The role of Boolean functions in hiding sums as trapdoors for some block ciphers

R. Aragona* [1], M. Calderini** [1] and M. Sala# [1]

[1]Department of Mathematics, University of Trento, Italy

Abstract

Most modern block ciphers are built using components whose cryptographic strength is evaluated in terms of their resistance to attacks on the whole cipher. In particular, differential properties of vectorial Boolean functions are studied for the S-Boxes to thwart differential cryptanalysis. Little is known on similar properties to avoid trapdoors in the design of the block cipher. In this paper we present a form of trapdoors coming from alternative vector space structures, which we call hidden sums, and give a characterization on the Boolean function S-Box to avoid any such hidden sum. We also study some properties of this new class of vectorial Boolean functions, which we call anti-crooked, and provide a toy cipher with a hidden sum trapdoor.

Keywords: Group action, Boolean function, iterated block cipher, trapdoor

1 Introduction

Most modern block ciphers are built using components whose cryptographic strength is evaluated in terms of the resistance offered to attacks on the whole cipher. In particular, differential properties of Boolean functions are studied for the S-Boxes to thwart differential cryptanalysis ([3, 21]).

Little is known on similar properties to avoid trapdoors in the design of the block cipher. In [6] the authors investigate the minimal properties for the S-Boxes (and the mixing layer) of an AES-like cipher (more precisely, a *translation-based cipher*, or *tb cipher*) to thwart the trapdoor coming from the imprimitivity action, first noted in [23]. Later, in [7] the authors provide stronger conditions on the S-Boxes of a tb cipher that avoid attacks coming from any group action. This result has been generalized to tb ciphers over any field in [1].

In this paper we present a form of trapdoors coming from alternative vector space structure, which we call *hidden sums* and study from a group action point of view in Section 2. In Section 3 we present a class of block cipher, which is a generalization of standard AES-like ciphers and we are able to provide a characterization on the Boolean function S-Box to avoid any such hidden sum, which can be dangerous cryptographic trapdoors. After having studied some properties of this new class of vectorial Boolean functions in Section 4, which we call *anti-crooked*, in Section 5 we provide a toy cipher with a hidden-sum trapdoor and show that it can be thus broken with attacks which are: much faster than brute force, independent of the number of rounds, independent of the key-schedule.

*riccardo.aragona@unitn.it, **marco.calderini@unitn.it, #maxsalacodes@gmail.com

2 On hidden sums

Let $p \geq 2$ be a prime and $\mathbb{F}_p$ be the finite field with p elements. Let $V = (\mathbb{F}_p)^d$. As observed by Li [19], the symmetric group $\mathrm{Sym}(V)$ will contain many isomorphic copies of the affine group $\mathrm{AGL}(V)$, which are its conjugates in $\mathrm{Sym}(V)$. So there are several structures $(V,\circ)$ of an $\mathbb{F}_p$-vector space on the set V, where $(V,\circ)$ is the abelian additive group of the vector space. Each of these structure will yield in general a different copy $\mathrm{AGL}(V,\circ)$ of the affine group within $\mathrm{Sym}(V)$. Any $\mathrm{AGL}(V,\circ)$ consists of the maps $x \mapsto g(x) \circ v$, where $g \in \mathrm{GL}(V,\circ)$, and $v \in V$.

Assumption 1. *Let G be a subgroup of* $\mathrm{Sym}(V)$.

- *G is contained in the affine subgroup* $\mathrm{AGL}(V,\circ)$
- *G contains T, an abelian regular subgroup*

For the rest of this section we consider the previous assumption and we call any $\circ$ a *hidden sum* for G. Since T is regular, for each $x \in V$ there exists unique $\sigma_x \in T$ such that $\sigma_x(0) = x$, therefore $T = T_V = \{\sigma_y \mid y \in V\}$. Since T is abelian regular subgroup contained in G, we obtain that T is an abelian regular subgroup of $\mathrm{AGL}(V,\circ)$. By [5, 8] we can define simultaneously both a structure of an associative, commutative, nilpotent ring $(V,\circ,\cdot)$ on V and an operation $\square$ on V such that $(V,\square)$ is an abelian p-group. The two operations are linked by

$$x \square y = x \circ y \circ xy \tag{1}$$

and $T = T_V$ is isomorphic to $(V,\square)$, that is $\sigma_y : x \mapsto x \square y$.

Lemma 2.1. *For each $u \in V$, the set uV is a subgroup of V with respect to both $\circ$ and $\square$.*

Proof. Since the distributive property between $\circ$ and $\cdot$ holds in $(V,\circ,\cdot)$, then we have $uv_1 \circ uv_2 = u(v_1 \circ v_2)$ for every $v_1, v_2 \in V$ and so uV is $\circ$-subgroup of V. The set uV is also a $\square$-subgroup of V since

$$uv_1 \square uv_2 = uv_1 \circ uv_2 \circ u^2 v_1 v_2 = u(v_1 \circ v_2 \circ uv_1 v_2) \in uV.$$

□

Since $T < \mathrm{AGL}(V,\circ)$, for $y \in V$ there is $\kappa_y \in \mathrm{GL}(V,\circ)$ such that

$$\sigma_y(x) = x \square y = \kappa_y(x) \circ y \tag{2}$$

for all $x \in V$.

We denote by $\boxminus t$ and $\ominus t$ the opposite of t with respect to $\square$ and $\circ$, respectively. By (2) we can write

$$x = x \boxminus y \square y = \kappa_y(x \boxminus y) \circ y = \kappa_y(\kappa_{\boxminus y}(x) \circ (\boxminus y)) \circ y$$

for all $x \in V$ and so we obtain

$$\kappa_{\boxminus y} = \kappa_y^{-1}. \tag{3}$$

In fact the map $(V,\square) \to T$, $y \mapsto \sigma_y$, is an isomorphism and we now show that the related map $(V,\square) \to \mathrm{GL}(V,\circ)$, $y \mapsto \kappa_y$, is a group homomorphism.

Proposition 2.2. *For every $x, y \in V$*

$$\kappa_{x \square y} = \kappa_y \kappa_x.$$

Therefore $\kappa_V = \{k_x \mid x \in V\}$ is a p-group and so it acts unipotently on $(V,\circ)$.

Proof. For every $x,y,z \in V$, we have

$$(x \square y) \square z = (\kappa_y(x) \circ y) \square z = \kappa_z(\kappa_y(x)) \circ \kappa_z(y) \circ z$$

and also

$$x \square (y \square z) = x \square (\kappa_z(y) \circ z) = \kappa_{\kappa_z(y)\circ z}(x) \circ \kappa_z(y) \circ z,$$

so from associativity we obtain

$$\kappa_{y \square z} = \kappa_{\kappa_z(y)\circ z} = \kappa_z \kappa_y.$$

It follows that κ_V is a group, image of the homomorphism which sends $y \in V$ to $\kappa_y \in \kappa_V$, so κ_V is a p-group and thus acts unipotently on $(V,\circ)$.

□

By the previous proposition and some well-known results on unipotent groups (see for instance [15]), we note that there exists $y \in V \setminus \{0\}$ such that $\kappa_x(y) = y$ for each $x \in V$ and so we can consider $U = \{y \in V \mid x \square y = x \circ y \text{ for all } x \in V\} \neq \{0\}$. We observe that U is a subgroup of both the structures $(V,\circ)$ and $(V,\square)$.

Let $a \in V \setminus \{0\}$, we define the derivative of ρ with respect to a and $\square$ as $D_a(\rho,\square) : x \mapsto \rho(x \square a) \boxminus \rho(x)$.

Definition 2.3. *A permutation $\rho \in \mathrm{Sym}(V)$ is called anti-crooked (AC) with respect to $\square$ if for each $a \in V \setminus \{0\}$ the set*

$$\mathrm{Im}(D_a(\rho,\square)) = \{\rho(x \square a) \boxminus \rho(x) \mid x \in V\}$$

***is not** a $\square$-coset of V. When we say that ρ is AC without specifying the sum, we mean that ρ is a vectorial Boolean function which is AC with respect to the standard sum in $(\mathbb{F}_2)^d$.*

Remark 2.4. *Compare the previous definition with the classical definition of crooked Boolean function [18], that is, for each $a \in V \setminus \{0\}$ the set*

$$\mathrm{Im}(D_a(\rho,+)) = \{\rho(x+a) + \rho(x) \mid x \in V\}$$

***is** a $+$-coset of V.*

Let $1_G \in G$ be the identity of G. We conclude this section with the main theorem linking the AC property with the existence of hidden sums with respect to G.

Theorem 2.5. *If there exists $\rho \in G \setminus \{1_G\}$ AC with respect to $\square$, then G is not contained in any copy of the affine group in* $\mathrm{Sym}(V)$.

Proof. By contradiction G is contained in an affine subgroup $\mathrm{AGL}(V,\circ)$ of $\mathrm{Sym}(V)$ and there is an AC $\rho \in G$. So G satisfies Assumption 1 and our preliminary results hold. Since ρ is $\circ$-affine and $\kappa_y \in \mathrm{GL}(V,\circ)$, by (3) for $a \in U$ and $x \in V$ it follows that

$$\begin{aligned}\rho(x \square a) \boxminus \rho(x) = \rho(x) \circ \rho(a) \boxminus \rho(x) &= \kappa^{-1}_{\rho(x)}(\rho(x)) \circ \kappa^{-1}_{\rho(x)}(\rho(a)) \circ (\boxminus \rho(x)) \\ &= \kappa^{-1}_{\rho(x)}(\rho(a)) \circ \big(\rho(x) \boxminus \rho(x)\big) = \kappa^{-1}_{\rho(x)}(\rho(a))\end{aligned}$$

Let $W = \{\kappa^{-1}_{\rho(x)}(\rho(a)) \mid x \in V\}$ be the image of $D_a(\rho,\square)$. Since $\kappa_x^{-1} = \kappa_{\boxminus x}$ for every $x \in V$ and ρ is a permutation we have $W = \{\kappa_x(\rho(a)) \mid x \in V\}$. For the sake of simplicity, we set $\rho(a) = \overline{a}$. By (1) and (2) we have $\kappa_x(\overline{a}) = \overline{a} \circ x\overline{a}$ and

$$W = \{\kappa_x(\overline{a}) \mid x \in V\} = \{\overline{a} \circ \overline{a}x \mid x \in V\}.$$

Substituting x by $\kappa_y^{-1}(x)$ in (2) we also obtain that $x \circ y = \kappa_y^{-1}(x) \square y$ for every $x, y \in V$, and so

$$\overline{a} \circ \overline{a}x = \kappa_{\overline{a}}^{-1}(\overline{a}x) \square \overline{a}.$$

We also have that $\kappa_{\overline{a}}^{-1}(\overline{a}x) = \overline{a}x(\boxminus\overline{a}) \circ \overline{a}x = \overline{a}(x(\boxminus\overline{a}) \circ x)$ lies in $\overline{a}V$. Then $W = \overline{a} \square \overline{a}V$. Hence, by Lemma 2.1, $\mathrm{Im}(D_a(\rho, \square))$ is a $\square$-coset of V for any $a \in U$, which contradicts the assumption that ρ is AC with respect to $\square$. $\square$

3 Regularity-based block ciphers over $\mathbb{F}_p$

We introduce a block cipher defined over a finite field $\mathbb{F}_p$, where the key action is not necessarily defined via the usual translations on the message space V, rather it can be abelian regular group acting on V. Indeed there are many block cipher where the keys act in a less traditional way (e.g. GOST [16], SAFER [20], Kalyna [22]).

Let $\mathscr{C} = \{\varphi_k \mid k \in \mathscr{K}\}$ be a block cipher for which the plaintext space $V = (\mathbb{F}_p)^d$, for some $d \in \mathbb{N}$, coincides with the ciphertext space, where any encryption function $\varphi_k \in \mathrm{Sym}(V)$ and $\mathscr{K}$ is the key space. Suppose that any φ_k is the composition of l round functions, that is, permutations $\varphi_{k,1}, \ldots, \varphi_{k,l}$, where each $\varphi_{k,h}$ is determined by a session key $k \in \mathscr{K}$ and the round index h. For each h define the group generated by the h-round functions

$$\Gamma_h(\mathscr{C}) = \langle \varphi_{k,h} \mid k \in \mathscr{K} \rangle \leq \mathrm{Sym}(V),$$

and the group generated by all $\Gamma_h(\mathscr{C})$'s

$$\Gamma_\infty(\mathscr{C}) = \langle \Gamma_h(\mathscr{C}) \mid h = 1, \ldots, l \rangle = \langle \varphi_{k,h} \mid k \in \mathscr{K}, h = 1, \ldots, l \rangle$$

Clearly $\Gamma_h(\mathscr{C}) \subseteq \Gamma_\infty(\mathscr{C})$ for each h.
In the literature, "round" often refers either to the "round index" or to the "round function".

We also assume that V is the Cartesian product

$$V = V_1 \times \cdots \times V_n \tag{4}$$

where $n > 1$, and the V_i's are subspaces of V with $\dim_{\mathbb{F}_p}(V_i) = m > 1$, for each $i \in \{1, \ldots, n\}$, so $d = mn$.

Let $T = T_1 \times \ldots \times T_n$ be an abelian subgroup of $G = \Gamma_\infty(\mathscr{C})$ such that T_i is abelian and acts regularly on V_i. Thanks to the results in the previous section, we have $T_i = T_{V_i} = \{\sigma_{i,v} \mid v \in V_i\}$ and there is an operation $\square_i$ on V_i such that $(V_i, \square_i)$ is an abelian group and T_i is isomorphic to $(V_i, \square_i)$. So we can define the regular action of T on V as the following parallel action

$$\sigma_v(w) = w \square v = (w_1 \square_1 v_1, \ldots, w_n \square_n v_n)$$

where $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ with $v_i, w_i \in V_i$.

Definition 3.1. *An element $\gamma \in \mathrm{Sym}(V)$ is called a* bricklayer transformation *with respect to* (4) *if γ acts on an element $v = (v_1, \cdots, v_n)$, with $v_i \in V_i$, as*

$$\gamma(v) = (\gamma_1(v_1), \cdots, \gamma_n(v_n)),$$

for some $\gamma_i \in \mathrm{Sym}(V_i)$. Each γ_i is called a brick.

Clearly any $\gamma \in T$ is a bricklayer transformation. Now we generalise the definition of translation based cipher $\mathscr{C}$ over $\mathbb{F}_p$ (Definition 3.3 in [1]) to include more general key actions.

Definition 3.2. *A block cipher $\mathscr{C} = \{\varphi_k \mid k \in \mathscr{K}\}$ over $\mathbb{F}_p$ is called* regularity based (rb) *if each encryption function φ_k is the composition of l round functions $\varphi_{k,h}$, for $k \in \mathscr{K}$, and $h = 1, \ldots, l$, where in turn each round function $\varphi_{k,h}$ can be written as a composition $\sigma_{\phi(k,h)}\lambda_h\gamma_h$ of three permutations acting on V, that is,*

- γ_h *is a bricklayer transformation not depending on* k *and* $\gamma_h(0) = 0$,
- λ_h *is a group automorphism of* $(V, \square)$ *not depending on* k,
- $\phi : \mathscr{K} \times \{1, \ldots, l\} \to V$ *is the key scheduling function, so that* $\phi(k,h)$ *is the* h-round key, *given the session key* k;
- *for at least one round index* h_0 *we have that the map* $\mathscr{K} \to V$ *given by* $k \mapsto \phi(k, h_0)$ *is surjective, that is, every element of* V *occurs as an* h_0*-round key.*

Remark 3.3. *A translation based cipher is a particular case of an rb cipher when* $T = (T, +)$ *is the usual translation group. Note however that we drop the assumption on the properness of the mixing layer, since it has to be considered only when dealing with a primitive action.*

We now work in the group $\Gamma_h(\mathscr{C})$, for a fixed h, omitting for simplicity the indices h for the various functions. Write $\rho = \lambda\gamma$. Since for h_0 the map $\mathscr{K} \to V$ given by $k \mapsto \phi(k, h_0)$ is surjective, note that

$$\Gamma_{h_0}(\mathscr{C}) = \langle \rho, T \rangle. \tag{5}$$

We are ready to prove the main result of this paper.

Theorem 3.4. *If* $\mathscr{C}$ *is an rb cipher and* γ_i *is AC with respect to* $\square_i$ *for any* $i \in \{1, \ldots, n\}$, *then* $\Gamma_{h_0}(\mathscr{C})$ *and* $\Gamma_\infty(\mathscr{C})$ *are not contained in any copy of the affine group of* $\mathrm{Sym}(V)$.

Proof. By contradiction $G = \Gamma_{h_0}(\mathscr{C})$ is contained in a copy $\mathrm{AGL}(V, \circ)$ of the affine group of $\mathrm{Sym}(V)$. By (5) we are under Assumption 1 and by the proof of Theorem 2.5 we obtain that if $a \neq 0$ and $a \in U \neq \{0\}$, with

$$U = \{y \in V \mid x \square y = x \circ y \text{ for all } x \in V\},$$

then $\mathrm{Im}(D_a(\rho, \square))$ is a $\square$-coset of V.

Since λ is an automorphism of $(V, \square)$, applying λ^{-1} to $\mathrm{Im}(D_a(\rho, \square))$, by definition of $\square$ and γ we obtain

$$\{\gamma(x \square a) \boxminus \gamma(x) \mid x \in V\} = c \square bV = (c_1 \square b_1 V_1) \times \ldots \times (c_n \square b_n V_n)$$

for some $b = (b_1, \ldots, b_n)$ and $c = (c_1, \ldots, c_n)$ in V.
Choosing one non-zero $a \in U$, then there is a non-zero component $a_i \in V_i$ of a, and we have that the projection

$$\{\gamma_i(x_i \square a_i) \boxminus \gamma_i(x_i) \mid x_i \in V_i\}$$

is a $\square_i$-coset of V_i. So we obtain a contradiction since γ_i is AC.

□

4 On anti crooked functions

In this section we consider the interesting case $p = 2$, and we give some properties on the anti-crookedness of a Boolean function with respect to the usual structure $(V, +)$.

Let $\mathbb{F} = \mathbb{F}_2$. Let $m \geq 1$, any vectorial Boolean function (vBf) f from $\mathbb{F}^m$ to $\mathbb{F}^m$ can be expressed uniquely as a univariate polynomial in $\mathbb{F}_{2^m}[x]$. When f is also invertible we call it a vBf permutation. We denote the derivative $D_a f = D_a(f, +)$

Definition 4.1. *Let* $m, n \geq 1$. *Let* f *be a vBf from* $\mathbb{F}^m$ *to* $\mathbb{F}^n$, *for any* $a \in \mathbb{F}^m$ *and* $b \in \mathbb{F}^n$ *we define*

$$\delta_f(a,b) = |\{x \in \mathbb{F}^m \mid D_a f(x) = b\}|.$$

The differential uniformity *of* f *is*

$$\delta(f) = \max_{a \in \mathbb{F}^m} \max_{b \in \mathbb{F}^n a \neq 0} \delta_f(a,b).$$

f is said δ-differential uniform *if* $\delta = \delta(f)$.
Those functions such that $\delta(f) = 2$ *are said* almost perfect nonlinear (APN).

We restrict from now on to the case $m = n$, any times we write that f is a vBf, we will implicit mean $f : \mathbb{F}^m \to \mathbb{F}^m$.

We recall the following definition presented recently in [6].

Definition 4.2. *Let f be a vBf. f is* weakly-APN *if*

$$|\mathrm{Im}(D_a f)| > \frac{2^{m-1}}{2}, \quad \forall a \in \mathbb{F}^m \setminus \{0\}.$$

The notion of weakly-APN function was introduced as a necessary condition to avoid a subtle trapdoor coming from imprimitive actions (see [6]).

A direct check shows that an APN function is a weakly-APN. However, functions that are weakly-APN but not APN are of interest as shown in the following.

Theorem 4.3. *Let f be a vBf on $\mathbb{F}_{2^m}$ that is weakly-APN but not APN. Then, there exists $a \in \mathbb{F}_{2^m}$ nonzero such that* $\mathrm{Im}(D_a f)$ *is not a coset of a subspace $W \subseteq \mathbb{F}_{2^m}$.*

Proof. By contradiction suppose that for all $a \neq 0$ we have $\mathrm{Im}(D_a f) = w + W$ for some $w \in \mathbb{F}_{2^m}$ and W vector space. Since f is weakly-APN, $|\mathrm{Im}(D_a f)|$ is strictly larger than 2^{m-2}, thus $|W| \geq 2^{m-1}$ and $\dim_{\mathbb{F}}(W) = m-1$. But then $D_a f$ is a 2-to-1 function for all $a \neq 0$, which means that f is APN, and this contradicts our hypothesis. In other words, there exists a such that $\mathrm{Im}(D_a f)$ is not a coset. □

Consider the following lemma for a power function (not necessarily a permutation).

Lemma 4.4. *Let us consider $\mathbb{F}_{2^m}$ as a vector space over $\mathbb{F}$. Let $f(x) = x^d$. If there exists $a \in \mathbb{F}_{2^m}$, $a \neq 0$, such that* $\mathrm{Im}(D_a f)$ *is a coset of a subspace of $\mathbb{F}_{2^m}$, then* $\mathrm{Im}(D_{a'} f)$ *is a coset of subspace of $\mathbb{F}_{2^m}$ for all $a' \neq 0$.*

Proof. We have $\mathrm{Im}(D_a f) = w + W$ where W is a $\mathbb{F}$-vector subspace of $\mathbb{F}_{2^m}$ for some $w \in \mathbb{F}_{2^m}$. Now, let $a' \in \mathbb{F}_{2^m}$, $a' \neq 0$, we have

$$D_{a'} f(x) = (x+a')^d + x^d = \left(\frac{a'}{a}\right)^d \left[\left(x\frac{a}{a'} + a\right)^d + \left(x\frac{a}{a'}\right)^d\right] = \left(\frac{a'}{a}\right)^d D_a f\left(x\frac{a}{a'}\right).$$

So we have $\mathrm{Im}(D_{a'} f) = \left(\frac{a'}{a}\right)^d \mathrm{Im}(D_a f) = \left(\frac{a'}{a}\right)^d w + \left(\frac{a'}{a}\right)^d W = w' + W'$. Since $W' = (a'/a)^d W$ is again an $\mathbb{F}$-vector subspace of $\mathbb{F}_{2^m}$, our claim is proved. □

Thanks to Lemma 4.4, for power functions we can strengthen Theorem 4.3.

Corollary 4.5. *Let f be a vBf permutation on $\mathbb{F}_{2^m}$ that is weakly-APN but not APN. If $f(x) = x^d$, then f is AC.*

Remark 4.6. *Given an arbitrary vBf there are three possible cases: f is either crooked or anti-crooked or neither. However, Lemma 4.4 shows that for a power function there are only* ***two*** *possible cases: f is either crooked or anti-crooked.*

We want now to investigate condition that guaranty the anti-crookedness of a Boolean function.

A vBf can also be represented by m Boolean functions of m variables, the combinations of those functions are called components. We denote by $< f, v >$ the combination corresponding to v. We recall the following non-linearity measure, as introduced in [12]:

$$\hat{n}(f) := \max_{a \in \mathbb{F}^m \setminus \{0\}} |\{v \in \mathbb{F}^m \setminus \{0\} : \deg(< D_a f, v >) = 0\}|.$$

For all $a \in \mathbb{F}^m \setminus \{0\}$, let V_a be the vector space

$$V_a = \{v \in \mathbb{F}^m \setminus \{0\} : \deg(< D_a f, v >) = 0\} \cup \{0\}.$$

By definition, if $t = \max_{a \in \mathbb{F}^m \setminus \{0\}} \dim(V_a)$, then $\hat{n}(f) = 2^t - 1$.

Proposition 4.7. *Let f be a vBf and $a \in \mathbb{F}^m \setminus \{0\}$. Then $f(a) + V_a^{\perp}$ is the smallest affine subspace of $\mathbb{F}^m$ containing $\mathrm{Im}(D_a f)$. In particular, $\hat{n}(f) = 0$ if and only if for any $a \in \mathbb{F}^m \setminus \{0\}$ there is no proper affine subspace of $\mathbb{F}^m$ containing $\mathrm{Im}(D_a f)$.*

Proof. Let $a \in \mathbb{F}^m \setminus \{0\}$. Note that $V_a = \{v \in \mathbb{F}^m :< D_a f, v >$ is constant$\}$. Let $x \in \mathbb{F}^m$, then $D_a f(x) = f(a) + w$, for some $w \in \mathbb{F}^m$, and $< D_a f(x), v >= c \in \mathbb{F}$ for all $v \in V_a$. In particular $c =< D_a f(0), v >=< f(a), v >$ and so $< w, v >= 0$, that is, $w \in V_a^{\perp}$. Then we have $\mathrm{Im}(D_a f) \subseteq f(a) + V_a^{\perp}$. Now, let A be an affine subspace containing $\mathrm{Im}(D_a f)$, then $A = f(a) + V$, for some vector subspace V in $\mathbb{F}^m$. For all $v \in V^{\perp}$, we have $< D_a f, v >=< f(a), v >= c \in \mathbb{F}$ and so, by definition, $V^{\perp} \subseteq V_a$. Then A contains $f(a) + V_a^{\perp}$.
Finally, $\hat{n}(f) = 0$ if and only if $V_a = \{0\}$ for all $a \in \mathbb{F}^m \setminus \{0\}$, and so our claim follows. □

Obviously, for any affine subspace W, $\mathrm{Im}(D_a f) \not\subset W \implies \mathrm{Im}(D_a f) \neq W$ and so we have the next corollary.

Corollary 4.8. *Let f be a vBf. If $\hat{n}(f) = 0$ then f is AC.*

Coming back to power functions it is important to recall a result by Kyureghyan.

Theorem 4.9 ([18])**.** *The only crooked APN power functions in $\mathbb{F}_{2^n}$ are those with exponent $2^i + 2^j$, $\gcd(i - j, n) = 1$.*

Recalling that the known exponents of APN power functions (up to factor 2^i) are

$$2^k + 1, \quad \gcd(k, m) = 1 \quad \text{(Gold's exponent [2, 13])}$$
$$2^{2k} - 2^k + 1, \quad \gcd(k, m) = 1 \quad \text{(Kasami's exponent [17])}$$
$$2^{4k} + 2^{3k} + 2^{2k} + 2^k - 1, \quad m = 5k \quad \text{(Dobbertin's functions [11])}$$

if $m = 2l + 1$ also

$$2^l + 3 \quad \text{(Welch's exponent [9, 4, 14])}$$
$$2^l + 2^{\frac{l}{2}} - 1 \quad \text{if } l \text{ is even and}$$
$$2^l + 2^{\frac{3l+1}{2}} - 1 \quad \text{if } l \text{ is odd} \quad \text{(Niho's exponent [10, 14])}$$
$$2^m - 2 \quad \text{(patched inversion [21])}$$

This implies that the only crooked power functions, among the known maps, are those with Gold's exponent. Thanks to Remark 4.6 we have:

Corollary 4.10. *Let x^d be one of the APN power functions above, with d not a Gold's exponent, then x^d is AC. In particular the power function x^{2^m-2} is AC for all $m \geq 3$.*

Proof. It follows directly from Lemma 4.4 and the theorem above. For the case of the patched inversion, from Corollary 4.5, it is AC also in even dimension. □

Having examined some anti-crooked functions we would like to show some properties of this notion.

Lemma 4.11. *If f is AC then f^{-1} is not necessarily AC.*

Proof. We provide an explicit example $f : \mathbb{F}^6 \to \mathbb{F}^6$ defined by $f(x) = x^{49}$, then $f^{-1}(x) = x^5$. A computer check shows that f is anti-crooked while f^{-1} is not. In particular, $\mathrm{Im}(D_{e^6}(f^{-1}))$ is an affine subspace of dimension 4, where e is a primitive element of $\mathbb{F}_{64}$ such that $e^6 = e^4 + e^3 + e + 1$. □

We recall that two vBf's f and f' are called CCZ-equivalent if their graphs $G_f = \{(x, f(x)) \mid x \in \mathbb{F}^n\}$ and $G_{f'} = \{(x, f'(x)) \mid x \in \mathbb{F}^n\}$ are affine equivalent. We recall also that f and f' are called EA-equivalent if there exist three affine functions g, g' and g'' such that $f' = g' \circ f \circ g + g''$.
Lemma 4.11 and the well-known fact that a vBf f is CCZ-equivalent to f^{-1} imply the following result.

Corollary 4.12. *The anti-crookedness is not CCZ invariant.*

On the other hand, surprisingly anti-crookedness behaves well with EA invariance, as shown below.

Proposition 4.13. *The anti-crookedness is EA invariant.*

Proof. Let f be a vBf anti-crooked, and let g be a vBf such that f and g are EA equivalent. Then, there exist three affinities $\lambda_1, \lambda_2, \lambda_3$ such that $g = \lambda_1 f \lambda_2 + \lambda_3$. Without loss of generality we can suppose $f(0) = g(0) = 0$ and $\lambda_i(0) = 0$ for all $i = 1, 2, 3$. Then

$$\begin{aligned} D_a g &= \lambda_1 f \lambda_2(x+a) + \lambda_1 f \lambda_2(x) + \lambda_3(x+a) + \lambda_3(x) \\ &= \lambda_1(f(\lambda_2(x) + \lambda_2(a)) + f(\lambda_2(x)) + \lambda_1^{-1}\lambda_3(a)), \end{aligned}$$

which implies

$$\mathrm{Im}(D_a g) = \lambda_1(\lambda_1^{-1}\lambda_3(a) \mathrm{Im}(D_a f)),$$

thus g is AC if and only if f is AC. □

5 A block cipher with a hidden sum

In this section we give an example, in a small dimension, of a translation based block cipher in which it is possible to embed a hidden-sum trapdoor. The underlying field is binary and all involved functions are vBf's.

Let $m = 3$, $n = 2$, then $d = 6$ and we have the message space $V = (\mathbb{F}_2)^6$. The mixing layer of our toy cipher is given by the matrix

$$\lambda = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Note that λ is a proper mixing layer (see Definition 3.2 [1]). The bricklayer transformation $\gamma = (\gamma_1, \gamma_2)$ of our toy cipher is given by two identical S-boxes

$$\gamma_1 = \gamma_2 = \alpha^5 x^6 + \alpha x^5 + \alpha^2 x^4 + \alpha^5 x^3 + \alpha x^2 + \alpha x$$

where α is a primitive element of $\mathbb{F}_{2^3}$ such that $\alpha^3 = \alpha + 1$.
The S-box γ_1 is 4-differential uniform but it is not anti-crooked, since $\mathrm{Im}(D_{\alpha^2}\gamma_1)$ is an affine subspace (of dimension 1).

We show now the existence of a hidden-sum trapdoor for our toy cipher. We consider the hidden sum $\circ$ over $V_1 = V_2 = (\mathbb{F}_2)^3$ induced by the elementary abelian regular group $T_\circ = \langle \tau_1, \tau_2, \tau_3 \rangle$, where

$$\tau_1(x) = x \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + e_1,\ \tau_2(x) = x \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + e_2,\ \tau_3(x) = x \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + e_3, \quad (6)$$

with $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. In other words, $\tau_i(x) = x \circ e_i$ for any $1 \leq i \leq 3$. Obviously $T = T_\circ \times T_\circ$ is an elementary abelian group inducing the hidden sum $(x_1,x_2) \circ' (y_1,y_2) = (x_1 \circ y_1, x_2 \circ y_2)$ on $V = V_1 \times V_2$. We denote
$e'_1 = (1,0,0,0,0,0), \ldots, e'_6 = (0,0,0,0,0,1)$ and we claim the following.

Theorem 5.1. *$\langle T_+, \lambda\gamma \rangle \subseteq AGL(V, \circ')$, where T_+ is the translation group with respect to $+$.*

Proof. By a computer check it results that $\sigma_{e'_i} \in \mathrm{AGL}(V, \circ')$ for all $1 \leq i \leq 6$ where $\sigma_{e'_i}(x) = x + e'_i$. Since these generate T_+, this implies $T_+ \subseteq \mathrm{AGL}(V, \circ')$.
By another computer check the map $\lambda\gamma$ lies in $\mathrm{AGL}(V, \circ')$ □

Thanks to the previous theorem, $\circ'$ is a hidden sum for our toy cipher, but it remains to verify whether it is possible to use it to attack the toy cipher with an attack that costs less than brute force. We have not discussed the key schedule and the number of rounds yet. We have in mind a cipher where the number of rounds is so large to make any classical attack useless (such as differential cryptanalysis) and the key scheduling offer no weakness. Therefore, the hidden sum will actually be essential to break the cipher only if the attack that we build will cost significantly less than 64 encryptions, considering that the key space is $(\mathbb{F}_2)^6$.

Remark 5.2. *Given a sum $\square$, the vectors e_1, e_2, e_3 may not be a linear basis of $(V_1, \square)$. For this specific sum $\circ$, the vectors e_1, e_2, e_3 actually form a basis for $(V_1, \circ)$ as can be checked by computer. Let $x = (x_1, x_2, x_3) \in V_1$, from* (6) *we can simply write*

$$\tau_1(x) = (x_1 + 1, x_2 + x_3, x_3),\ \tau_2(x) = (x_1, x_2 + 1, x_3),\ \tau_3(x) = (x_1, x_1 + x_2, x_3 + 1).$$

Let us write x as a linear combination of e_1, e_2 and e_3 w.r.t. to the sum $\circ$, i.e. $x = \lambda_1 e_1 \circ \lambda_2 e_2 \circ \lambda_3 e_3$. We claim that $\lambda_1 = x_1$, $\lambda_3 = x_3$ and $\lambda_2 = \lambda_1\lambda_3 + x_2$. To show that, we adopt the following notation: if $b \in \mathbb{F}_2$, we can write $\tau_v^b(x)$ denoting either $\tau_v(x)$ (when $b = 1$) or x (when $b = 0$). Then

$$\begin{aligned} x &= (\lambda_1 e_1 \circ \lambda_2 e_2) \circ \lambda_3 e_3 = \tau_3^{\lambda_3}(\lambda_1 e_1 \circ \lambda_2 e_2) = \tau_3^{\lambda_3}(\tau_2^{\lambda_2}(\lambda_1 e_1)) = \tau_2^{\lambda_2}(\tau_3^{\lambda_3}(\lambda_1 e_1)) \\ &= \tau_2^{\lambda_2}(\tau_3^{\lambda_3}((\lambda_1, 0, 0))) = \tau_2^{\lambda_2}((\lambda_1, \lambda_3\lambda_1, \lambda_3)) = (\lambda_1, \lambda_1\lambda_3 + \lambda_2, \lambda_3). \end{aligned}$$

So

$$(x_1, x_2, x_3) = x = (\lambda_1, \lambda_1\lambda_3 + \lambda_2, \lambda_3)$$

and our claim is proved.

Thanks to the previous remark we can find the coefficients of a vector $v' = (v, u) \in V$ with respect to $\circ'$ by using the following algorithm separately on the two bricks of w.

Algorithm 1.
Input: *vector $x \in \mathbb{F}_2^3$*
Output: *coefficients λ_1, λ_2 and λ_3.*
[1] $\lambda_1 \leftarrow x_1$;
[2] $\lambda_3 \leftarrow x_3$;
[3] $\lambda_2 \leftarrow \lambda_1\lambda_3 + x_2$;
return $\lambda_1, \lambda_2, \lambda_3$.

Let $v' = (v,u) \in V$, we write

$$v = \lambda_1^v e_1 \circ \lambda_2^v e_2 \circ \lambda_3^v e_3 \text{ and } u = \lambda_1^u e_1 \circ \lambda_2^u e_2 \circ \lambda_3^u e_3.$$

We denote by

$$[v'] = [\lambda_1^v, \lambda_2^v, \lambda_3^v, \lambda_1^u, \lambda_2^u, \lambda_3^u]$$

the vector with the coefficients obtained from the bricks of v using Algorithm 1.

Let $\varphi = \varphi_k$ be the encryption function, with a given unknown session key k. We want to mount two attacks by computing the matrix M and the translation vector t defining $\varphi \in \mathrm{AGL}(V, \circ')$, which exist thanks to Theorem 5.1.
Assume we can call the encryption oracle. Then M can be computed from the 7 ciphertexts $\varphi(0), \varphi(e_1'), \ldots, \varphi(e_6')$, since the translation vector is $[t] = [\varphi(0)]$ and the $[\varphi(e_i')] + [t]$'s represent the matrix rows. In other words, we will have

$$[\varphi(v')] = [v'] \cdot M + [t], \quad [\varphi^{-1}(v')] = ([v'] + [t]) \cdot M^{-1},$$

for all $v' \in V$, where the product row by column is the standard scalar product. The knowledge of M and M^{-1} provides a global deduction (reconstruction), since it becomes trivial to encrypt and decrypt. However, we have an alternative depending on how we compute φ^{-1}:

- if we compute M^{-1} from M, by applying for example Gaussian reduction, we will need only our 7 initial encryptions;
- else we can compute M^{-1} from the action of ϕ^{-1}, assuming we can call the decryption oracle, simply by performing the 7 decryptions $\varphi^{-1}(e_i')$ and $\varphi^{-1}(0)$; indeed, the rows of M^{-1} will obviously be $[\varphi^{-1}(e_i')] + [\varphi^{-1}(0)]$.

The first attack requires more binary operations, since we need a matrix inversion, but only 7 encryptions. The second attack requires both 7 encryptions and 7 decryptions, but less binary operations. The first attack is a chosen-plaintext attack, while the second is a chosen-plaintext/chosen-ciphertext attack. Both obtain the same goal, that is, the complete reconstruction of the encryption and decryption functions. Note that, since an encryption/decryption will cost a huge number of binary operations in our assumptions (we are supposing that many rounds are present), the first attack is more dangerous and its cost is approximately that of 7 encryptions, while the cost of the second attack is approximately 14 encryptions (being the cost of an encryption close to the cost of a decryption).

References

[1] R. Aragona, A. Caranti, F. Dalla Volta, M. Sala, *On the group generated by the round functions of translation based ciphers over arbitrary finite fields*, Finite Fields and Their Applications 25 (2014), 293–305.

[2] T. Bending, D. Fon-Der-Flaass, *Crooked functions, bent functions, and distance regular graphs*, Electron. J. Comb. 5 (1998), 1–4.

[3] E. Biham, A.Shamir *Differential cryptanalysis of DES-like cryptosystems*. J. Cryptol 4(1) (1991), 3–72.

[4] A. Canteaut, P. Charpin, H. Dobbertin, *Binary m-sequences with three-valued crosscorrelation: a proof of Welch's conjecture*,IEEE Trans.Inform.Theory 46 (2000), 4–8.

[5] A. Caranti, F. Dalla Volta, M. Sala, *Abelian regular subgroups of the affine group and radical rings*, Publ. Math. (Debr.) 69 (3) (2006), 297-308.

[6] A. Caranti, F. Dalla Volta, and M. Sala, *On some block ciphers and imprimitive groups*, Appl. Algebra Engrg. Comm. Comput. 20 (5-6) (2009), 339–350.

[7] A. Caranti, F. Dalla Volta, M. Sala, An application of the O'Nan-Scott theorem to the group generated by the round functions of an AES-like cipher, Des. Codes Cryptogr. 52 (3) (2009), 293–301.

[8] S. Featherstonhaugh, A. Caranti, L. Childs. *Abelian Hopf Galois structures on prime-power Galois field extensions*, Trans. Amer. Math. Soc. 364 (7) (2012), 3675–3684.

[9] H. Dobbertin, *Almost perfectly nonlinear power functions on GF*(2^n)*: the Welch case*, IEEE Transactions on Information Theory 45 (1999), 1271–1275.

[10] H. Dobbertin, *Almost perfectly nonlinear power functions on GF*(2^n)*: the Niho case*, Information and Computation 151 (1999), 57–72.

[11] H. Dobbertin, *Almost perfect nonlinear functions on GF*(2^n)*: a new case for n divisible by 5*, in Finite Fields and Applications, D. Jungnickel and H.Niederreiter (Eds.), Springer, Berlin 2001, 113–121.

[12] C. Fontanari, V. Pulice, A. Rimoldi, M. Sala, *On weakly APN function and 4-bit S-boxes*, Finite Fields and Appl. 18 (2012), 522–528.

[13] R. Gold, *Maximal recursive sequences with 3-valued recursive crosscorrelation functions*, IEEE Trans. Inform. Theory 14 (1968), 154–156.

[14] H. D. L. Hollmann, Q. Xing, *A proof of the Welch and Niho conjectures on cross-correlations of binary m-sequences*, Finite Fields Appl. 7 (2001), 253–286.

[15] J. E. Humphreys, *Linear algebraic groups*. Vol. 430, Springer,1975.

[16] IETF, *RFC 5830: GOST 28147-89 encryption, decryption and MAC algorithms*, March 2010.

[17] T. Kasami,*The weight enumerators for several classes of subcodes of the second order binary Reed- Muller codes*, Inform. Control. 18 (1971), 369–394.

[18] G. Kyureghyan, *Crooked maps in* $\mathbb{F}_{2^n}$, Finite Fields and their applications 13 (3) (2007), 713–726.

[19] C. H. Li, *The finite primitive permutation groups containing an abelian regular subgroup*, Proc. Lond. Math. Soc. 87 (3) (2003), 725–747.

[20] J. L. Massey, *SAFER K-64: A byte-oriented block-ciphering algorithm*, Fast Software Encryption, LNCS vol. 809, (1994), 1–17.

[21] K. Nyberg, *Differentially uniform mappings for cryptography*, Advances in Cryptology, EUROCRYPT '93, Lecture Notes in Computer Science 765 (1994), 55–64.

[22] R. Oliynykov, I. Gorbenko, V. Dolgov, V. Ruzhentsev, *Results of Ukrainian national public cryptographic competition*, Tatra Mountains Mathematical Publications, 47(1) (2010), 99–113.

[23] K. G. Paterson, *Imprimitive permutation groups and trapdoors in interated block ciphers*, in: Fast Software Encryption, in: LNCS, vol. 1636, Springer, Berlin, (1999), 201–214.