l(x, y_l)$ for $l < \lambda$ such that:
 - (a) $\{\phi_l(x, a_{l,j})\}_{j < \omega}$ is inconsistent for every $l < \lambda$.
 - (b) $p(x) \cup \{\phi_l(x, a_{l,f(l)})\}_{l < \lambda}$ is consistent, for any $f : \lambda \rightarrow \omega$.

Definition 4.6. T is called *strong* if there is no inp-pattern of infinite depth in it. Clearly, if T is strong then it is NTP_2 .

Fact 4.7. (1) In the definition of strong it is enough to look at types in one variable. [11, Theorem 2.5]

- (2) If $(\bar{a}_l, \phi_{l,0}(x, y_{l,0}) \vee \phi_{l,1}(x, y_{l,1}), k_l)_{l < \lambda}$ is an inp-pattern, then $(\bar{a}_l, \phi_{l,f(l)}(x, y_{l,f(l)}), k_l)_{l < \lambda}$ is an inp-pattern for some $f : \lambda \rightarrow \{0, 1\}$. [11, Lemma 7.1]

Lemma 4.8. (Folklore) Let $(a_l)_{l \in \omega}$ be an indiscernible sequence over E . Then $\bigcap_{l \in \omega} \text{dcl}(E(a_l)) = \text{dcl}(E(a_0)) \cap \text{dcl}(E(a_1))$.

Proof. Clearly $\bigcap_{l \in \omega} \text{dcl}(E(a_l)) \subseteq \text{dcl}(E(a_0)) \cap \text{dcl}(E(a_1))$.

Let $\alpha \in \text{dcl}(E(a_0)) \cap \text{dcl}(E(a_1))$. Let $\varphi_1(x, a_0)$ and $\varphi_2(x, a_1)$ be the formulas that define α with parameters in $E(a_0)$ and $E(a_1)$ respectively.

Then $\mathbb{M} \models \varphi_1(\alpha, a_0) \wedge \varphi_2(\alpha, a_1) \wedge \exists^{=1} x \varphi_1(x, a_0) \wedge \exists^{=1} x \varphi_2(x, a_1)$.

By indiscernibility $\mathbb{M} \models \exists x (\varphi_1(x, a_0) \wedge \varphi_2(x, a_k)) \wedge \exists^{=1} x \varphi_1(x, a_0) \wedge \exists^{=1} x \varphi_2(x, a_k)$, for all $k \geq 1$. Since $\mathbb{M} \models \varphi_1(\alpha, a_0) \wedge \exists^{=1} x \varphi_1(x, a_0)$, we get that $\mathbb{M} \models \varphi_2(\alpha, a_k)$, for all $k \geq 1$. Then $\alpha \in \text{dcl}(E(a_k))$ for all $k \geq 1$, since $\mathbb{M} \models \exists^{=1} x \varphi_2(x, a_k)$. \square

Note that in particular this implies for all $0 < l < j$ in \mathbb{N} :

$$\text{dcl}(E(a_l)) \cap \text{dcl}(E(a_j)) = \text{dcl}(E(a_0)) \cap \text{dcl}(E(a_1)).$$

Lemma 4.9. (Folklore) Let $(a_l)_{l \in \omega}$ be an indiscernible sequence over E . Let $F = \text{dcl}(E(a_0)) \cap \text{dcl}(E(a_1))$. Then $(a_l)_{l \in \omega}$ is indiscernible over F .

Proof. If $k_0 < \dots < k_r$, we want to show that:

$$\text{tp}(a_0, \dots, a_r/F) = \text{tp}(a_{k_0}, \dots, a_{k_r}/F).$$

Let $\bar{\beta} \subseteq F$, and $\varphi(\bar{x}, \bar{\beta}) \in \text{tp}(a_0, \dots, a_r/F)$. Then $\bar{\beta} \subseteq \text{dcl}(E(a_0)) \cap \text{dcl}(E(a_1))$. By Lemma 4.8, $\bar{\beta} \subseteq \text{dcl}(E(a_k))$ for all $k \in \omega$.

Let $\psi(\bar{x}, a_0)$ the formula that defines $\bar{\beta}$ over $E(a_0)$. Then by indiscernibility $\psi(\bar{x}, a_k)$ defines $\bar{\beta}$ over $E(a_k)$ for all $k \in \omega$. Let $k > k_r$, Then

$$\mathbb{M} \models \exists \bar{x} (\varphi(a_0, \dots, a_r, \bar{x}) \wedge \psi(\bar{x}, a_k)).$$

Then by indiscernibility:

$$\mathbb{M} \models \exists \bar{x} (\varphi(a_{k_0}, \dots, a_{k_r}, \bar{x}) \wedge \psi(\bar{x}, a_k)).$$

As $\bar{\beta}$ is the only tuple that satisfies $\psi(\bar{x}, a_k)$, we have $\mathbb{M} \models \varphi(a_{k_0}, \dots, a_{k_r}, \bar{\beta})$. \square

4.1 Independence property, NTP_2 and strength

Theorem 4.10. *Let M be a PRC field which is neither algebraically closed nor real closed. Then $\text{Th}_{\mathcal{L}_{\mathcal{R}}}(M)$ has the independence property. More generally $\text{Th}_{\mathcal{L}_{\mathcal{R}}}(M)$ has the IP_n property, for all $n \in \mathbb{N}$ (see Definition 2.1 of [22]).*

Proof. By Properties 2.5 (3) $M(\sqrt{-1})$ is a PRC field. Since $M(\sqrt{-1})$ has no orderings, it is a PAC field. Observe that as M is not real closed, $M(\sqrt{-1})$ is neither separably closed nor real closed. By Corollary 6.5 of [17] $M(\sqrt{-1})$ has the IP, more generally by Corollary 7.4 of [22] $M(\sqrt{-1})$ has the IP_n property. Then M has the IP_n property, since $M(\sqrt{-1})$ is interpretable in M . \square

Lemma 4.11. *Let \mathcal{L} be a language and let T be an \mathcal{L} -theory. Suppose that for any \mathcal{L} -atomic formula $\varphi(\bar{x}, \bar{y})$, there are $\mathcal{L}' \subseteq \mathcal{L}$ and an \mathcal{L}' -theory T' such that: $\varphi(\bar{x}, \bar{y})$ is an \mathcal{L}' -formula, T' is NIP and for all $M \models T$, there is $M' \models T'$ such that $M|_{\mathcal{L}'} \subseteq M'$. Then in T every quantifier-free \mathcal{L} -formula is NIP.*

Proof. Every quantifier-free formula $\phi(\bar{x}, \bar{y})$ is a Boolean combination of atomic formulas. Since every Boolean combination of NIP formulas is NIP (Lemma 2.9 of [42]), it is enough to show that every atomic formula is NIP.

Let $\varphi(\bar{x}, \bar{y})$ be an atomic formula and let M be a model of T . Let \mathcal{L}' , T' and M' satisfy the hypothesis of the Lemma for the formula $\varphi(\bar{x}, \bar{y})$. Suppose that $\varphi(\bar{x}, \bar{y})$ has the independence property. Then for all $m \in \mathbb{N}$, there is a family of tuples $\{b_l : l < m\}$ and $\{a_A : A \in \mathcal{P}(m)\}$ in M such that $M \models \varphi(a_A, b_l)$ if and only if $l \in A$. Then $M' \models \varphi(a_A, b_l)$ if and only if $l \in A$, since $M|_{\mathcal{L}'} \subseteq M'$. That contradicts the fact that T' is NIP. \square

Corollary 4.12. *Let $n \geq 1$. In n -PRC every quantifier-free \mathcal{L}_n -formula is NIP.*

Proof. By Lemma 4.11 using the fact that in n -PRC the atomic formulas are of the form $p(\bar{x}, \bar{y}) >_i 0$, with $i \in \{1, \dots, n\}$ and $p(\bar{x}, \bar{y}) \in \mathbb{Q}[\bar{x}, \bar{y}]$, and that RCF is NIP. \square

Fact 4.13. [42, Proposition 2.8] *The formula $\varphi(x, y)$ is NIP if and only if for any indiscernible sequence $(a_l : l \in \mathbb{N})$ and any tuple b , there is some $k \in \mathbb{N}$ such that $\varphi(a_l, b) \leftrightarrow \varphi(a_j, b)$, for all $l, j \geq k$.*

Lemma 4.14. *Let $n \geq 1$ and let M be a bounded PRC field with exactly n orders, which is not real closed. Let $T := Th_{\mathcal{L}_n}(M)$ (see 3.3 for the definition of \mathcal{L}_n). Let $E = \text{acl}(E) \subset M$ and $(a_j)_{j \in \omega}$ an indiscernible sequence over E . Let $\phi(x, \bar{y})$ be an \mathcal{L}_n -formula and I a multi-interval definable over E such that $\{x \in I : M \models \phi(x, a_0)\}$ is multi-dense in I . Then for all multi-intervals $J \subseteq I$ definable over E , $\{x \in J\} \cup \{\phi(x, a_0) \wedge \phi(x, a_1)\}$ is consistent.*

Proof.

Claim 1. We can suppose that $E(a_0)^{\text{alg}} \cap E(a_1)^{\text{alg}} = E^{\text{alg}}$.

Proof. Denote by $A_0 := \text{acl}(E(a_0))$ and by $A_1 := \text{acl}(E(a_1))$.

By Lemma 4.9 and the fact that $\text{acl} = \text{dcl}$ we can suppose that $A_0 \cap A_1 = E$.

Since $A_0 A_1$ is a regular extension of A_0 and of A_1 , by Lemma 2.1 of [7] $A_0^{\text{alg}} \cap A_1^{\text{alg}} = (A_0 \cap A_1)^{\text{alg}}$.

As $A_0^{\text{alg}} = E(a_0)^{\text{alg}}$, $A_1^{\text{alg}} = E(a_1)^{\text{alg}}$ and $A_0 \cap A_1 = E$, we get that $E(a_0)^{\text{alg}} \cap E(a_1)^{\text{alg}} = E^{\text{alg}}$. \square

Let $J = \bigcap_{i=1}^n (J^i \cap M)$ be a multi-interval definable over E such that $J \subseteq I$.

Claim 2. There exists \tilde{c} in some elementary extension N of M such that $\tilde{c} \in J$, $\tilde{c} \notin \text{acl}(Ea_j : j \in \omega)$ and for all $j \in \omega$, $\text{qftp}_{\mathcal{L}_n}(\tilde{c}, a_0/E) = \text{qftp}_{\mathcal{L}_n}(\tilde{c}, a_j/E)$.

Proof. By compactness it is enough to show that if $\psi_1(x, y), \dots, \psi_m(x, y)$ are quantifier-free $\mathcal{L}_n(E)$ -formulas, then the type:

$$\{x \in J\} \cup \{p(x) \neq 0\}_{p \in E(a_j : j \in \omega)[x], p \neq 0} \cup \{\psi_l(x, a_0) \leftrightarrow \psi_l(x, a_j)\}_{0 < j, 1 \leq l \leq m}$$

is consistent.

Let $d \in J$, $d \notin \text{acl}(Ea_j : j \in \omega)$; by Corollary 4.12 and Fact 4.13, for all $1 \leq l \leq m$ there exists $k_l \in \mathbb{N}$ such that $\psi_l(d, a_{j_1}) \leftrightarrow \psi_l(d, a_{j_2})$ for all $k_l \leq j_1 < j_2$. Let $k = \max\{k_1, \dots, k_m\}$.

Then d realizes the type:

$$\{x \in J\} \cup \{p(x) \neq 0\}_{p \in E(a_j : j \in \omega)[x], p \neq 0} \cup \{\psi_l(x, a_{j_1}) \leftrightarrow \psi_l(x, a_{j_2})\}_{k \leq j_1 < j_2, 1 \leq l \leq m}$$

Since J is definable with parameters in E , $p(x) \in E(a_j : j \in \omega)[x]$ and $(a_j)_{j \in \omega}$ is indiscernible over E , it follows that the type:

$$\{x \in J\} \cup \{p(x) \neq 0\}_{p \in E(a_j : j \in \omega)[x], p \neq 0} \cup \{\psi_l(x, a_0) \leftrightarrow \psi_l(x, a_j)\}_{0 < j, 1 \leq l \leq m}$$

is consistent. \square

Claim 3. There exists c in some elementary extension N of M such that $c \in J$, $c \notin \text{acl}(Ea_j : j \in \omega)$, $N \models \phi(c, a_0)$, and for all $j \in \omega$, $\text{qftp}_{\mathcal{L}_n}(c, a_0/E) = \text{qftp}_{\mathcal{L}_n}(c, a_j/E)$.

Proof. By compactness it is enough to show that if $\psi_1(x, y), \dots, \psi_m(x, y)$ are quantifier-free $\mathcal{L}_n(E)$ -formulas, then the type:

$$\{x \in J\} \cup \{p(x) \neq 0\}_{p \in E(a_j : j \in \omega)[x], p \neq 0} \cup \{\psi_l(x, a_0) \leftrightarrow \psi_l(x, a_j)\}_{0 < j, 1 \leq l \leq m} \cup \{\phi(x, a_0)\}$$

is consistent. Define $\theta(x, y_1, y_2) := \bigwedge_{l=1}^m \psi_l(x, y_1) \leftrightarrow \psi_l(x, y_2)$.

Let $N \succeq M$ be $|E|^+$ -saturated, and $\tilde{c} \in N$ satisfying claim 2.

Then $N \models \theta(\tilde{c}, a_0, a_j)$, for all $0 < j \in \omega$. As $\theta(x, a_0, a_j)$ is a quantifier-free \mathcal{L}_n -formula and $\tilde{c} \notin \text{acl}(E(a_j) : j \in \omega)$, there exist for all $i \in \{1, \dots, n\}$, $<_i$ -intervals $B_j^i \subseteq N^{(i)}$ qf-definable over $E(a_0, a_j)$, such that $\tilde{c} \in \bigcap_{i=1}^n (B_j^i \cap N)$ and $\bigcap_{i=1}^n (B_j^i \cap N) \subseteq \theta(N, a_0, a_j)$.

Let $1 \leq i \leq n$; as N is $|E|^+$ -saturated, there exists an $<_i$ -interval $B^i \subseteq N^{(i)}$ containing \tilde{c} , such that $B^i \cap N \subseteq \bigcap_{j \in \omega} (B_j^i \cap N)$.

As $\tilde{c} \in J := \bigcap_{i=1}^n (J^i \cap N)$, we can assume that $B^i \subseteq J^i$, for all $1 \leq i \leq n$.

Then $B := \bigcap_{i=1}^n (B^i \cap N) \subseteq J \subseteq I$ and $B \subseteq \bigcap_{l=1}^m \theta(N, a_0, a_j)$, for all $j \in \omega$.

By multi-density of $\phi(x, a_0)$ in I and saturation, there exists $c \in B$, $c \notin \text{acl}(Ea_j : j \in \omega)$, such that $N \models \phi(c, a_0)$. As $c \in B$, $\psi_l(c, a_0) \leftrightarrow \psi_l(c, a_j)$, for all $j \in \omega$, $1 \leq l \leq m$. □

By Corollary 3.22 there is c^* in some elementary extension N^* of N , such that $\text{tp}(c^*, a_0/E) = \text{tp}(c^*, a_1/E)$ and $\text{tp}(c^*, a_0/E) = \text{tp}(c, a_0/E)$. So $N^* \models \phi(c^*, a_0) \wedge \phi(c^*, a_1)$ and since $c \in J$, J is definable with parameters in E , and $\text{tp}(c^*/E) = \text{tp}(c/E)$, we obtain that $c^* \in J$. Then c^* realizes $\{x \in J\} \cup \{\phi(x, a_0) \wedge \phi(x, a_1)\}$. □

Theorem 4.15. *Let $n \geq 1$ and let M be a bounded PRC field with exactly n orders which is not real closed, and let $T := \text{Th}_{\mathcal{L}_n}(M)$. Let $E = \text{acl}(E) \subset M$ and $(a_j)_{j \in \omega}$ an indiscernible sequence over E . Let $\phi(x, \bar{y})$ be an $\mathcal{L}_n(E)$ -formula and $I := \bigcap_{i=1}^n I^i$ a multi-interval definable over E such that $\{x \in I : M \models \phi(x, a_0)\}$ is multi-dense in I . Then the type $p(x) := \{\phi(x, a_j)\}_{j \in \omega}$ is consistent.*

Proof. Define $\psi(x, y_1, y_2) := \phi(x, y_1) \wedge \phi(x, y_2)$. By Theorem 3.13 there are a finite set $B \subseteq M$, $m \in \mathbb{N}$ and I_1, \dots, I_m , with $I_j := \bigcap_{i=1}^n I_j^i$ a multi-interval such that:

$$(1) \quad \psi(M, a_0, a_1) \subseteq \bigcup_{j=1}^m I_j \cup B,$$

(2) $\{x \in I_j : M \models \psi(x, a_0, a_1)\}$ is multi-dense in I_j , for all $1 \leq j \leq m$.

Denote by $E^{(i)} := E^{alg} \cap M^{(i)}$.

Claim. There is $j \in \{1, \dots, m\}$ such that for all $i \in \{1, \dots, n\}$, $|I_j^i \cap E^{(i)}| \geq 2$.

Proof. Suppose that for all $j \in \{1, \dots, m\}$, there is $i_j \in \{1, \dots, n\}$ such that $|I_j^{i_j} \cap E^{(i_j)}| \leq 1$. So we can find an $<_{i_j}$ -interval $J^{i_j} \subseteq I$ with extremities in $E^{(i_j)}$ such that $J^{i_j} \cap I^{i_j} = \emptyset$. Observe that if $i_j = i_l$, for some $j, l \in \{1, \dots, m\}$, then we can choose $J^{i_j} = J^{i_l}$.

Let $J := \bigcap_{j=1}^m J^{i_j} \cap \bigcap_{i \neq i_j} M^{(i)}$. Then J is a multi interval, definable over E , and $J \subseteq I$. By

Lemma 4.14 there exists $c \in J$ such that $M \models \psi(c, a_0, a_1)$. This implies that $J \cap \bigcup_{j=1}^m I_j \neq \emptyset$.

Thus there is $j \in \{1, \dots, m\}$ such that $J \cap I_j \neq \emptyset$, whence $J^{i_j} \cap I_j^{i_j} \neq \emptyset$ which gives the desired contradiction. \square

Take $x_1^i \neq x_2^i \in I_j^i \cap E^{(i)}$ for each $1 \leq i \leq n$. Define $y_i := \min_{<_i} \{x_1, x_2\}$, $z_i := \max_{<_i} \{x_1, x_2\}$ and let $\tilde{I}_1 := \bigcap_{i=1}^n ((y_i, z_i)_i \cap M)$. Then \tilde{I}_1 is a multi-interval definable over E .

Define $b_j := (a_{2j}, a_{2j+1})$; then $(b_j)_{j \in \omega}$ is indiscernible over E . Then $\{x \in \tilde{I}_1 : M \models \psi(x, b_0)\}$ is multi-dense in \tilde{I}_1 , since $\tilde{I}_1 \subseteq I_j$.

Repeating this process with the formula $\psi(x, b_0)$ and the multi-interval \tilde{I}_1 , we find a multi-interval \tilde{I}_2 , definable over E such that that $\{x \in \tilde{I}_2 : M \models \phi(x, a_0) \wedge \phi(x, a_1) \wedge \phi(x, a_2) \wedge \phi(x, a_3)\}$ is multi-dense in \tilde{I}_2 , etc. This shows that $p(x)$ is finitely consistent. \square

Theorem 4.15 can be easily generalized to several variables:

Theorem 4.16. *Let $n \geq 1$ and let M be a bounded PRC field with exactly n orders, which is not real closed. Let $T := Th_{\mathcal{L}_n}(M)$. Let $E = \text{acl}(E) \subset M$ and $(a_j)_{j \in \omega}$ an indiscernible sequence over E . Let $\phi(x_1, \dots, x_r, \bar{y})$ be an $\mathcal{L}_n(E)$ -formula and C a multi-cell in M^r definable over E such that $\{(x_1, \dots, x_r) \in C : \phi(x_1, \dots, x_r, a_0)\}$ is multi-dense in C . Then $p(x_1, \dots, x_r) := \{\phi(x_1, \dots, x_r, a_j)\}_{j \in \omega}$ is consistent.*

Proof. We will first show that Lemma 4.14 can be generalized to several variables.

Claim. If $J \subseteq C$ is a multi-box in M^r definable over E , then there exists $c = (c_1, \dots, c_r)$ in some elementary extension N of M such that: $c \in J$, $\text{trdeg}(E(c)/E) = r$, $N \models \phi(c, a_0)$ and for all $j \in \omega$, $\text{qftp}_{\mathcal{L}_n}(c, a_0/E) = \text{qftp}_{\mathcal{L}_n}(c, a_j/E)$.

Proof. Let $J := \bigcap_{i=1}^n (J^i \cap M^r) \subseteq C$ be a multi-box in M^r definable over E . As in Claim 1 of

Lemma 4.14 we can suppose that $E(a_0)^{alg} \cap E(a_1)^{alg} = E^{alg}$.

Let $\psi_1(\bar{x}, \bar{y}), \dots, \psi_m(\bar{x}, \bar{y})$ be quantifier-free $\mathcal{L}_n(E)$ -formulas, and consider the type:

$$q(\bar{x}) := \{\bar{x} \in J\} \cup \{p(\bar{x}) \neq 0\}_{p \in E(a_j: j \in \omega)[\bar{x}], p \neq 0} \cup \{\psi_l(\bar{x}, a_0) \leftrightarrow \psi_l(\bar{x}, a_j)\}_{0 < j, 1 \leq l \leq m} \cup \{\phi(\bar{x}, a_0)\}$$

By compactness it is enough to show that $q(\bar{x})$ is consistent.

Define $\theta(\bar{x}, y_1, y_2) := \bigwedge_{l=1}^m \psi_l(\bar{x}, y_1) \leftrightarrow \psi_l(\bar{x}, y_2)$, for $1 \leq l \leq m$.

Exactly as in Claim 2 of Lemma 4.14 there exists $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_r)$ in some elementary extension N of M such that: $\tilde{c} \in J$, $\text{trdeg}(E(\tilde{c})/E) = r$, and for all $j \in \omega$, $\text{qftp}(\tilde{c}, a_0/E) = \text{qftp}(\tilde{c}, a_j/E)$. Suppose that N is E^+ -saturated.

Then $N \models \bigcap_{l=1}^m \theta(\tilde{c}, a_0, a_j)$, for all $j \in \omega$.

As $\theta(\bar{x}, a_0, a_j)$ is quantifier-free, using cell decomposition in each real closure $N^{(i)}$ for all $1 \leq i \leq n$ and the fact that $\text{trdeg}(E(\tilde{c})/E) = r$, there exists for all $i \in \{1, \dots, n\}$, $<_i$ -open cells $B_j^i \subseteq (N^{(i)})^r$ definable over $E(a_0, a_j)$, such that $\tilde{c} \in \bigcap_{i=1}^n (B_j^i \cap N^r)$ and $\bigcap_{i=1}^n (B_j^i \cap N^r) \subseteq \theta(N, a_0, a_j)$. By saturation and the fact that each $(B_j^i \cap N^r)$ is $<_i$ -open (in the product topology), there exists an $<_i$ -box B^i containing \tilde{c} , such that $B^i \cap N^r \subseteq \bigcap_{j \in \omega} (B_j^i \cap N^r)$.

As $\tilde{c} \in J^i$, for all $1 \leq i \leq n$, we can assume by taking the intersection that $B^i \subseteq J^i$.

Then $B := \bigcap_{i=1}^n (B^i \cap N^r) \subseteq J \subseteq C$ and $B \subseteq \bigcap_{l=1}^m \theta(N, a_0, a_j)$, for all $j \in \omega$.

By multi-density of $\phi(\bar{x}, a_0)$ in C and saturation, there exists $c = (c_1, \dots, c_r) \in B$ such that $\text{trdeg}(E(c)/E) = r$, and $N \models \phi(c, a_0)$. As $c \in B$, we get that $\psi_l(c, a_0) \leftrightarrow \psi_l(c, a_j)$, for all $j \in \omega$, $1 \leq l \leq m$. Then c realizes $q(\bar{x})$. \square

As in Lemma 4.14 using Theorem 3.22, for all multi-box $J \subseteq C$ definable over E , $\{\bar{x} \in J\} \cup \{\phi(\bar{x}, a_0) \wedge \phi(\bar{x}, a_1)\}$ is consistent.

Define $\psi(\bar{x}, y_1, y_2) := \phi(\bar{x}, y_1) \wedge \phi(\bar{x}, y_2)$. By Theorem 3.17 there are a set $V \subseteq M^r$, $m \in \mathbb{N}$, and C_1, \dots, C_m with $C_j = \bigcap_{i=1}^n (C_j^i \cap M^r)$ a multi-cell such that:

$$(1) \quad \psi(M, a_0, a_1) \subseteq \bigcup_{j=1}^m C_j \cup V,$$

(2) the set V is contained in some proper Zariski closed subset of M^r , which is definable over $\text{acl}(a_0, a_1)$,

(3) $\{x \in C_j : M \models \psi(\bar{x}, a_0, a_1)\}$ is multi-dense in C_j for all $1 \leq j \leq m$,

(4) the set $C_j^i \cap M^r$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}(a_0, a_1)$ -formula, for all $1 \leq j \leq m$.

If $J \subseteq C$ is a multi-box in M^r definable over E , then there is $\bar{x} \in J$ such that $\phi(\bar{x}, a_0) \wedge \phi(\bar{x}, a_1)$. So there exists $j \leq m$ such that $J \cap C_j \neq \emptyset$. As in Theorem 4.15 there exists $j \leq m$ and multi-cell $J \subseteq C_j$, definable over E such that $\psi(\bar{x}, a_0, a_1)$ is multi-dense in J . The rest of the proof is as in Theorem 4.15. \square

Definition 4.17. ([34, Definition 2.1])

- (1) We say that a formula $\delta(x, a)$ *strongly divides* over A if $\text{tp}(a/A)$ is non algebraic and $\{\delta(x, a')\}_{a' \models \text{tp}(a/A)}$ is k -inconsistent for some $k \in \mathbb{N}$.
- (2) A formula $\delta(x, a)$ *\mathfrak{p} -divides* (*thorn divides*) over A if we can find some tuple c such that $\delta(x, a)$ strongly divides over Ac .
- (3) A formula *\mathfrak{p} -forks* (*thorn forks*) over A if it implies a (finite) disjunction of formulas which \mathfrak{p} -divide over A .
- (4) The type $p(x)$ *\mathfrak{p} -divides* if there is a formula in $p(x)$ which \mathfrak{p} -divides; similarly for \mathfrak{p} -forking.
- (5) We say that a is *\mathfrak{p} -independent of b* over A , denoted by $a \perp_A^{\mathfrak{p}} b$, if $\text{tp}(a/Ab)$ does not \mathfrak{p} -fork over A .

Observe that in (1), k -inconsistency means: if a_1, \dots, a_k realize the $\text{tp}(a/A)$ and the sets $\delta(M, a_1), \dots, \delta(M, a_k)$ are distinct, then their intersection is empty.

Definition 4.18. [18, Theorem 3.7] A theory is called *rosy* if there is some κ such that there are no \mathfrak{p} -forking chains of length κ . That is for all b one can not find $(a_j)_{j \in \kappa}$ such that for $\alpha < \kappa$ one has $\text{tp}(b/(a_j)_{j \leq \alpha})$ \mathfrak{p} -forks over $(a_j)_{j < \alpha}$.

Theorem 4.19. *If M is an unbounded PAC field, then $\text{Th}_{\mathcal{L}_{\mathcal{R}}}(M)$ has TP_2 and is not rosy.*

Proof. Suppose that M is an unbounded PAC field which is sufficiently saturated. As in the proof of Theorem 3.9 of [6], we can assume that there are infinitely many finite algebraic extensions $\{L_j\}_{j \in \omega}$ of M , which are linearly disjoint over M and with Galois group over M isomorphic to some fixed simple group G . Let $r = |G|$. For each $j \in \omega$, take α_j such that $L_j = M(\alpha_j)$. Let $g(\bar{Y}, X) \in \mathbb{Z}[\bar{Y}, X]$ and $\bar{a}_j := (a_{1j}, \dots, a_{rj}) \in M^r$ be such that $g(\bar{a}_j, X) = X^r + a_{1j}X^{r-1} + \dots + a_{rj}$ is the minimal polynomial of α_j over M . Define \tilde{L} as the field composite of $\{L_j : j \in \omega\}$.

Choose an element t transcendental over M and $n > 4$ such that G embeds into A_n (the group of even permutations of n elements). Observe that such an n exists: G embeds into S_r , the group of permutations on r elements, which in turn embeds into A_{2r} .

Let k be the prime field of M . Let $\{b_l : l \in \omega\} \subseteq M$ be algebraically independent over k . As in Lemma 3.8 of [6], using Theorem A of [38], we can find an algebraic extension E_1 of $k(b_1, t)$ such that $E_1/k(b_1)$ and $E_1/k(t)$ are regular, and $\text{Gal}(E_1/k(b_1, t)) \cong A_n$. Take β_1 integral over $k[b_1, t]$ such that $E_1 = k(b_1, t)(\beta_1)$. Let $p(b_1, t, X) \in k[b_1, t, X]$ be the minimal polynomial of

β_1 over $k(b_1, t)$. For all $l \in \omega$ define E_l as the field extension of $k(b_l, t)$ generated by a root of $p(b_l, t, X)$.

Since E_l is a regular extension of $k(b_l)$, and is algebraically independent from M over $k(b_l)$ the field $M_l := ME_l$ is a regular extension of M and $\text{Gal}(M_l/M(t)) \cong A_n$. Define \widetilde{M} as the field composite of $\{M_l : l \in \omega\}$. As in Lemma 3.8 of [6], \widetilde{M} is a regular extension of M .

Let $\varphi(t, b_l, \bar{a}_j)$ be the formula that says ‘‘The extension generated by a root of $p(b_l, t, X)$ contains a root of $g(\bar{a}_j, X) = 0$ ’’. Note that such a formula exists, because in M we can interpret finite Galois extensions of M (see Appendix 1 of [7] for more details).

Observe that for all $l \in \omega$, $\{\varphi(t, b_l, \bar{a}_j) : j \in \omega\}$ is $(\frac{n!}{2r} + 1)$ -inconsistent: Otherwise there would exist $l \in \omega$ and j_1, \dots, j_s with $s = (\frac{n!}{2r} + 1)$ such that $M_l \supseteq L_{j_1} \cdots L_{j_s}$. But $[M_l : M(t)] = \frac{n!}{2}$, $[L_j : M] = r$ for all $l, j \in \omega$ and $\{L_j\}_{j \in \omega}$ is linearly disjoint over M . Then $sr < \frac{n!}{2}$ which contradicts the definition of s .

Claim. If $f : \omega \rightarrow \omega$, then $\{\varphi(t, b_l, \bar{a}_{f(l)}) : l \in \omega\}$ is consistent.

Proof. For each l , fix an embedding $h_l : \text{Gal}(L_{f(l)}/M) \rightarrow \text{Gal}(M_l/M(t))$ and let $S_l := \{(\sigma, h_l(\sigma)) : \sigma \in \text{Gal}(L_{f(l)}/M)\}$. Then S_l is a subgroup of $\text{Gal}(L_{f(l)}M_l/M(t)) \cong \text{Gal}(L_{f(l)}/M) \times \text{Gal}(M_l/M(t))$, because $M^{alg} \cap M_l = M$. Define $P_l := \text{Fix}(S_l) \subseteq L_{f(l)}M_l$. By definition of S_l , $L_{f(l)}M_l = L_{f(l)}P_l = M_lP_l$.

Then $\text{Gal}(\widetilde{LM}/M(t)) \cong \prod_{j \in \omega} \text{Gal}(L_j/M) \times \prod_{l \in \omega} \text{Gal}(M_l/M(t))$.

Define $S := \{((\sigma_j)_{j \in \omega}, (\tau_l)_{l \in \omega}) \in \text{Gal}(\widetilde{LM}/M(t)) : \forall l \in \omega, \tau_l = h_l(\sigma_{f(l)})\}$. If \widetilde{P} is the field composite of $\{P_l : l \in \omega\}$, then $\widetilde{P} = \text{Fix}(S)$.

Note that $\text{Gal}(\widetilde{P}/M(t))$ projects onto $\text{Gal}(\widetilde{L}/M)$. Indeed, let $(\sigma_j)_{j \in \omega} \in \text{Gal}(\widetilde{L}/M)$ and for each $l \in \omega$ let $\tau_l = h_l(\sigma_{f(l)})$. Then $((\sigma_j)_{j \in \omega}, (\tau_l)_{l \in \omega}) \in \text{Gal}(\widetilde{P}/M(t))$ is an extension of $(\sigma_j)_{j \in \omega}$.

This implies that \widetilde{P} is a regular extension of M . Then by Fact 2.3 there exists an elementary extension M^* of M such that $\widetilde{P} \subseteq M^*$.

Since $\widetilde{P}M_l \supseteq L_{f(l)}$ and $\widetilde{P} \subseteq M^*$, it follows that $M^* \models \exists t \bigwedge_{l \in \omega} \varphi(t, b_l, a_{f(l)})$. Then $\{\varphi(t, b_l, a_{f(l)}) : l \in \omega\}$ is consistent in M , since M^* is an elementary extension of M . \square

Therefore the formula $\varphi(t; x, y)$ has TP_2 .

To see that $\text{Th}_{\mathcal{L}_{\mathcal{R}}}(M)$ is not rosy we need to show that for any cardinal κ , there exists c and $(d_l)_{l < \kappa}$ such that for all $\alpha < \kappa$, $c \not\prec_{(d_l)_{l < \alpha}}^b d_\alpha$.

Let κ be a cardinal, construct as before $(b_l)_{l \in \kappa}$ and $(\bar{a}_j)_{j \in \kappa}$. Let $\phi(t, b_l, \bar{a}_j) := \varphi(t, b_l, \bar{a}_j) \wedge$ ‘‘the extension generated by a root of $g(\bar{a}_j, X)$ is Galois, with Galois group isomorphic to G ’’. Let $\alpha < \kappa$.

We claim that $\{\phi(t, b_\alpha, \bar{a}) : \bar{a} \models \text{tp}(\bar{a}_\alpha/b_l, a_l : l < \alpha)\}$ is inconsistent: observe that if $\beta_1, \beta_2 \models \text{tp}(\bar{a}_\alpha/b_l, a_l : l < \alpha)$ and $\phi(t, b_\alpha, \beta_1) \neq \phi(t, b_\alpha, \beta_2)$, then $L_{\beta_1} \neq L_{\beta_2}$, where L_{β_1} and L_{β_2} are extensions generated by a root of $g(\beta_1, X) = 0$ and $g(\beta_2, X) = 0$ respectively; but as above M_α can only contain finitely many distinct sub extensions with Galois group G .

By definitions and properties of rosy theories, the formula $\phi(t, b_\alpha, \bar{a}_\alpha)$ strongly divides over $(b_j a_l : j \leq \alpha, l < \alpha)$. Let $(d_l)_{l \in \kappa} := (b_l, \bar{a}_l)_{l \in \kappa}$. As in the Claim, $\{\phi(t, d_l) : l \in \kappa\}$ is consistent, and we let c realize $\{\phi(t, d_l) : l \in \kappa\}$. Since $\phi(t, d_\alpha) \in \text{tp}(c/(d_l)_{l < \alpha})$; $c \not\prec_{(d_l)_{l < \alpha}}^b d_\alpha$. \square

Corollary 4.20. *If M is an unbounded PRC field, then $\text{Th}_{\mathcal{L}_{\mathcal{R}}}(M)$ is not rosy.*

Proof. As before $M(\sqrt{-1})$ is a PAC field. As $M(\sqrt{-1})$ is unbounded and interpretable in M , by Theorem 4.19 it is not rosy, and so M is not rosy. \square

Fact 4.21. *The theory of real closed fields (RCF) is dp-minimal ([42, Theorem A.6]) and the dp-rank coincides with the burden (dp-rank coincides with bdn in any NIP theory, see [1]). In particular RCF is strong. Since dp-rank is sub-additive (Corollary 4.2 of [31]) the burden satisfies the following: if $r \in \mathbb{N}$ and $p(x_1, \dots, x_r) := \{x_1 = x_1, \dots, x_r = x_r\}$, then $\text{bdn}(p(x_1, \dots, x_r)) = r$.*

Theorem 4.22. *Let $n \geq 1$, let M be a bounded PRC field with exactly n orders. Then $\text{Th}_{\mathcal{L}_{\mathcal{R}}}(M)$ is strong and $\text{bdn}(\{x = x\}) = n$.*

Proof. If M is real closed, by Fact 4.21 $\text{Th}_{\mathcal{L}_{\mathcal{R}}}(M)$ is strong of burden 1. Suppose that M is not real closed. By Lemma 3.5 it is enough to show that $T = \text{Th}_{\mathcal{L}_n}(M)$ is strong of burden n . We can suppose that M is sufficiently saturated. For $l \in \{0, \dots, n-1\}$, define the formula $\varphi_l(x, y) := y <_{l+1} x <_{l+1} y + 1$. Take $((a_{l,j})_{j \in \omega})_{l \leq n-1}$, such that $a_{l,j+1} = a_{l,j} + 1$. Using the Approximation Theorem (3.1), $(\bar{a}_l, \varphi_l(x, y), 2)_{0 \leq l < n}$, with $\bar{a}_l = (a_{l,j})_{j \in \omega}$, is an inp-pattern of depth n . It follows that the burden is greater than or equal to n .

Suppose that there is an inp-pattern $(\bar{a}_l, \phi_l(x, y), k_l)_{0 \leq l < n+1}$ of depth $n+1$; by compactness we can take $\bar{a}_l := (a_{l,j})_{j \in \kappa}$, with κ a sufficiently large cardinal. We can suppose that for all $0 \leq l < n+1$, $\phi_l(x, y)$ has parameters in a countable set E with $E = \text{acl}(E) \subseteq M$ so that $G(E) \cong G(M)$.

By Fact 4.5 we can suppose that the array $(\bar{a}_l)_{l < n+1}$ has rows mutually indiscernible over E . It follows from Fact 4.7, Theorem 3.13 and indiscernibility, that we can suppose that for all

$0 \leq l < n+1, j < \kappa$ there is a multi-interval $I_{l,j} = \bigcap_{i=1}^n (I_{l,j}^i \cap M)$ such that:

- (a) $\phi_l(M, a_{l,j}) \subseteq I_{l,j}$,
- (b) $\{x \in I_{l,j} : M \models \phi_l(x, a_{l,j})\}$ is multi-dense in $I_{l,j}$,
- (c) $I_{l,j}^i \cap M$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}$ -formula with parameters in $E(a_{l,j})$.

As $(\bar{a}_l, \phi_l(x, y), k_l)_{0 \leq l < n+1}$ is an inp-pattern, for all $f : \{0, \dots, n\} \rightarrow \kappa$, $\bigcap_{l=0}^n I_{l,f(l)} \neq \emptyset$.

Claim. There exists $0 \leq l \leq n$ such that $\bigcap_{j \in \kappa} I_{l,j}^i \neq \emptyset$, for all $1 \leq i \leq n$.

Proof. Define for all $i \in \{1, \dots, n\}$, $A_i := \{l \in \{0, \dots, n\} : \bigcap_{j \in \kappa} I_{l,j}^i = \emptyset\}$. As the burden of RCF is 1 (Fact 4.21), $|A_i| \leq 1$ for all $i \in \{1, \dots, n\}$. Then $|\bigcup_{i=1}^n A_i| \leq n$, so there is $l \in \{0, \dots, n\}$ such that $l \notin \bigcup_{i=1}^n A_i$. Then $\bigcap_{j \in \kappa} I_{l,j}^i \neq \emptyset$, for all $1 \leq i \leq n$. \square

Let $l \in \{0, \dots, n\}$ satisfy the claim. We will only consider the row l of the array so we denote $(a_j)_{j \in \omega} := (a_{l,j})_{j \in \omega}$.

It follows by saturation that for all $i \in \{1, \dots, n\}$ there exists a non-empty $<_i$ -open interval $I^i \subseteq M^{(i)}$ such that:

$$I^i \cap M \subseteq \bigcap_{j \in \kappa} (I_{l,j}^i \cap M).$$

By density of M in every real closure (Fact 2.5) we can suppose that $I^i := (c_i, d_i)_i$, with $c_i, d_i \in M$. Let $I := \bigcap_{i=1}^n (I^i \cap M)$; by the Approximation Theorem (3.1) $I \neq \emptyset$. Let $t > k_l$, we can suppose that κ is large enough. By Erdős-Rado we can find a countable sequence $(b_j)_{j \in \omega}$, indiscernible over $E' := \text{acl}(E(c_i, d_i) : 1 \leq i \leq n)$ and such that the first t elements are in $(a_j)_{j \in \kappa}$. So we have that $\{\phi_l(x, b_j)\}_{j \in \omega}$ is k_l -inconsistent.

Observe that for all $j \in \omega$, $\{x \in I : M \models \phi_l(x, b_j)\}$ is multi-dense in I . Since I is $\mathcal{L}_n(E')$ -definable and $(b_j)_{j \in \omega}$ is indiscernible over E' , by Theorem 4.15 $\{\phi_l(x, b_j)\}_{j \in \omega}$ is consistent. This contradicts the k_l -inconsistency. \square

Theorem 4.23. *Let M be a PRC field. Then $Th_{\mathcal{L}_{\mathcal{R}}}(M)$ is NTP_2 if and only if M is bounded.*

Proof. (\Rightarrow) Suppose by contradiction that M is unbounded. Then $M(\sqrt{-1})$ is a PAC field. As $M(\sqrt{-1})$ is unbounded and interpretable in M , by Theorem 4.19 it has TP_2 , and so M has TP_2 . This contradicts the fact that $Th_{\mathcal{L}_{\mathcal{R}}}(M)$ is NTP_2 .

(\Leftarrow) Let M be a bounded PRC field. By Remark 3.2 there exists $n \in \mathbb{N}$ such that M has exactly n different orders. If $n = 0$, M is a PAC field, so by Corollary 4.8 of [8] $Th(M)$ is simple and therefore it is NTP_2 . If $n \geq 1$ by Theorem 4.22 $Th(M)$ is strong and so is NTP_2 . \square

4.2 Burden of types in PRC fields

Let $n \geq 1$ and fix a bounded PRC field K with exactly n orders and which is not real closed. Let $T := Th_{\mathcal{L}_n}(K)$ (see 3.3). For the rest of the section we are going to work inside a monster model $(M, <_1, \dots, <_n)$ of T .

Lemma 4.24. *Let $r \in \mathbb{N}$. Let $p(\bar{x}) := \{x_1 = x_1 \wedge \dots \wedge x_r = x_r\}$. Then $\text{bdn}(p(\bar{x})) = nr$.*

Proof. The proof is a generalization of the proof of Theorem 4.22 to several variables. For all $(i, l) \in \{1, \dots, n\} \times \{1, \dots, r\}$, define the formula $\varphi_{(i,l)}(x_1, \dots, x_r, y) := y <_i x_l <_i y + 1$. For each (i, l) take $(a_{l,j}^i)_{j \in \omega}$ such that $a_{l,j+1}^i = a_{l,j}^i + 1$. By Remark 3.15 $(\bar{a}_l^i, \varphi_{(i,l)}(x_1, \dots, x_r, y), 2)_{1 \leq i \leq n, 1 \leq l \leq r}$, with $\bar{a}_l^i = (\bar{a}_{l,j}^i)_{j \in \omega}$ is an inp-pattern of depth nr in $p(\bar{x})$. It follows that $\text{bdn}(p(\bar{x})) \geq nr$.

Suppose that there is an inp-pattern $(\bar{a}_l, \phi_l(x_1, \dots, x_r, \bar{y}), k_l)_{0 \leq l \leq nr}$ of depth $nr + 1$.

By compactness we can take $\bar{a}_l := (a_{l,j})_{j \in \kappa}$, with κ a sufficiently large cardinal. We can suppose that for all $0 \leq l \leq nr$, $\phi_l(x_1, \dots, x_r, \bar{y})$ has parameters in a countable set E with $E = \text{acl}(E) \subseteq M$ and $G(E) \cong G(M)$. By Fact 4.5 we can suppose that the array $(\bar{a}_l)_{0 \leq l \leq nr}$ has rows mutually indiscernible over E .

It follows from Fact 4.7, Theorem 3.17 and indiscernibility that we can suppose that for all $0 \leq l \leq nr$, $j < \kappa$ there is $C_{l,j} = \bigcap_{i=1}^n (C_{l,j}^i \cap M^r)$ a multi-cell in M^r such that:

- (a) $\phi_l(M, a_{l,j}) \subseteq C_{l,j}$
- (b) $\{(x_1, \dots, x_r) \in C_{l,j} : M \models \phi_l(x_1, \dots, x_r, a_{l,j})\}$ is multi-dense in $C_{l,j}$.
- (c) $C_{l,j}^i \cap M^r$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}$ -formula with parameters in $E(a_{l,j})$.

Claim. There exists $0 \leq l \leq nr$ such that $\bigcap_{j \in \kappa} C_{l,j}^i \neq \emptyset$, for all $1 \leq i \leq n$.

Proof. Define for all $i \in \{1, \dots, n\}$, $A_i := \{l \in \{0, \dots, nr\} : \bigcap_{j \in \kappa} C_{l,j}^i = \emptyset\}$. The fact that the burden in the theory of real closed fields of the type $\{x_1 = x_1 \wedge \dots \wedge x_r = x_r\}$ is r (Fact 4.21), implies that for all $i \in \{1, \dots, n\}$, $|A_i| \leq r$. Then $|\bigcup_{i=1}^n A_i| \leq nr$, so there is $l \in \{0, \dots, nr\}$ such

that $l \notin \bigcup_{i=1}^n A_i$. Then $\bigcap_{j \in \kappa} C_{l,j}^i \neq \emptyset$, for all $1 \leq i \leq n$. □

Let $l \in \{0, \dots, nr\}$ satisfy the claim. We will only consider the row l of the array so we denote $(a_j)_{j \in \omega} := (a_{l,j})_{j \in \omega}$. It follows by saturation that for all $i \in \{1, \dots, n\}$ there exists an $<_i$ -box U^i in $(M^{(i)})^r$ such that:

$$U^i \cap M^r \subseteq \bigcap_{j \in \kappa} C_{l,j}^i \cap M^r$$

By density of M in every real closure (Fact 2.5) we can suppose that U^i is definable with parameters \bar{c}_i in M . Denote by $U := \bigcap_{i=1}^n (U^i \cap M^r)$.

Let $t > k_l$, we can suppose that κ is large enough. By Erdős-Rado we can find a countable sequence $(b_j)_{j \in \omega}$, indiscernible over $E' := \text{acl}(E(\bar{c}_i) : i \leq n)$ and such that the first t elements are in $(a_j)_{j \in \kappa}$. So we have that $\{\phi_l(x, b_j)\}_{j \in \omega}$ is k_l -inconsistent.

Observe that for all $j \in \omega$, $\{\bar{x} \in U : M \models \phi_l(\bar{x}, b_j)\}$ is multi-dense in U . Since U is $\mathcal{L}_n(E')$ -definable and $(b_j)_{j \in \omega}$ is indiscernible over E' , by Theorem 4.16 $\{\phi_l(\bar{x}, b_j) : j \in \omega\}$ is consistent. This is a contradiction with the k_l -inconsistency. \square

Lemma 4.25. *Let $r \in \mathbb{N}$, $A \subseteq M$ and $\bar{a} := (a_1, \dots, a_r) \in M^r$ such that $\text{trdeg}(A(\bar{a})/A) = r$. Then $\text{bdn}(\bar{a}/A) = nr$.*

Proof. Let $p(\bar{x}) := \{x_1 = x_1 \wedge \dots \wedge x_r = x_r\}$. By Lemma 4.24 $\text{bdn}(p(\bar{x})) = nr$. As $p(\bar{x}) \subseteq \text{tp}(\bar{a}/A)$, we obtain that $\text{bdn}(\bar{a}/A) \leq nr$. We will show that $\text{bdn}(\bar{a}/A) \geq nr$.

Since $\text{tp}(a_1/A(a_2, \dots, a_r))$ is not algebraic, there is a sequence $(b_{1,j})_{j \in \omega}$ in M such that:

- (1) $\text{tp}(b_{1,j}/A(a_2, \dots, a_r)) = \text{tp}(a_1/A(a_2, \dots, a_r))$, for all $j \in \omega$,
- (2) $(b_{1,j})_{j \in \omega}$ is indiscernible over $A(a_2, \dots, a_r)$.

Then using exchange property of acl , we have that $\text{tp}(a_2/A(a_3, \dots, a_r, b_{1,j} : j \in \omega))$ is not algebraic. By induction we can find for all $l \in \{1, \dots, r\}$ a sequence $(b_{l,j})_{j \in \omega}$ such that:

- (1) $\text{tp}(b_{l,j}/A(a_{l+1}, \dots, a_r, b_{1,j}, \dots, b_{l-1,j} : j \in \omega)) = \text{tp}(a_l/A(a_{l+1}, \dots, a_r, b_{1,j}, \dots, b_{l-1,j} : j \in \omega))$, for all $j \in \omega$,
- (2) $(b_{l,j})_{j \in \omega}$ is indiscernible over $A(a_{l+1}, \dots, a_r, b_{1,j}, \dots, b_{l-1,j} : j \in \omega)$.

Claim 1. For all $j_1, \dots, j_r \in \omega$, $\text{tp}(b_{1,j_1}, \dots, b_{r,j_r}/A) = \text{tp}(a_1, \dots, a_r/A)$.

Proof. If $\varphi(x_1, \dots, x_r) \in \text{tp}(a_1, \dots, a_r/A)$, then $\varphi(x_1, a_2, \dots, a_r) \in \text{tp}(a_1/Aa_2, \dots, a_r)$. Then $\varphi(b_{1,j_1}, x_2, a_3, \dots, a_r) \in \text{tp}(a_2/Aa_3, \dots, a_r, b_{1,j} : j \in \omega) = \text{tp}(b_{2,j_2}/Aa_3, \dots, a_r, b_{1,j} : j \in \omega)$, since $\text{tp}(a_1/Aa_2, \dots, a_r) = \text{tp}(b_{1,j_1}/Aa_2, \dots, a_r)$. Iterate the procedures to get the result. \square

Fix $(i, l) \in \{1, \dots, n\} \times \{1, \dots, r\}$. There is a sequence $(I_{l,j}^i)_{j \in \omega}$ of $<_i$ -open intervals in M such that $b_{l,j} \in I_{l,j}^i$, and if $j_1 \neq j_2$, then $I_{l,j_1}^i \cap I_{l,j_2}^i = \emptyset$. Let $\alpha_{l,j}^i$ be the pair of extremities of $I_{l,j}^i$ and $\varphi_{(i,l)}(x_1, \dots, x_r, \bar{y})$ the $\mathcal{L}^{(i)}$ -formula such that:

$$M \models \varphi_{(i,l)}(x_1, \dots, x_r, \alpha_{l,j}^i) \leftrightarrow x_l \in I_{l,j}^i$$

Claim 2. Let $\beta_l^i := (\alpha_{l,j}^i)_{j \in \omega}$. Then $(\beta_l^i, \varphi_{(i,l)}(\bar{x}, \bar{y}), 2)_{1 \leq i \leq n, 1 \leq l \leq r}$ is an inp-pattern of depth nr in $\text{tp}(\bar{a}/A)$.

Proof. (1) Fix $(i, l) \in \{1, \dots, n\} \times \{1, \dots, r\}$. Then $\{\varphi_{(i,l)}(\bar{x}, \alpha_{l,j}^i)\}_{j \in \omega}$ is 2-inconsistent: Clear from the fact that $I_{l,j_1}^i \cap I_{l,j_2}^i = \emptyset$.

(2) Let $f : \{1, \dots, n\} \times \{1, \dots, r\} \rightarrow \omega$. Then $\{\varphi_{(i,l)}(\bar{x}, \alpha_{l,f(i,l)}^i)\}_{1 \leq i \leq n, 1 \leq l \leq r} \cup \text{tp}(\bar{a}/A)$ is consistent:

$$\{\bar{x} : \varphi_{(i,l)}(\bar{x}, \alpha_{l,f(i,l)}^i)\}_{1 \leq i \leq n, 1 \leq l \leq r} = \{(x_1, \dots, x_r) : x_l \in I_{l,f(i,l)}^i\}_{1 \leq i \leq n, 1 \leq l \leq r} =$$

$$\{(x_1, \dots, x_r) : (x_1, \dots, x_r) \in \bigcap_{i=1}^n (I_{1,f(i,1)}^i \times \dots \times I_{r,f(i,r)}^i)\}.$$

Let $\bar{b}_i := (b_{1,f(i,1)}, \dots, b_{r,f(i,r)})$ and $C^i := I_{1,f(i,1)}^i \times \dots \times I_{r,f(i,r)}^i$. By definition $\bar{b}_i \in C^i$ and by Claim 1 $\text{tp}(\bar{b}_i/A) = \text{tp}(\bar{a}/A)$. Then by Lemma 3.18 $q(\bar{x}) := \{x \in \bigcap_{i=1}^n C^i\} \cup \{\text{tp}(\bar{a}/A)\}$ is consistent. □

Claim 2 implies that $\text{bdn}(\bar{a}/A) \geq nr$. Hence $\text{bdn}(\bar{a}/A) = nr$. □

Theorem 4.26. *Let $r \in \mathbb{N}$ and $\bar{a} := (a_1, \dots, a_r) \in M^r$. Then $\text{bdn}(\bar{a}/A) = n \cdot \text{trdeg}(A(\bar{a})/A)$. Therefore the burden is additive (i.e. $\text{bdn}(\bar{a}\bar{b}/A) = \text{bdn}(\bar{a}/A) + \text{bdn}(\bar{b}/A\bar{a})$).*

Proof. Let $k = \text{trdeg}(A(\bar{a})/A)$. As before we can easily build an inp-pattern of depth nk in $\text{tp}(\bar{a}/A)$, and so $\text{bdn}(\bar{a}/A) \geq nk$.

Suppose without loss of generality that $\{a_1, \dots, a_k\}$ is a transcendence basis of $A(\bar{a})/A$. Then $\text{bdn}(a_{k+1}, \dots, a_r/A(a_1, \dots, a_k)) = 0$, since $a_{k+1}, \dots, a_r \in \text{acl}(A(a_1, \dots, a_k))$. By Lemma 4.25, $\text{bdn}(a_1, \dots, a_k/A) = nk$, and by 4.3 $\text{bdn}(\bar{a}/A) \leq nk$.

Since the transcendence degree is additive, the burden is additive. □

4.3 Resilience of PRC fields

Definition 4.27. [10, Definition 4.8] Let \mathcal{L} be a language and let T be a complete \mathcal{L} -theory. We say that T is *resilient* if we cannot find indiscernible sequences $\bar{a} = (a_j)_{j \in \mathbb{Z}}$, $\bar{b} = (b_l)_{l \in \mathbb{Z}}$, and a formula $\phi(x, y)$ such that:

- (1) $a_0 = b_0$,
- (2) \bar{b} is indiscernible over $(a_j)_{j \neq 0}$,
- (3) $\{\phi(x, a_j)\}_{j \in \mathbb{Z}}$ is consistent,
- (4) $\{\phi(x, b_l)\}_{l \in \mathbb{Z}}$ is inconsistent.

Remark 4.28. (1) It follows by compactness that we get an equivalent definition if we replace \mathbb{Z} by κ for the sequence (b_l) , where $\kappa \geq \omega$.

- (2) [12] If $\bar{a} = (a_j)_{j \in \mathbb{Z}}$, $\bar{b} = (b_l)_{l \in \mathbb{Z}}$ and $\phi(x, y)$ satisfy the conditions of Definition 4.27 and $\phi = \phi_1 \vee \phi_2$, then there is $t \in \{1, 2\}$, $\bar{a}' = (a'_j)_{j \in \mathbb{Z}}$ and $\bar{b}' = (b'_j)_{j \in \mathbb{Z}}$ such that $\phi_t(x, y)$, \bar{a}' and \bar{b}' satisfy the conditions of Definition 4.27.

- (3) [12] If T is not resilient we can find \bar{a} , \bar{b} and $\phi(x, y)$ with $|x| = 1$ satisfying the conditions of definition 4.27.

Fact 4.29. [10, Proposition 4.11]

- (1) If T is NIP, then it is resilient.
- (2) If T is simple, then it is resilient.
- (3) If T is resilient, then it is NTP₂.

Theorem 4.30. Let $n \in \mathbb{N}$, let M be a bounded PRC field with exactly n orders and let $T := Th_{\mathcal{L}_n}(M)$. Then T is resilient

Proof. If $n = 0$ M is PAC, so T is simple and by Fact 4.29 it is resilient. If M is real closed, then T is NIP and by Fact 4.29 it is resilient. Suppose that M is neither PAC nor real closed. Suppose that M is sufficiently saturated. Let κ be a sufficiently large cardinal. Suppose by contradiction that there exists $E \subseteq M$, an $\mathcal{L}_n(E)$ -formula $\phi(x, y)$, and indiscernible sequences over E , $\bar{a} = (a_j)_{j \in \mathbb{Z}}$, $\bar{b} = (b_l)_{l \in \kappa}$ such that:

- (1) $a_0 = b_0$,
- (2) \bar{b} is indiscernible over $(E(a_j) : j \in \mathbb{Z} \setminus \{0\})$,
- (3) $\{\phi(x, a_j)\}_{j \in \mathbb{Z}}$ is consistent,
- (4) $\{\phi(x, b_l)\}_{l \in \kappa}$ is inconsistent.

We can suppose that $E = \text{acl}(E)$ and by Remark 4.28 that $|x| = 1$.

By Theorem 3.13, $\phi(M, a_0) = \bigvee_{t=0}^m \phi_t(M, a_0)$ where $\phi_0(M, a_0)$ is finite, and for each $\phi_t(M, a_0)$,

$t > 0$, there is a multi-interval $I_t(a_0) = \bigcap_{i=1}^n I_t^i(a_0) \cap M$ such that:

- (1) $\phi_t(M, a_0) \subseteq I_t(a_0)$,
- (2) $\{x \in I_t(a_0) : M \models \phi(x, a_0)\}$ is multi-dense in $I_t(a_0)$ for all $1 \leq t \leq m$,
- (3) the set $I_t^i(a_0) \cap M$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}(Ea_0)$ -formula, for all $1 \leq t \leq m$ and $1 \leq i \leq n$.

By indiscernability, the same is true for all a_j , as $a_0 = b_0$ and $(b_l)_{l \in \kappa}$ is indiscernible over E , the same is also true for all b_l . By Remark 4.28(2), we may assume $\phi(M, a_0) = \phi_t(M, a_0)$, for some $1 \leq t \leq m$. If $t = 0$ there is nothing to prove so suppose that $t > 0$. Denote by $I(a_j) = I_t(a_j)$ and $I(b_l) = I_t(b_l)$.

As $\{\phi(x, a_j)\}_{j \in \mathbb{Z}}$ is consistent, $\bigcap_{j \in \mathbb{Z}} I(a_j) \neq \emptyset$. Then for all $i \in \{1, \dots, n\}$, $\bigcap_{j \in \mathbb{Z}} I^i(a_j) \neq \emptyset$. Since $Th(M^{(i)})$ is NIP, by Fact 4.29 it is resilient. This implies that for all $i \in \{1, \dots, n\}$, $\bigcap_{l \in \kappa} I^i(b_l) \neq \emptyset$. By density of M in each real closure (Fact 2.5) and saturation of M , there exists a non-empty $<_i$ -open interval $I^i \subseteq M^{(i)}$, with extremities in M such that $I^i \subseteq \bigcap_{l \in \kappa} I^i(b_l)$. Let

$I := \bigcap_{i=1}^n I^i \cap M$. Let c_i, d_i be the extremities of I^i .

Let k be such that $\{\phi(x, b_l)\}_{l \in \kappa}$ is k -inconsistent. By Erdős-Rado we can find a countable sequence $(c_l)_{l \in \omega}$, indiscernible over $E' := \text{acl}(Ec_i, d_i : i \leq n)$ and such that the first k elements are in $(b_l)_{l \in \kappa}$. So we have that $\{\phi_l(x, c_l)\}_{l \in \omega}$ is k -inconsistent.

Observe that for all $l \in \omega$, $\{x \in I : M \models \phi(x, c_l)\}$ is multi-dense in I . Since I is $\mathcal{L}_n(E')$ -definable and $(c_j)_{j \in \omega}$ is indiscernible over E' , by Theorem 4.15 $\{\phi(x, c_l)\}_{l \in \omega}$ is consistent. This contradicts the inconsistency and shows that T is resilient. \square

4.4 Forking and dividing in bounded PRC fields

Definition 4.31. We fix a theory T and a monster model \mathbb{M} of T . Let $A \subseteq \mathbb{M}$ be a small subset and let a be a tuple in \mathbb{M} .

- (1) We say that *the formula $\psi(x, a)$ divides over A* if there exists $k \in \mathbb{N}$ and an indiscernible sequence over A , $(a_j)_{j \in \omega}$, such that: $a_0 = a$ and $\{\psi(x, a_j) : j \in \omega\}$ is k -inconsistent.
- (2) We say that *the formula $\phi(x, a)$ forks over A* if there is a number $m \in \mathbb{N}$ and formulas $\psi_j(x, a_j)$ for $j < m$ such that $\phi(x, a) \vdash \bigvee_{j < m} \psi_j(x, a_j)$ and $\psi_j(x, a_j)$ divides over A for every $j < m$.
- (3) A *type p forks (divides) over A* if it implies a formula which forks (divides) over A .
- (4) We say that A is an *extension base* if for all tuples a in \mathbb{M} , $\text{tp}(a/A)$ does not fork over A .

Denote by $a \perp_A b$ if $\text{tp}(a/Ab)$ does not fork over A .

Fact 4.32. *The following properties are satisfied by the relation \perp in any theory:*

- (1) [14, Remark 2.14] *if $a \perp_{Ab} c$ and $b \perp_{Ac} c$, then $ab \perp_{Ac}$,*
- (2) [14, Lemma 3.21] *if forking equals dividing over A , then $a \perp_A b$ iff $a \perp_{\text{acl}(A)} b$ iff $\text{acl}(Aa) \perp_{Ab}$ iff $a \perp_A \text{acl}(Ab)$.*

Corollary 4.33. *Suppose that for any set A , every 1-type over A does not fork over A ; then every A is an extension base.*

Proof. This follows immediately from (1) of Fact 4.32 by induction on the arity of the type. \square

Fact 4.34. [14, Theorem 1.1, Corollary 1.3] *Let T be a NTP_2 theory.*

- (1) *Forking equals dividing over any extension base (in particular over any model).*
- (2) *If all sets are extensions bases, then forking equals dividing.*

Fact 4.35. [41, Corollary 2.6] *In the theory of real closed fields all sets are extensions bases and forking equals dividing.*

Notation 4.36. Let $n \geq 1$, as in 3.3 we fix a bounded PRC field K , which is not real closed and has exactly n orders and let $T := Th_{\mathcal{L}_n}(K)$. Let M be a monster model of T , let a be a tuple of M and $A, B \subseteq M$. Denote by $a \perp_A^i B$ if $\text{tp}^{M^{(i)}}(a/AB)$ does not fork over A and by $a \perp_A^{ACF} B$ if a is *ACF*-independent of B over A .

Theorem 4.37. *In T all sets are extension bases and forking equals dividing.*

Proof. By Theorem 4.23 T is NTP_2 , and by Fact 4.34 it is enough to show that all sets are extensions bases. Suppose by contradiction that there exists $A \subseteq M$ and a tuple a in M such that $\text{tp}(a/A)$ forks over A . We can suppose that $a \notin \text{acl}(A)$. By Corollary 4.33 we can also suppose that $|a| = 1$.

Then there are $\phi(x) \in \text{tp}(a/A)$, $m \in \mathbb{N}$ and $\psi_j(x, a_j)$ for $j < m$ such that: $\phi(x) \vdash \bigvee_{j < m} \psi_j(x, a_j)$ and $\psi_j(x, a_j)$ divides over A for every $j < m$. Observe that since $a \notin \text{acl}(A)$ and $\phi(x) \in \text{tp}(a/A)$, we obtain that $|\phi(M)| = \infty$ and therefore there is $j < m$ such that $|\psi_j(M, a_j)| = \infty$.

For each $j < m$, by Theorem 3.13 there are a finite set $A_j \subseteq \psi_j(M, a_j)$, $t_j \in \mathbb{N}$ and multi-intervals I_1, \dots, I_{t_j} , definable with parameters in $\text{acl}(A(a_j))$ such that: $A_j \subseteq \text{acl}(Aa_j)$, $\psi_j(M, a_j) \subseteq \bigcup_{l=1}^{t_j} I_l \cup A_j$ and $\{x \in I_l : M \models \psi_j(x, a_j)\}$ is multi-dense in I_l , for all $l \leq t_j$. Then

$\psi_j(x, a_j)$ is equivalent to $\bigvee_{l=1}^{t_j} (\psi_j(x, a_j) \wedge x \in I_l) \vee x \in A_j$. Observe that since $\psi_j(x, a_j)$ divides over A , $\psi_j(x, a_j) \wedge x \in I_l$ divides over A for all $l \leq t_j$.

Therefore we can suppose that $\phi(x) \vdash \bigvee_{j < m} \psi_j(x, a_j) \vee x \in B$ where the following is satisfied:

- (1) B is a finite subset of $\bigcup_{j < m} \text{acl}(Aa_j)$,
- (2) for all $j < m$ $\psi_j(x, a_j)$ divides over A ,
- (3) for all $j < m$ there is a multi-interval I_j , definable in M with parameters in $\text{acl}(A(a_j))$ such that: $\psi_j(M, a_j) \subseteq I_j$ and $\{x \in I_j : M \models \psi_j(x, a_j)\}$ is multi-dense in I_j .

Claim. There is $j < m$ and a multi-interval J definable over A such that $J \subseteq I_j$.

Proof. As $a \notin \text{acl}(A)$ and $\phi(x) \in \text{tp}(a/A)$, by Proposition 3.12 there exists a multi-interval I , definable over A such that $a \in I$ and $\{x \in I : M \models \phi(x)\}$ is multi-dense in I .

Let $J \subseteq I$ be a multi-interval definable over A . Since B is a finite set, $\phi(x) \vdash \bigvee_{j < m} \psi_j(x, a_j) \vee x \in B$, and by multi-density of $\phi(x)$ in I , it follows that $J \cap \bigcup_{j < m} I_j \neq \emptyset$. There are infinitely many multi-intervals $J \subseteq I$ definable over A : for all $i \in \{1, \dots, n\}$, let I^i be an $<_i$ -open interval in $M^{(i)}$ such that $I = \bigcap_{i=1}^n I^i \cap M$. Since I is definable over A , then $I^i = (a^i, b^i)_i := \{x \in M^{(i)} : a^i <_i x <_i b^i\}$, with $a^i, b^i \in \text{acl}^{M^{(i)}}(A)$. Let $m \in \mathbb{N}$ be such that $(a^i + \frac{1}{m}, b^i - \frac{1}{m})_i \subset (a^i, b^i)_i$, for all $1 \leq i \leq n$. Then for all $k \geq m$, $\bigcap_{i=1}^n (a^i + \frac{1}{k}, b^i - \frac{1}{k})_i$ is a multi-interval in I , definable over A .

Then there exists $j < m$, such that $|I_j \cap \text{acl}(A)| = \infty$. Thus there is a multi-interval J , definable over A such that $J \subseteq I_j$. □

As $\psi_j(x, a_j)$ divides over A , there are $k \in \mathbb{N}$ and an indiscernible sequence over A , $(a_{j,l})_{l \in \omega}$ such that $a_{j,0} = a_j$ and $\{\psi_j(x, a_{j,l}) : l \in \omega\}$ is k -inconsistent. As $J \subseteq I_j$ and $\{x \in I_j : M \models \psi_j(x, a_{j,0})\}$ is multi-dense in I_j , we have that $\{x \in J : M \models \psi_j(x, a_{j,0})\}$ is multi-dense in J . As J is definable over A , by Theorem 4.15 $\{\psi_j(x, a_{j,l}) : l \in \omega\}$ is consistent. This contradicts the k -inconsistency. □

Theorem 4.38. *Let M be a model of T , let a be a tuple in M and $A \subseteq B \subseteq M$. Then $a \perp_A B$ if and only if $a \perp_A^i B$, for all $1 \leq i \leq n$.*

Proof. We can suppose that M is sufficiently saturated, and that $A = \text{dcl}(A)$, $B = \text{dcl}(B)$.

(\Leftarrow): Suppose that $a \perp_A^i B$, for all $1 \leq i \leq n$. Observe that $a \perp_A^i B$ implies $a \perp_A^{ACF} B$. By Fact 4.32 and Fact 4.35 we can suppose that $\text{trdeg}(A(a)/A) = \text{trdeg}(B(a)/B) = |a|$.

Suppose by contradiction that $\text{tp}(a/B)$ forks over A . Then there exists $b \subseteq B$ and $\phi(x, b) \in \text{tp}(a/B)$ such that $\phi(x, b)$ forks over A . By Theorem 4.37, $\phi(x, b)$ divides over A . Let κ be a sufficiently large cardinal; by compactness there exists $k \in \mathbb{N}$ and an indiscernible sequence over A , $(b_j)_{j \in \kappa}$ such that $b_0 = b$ and $\{\phi(x, b_j) : j \in \kappa\}$ is k -inconsistent.

As $M \models \phi(a, b)$ and $\text{trdeg}(B(a)/B) = |a|$, by Theorem 3.17 there is a multi-cell $C := \bigcap_{i=1}^n (C^i \cap M^{|a|})$, such that C^i is quantifier-free $\mathcal{L}^{(i)}(b)$ -definable, $a \in C$ and $\{x \in C : M \models \phi(x, b)\}$ is multi-dense in C . Let $\psi(x, b)$ be a quantifier free \mathcal{L}_n -formula such that $M \models x \in C \leftrightarrow \psi(x, b)$ and let $C_j = \bigcap_{i=1}^n (C_j^i \cap M^{|a|})$ be the multi-cell definable in M by the formula $\psi(x, b_j)$. Then by indiscernibility; for all $j \in \kappa$, $\{x \in C_j : M \models \phi(x, b_j)\}$ is multi-dense in C_j .

By hypothesis for all $1 \leq i \leq n$, $\text{tp}^{M^{(i)}}(a/B)$ does not divide over A . As “ $x \in C^i$ ” $\in \text{tp}^{M^{(i)}}(a/B)$, it follows that “ $x \in C^i$ ” does not divide over A . This implies that $\{x \in C_j^i : j \in \kappa\}$ is consistent.

Since for all $j \in \kappa$, C_j^i is $<_i$ -open in $M^{|a|}$, by saturation of M and density of M in each real closure $M^{(i)}$, there exists a multi-box $D^i \subseteq (M^{(i)})^{|a|}$ definable over M , such that $D^i \cap M^{|a|} \subseteq \bigcap_{j \in \kappa} (C_j^i \cap M^{|a|})$. Let $\alpha_i \subseteq M$ such that D^i is $\mathcal{L}^{(i)}(\alpha_i)$ -definable.

As before using Erdős-Rado we can find a countable sequence $(c_j)_{j \in \omega}$, indiscernible over $A' = \text{acl}(A(\alpha_i)_{i \leq n})$ and such that the first k elements are in $(b_j)_{j \in \kappa}$. So we have that $\{\phi(x, c_j)\}_{j \in \omega}$ is k -inconsistent. Then $(c_j)_{j \in \omega}$ is indiscernible over A' , $D := \bigcap_{i=1}^n (D^i \cap M^{|a|})$ is definable over A' and for all $j \in \omega$, $\{x \in D : M \models \phi(x, c_j)\}$ is multi-dense in D . Then by Theorem 4.16 $\{\phi(x, c_j) : j \in \omega\}$ is consistent. This contradicts the k -inconsistency.

(\Rightarrow) Suppose that $a \perp_A B$; this implies that for all $i \in \{1, \dots, n\}$, $\text{tp}_{\mathcal{L}^{(i)}}^M(a/B)$ does not fork over A . By Fact 4.32 we can suppose that $\text{acl}(A) = A$. Suppose by contradiction that there exists $i \in \{1, \dots, n\}$ such that $\text{tp}_{\mathcal{L}^{(i)}}^{M^{(i)}}(a/B)$ forks over A . Then by Fact 4.35 there is $\phi(x, b) \in \text{tp}_{\mathcal{L}^{(i)}}^{M^{(i)}}(a/B)$ which divides over A . By quantifier elimination in RCF , we can suppose that $\phi(x, b)$ is a quantifier-free $\mathcal{L}^{(i)}$ -formula.

Then there exists $k \in \mathbb{N}$ and an $\mathcal{L}^{(i)}$ -indiscernible sequence over A , $(b_j)_{j \in \omega} \subseteq M^{(i)}$ such that: $b_0 = b$ and $\{\phi(x, b_j) : j \in \omega\}$ is k -inconsistent. By Lemma 5.35 of [42] we can suppose that $b_j \perp_A^i b_0 \dots b_{j-1}$ for all $j \in \omega$, and so $b_j \perp_A^{ACF} b_0 \dots b_{j-1}$, for all $j \in \omega$.

Claim. Each $<_t$ extends to an order on $A(b_j : j \in \omega)$.

Proof. We have that $b_0 = b$, $b \in M$, and $(b_j)_{j \in \omega}$ is $\mathcal{L}^{(i)}$ -indiscernible. Then b and b_j satisfy the same algebraic formulas over A . This implies that we can extend each order $<_t$ in $A(b_j)$, since $A(b)$ is $<_t$ -ordered. Suppose that each $<_t$ extends to an order on $A(b_0, \dots, b_m)$. As $b_{m+1} \perp_A^{ACF} b_0 \dots b_m$ and $A(b_{m+1})/A$ is regular, $A(b_{m+1})$ is linearly disjoint of $A(b_0, \dots, b_m)$ over A . By Amalgamation theorem for ordered fields (2.2) each $<_t$ extends to $A(b_0, \dots, b_{m+1})$. \square

Since M/A and $A(b_j : j \in \omega)/A$ are regular, we can suppose that they are linearly disjoint over A . By the Amalgamation Theorem for ordered fields (2.2), $M(b_j : j \in \omega)$ is a totally real regular extension of M . So by Fact 2.3 M is existentially closed in $M(b_j : j \in \omega)$. Then by saturation there exists an $\mathcal{L}^{(i)}$ -indiscernible sequence $(b'_j)_{j \in \omega} \subseteq M$ over A such that: $\text{tp}(b'_j/A) = \text{tp}(b/A)$ for all $j \in \omega$ and $\{\phi(x, b'_j) : j \in \omega\}$ is k -inconsistent. As $\phi(x, b) \in \text{tp}_{\mathcal{L}^{(i)}}^M(a/B)$, this implies that $\text{tp}_{\mathcal{L}^{(i)}}^M(a/B)$ fork over A . This is a contradiction. \square

4.5 Lascar types

Definition 4.39. Let T be an \mathcal{L} -theory and \mathbb{M} be a model monster of T . Let a and b be tuples in \mathbb{M} and $A \subseteq \mathbb{M}$. We write $\text{Lstp}(a/A) = \text{Lstp}(b/A)$ (a and b have the same Lascar strong type

over A) if there are $n \in \omega$, $a = a_0, \dots, a_n = b$ such that a_i, a_{i+1} start a A -indiscernible sequence for each $i < n$.

We let $d_A(a, b)$ be the *Lascaar distance*, that is the smallest n as in the definition or ∞ if it does not exist.

Fact 4.40. [24, Lemma 2.9] *If T is NIP and A is an extension base, then $\text{Lstp}(a/A) = \text{Lstp}(b/A)$ if and only if $d_A(a, b) \leq 2$.*

Lemma 4.41. *Let M be a sufficiently saturated bounded PRC field with exactly n orders. We consider M as an \mathcal{L}_n -structure. Let $A \subseteq M$ and let $(a_j)_{j \in \omega}$ be a quantifier-free \mathcal{L}_n -indiscernible sequence over A such that $a_j \downarrow_A^{ACF} a_0, \dots, a_{j-1}$ and $\text{tp}(a_0/A) = \text{tp}(a_1/A)$. Then there is an \mathcal{L}_n -indiscernible sequence $(c_j)_{j \in \omega}$ over A such that $\text{tp}_{\mathcal{L}_n}(c_l, c_j/A) = \text{tp}_{\mathcal{L}_n}(a_0, a_1/A)$ for all $l < j$.*

Proof. Let $p(x, y) = \text{tp}_{\mathcal{L}_n}(a_0, a_1/A)$.

Claim. There is a sequence $(b_j)_{j < \omega}$ such that $\text{qftp}_{\mathcal{L}_n}((b_j)_{j < \omega}/A) = \text{qftp}_{\mathcal{L}_n}((a_j)_{j < \omega}/A)$, and for every $l < j$, (b_l, b_j) realizes p .

Proof. We will construct the sequence $(b_j)_{j < \omega}$ by induction on $j \geq 1$. For $j = 1$, there is nothing to prove, we take $b_0 = a_0$, $b_1 = a_1$. Assume b_0, \dots, b_j constructed; it suffices to find b_{j+1} such that $\text{qftp}_{\mathcal{L}_n}(b_0, \dots, b_{j+1}/A) = \text{qftp}_{\mathcal{L}_n}(a_0, \dots, a_{j+1}/A)$, $\text{tp}_{\mathcal{L}_n}(b_{j+1}/A(b_0, \dots, b_{j-1})) = \text{tp}_{\mathcal{L}_n}(b_j/A(b_0, \dots, b_{j-1}))$ and $\text{tp}_{\mathcal{L}_n}(b_j, b_{j+1}/A) = p$.

Let c be such that $\text{qftp}_{\mathcal{L}_n}(b_0, \dots, b_j, c/A) = \text{qftp}_{\mathcal{L}_n}(a_0, \dots, a_j, a_{j+1}/A)$. Let $B := A(b_0, \dots, b_{j-2})$, c realizes $\text{qftp}_{\mathcal{L}_n}(c/Bb_j) \cup \text{qftp}_{\mathcal{L}_n}(b_j/Bb_{j-1})$. By Theorem 3.21, $\text{tp}_{\mathcal{L}_n}(c/Bb_j) \cup \text{tp}_{\mathcal{L}_n}(b_j/Bb_{j-1}) \cup \text{qftp}_{\mathcal{L}_n}(c/Bb_{j-1}b_j)$ is realized by some c^* . Then $\text{tp}_{\mathcal{L}_n}(c^*/Bb_{j-1}) = \text{tp}_{\mathcal{L}_n}(b_j/Bb_{j-1})$.

Since $\text{qftp}_{\mathcal{L}_n}(b_j, c^*/B) = \text{qftp}_{\mathcal{L}_n}(b_{j-1}, c^*/B)$, by Corollary 3.22 there is b_{j+1} such that:

- (1) $\text{qftp}_{\mathcal{L}_n}(b_{j+1}/Bb_{j-1}b_j) = \text{qftp}_{\mathcal{L}_n}(c^*/Bb_{j-1}b_j)$,
- (2) $\text{tp}_{\mathcal{L}_n}(b_{j-1}, b_{j+1}/B) = \text{tp}_{\mathcal{L}_n}(b_{j-1}, c^*/B)$,
- (3) $\text{tp}_{\mathcal{L}_n}(b_j, b_{j+1}/B) = \text{tp}_{\mathcal{L}_n}(b_{j-1}, b_j/A)$.

Then $\text{qftp}_{\mathcal{L}_n}(b_{j+1}/Bb_{j-1}b_j) = \text{qftp}_{\mathcal{L}_n}(a_{j+1}/Aa_0, \dots, a_j)$, $\text{tp}_{\mathcal{L}_n}(b_{j-1}, b_{j+1}/B) = \text{tp}_{\mathcal{L}_n}(b_{j-1}, b_j/B)$, and $\text{tp}_{\mathcal{L}_n}(b_j, b_{j+1}/A) = \text{tp}_{\mathcal{L}_n}(b_0, b_1/A)$. \square

By compactness we can extend the sequence of the claim to $(b_j)_{j \in \kappa}$ with κ a sufficiently large cardinal. Since κ is large enough, we can find an \mathcal{L}_n -indiscernible sequence $(c_j)_{j \in \omega}$ over A such that $\text{tp}_{\mathcal{L}_n}(c_0, c_1/A) = \text{tp}_{\mathcal{L}_n}(b_0, b_1/A) = \text{tp}_{\mathcal{L}_n}(a_0, a_1/A)$. \square

Theorem 4.42. *Let $n \geq 1$ and let M be a sufficiently saturated bounded PRC field with exactly n orders. We consider M as an \mathcal{L}_n -structure. Let a, b be tuples in M and $A \subseteq M$. Then $\text{Lstp}_{\mathcal{L}_n}(a/A) = \text{Lstp}_{\mathcal{L}_n}(b/A)$ if and only if $d_A(a, b) \leq 2$ if and only if $\text{tp}_{\mathcal{L}_n}(a/A) = \text{tp}_{\mathcal{L}_n}(b/A)$.*

Proof. Suppose that $\text{tp}_{\mathcal{L}_n}(a/A) = \text{tp}_{\mathcal{L}_n}(b/A)$. Then by Fact 4.40, for all $i \in \{1, \dots, n\}$, $\text{Lstp}_{\mathcal{L}^{(i)}}^{M^{(i)}}(a/A) = \text{Lstp}_{\mathcal{L}^{(i)}}^{M^{(i)}}(b/A)$ and $d_A(a, b) \leq 2$ (in $M^{(i)}$). Then there are $a = a_0^i, a_1^i, a_2^i = b$ such that a_0^i, a_1^i and a_1^i, a_2^i start an $\mathcal{L}^{(i)}$ -indiscernible sequence (in $M^{(i)}$) over A , for each $1 \leq i \leq n$.

By the proof of Proposition 5.25 of [42] and since $\text{Th}(M^{(i)})$ is NIP, we can suppose that $a_0^i \downarrow_A^{ACF} a_1^i$ and that $a_1^i \downarrow_A^{ACF} a_2^i$.

For each $i \in \{1, \dots, n\}$, let $(c_j^i)_{j \in \omega}$ be an $\mathcal{L}^{(i)}$ -indiscernible sequence (in $M^{(i)}$) over A , such that $c_0^i = a$, $c_1^i = a_1^i$ and $c_j^i \downarrow_A^{ACF} c_0^i, \dots, c_{j-1}^i$ (see proof of Proposition 5.25 of [42]).

For each $i \geq 2$, let $\phi_i : A(c_j^i : j \in \omega) \rightarrow A(c_j^1 : j \in \omega)$ be the bijection such that $\phi_i|_A$ is the identity and $\phi_i(c_j^i) = c_j^1$. Equip $A(c_j^1 : j \in \omega)$ with the unique \mathcal{L}_n -structure which makes each ϕ_i ($i \geq 2$) into an $\mathcal{L}^{(i)}$ -isomorphism (this not necessary coincide with the \mathcal{L}_n -structure inherited from M). It follows that $A(c_j^1 : j \in \omega)/A$ is a totally real regular extension, and so there is $M^* \succeq M$ which contains an $\mathcal{L}_n(A)$ -isomorphic copy of $A(c_j^1 : j \in \omega)$ (with this new \mathcal{L}_n -structure).

Since the sequence $(c_j^1)_{j \in \omega}$ is quantifier-free \mathcal{L}_n -indiscernible over A , by Lemma 4.41 we can find an \mathcal{L}_n -indiscernible sequence $(d_j)_{j \in \omega}$ over A , such that $\text{tp}_{\mathcal{L}_n}(d_0, d_1/A) = \text{tp}_{\mathcal{L}_n}(c_0^1, c_1^1/A) = \text{tp}_{\mathcal{L}_n}(a, c_1^1/A)$.

In the same way there is an \mathcal{L}_n -indiscernible sequence $(e_j)_{j \in \omega}$ over A such that $\text{tp}_{\mathcal{L}_n}(e_0, e_1/A) = \text{tp}_{\mathcal{L}_n}(c_1^1, b/A)$. This implies that $d_A(a, b) \leq 2$. The rest of the assertions are clear. \square

Remark 4.43. Chernikov and Ben Yaacov showed in [10] the following Independence Theorem for NTP₂ theories (Theorem 3.3 of [10]): *Let T be NTP₂ and A an extension base. Assume that $c \downarrow_A ab$, $a \downarrow_A bb'$ and $\text{Lstp}(b/A) = \text{Lstp}(b'/A)$. Then there is c' such that $c' \downarrow_A ab'$, $\text{tp}(c'a/A) = \text{tp}(ca/A)$ and $\text{tp}(c'b'/A) = \text{tp}(cb'/A)$.*

It is easy to see that in the case of PRC bounded fields the Amalgamation Theorem (Theorem 3.21) implies the independence theorem for NTP₂ theories: Suppose that M is a bounded PRC field with exactly n orders. We consider M as an \mathcal{L}_n -structure. Let $A \subseteq M$, and let $c \downarrow_A ab$, $a \downarrow_A bb'$ and $\text{Lstp}_{\mathcal{L}_n}(b/A) = \text{Lstp}_{\mathcal{L}_n}(b'/A)$. Let $\phi \in \text{Aut}(M/A)$ such that $\phi(b) = b'$.

Observe that as $\text{Th}(M^{(i)})$ is NIP, Theorem 3.3 of [10] is true in each $M^{(i)}$, for all $i \in \{1, \dots, n\}$. This implies that for all $i \in \{1, \dots, n\}$, there is $c'_i \downarrow_A^i ab'$ such that c'_i realizes $\text{tp}_{\mathcal{L}^{(i)}}^{M^{(i)}}(c/Aa) \cup \text{tp}_{\mathcal{L}^{(i)}}^{M^{(i)}}(\phi(c)/Ab')$.

It follows that there is $c' \in M$ which realizes $\text{qftp}_{\mathcal{L}_n}^M(c/Aa) \cup \text{qftp}_{\mathcal{L}_n}^M(\phi(c)/Ab')$ and such that $c' \downarrow_A^i ab'$ for all $i \in \{1, \dots, n\}$. In particular, $c \downarrow_A^{ACF} ab'$.

By Theorem 3.21 there is $d \downarrow_A^{ACF} ab'$ realizing $\text{tp}_{\mathcal{L}_n}^M(c/Aa) \cup \text{tp}_{\mathcal{L}_n}^M(\phi(c)/Ab')$. Then $\text{tp}(da/A) = \text{tp}(ca/A)$ and $\text{tp}(db'/A) = \text{tp}(\phi(c)b'/A) = \text{tp}(cb'/A)$.

5 Preliminaries on pseudo p -adically closed fields

5.1 p -adically closed fields

Definition 5.1. Let (M, v) be a valued field. The valuation v is called *p -adic* if the residue field is \mathbb{F}_p and $v(p)$ is the smallest positive element of the value group $v(M)$. A field M which admits a p -adic valuation is called *formally p -adic*; it must be of characteristic 0.

If (M, v) is a valued field and v is a p -adic valuation on M , then we say that (M, v) is a *p -adically valued field*. A p -adically valued field (M, v) which has no proper p -adically valued algebraic extension is called *p -adically closed*. A *p -adic closure* of (M, v) is an algebraic extension $(\overline{M}^p, \overline{v}^p)$, which is p -adically closed. Each p -adically valued field (M, v) has a *p -adic closure*, but as opposed to real closures, in the p -adic case, in general the p -adic closure is not unique. However we have the following criterion:

If (L_1, v_1) and (L_2, v_2) are p -adic closures of (M, v) , then (L_1, v_1) and (L_2, v_2) are isomorphic over M if and only if for each $n \in \mathbb{N}$, $L_1^n \cap M = L_2^n \cap M$ ([36, page 57]).

5.2. Quantifier elimination: Let $\mathcal{L}_{Mac} := \mathcal{L}_{\mathcal{R}} \cup \{O_v\} \cup \{P_m : m > 1\}$, where $\mathcal{L}_{\mathcal{R}}$ is the language of rings, and O_v and P_m are unary relation symbols. If we interpret O_v as the valuation ring and P_m as the set of m -th powers, then we can axiomatize the class of p -adically closed fields (*p CF fields*) in the language \mathcal{L}_{Mac} . By Theorem 1 of [33] the theory of p CF fields admits quantifier elimination in \mathcal{L}_{Mac} .

Observe that if (M, v) is a p CF field, then the valuation ring O_v is quantifier-free definable in $\mathcal{L}_{\mathcal{R}} \cup \{P_m : m > 1\}$:

(*) $M \models O_v(a)$ if and only if $M \models P_m(1 + pa^m)$, where $m = 2$ if $p \neq 2$ and $m = 3$ if $p = 2$.

It follows that O_v is $\mathcal{L}_{\mathcal{R}}$ -definable.

Fact 5.3. Let (M, v) be a p CF field and let $A \subseteq M$. Then $A^{alg} \cap M = acl(A) = dcl(A) \prec M$.

Proof. By Proposition 3.4 of [44] $acl(A) = dcl(A)$, and its proof shows that $A^{alg} \cap M \models p$ CF. Since p CF is model complete (in the language of rings), then $A^{alg} \cap M \prec M$. As $A \subseteq A^{alg} \cap M$, then $acl(A) \subseteq A^{alg} \cap M$. \square

Definition 5.4. Let (M, v) be a p -adically closed field, and denote by $v(M)$ the value group. A *1-cell* in M is either a singleton or a set of the form

$$\{x \in M : \gamma_1 < v(x - a) < \gamma_2 \wedge P_n(\lambda(x - a)) \wedge \lambda(x - a) \neq 0\}$$

where $\gamma_1, \gamma_2 \in v(M) \cup \{-\infty, +\infty\}$, $a \in M$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{Z}$. Observe that if V is a 1-cell which is not a singleton, then V is open in the topology generated by the valuation, and we call V an *open 1-cell*.

Fact 5.5. [21, Lemma 4.3]/[39, Lemma 4.1] Let M be a p -adically closed field and $A \subseteq M$. Then every A -definable subset of M is a finite union of disjoint 1-cells definable with parameters in A .

5.2 Pseudo p -adically closed fields

Definition 5.6. A field extension N/M is called *totally p -adic* if for every p -adic closure \overline{M}^p of M , there exists a p -adic closure \overline{N}^p of N such that $\overline{M}^p \subseteq \overline{N}^p$. Observe that this is equivalent to: each p -adic valuation on M can be extended to a p -adic valuation on N .

Fact 5.7. [25, Lemma 13.9] *For a field M the following are equivalent:*

- (1) *M is existentially closed (relative to $\mathcal{L}_{\mathcal{R}}$) in every totally p -adic regular extension.*
- (2) *Every non-empty absolutely irreducible variety V defined over M has an M -rational point, provided that it has a simple rational point in each p -adic closure of M .*

Definition 5.8. A field M of characteristic 0 that satisfies the conditions of Fact 5.7 is called *pseudo p -adically closed* (PpC). By Lemma 10.1 of [29] the class of PpC fields is elementary in the language $\mathcal{L}_{\mathcal{R}}$.

Fact 5.9. [29, Theorem 10.8] *Let M be a PpC field and let v be a p -adic valuation on M . Then:*

- (1) *The p -adic closure of M with respect to v is exactly its henselization. In particular all p -adic closures of M with respect to v are M -isomorphic.*
- (2) *M is dense in the p -adic closure \overline{M}^p with respect to v .*
- (3) *If v_1 and v_2 are distinct p -adic valuations on M , then v_1 and v_2 are independent (i.e. v_1 and v_2 generate different topologies).*

Lemma 5.10. *Let M be a PpC field and $A \subset M$. Then $A^{alg} \cap M = acl^M(A)$. If in addition M has a definable p -adic valuation, then $acl^M(A) = dcl^M(A)$.*

Proof. The proof is identical to the one of Lemma 2.6, we only need to replace the amalgamation Theorem of orders by Lemma 4 of [32]. □

5.3 The theory of PpC fields with n p -adic valuations

As in the case of PRC fields we are interested in bounded pseudo p -adically closed fields and these fields have finitely many p -adic valuations: If M is a PpC field with infinitely many p -adic valuations, by 5.2 (*) if $p \neq 2$ ($p = 2$), M would have infinitely many extensions of degree 2 (resp degree 3), and so M would not be bounded. For this reason we will restrict our attention to pseudo p -adically closed fields with exactly n p -adic valuations, for a fixed $n \in \mathbb{N}$.

Definition 5.11. Let M be a field and let v_1, \dots, v_n be n p -adic valuations on M . The field (M, v_1, \dots, v_n) is *n -pseudo p -adically closed* (n -PpC) if:

- (1) M is a PpC field,

- (2) if $i \neq j$, then v_i and v_j are different valuations on M ,
- (3) v_1, \dots, v_n are the only p -adic valuations of M .

Notation 5.12. If (M, v_1, \dots, v_n) is an n -PpC field, we denote by $M^{(i)}$ a fixed p -adic closure of M respect to v_i . Recall that by Fact 5.9 $M^{(i)}$ is unique up to isomorphism.

Fact 5.13. [19, Lemma 3.6] *Let (M, v_1, \dots, v_n) be an n -PpC field and let V be an absolutely irreducible variety defined over M . For each $1 \leq i \leq n$, let $q_i \in V(M^{(i)})$ be a simple point. Then V contains an M -rational point q , arbitrary v_i -close to q_i , for all $i \in \{1, \dots, n\}$.*

Fact 5.14. [29, Proposition 10.4] *Let (M, v_1, \dots, v_n) and (N, w_1, \dots, w_n) be n -PpC fields. Let L be a common subfield of M and N . Suppose that there exists an isomorphism $\varphi : G(N) \rightarrow G(M)$ such that $\text{res}_{L^{\text{alg}}} \varphi(\sigma) = \text{res}_{L^{\text{alg}}} \sigma$ for each $\sigma \in G(N)$. Suppose further that $M^{(i)}$ (resp $N^{(i)}$) is a p -adic closure of M (resp N) with respect to v_i (resp w_i) such that $\varphi(G(N^{(i)})) = G(M^{(i)})$, for $i \leq n$. Then $(M, v_1, \dots, v_n) \equiv_L (N, w_1, \dots, w_n)$.*

Corollary 5.15. *Let $(M, v_1, \dots, v_n) \subseteq (N, w_1, \dots, w_n)$ be two n -PpC fields. If $\text{res} : G(N) \rightarrow G(M)$ is an isomorphism, then $(M, v_1, \dots, v_n) \prec (N, w_1, \dots, w_n)$.*

6 Bounded pseudo p -adically closed field

In this section we show that the strategies used and the results obtained in section 3 for PRC bounded fields can be generalized without much difficulty for PpC bounded fields. The biggest difference is the need to extend the language, since it is necessary to distinguish the n -th powers in each p -adic closure with respect to each p -adic valuation. For this we work with a generalization of the language of Macintyre for fields with n p -adic valuations.

Lemma 6.1. *Let K be a bounded PpC field, and let K_0 be a countable elementary substructure of K . Let \mathcal{L} be the language of rings with constant symbols for the elements of K_0 . Then K has only finitely many p -adics valuations and each one is definable by an existential \mathcal{L} -formula.*

Proof. Let v be a p -adic valuation on K , and let \overline{K}^p be a p -adic closure of M with respect to v . Let $\{P_m\}_{m \in \mathbb{N}}$ be such that $K \models P_m(a)$ if and only if $\overline{K}^p \models \exists y(y^m = a \wedge a \neq 0)$. By Fact 5.9 (1) \overline{K}^p is the henselization of K with respect to the valuation v . Let K_m be the composite field of all the extensions of K of degree m . Observe that as K is bounded, K_m is a finite extension of K . In K we can interpret without quantifiers in the language \mathcal{L} , the structure $(K_m, +, \cdot, G)$ with $G = \{\sigma|_{K_m} : \sigma \in G(\overline{K}^p)\}$, and then:

$$K \models P_m(a) \text{ if and only if } K_m \models \exists y(y^m = a \wedge a \neq 0 \wedge \forall \sigma \in G(\sigma(y) = y)).$$

This implies that for all $m \in \mathbb{N}$, P_m is definable by an existential \mathcal{L} -formula, and then by 5.2(*) v is definable by an existential \mathcal{L} -formula. As all the p -adic valuations are independent and are definable in K_2 or K_3 , K has only finitely many p -adic valuations. \square

Notation 6.2. We fix a bounded PpC field K , which is not p -adically closed and a countable elementary substructure K_0 of K . Then $K_0^{alg}K = K^{alg}$ and $G(K_0) \cong G(K)$.

Since K is bounded, by Lemma 6.1 there exists $n \in \mathbb{N}$ such that K has exactly n distinct p -adic valuations. Thus K is an n -PpC field. We will suppose that $n \geq 1$.

In this section we will work over K_0 , thus we denote by \mathcal{L} the language of rings with constant symbols for the elements of K_0 , $\mathcal{L}^{(i)} := \mathcal{L} \cup \{O_i\} \cup \{P_m^i\}_{m \in \mathbb{N}, m > 1}$ and $\mathcal{L}_n := \mathcal{L} \cup \{O_i\}_{i \leq n} \cup \{P_m^i\}_{i \leq n, m \in \mathbb{N}, m > 1}$, where O_i and P_m^i are unary relation symbols. We interpret O_i as the valuation ring corresponding to v_i and define P_m^i so that:

$$K \models P_m^i(a) \text{ if and only if } K^{(i)} \models \exists y y^m = a \wedge a \neq 0,$$

where $K^{(i)}$ is a p -adic closure of K with respect to K .

We let $T := Th_{\mathcal{L}_n}(K)$. If M is a model of T we denote by $M^{(i)}$ the p -adic closure of M with respect to v_i . As in 3.4, using Corollary 5.15 we obtain that T is model complete. Observe that by Lemma 6.1 the predicates P_m^i and the valuation ring O_i are definable in the language \mathcal{L} by an existential formula.

Corollary 6.3. $Th_{\mathcal{L}}(K)$ is model complete.

6.4. Types: As in 3.7, using Fact 5.14, we have a good description of the types in T : Let M be a model of T , A a subfield of M and a, b tuples in M . Then $\text{tp}(a/A) = \text{tp}(b/A)$ if and only if there is an \mathcal{L} -isomorphism Φ between $\text{acl}(A(a))$ and $\text{acl}(A(b))$, which sends a to b and is the identity on A .

6.1 Density theorem for bounded PpC fields

6.1.1 Density theorem for one variable definable sets

Definition 6.5. Let (M, v_1, \dots, v_n) be a model of T .

- (1) A subset of M of the form $C = \bigcap_{i=1}^n (C^i \cap M)$, with C^i a non-empty open 1-cell in $M^{(i)}$ (see definition 5.4), is called a *multi-1-cell*. By 3.1 (Approximation Theorem) and Fact 5.9(2) every multi-1-cell is non-empty.
- (2) A multi-1-cell $C = \bigcap_{i=1}^n (C^i \cap M)$ such that each C^i is a non-empty v_i -ball in $M^{(i)}$, is called a *multi-ball*.
- (3) A definable subset S of a multi-1-cell $C = \bigcap_{i=1}^n (C^i \cap M)$ is called *multi-dense* in C if for any multi-ball $J \subseteq C$, $J \cap S \neq \emptyset$. Note that multi-density implies v_i -density in C^i , for all $i \in \{1, \dots, n\}$.

Remark 6.6. Observe that if (M, v) is a p -adically valued field and $(\overline{M}^p, \overline{v}^p)$ is a p -adic closure of (M, v) , then for any 1-cell C in \overline{M}^p definable with parameters $\bar{a} \subseteq M$, the set $C \cap M$ is definable in M by a quantifier-free $\mathcal{L}_{Mac}(\bar{a})$ -formula, where $\mathcal{L}_{Mac} = \mathcal{L}_{\mathcal{R}} \cup \{O_v\} \cup \{P_m : m \in \mathbb{N}\}$.

Proof. As in Remark 3.11 using the quantifier elimination of $Th(\overline{M}^p)$ (Fact 5.2) and Fact 5.3 which says $\text{acl}^{\overline{M}^p} = \text{dcl}^{\overline{M}^p}$. \square

Proposition 6.7. *Let (M, v_1, \dots, v_n) be a model of T . Let $\phi(x, \bar{y})$ be an \mathcal{L}_n -formula, \bar{a} a tuple in M and $b \in M$ such that $M \models \phi(b, \bar{a})$ and $b \notin \text{acl}(\bar{a})$. Then there is a multi-1-cell*

$$C = \bigcap_{i=1}^n (C^i \cap M) \text{ such that:}$$

- (1) $b \in C$,
- (2) $\{x \in C : M \models \phi(x, \bar{a})\}$ is multi-dense in C ,
- (3) the set $C^i \cap M$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}(\bar{a})$ -formula, for all $1 \leq i \leq n$.

Proof. As in the proof of Proposition 3.12 using Theorem 6.1 and the fact that T is model complete, we can find $d \in \mathbb{N}$, $\bar{y}_0 \in M^d$ and an absolutely irreducible variety V defined over $\text{acl}(\bar{a})$, such that $(b, \bar{y}_0) \in V^{sim}(M)$ and $\{x \in M : \exists \bar{y}(x, \bar{y}) \in V(M)\} \subseteq \phi(M, \bar{a})$.

For each $i \in \{1, \dots, n\}$ we define:

$$A_i := \{x \in M^{(i)} : \exists (y_1, \dots, y_d) \in (M^{(i)})^d : (x, y_1, \dots, y_d) \text{ is a simple point of } V\}.$$

Observe that A_i is $\mathcal{L}^{(i)}(\bar{a})$ -definable in $M^{(i)}$ and $b \in A_i$. By Lemma 5.5 there exists a 1-cell C^i in $M^{(i)}$, $\mathcal{L}^{(i)}(\bar{a})$ -definable, such that $b \in C^i$ and $C^i \subseteq A_i$.

As $b \notin \text{acl}(\bar{a})$, C^i is a v_i -open set. Define $C := \bigcap_{i=1}^n (C^i \cap M)$ and $S := \{x \in C : M \models \phi(x, \bar{a})\}$.

As in the proof of Proposition 3.12, using Fact 5.13, we obtain that S is multi-dense in C . \square

Theorem 6.8. *Let (M, v_1, \dots, v_n) be a model of T , let $\phi(x, \bar{y})$ be an \mathcal{L}_n -formula and let \bar{a} be a tuple in M . Then there are a finite set $A \subseteq \phi(M, \bar{a})$, $m \in \mathbb{N}$ and C_1, \dots, C_m , with*

$$C_j = \bigcap_{i=1}^n (C_j^i \cap M) \text{ a multi-1-cell such that:}$$

- (1) $A \subseteq \text{acl}(\bar{a})$,
- (2) $\phi(M, \bar{a}) \subseteq \bigcup_{j=1}^m C_j \cup A$,
- (3) $\{x \in C_j : M \models \phi(x, \bar{a})\}$ is multi-dense in C_j for all $1 \leq j \leq m$,

- (4) the set $C_j^i \cap M$ is definable in M by a quantifier-free $\mathcal{L}^{(i)}(\bar{a})$ -formula, for all $1 \leq j \leq m$ and $1 \leq i \leq n$.

Proof. Exactly the same proof as in Theorem 3.13, replacing o-minimality of each real closure by Fact 5.5 and Fact 2.8 by Fact 5.13. \square

6.1.2 Density theorem for several variable definable sets

Definition 6.9. Let (M, v_1, \dots, v_n) be a model of T and $r \in \mathbb{N}$.

- (1) A subset of M^r of the form $U = \bigcap_{i=1}^n (U^i \cap M^r)$ with U^i a non-empty v_i -open set in $(M^{(i)})^r$ is called a *multi-open set in M^r* (or only *multi-open set* when r is clear). Observe that by Remark 3.15 and density of M in each $M^{(i)}$ (Fact 5.9(2)) every multi-open set is not empty.

- (2) A definable subset S of a multi-open set $U = \bigcap_{i=1}^n (U^i \cap M^r)$ is called *multi-dense in U* if for any multi-open $V \subseteq U$, $V \cap S \neq \emptyset$.

Note that multi-density in U implies v_i -density in U^i , for all $i \in \{1, \dots, n\}$.

Fact 6.10. [33, 2.1] Let (M, v) be a p -adically closed field. Let $r \in \mathbb{N}$ and $\phi(x_1, \dots, x_r, \bar{a})$ be an \mathcal{L}_{Mac} -formula. Then $\phi(M, \bar{a})$ is a finite union of $\mathcal{L}_{Mac}(\bar{a})$ -definable sets each of which is v -open in M^r (in the product topology) or is of the form $\{(x_1, \dots, x_r) : x \in U \wedge p(x_1, \dots, x_r) = 0\}$, where $p(x_1, \dots, x_r) \in \text{acl}(\bar{a})[\bar{x}]$, $p \neq 0$ and U is v -open in M^r . Observe that by quantifier elimination in the theory of p -adically closed fields these sets are in fact quantifier-free $\mathcal{L}_{Mac}(\bar{a})$ -definable.

Theorem 3.17 of section 3.1 is generalized without difficulty to the class of bounded PpC fields. In the proof we only need to replace o-minimality by Fact 6.10, real closures by p -adic closures, $<_i$ -open cells in M^d by v_i -open sets in M^d , and multi-cells by multi-open sets. We thus obtain:

Theorem 6.11. Let (M, v_1, \dots, v_n) be a model of T and let $r \in \mathbb{N}$. Let $\phi(x_1, \dots, x_r, \bar{y})$ be an \mathcal{L}_n -formula and \bar{a} be a tuple in M . Then there are a set V , $m \in \mathbb{N}$, and U_1, \dots, U_m with $U_j = \bigcap_{i=1}^n (U_j^i \cap M^r)$ a multi-open set such that:

$$(1) \phi(M, \bar{a}) \subseteq \bigcup_{j=1}^m U_j \cup V,$$

- (2) the set V is contained in some proper Zariski closed subset of M^r , which is definable over $\text{acl}(\bar{a})$,

- (3) $\{x \in U_j : \phi(\bar{x}, \bar{a})\}$ is multi-dense in U_j for all $1 \leq j \leq m$,

- (4) the set U_j^i is definable in M by a quantifier-free $\mathcal{L}^{(i)}(\bar{a})$ -formula, for all $1 \leq j \leq m$, $1 \leq t \leq l_i$.

Lemma 6.12. *Let (M, v_1, \dots, v_n) be a model of T . Let $A \subseteq M$ and let \bar{a} be a tuple of M such that $\text{trdeg}(A(\bar{a})/A) = |\bar{a}|$. For all $i \in \{1, \dots, n\}$, let $\bar{b}_i \in M^{|\bar{a}|}$ be such that $\text{qftp}_{\mathcal{L}^{(i)}}(\bar{b}_i/A) = \text{qftp}_{\mathcal{L}^{(i)}}(\bar{a}/A)$, and let U^i be a non-empty $<_i$ -open set in $(M^{(i)})^{|\bar{a}|}$ such that $\bar{b}_i \in U^i$. Then the type $p(\bar{x}) := \{\bar{x} \in \bigcap_{i=1}^n U^i\} \cup \text{tp}_{\mathcal{L}_n}(\bar{a}/A)$ is consistent.*

Proof. As in Lemma 3.18, replace Theorem 3.17 by Theorem 6.11. \square

6.2 Amalgamation theorems for bounded PpC fields

Proposition 3.20 and Theorems 3.21 and Corollary 3.22 of section 3.2 can be easily generalized to the class of bounded PpC fields. We thus obtain:

Theorem 6.13. *Let (M, v_1, \dots, v_n) be a model of T . Let $E = \text{acl}(E) \subseteq M$. Let a_1, a_2, c_1, c_2 be tuples of M such that $E(a_1)^{\text{alg}} \cap E(a_2)^{\text{alg}} = E^{\text{alg}}$ and $\text{tp}_{\mathcal{L}_n}(c_1/E) = \text{tp}_{\mathcal{L}_n}(c_2/E)$. Assume that there is c ACF-independent of $\{a_1, a_2\}$ over E realizing $\text{qftp}_{\mathcal{L}_n}(c_1/E(a_1)) \cup \text{qftp}_{\mathcal{L}_n}(c_2/E(a_2))$. Then $\text{tp}_{\mathcal{L}_n}(c_1/Ea_1) \cup \text{tp}_{\mathcal{L}_n}(c_2/Ea_2) \cup \text{qftp}_{\mathcal{L}_n}(c/E(a_1, a_2))$ is consistent.*

Corollary 6.14. *Let (M, v_1, \dots, v_n) be a model of T . Let $E = \text{acl}(E) \subseteq E$. Let a_1, a_2, d be tuples of M , such that $\text{tp}_{\mathcal{L}_n}(a_1/E) = \text{tp}_{\mathcal{L}_n}(a_2/E)$, d is ACF-independent of $\{a_1, a_2\}$ over E and $\text{qftp}_{\mathcal{L}_n}(d, a_1/E) = \text{qftp}_{\mathcal{L}_n}(d, a_2/E)$. Suppose that $E(a_1)^{\text{alg}} \cap E(a_2)^{\text{alg}} = E^{\text{alg}}$.*

Then there exists a tuple d^ in some elementary extension M^* of M such that:*

- (1) $\text{qftp}_{\mathcal{L}_n}(c^*/E(a_1, a_2)) = \text{qftp}_{\mathcal{L}_n}(c/E(a_1, a_2))$,
- (2) $\text{tp}_{\mathcal{L}_n}(d^*, a_1/E) = \text{tp}_{\mathcal{L}_n}(d^*, a_2/E)$,
- (3) $\text{tp}_{\mathcal{L}_n}(d^*, a_1/E) = \text{tp}_{\mathcal{L}_n}(d, a_1/E)$.

The proofs of Theorems 6.13 and 6.14 are exactly the same as those of Theorems 3.21 and 3.22 respectively. It is only required to replace the orders by p -adic valuations and real closures for p -adic closures. We also note the following:

6.15. Let (M, v_1, \dots, v_n) be a model of T . Denote by $M^{(i)}$ a fixed p -adic closure of M for the valuation v_i . Then:

- (1) If $A, B \subseteq M$ and $\Phi : A \rightarrow B$ is an \mathcal{L}_n -isomorphism, by Fact 5.3 for each $1 \leq i \leq n$ we can extend Φ uniquely to an $\mathcal{L}^{(i)}$ -isomorphism $\Phi^i : A^{(i)} \rightarrow B^{(i)}$, where $A^{(i)} = A^{\text{alg}} \cap M^{(i)}$ and $B^{(i)} = B^{\text{alg}} \cap M^{(i)}$.
- (2) Let L/M be an algebraic field extension. If there exists a conjugate H of $G(M^{(i)})$ such that $L \subseteq \text{Fix}(H)$, then the valuation v_i can be extended to a p -adic valuation on L . Since all the p -adic closures for the valuation v_i are isomorphic, we can extend the predicates P_m^i to L such that for all $a \in M$, $M \models P_m(a)$ if and only if $L \models P_m(a)$. Then L is an $\mathcal{L}^{(i)}$ -extension of M .

7 Independence property in PpC fields

Theorem 4.10 says that the complete theory of a PRC field which is neither real closed nor algebraically closed is not NIP. Contrary to PRC fields, the algebraic extensions of a PpC field are not necessarily PpC fields. So the proof of Theorem 4.10 cannot be generalized to PpC fields. For this reason, to prove that the theory of PpC fields is not NIP we will give an explicit example of a formula with the independence property.

Theorem 7.1. *Let $p > 2$ be a prime number. Let M be a bounded PpC field with two distinct p -adic valuations v_1, v_2 . Then the formula*

$$\phi(x, y) := \bigwedge_{i=1}^2 v_i(x) > 0 \wedge \bigwedge_{i=1}^2 v_i(y) > 0 \wedge \exists z \left(\bigwedge_{i=1}^2 v_i(z-1) > 0 \wedge z^2 = x + y + 1 \right)$$

has the independence property.

Proof. Recall that by Theorem 6.1, v_1 and v_2 are existentially \mathcal{L} -definable (see 6.2 for the definition of \mathcal{L}). So $\phi(x, y)$ is an \mathcal{L} -formula. Let $m \in \mathbb{N}$ and $k = m + 2^m$. Let $\Gamma := \mathbb{Z}^k$; then Γ is an ordered abelian group with the lexicographic order. Let t be an indeterminate, and let $M((t^\Gamma))$ be the set of elements of the form $\sum_{\gamma \in \Gamma} a_\gamma t^\gamma$, with $a_\gamma \in M$ and such that $\{\gamma \in \Gamma : a_\gamma \neq 0\}$

is well-ordered. Then $F := M((t^\Gamma))$ is a field and the t -adic valuation $v_t : F^* \rightarrow \Gamma$ given by $v_t(\sum_{\gamma \in \Gamma} a_\gamma t^\gamma) = \min\{\gamma \in \Gamma : a_\gamma \neq 0\}$ is such that (F, v_t) is Henselian.

For each $1 \leq r \leq m + 2^m$, let $z_r := (z_{1r}, \dots, z_{kr}) \in \Gamma$ such that $z_{jr} = 0$ if $j \neq r$ and $z_{rr} = 1$. For each $0 \leq j < m$, let $x_j = t^{z_{j+1}} \in F$. Let $(A_l)_{1 \leq l \leq 2^m}$ an enumeration of $\mathcal{P}(m)$, for each $1 \leq l \leq 2^m$ let $y_l = t^{z_{m+l}}$. Then the elements $\{x_j, y_l : 0 \leq j < m, 1 \leq l \leq 2^m\}$ are transcendental and algebraically independent over M , satisfying $v_t(x_j) > 0$ and $v_t(y_l) > 0$.

Define $M_0 := M(x_j, y_l : 0 \leq j < m, A_l \in \mathcal{P}(m))$ and $L := M_0(\sqrt{x_j + y_l + 1} : 0 \leq j < m, A_l \in \mathcal{P}(m))$. As $p \neq 2$, for all $0 \leq j < m$ and $A_l \in \mathcal{P}(m)$, 1 is a residual simple root of $z^2 = x_j + y_l + 1$; as (F, v_t) is Henselian there exists $z_{j,l} \in F$ such that $z_{j,l}^2 = x_j + y_l + 1$ and $v_t(z_{j,l} - 1) > 0$. Therefore $L \subseteq F$.

Let v be a p -adic valuation on M . Define the valuation w on F as follows: If $a = \sum a_\gamma t^\gamma$ and $v_t(a) = \gamma_0$, then $w(a) := (v(a_{\gamma_0}), \gamma_0)$. Then the value group of w is $v(M) \times \Gamma$, and it is ordered with the anti-lexicographic order. If $v(M)$ has a smallest positive element 1, then $(1, 0)$ is the smallest positive element of $w(F)$. This implies in particular that w is also a p -adic valuation on F . Therefore F is a totally p -adic extension of M , and as $L \subseteq F$ we obtain that L is a totally p -adic extension of M . In particular v_1, v_2 extend to p -adic valuations w_1, w_2 on L satisfying $w_i(x_j) > 0$, $w_i(y_l) > 0$, and $w_i(z_{j,l} - 1) > 0$ for all $i \in \{1, 2\}$, $0 \leq j < m$ and $A_l \in \mathcal{P}(m)$.

Claim. The fields $M_0(\sqrt{x_j + y_l + 1})$, with $(j, A_l) \in m \times \mathcal{P}(m)$, are linearly disjoint over M_0

Proof. Let $H := \langle x_j + y_l + 1 : 0 \leq j < m, A_l \in \mathcal{P}(m), (M_0^*)^2 \rangle / (M_0^*)^2$, a subgroup of $M_0^* / (M_0^*)^2$.

By Kummer theory, $L := M_0(H^{\frac{1}{2}})$ is a Galois extension of M_0 and $\text{Gal}(L/M_0) \cong H$. So we have that $[L : M_0] = |\text{Gal}(L/M_0)| = |H|$.

Since H is a $\mathbb{Z}/2\mathbb{Z}$ -vector space, it is sufficient to show that for all $C \subseteq m \times \mathcal{P}(m)$, $C \neq \emptyset$, $\prod_{(j,A_l) \in C} (x_j + y_l + 1)$ is not a square in M_0^* .

Suppose there are $z \in M_0^*$ and $C \subseteq m \times \mathcal{P}(m)$, $C \neq \emptyset$ such that:

$$\prod_{(j,A_l) \in C} (x_j + y_l + 1) = z^2.$$

We have that $R = M[x_j, y_l : 0 \leq j < m, A_l \in \mathcal{P}(m)]$ is integrally closed over M_0 , hence z belong to R and in R , $(x_j + y_l + 1) \mid z^2$, for all $(j, A_l) \in C$.

Let $(r, A_s) \in C$; since $x_r + y_s + 1$ is irreducible in R , $(x_r + y_s + 1)^2 \mid z^2$. Then,

$$(x_r + y_s + 1)^2 \mid \prod_{(j,A_l) \in C} (x_j + y_l + 1),$$

which gives us, by canceling,

$$(x_r + y_s + 1) \mid \prod_{(j,A_l) \neq (r,A_s), (j,A_l) \in C} (x_j + y_l + 1).$$

This contradicts the irreducibility of the $x_r + y_s + 1$ in M_0 .

Hence $|H| = 2^{m2^m}$, and $[L : M_0] = 2^{m2^m}$. \square

By the Claim, $\text{Gal}(L/M) \cong (\mathbb{Z}/2\mathbb{Z})^{m2^m}$, so there exists $\sigma \in \text{Gal}(L/M_0)$ such that $\sigma(z_{j,l}) = z_{j,l} \Leftrightarrow j \in A_l$. Observe that as w_2 is a p -adic valuation on L , $w_2 \circ \sigma$ is also a p -adic valuation on L extending v_2 . Replace w_2 by $w_2 \circ \sigma$. Then in (L, w_1, \dots, w_n) we have that:

$$\bigwedge_{i=1}^2 w_i(x_j) > 0 \wedge \bigwedge_{i=1}^2 w_i(y_l) > 0 \wedge [\exists z (\bigwedge_{i=1}^2 w_i(z-1) > 0 \wedge z^2 = x_j + y_l + 1)] \Leftrightarrow j \in A_l.$$

Observe that since each v_i is \mathcal{L} -definable, and \mathcal{L} contains constant symbols for a fixed submodel of M , each w_i is also \mathcal{L} -definable.

Since (L, w_1, \dots, w_n) is a totally p -adic regular extension of (M, v_1, \dots, v_n) , Lemma 6.1 and Fact 5.7 imply that (M, v_1, \dots, v_n) is existentially closed in (L, w_1, \dots, w_n) . Then there are a_j, b_l in M for every $0 \leq j < m$ and $A_l \in \mathcal{P}(m)$, such that $M \models \phi(a_j, b_l)$ if and only if $j \in A_l$. \square

Remark 7.2. The proof for $p = 2$ is similar, using instead the formula

$$\phi(x, y) := \bigwedge_{i=1}^2 v_i(x) > 0 \wedge \bigwedge_{i=1}^2 v_i(y) > 0 \wedge \exists z (\bigwedge_{i=1}^2 v_i(z-1) > 0 \wedge z^3 = x + y + 1)$$

and working in $M(\omega)$, with $\omega^2 + \omega + 1 = 0$.

Definition 7.3. Let M be a field and let \mathcal{M} be a family of separable algebraic extensions of M . Assume that \mathcal{M} is closed under the action of $G(M)$. We say that M is *pseudo \mathcal{M} -closed* (*PMC*) if every non-empty absolutely irreducible variety V defined over M with an \overline{M} -simple rational point for each $\overline{M} \in \mathcal{M}$, has an M -rational point.

Corollary 7.4. *Let M be a bounded PpC field which is not p -adically closed. Then $T = Th_{\mathcal{L}}(M)$ (see Notation 6.2) is not NIP.*

Proof. Since M is bounded there exists $n \in \mathbb{N}$ such that M has exactly n distinct p -adic valuations. If $n \geq 2$, by Theorem 7.1 and Remark 7.2; T has the *IP*. Suppose that $n = 1$, let v be the unique valuation on M ; by Theorem 6.1 v is definable. Since M is not p -adically closed it is not Henselian, and there exists a finite algebraic extension N of M such that N admits two distinct valuations v_1, v_2 , extending v . Observe that as \mathcal{L} contain constant symbols for a fixed submodel of M , in M we can interpret the structure (N, v_1, v_2) (see Appendix 1 of [7] for more details). Therefore it is enough to show that (N, v_1, v_2) has the *IP*.

Let $M^{(1)}$ be a fixed p -adic closure of (M, v) and let $\mathcal{M} := \{\sigma(M^{(1)}) : \sigma \in G(M)\}$. Then M is *PMC*. By Lemma 7.2 of [29] N is *PM(N)C*, where $\mathcal{M}(N) = \{\sigma(M^{(1)})N : \sigma \in G(M)\}$.

We do the case $p > 2$, the case $p = 2$ can be adapted similarly. Let $m \in \mathbb{N}$ and $k = m + 2^m$. Take Γ, x_j, y_l for $0 \leq j < m$ and $1 \leq l \leq 2^m$ as in the proof of Theorem 7.1. Let $N_0 := N(x_j, y_l : 0 \leq j < m, A_l \in \mathcal{P}(m))$ and $L := N_0(\sqrt{x_j + y_l + 1} : 0 \leq j < m, A_l \in \mathcal{P}(m))$.

Let $\{v_i\}_{i \in I}$ be the set of valuations in N extending v . Denote by $N^{(i)}$ the Henselian closure of (N, v_i) , and let $T_i := Th(N^{(i)})$. As in the proof of Theorem 7.1 we can find definable valuations w_i on L extending v_i such that:

- (1) $(L, w_i) \models (T_i)_{\forall}$
- (2) $L \models \bigwedge_{i=1}^2 w_i(x_j) > 0 \wedge \bigwedge_{i=1}^2 w_i(y_l) > 0 \wedge [\exists z(\bigwedge_{i=1}^2 w_i(z-1) > 0 \wedge z^2 = x_j + y_l + 1)]$ if and only if $j \in A_l$

Observe that N is existentially closed in L : we can suppose that $L = N(\bar{a})$, and let V be an absolutely irreducible variety such that \bar{a} is a generic point of V . Then V has a simple point in each $N^{(i)}$, for all $i \in I$, and as N is *PM(N)C*, it follows that V has an N -rational point.

Since the valuations are definable in N (in the language of rings expanded by constant symbols) and N is existentially closed in L , we obtain that (N, v_1, v_2) has the *IP*. □

8 PpC fields and their stability theoretic properties

Proposition 8.1. *Let $n \geq 1$. In n -PpC every quantifier-free \mathcal{L}_n -formula is NIP.*

Proof. By Lemma 4.11, since by 1.32(*) every atomic formula is of the form $P_m^i(p(\bar{x}, \bar{y}))$, with $i \in \{1, \dots, n\}$ and $p(\bar{x}, \bar{y}) \in \mathbb{Q}[\bar{x}, \bar{y}]$, and *pCF* is NIP. □

Notation 8.2. We work with the notation of 6.2. Let $n \geq 1$. Fix K a bounded PpC field with exactly n p -adic valuations, which is not p -adically closed and let $T := Th_{\mathcal{L}_n}(K)$. Let M be a monster model T and $M^{(i)}$ a fixed p -adic closure of M for the valuation v_i . Denote by $a \downarrow_A^i B$ if $\text{tp}^{M^{(i)}}(a/AB)$ does not fork over A and by $a \downarrow_A^{ACF} B$ if a is ACF-independent of B over A . Observe that $a \downarrow_A^i B$ implies $a \downarrow_A^{ACF} B$, for all $i \in \{1, \dots, n\}$.

Theorem 8.3. *Let $E = \text{acl}(E) \subseteq M$ and $(a_j)_{j \in \omega}$ an indiscernible sequence over E . Let $r \geq 1$, let $\phi(x_1, \dots, x_r, \bar{y})$ be an \mathcal{L}_n -formula and C a multi-open set in M^r definable over E , such that $\{(x_1, \dots, x_r) \in C : \phi(x_1, \dots, x_r, a_0)\}$ is multi-dense in C . Then $p(x_1, \dots, x_r) := \{\phi(x_1, \dots, x_r, a_j)\}_{j \in \omega}$ is consistent.*

Proof. The proof of Lemma 4.16 generalizes almost immediately to our context. The proof of the Claim is identical: replace Corollary 4.12 by Proposition 8.1 and observe that by Fact 6.10 if $\psi(x_1, \dots, x_r, \bar{a})$ is a quantifier-free \mathcal{L}_n -formula, $c \in M^r$, $M \models \psi(c, \bar{a})$ and $\text{trdeg}(\text{acl}(\bar{a})c/\text{acl}(\bar{a})) = r$, then there is for all $i \in \{1, \dots, n\}$, a v_i -open set C^i in $(M^{(i)})^r$, definable over $\text{acl}(\bar{a})$, such that: $\bigcap_{i=1}^n (C^i \cap M^r) \subseteq \psi(M, \bar{a})$ and $c \in \bigcap_{i=1}^n (C^i \cap M^r)$. The rest of the proof is identical, replace Theorem 3.22 by Theorem 6.14, Theorem 3.17 by Theorem 6.11, and multi-cells in M^r by multi-open sets in M^r . \square

Fact 8.4. [16, Theorem 6.6] *The theory of p -adically closed fields (pCF fields) is strong and if $r \in \mathbb{N}$, $\text{bdn}(x_1 = x_1 \wedge \dots \wedge x_r = x_r) = r$.*

Theorem 8.5. *The theory T is strong, resilient and $\text{bdn}(\{x = x\}) = n$. If $r \geq 1$, and $p(\bar{x}) := \{x_1 = x_1 \wedge \dots \wedge x_r = x_r\}$, then $\text{bdn}(p(\bar{x})) = nr$.*

Proof. We work in a monster model (M, v_1, \dots, v_n) of T . For $l \in \{0, \dots, n-1\}$, define the formula $\varphi_l(x, y) := v_{l+1}(x-y) > 0$. Find $((a_{l,j})_{j \in \omega})_{l \leq n-1}$, such that $\varphi_l(M, a_{l,j_1}) \cap \varphi_l(M, a_{l,j_2}) = \emptyset$, for all $j_1 \neq j_2$ and for all $0 \leq l \leq n-1$. Using the Approximation Theorem (3.1) we obtain that $(\bar{a}_l, \varphi_l(x, y), 2)_{l < n}$, with $\bar{a}_l = (a_{l,j})_{j \in \omega}$ is an inp-pattern of depth n . It follows that the $\text{bdn}(\{x = x\})$ is greater or equal to n . Observe that this can easily be generalized to several variables and then $\text{bdn}(p(\bar{x})) \geq nr$.

To show that $\text{bdn}(p(\bar{x})) \leq nr$ it is only necessary to replace in the proof of Theorem 4.24 multi-cell by multi-open set, Theorem 3.17 by Theorem 6.11 and Theorem 4.16 by Theorem 8.3.

The proof of resilient is as in Theorem 4.30, using the fact that the p -adically closed fields are NIP and replacing Theorem 3.13 by Theorem 6.11 and Theorem 4.15 by Theorem 8.3. \square

Corollary 8.6. *If M is a bounded PpC field, then $Th(M)$ is NTP_2 .*

Proof. This follows immediately from Theorem 8.5 using the fact that if M is bounded, then there exists $n \in \mathbb{N}$ such that M has exactly n p -adic valuations, and they are definable. \square

Observe that the proof of the converse of Theorem 4.23 cannot be generalized to bounded PpC fields, as algebraic extensions of PpC fields are not necessarily PpC.

Theorem 8.7. *Let $r \in \mathbb{N}$ and $\bar{a} := (a_1, \dots, a_r) \in M^r$. Then $\text{bdn}(\bar{a}/A) = n \cdot \text{trdeg}(A(\bar{a})/A)$. Therefore the burden is additive (i.e. $\text{bdn}(\bar{a}\bar{b}/A) = \text{bdn}(\bar{a}/A) + \text{bdn}(\bar{b}/A\bar{a})$).*

Proof. Exactly the same proof as for Lemma 4.25 and Theorem 4.26, replacing $<_i$ -open interval by 1-cell for the valuation v_i and Lemma 3.18 by Lemma 6.12. \square

Fact 8.8. *In the theory of p -adically closed fields (pCF) all sets (in the real sort) are extensions bases and forking equals dividing.*

Proof. If M is p -adically closed and $A \subseteq M$, then by Proposition 3.4 of [44] $\text{dcl}(A)$ is p -adically closed. Using Corollary 3.22 of [14] we obtain that forking equals dividing. \square

Theorem 8.9. *In T all sets are extensions bases and forking equals dividing, and if a is a tuple in M and $A \subseteq B \subseteq M$, then $a \perp_A B$ if and only if $a \perp_A^i B$, for all $1 \leq i \leq n$.*

Proof. The proof of Theorems 4.37 and 4.38 generalizes immediately to our context. We need to replace multi-intervals by multi-1-cells, multi-cells and multi-boxes by multi-open sets and real closures by p -adic closures. It is also necessary to change Theorems 3.13 and 3.17 by Theorems 6.8 and 6.11, Theorem 4.15 and 4.16 by Theorem 8.3 and the amalgamation theorem of orders 2.2 by Lemma 4 of [32]. \square

Theorem 8.10. *Let $n \geq 1$ and let M be a sufficiently saturated bounded PpC field with exactly n p -adique valuations. We consider M as a \mathcal{L}_n -structure. Let a, b tuples in M and $A \subseteq M$. Then $\text{Lstp}_{\mathcal{L}_n}(a/A) = \text{Lstp}_{\mathcal{L}_n}(b/A)$ if and only if $d_A(a, b) \leq 2$ if and only if $\text{tp}_{\mathcal{L}_n}(a/A) = \text{tp}_{\mathcal{L}_n}(b/A)$.*

Proof. As in Theorem 4.42, using the fact that if $M^{(i)}$ is the p -adic closure of M for the valuation v_i , then $\text{Lstp}_{\mathcal{L}^{(i)}}^{M^{(i)}}(a/A) = \text{Lstp}_{\mathcal{L}^{(i)}}^{M^{(i)}}(b/A)$ if and only if $d_A(a, b) \leq 2$ if and only if $\text{tp}_{\mathcal{L}^{(i)}}^{M^{(i)}}(a/A) = \text{tp}_{\mathcal{L}^{(i)}}^{M^{(i)}}(b/A)$. \square

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