SEIDEL ELEMENTS AND MIRROR TRANSFORMATIONS FOR TORIC STACKS

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Abstract. We give a precise relation between the mirror transformation and the Seidel elements for weak Fano toric Deligne-Mumford stacks. Our result generalizes the corresponding result for toric varieties proved by González and Iritani in [5].

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Contents

1. Introduction

In [5], González and Iritani gave a precise relation between the mirror map and the Seidel elements for a smooth projective weak Fano toric variety X. The goal of this paper is to generalize the main theorem of [5] to a smooth projective weak Fano toric Deligne-Mumford stack $\mathcal{X}$.

Let $\mathcal{X}$ be a smooth projective weak Fano toric Deligne-Mumford stack, the mirror theorem can be stated as an equality between the I-function and the J-function via a change of coordinates, called mirror map (or mirror transformation). We refer to [3] and section 4.1 of [6] for further discussions.

Key words and phrases. Seidel elements, mirror transformations, Batyrev relations, Weak Fano, toric Deligne-Mumford stacks.

Let Y be a monotone symplectic manifold. For a loop λ in the group of Hamiltonian symplectomorphisms on Y, Seidel [10] constructed an invertible element $S(\lambda)$ in (small) quantum cohomology counting sections of the associated Hamiltonian Y-bundle $E_\lambda \to \mathbb{P}^1$. The Seidel element $S(\lambda)$ defines an element in $Aut(QH(Y))$ via quantum multiplication and the map $\lambda \mapsto S(\lambda)$ gives a representation of $\pi_1(Ham(Y))$ on $QH(Y)$. The construction was extended to all symplectic manifolds by McDuff and Tolman in [9]. Let $D_1, \ldots, D_m$ be the classes in $H^2(X)$ Poincaré dual to the toric divisors. When the loop λ is a circle action, McDuff and Tolman [9] considered the Seidel element $\tilde{S}_j$ associated to an action λ_j that fixes the toric divisor D_j. The definition of Seidel representation and Seidel element were extended to symplectic orbifolds by Tseng-Wang in [11].

Given a circle action on X (resp. $\mathcal{X}$), the Seidel element in [5] (resp. [11]) is defined using the small quantum cohomology ring. In this paper, we need to define it, for smooth projective Deligne-Mumford stack, with deformed quantum cohomology to include the bulk deformations. For weak Fano toric Deligne-Mumford stack, the mirror theorem in [6] shows that the mirror map $\tau(y) \in H^{\leq 2}_{orb}(\mathcal{X})$, therefore, we will only need bulk deformations with $\tau \in H^{\leq 2}_{orb}(\mathcal{X})$.

We consider the Seidel element $\tilde{S}_j$ associated to the toric divisor D_j as well as the Seidel element $\tilde{S}_{m+j}$ corresponding to the box element s_j. The Seidel element in definition 2.2 shows that $S = q_0\tilde{S}$ is a pull-back of a coefficient of the J-function $J_{\mathcal{E}_j}$ of the associated orbifiber bundle $\mathcal{E}_j$, hence we can use the mirror theorem for $\mathcal{E}_j$ to calculate $\tilde{S}_j$ when $\mathcal{E}_j$ is weak Fano.

We extend the definition of the Batyrev element $\tilde{D}_j$ to weak Fano toric Deligne-Mumford stacks via partial derivatives of the mirror map $\tau(y)$. As analogues of the Seidel elements in B-model, the Batyrev elements can be explicitly computed from the I-function of $\mathcal{X}$. The following theorem states that the Seidel elements and the Batyrev elements only differ by a multiplication of a correction function.

Theorem 1.1. *Let X be a smooth projective toric Deligne-Mumford stack with $\rho^S \in cl\left(C^S(\mathcal{X})\right)$.*

(i) the Seidel element $\tilde{S}_j$ associated to the toric divisor D_j is given by

$$\tilde{S}_j\left(\tau(y)\right) = exp\left(-g_0^{(j)}(y)\right)\tilde{D}_j(y)$$

where $\tau(y)$ is the mirror map of $\mathcal{X}$ and the function $g_0^{(j)}$ is given explicitly in (40);

(ii) the Seidel element $\tilde{S}_{m+j}$ corresponding to the box element s_j is given by

$$\tilde{S}_{m+j}\left(\tau(y)\right) = exp\left(-g_0^{(m+j)}\right) y^{-D^{S\vee}_{m+j}}\tilde{D}_{m+j}(y),$$

where $\tau(y)$ is the mirror map of $\mathcal{X}$ and the function $g_0^{(m+j)}$ is given explicitly in (52).

It appears that the correction coefficients in the above theorem coincide with the instanton corrections in theorem 1.4 in [2]. This phenomenon also indicates the deformed quantum cohomology of the toric Deligne-Mumford stack $\mathcal{X}$ is isomorphic to the Batyrev ring given in [6].

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2. Seidel Elements and J-functions

2.1. Generalities.

In this section, we will fix our notation and construct the Seidel elements of smooth projective Deligne-Mumford stacks using τ-deformed quantum cohomology.

Let $\mathcal{X}$ be a smooth projective Deligne-Mumford stack, equipped with a $\mathbb{C}^\times$ action.

Definition 2.1. The associated orbifiber bundle of the $\mathbb{C}^\times$-action is the $\mathcal{X}$-bundle over $\mathbb{P}^1$

$$\mathcal{E} := \mathcal{X} \times \left(\mathbb{C}^2 \setminus \{0\}\right) / \mathbb{C}^\times \to \mathbb{P}^1,$$

where $\mathbb{C}^\times$ acts on $\mathbb{C}^2 \setminus \{0\}$ via the standard diagonal action.

Let $\phi_1, \ldots, \phi_N$ be a basis for the orbifold cohomology ring $H^*_{orb}(\mathcal{X}) := H^*(\mathcal{I}\mathcal{X}; \mathbb{Q})$ of $\mathcal{X}$, where $\mathcal{I}\mathcal{X}$ is the inertia stack of $\mathcal{X}$. Let $\phi^1, \ldots, \phi^N$ be the dual basis of $\phi_1, \ldots, \phi_N$ with respect to the orbifold Poincaré pairing. Furthermore, let $\hat{\phi}_1, \ldots, \hat{\phi}_M$ denote a basis for the orbifold cohomology $H^*_{orb}(\mathcal{E}) := H^*(\mathcal{I}\mathcal{E}; \mathbb{Q})$ of $\mathcal{E}$. Let $\hat{\phi}^1, \ldots, \hat{\phi}^M$ be the dual basis of $\hat{\phi}_1, \ldots, \hat{\phi}_M$ with respect to the orbifold Poincaré pairing.

We will use X to denote the coarse moduli space of $\mathcal{X}$ and use E to denote the coarse moduli space of $\mathcal{E}$. Then the $\mathbb{C}^\times$ action on $\mathcal{X}$ descends to the $\mathbb{C}^\times$ action on X with E being the associated bundle. Following [8] and [5], there is a (non-canonical) splitting

$$H^*(\mathcal{E}; \mathbb{Q}) \cong H^*(E; \mathbb{Q}) \cong H^*(X; \mathbb{Q}) \otimes H^*(\mathbb{P}^1; \mathbb{Q}) \cong H^*(\mathcal{X}; \mathbb{Q}) \otimes H^*(\mathbb{P}^1; \mathbb{Q}).$$

According to [5], there is a unique $\mathbb{C}^\times$-fixed component $F_{\max} \subset X^{\mathbb{C}^\times}$ such that the normal bundle of $F_{\max}$ has only negative $\mathbb{C}^\times$-weights. Let σ_0 be the section associated to a fixed point in $F_{\max}$. Following [5], there is a splitting defined by this maximal section.

(1)
$$H_2(\mathcal{E}; \mathbb{Z})/tors \cong H_2(E; \mathbb{Z})/tors \cong \mathbb{Z}[\sigma_0] \oplus (H_2(X, \mathbb{Z})/tors) \cong \mathbb{Z}[\sigma_0] \oplus (H_2(\mathcal{X}, \mathbb{Z})/tors).$$

Let $NE(X) \subset H_2(X; \mathbb{R})$ denote the Mori cone, i.e. the cone generated by effective curves and set

$$NE(X)_{\mathbb{Z}} := NE(X) \cap (H_2(X, \mathbb{Z})/tors).$$

Then, by lemma 2.2 of [5], we have

$$NE(E)_{\mathbb{Z}} = \mathbb{Z}_{\geq 0}[\sigma_0] + NE(X)_{\mathbb{Z}}. \tag{2}$$

Let $H_2^{sec}(E; \mathbb{Z})$ be the affine subspace of $H_2(E, \mathbb{Z})/tors$ which consists of the classes that project to the positive generator of $H_2(\mathbb{P}^1; \mathbb{Z})$, we set

$$NE(E)^{sec}_{\mathbb{Z}} := NE(E)_{\mathbb{Z}} \cap H_2^{sec}(E; \mathbb{Z}),$$

then we obtain

$$NE(E)^{sec}_{\mathbb{Z}} = [\sigma_0] + NE(X)_{\mathbb{Z}}. \tag{3}$$

We choose a nef integral basis $\{p_1,\dots,p_r\}$ of $H^2(\mathcal{X};\mathbb{Q})$, then there are unique lifts of $p_1,\dots,p_r$ in $H^2(\mathcal{E};\mathbb{Q})$ which vanish on $[\sigma_0]$. By abuse of notation, we also denote these lifts as $p_1,\dots,p_r$, these lifts are also nef. Let p_0 be the pullback of the positive generator of $H^2(\mathbb{P}^1;\mathbb{Z})$ in $H^2(\mathcal{E};\mathbb{Q})$. Therefore, $\{p_0,p_1,\dots,p_r\}$ is an integral basis of $H^2(\mathcal{E};\mathbb{Q})$.

Let $q_0,q_1,\dots,q_r$ be the Novikov variables of $\mathcal{E}$ dual to $p_0,p_1,\dots,p_r$ and $q_1,\dots,q_r$ be the Novikov variables of $\mathcal{X}$ dual to $p_1,\dots,p_r$. We denote the Novikov ring of $\mathcal{X}$ and the Novikov ring of $\mathcal{E}$ by

$$\Lambda_{\mathcal{X}} := \mathbb{Q}[[q_1,\dots,q_r]] \quad \text{and} \quad \Lambda_{\mathcal{E}} := \mathbb{Q}[[q_0,q_1,\dots,q_r]],$$

respectively. For each $d\in NE(X)_{\mathbb{Z}}$, we write

$$q^d := q_1^{\langle p_1,d\rangle}\cdots q_r^{\langle p_r,d\rangle}\in\Lambda_{\mathcal{X}};$$

and for each $\beta\in NE(E)_{\mathbb{Z}}$, we write

$$q^\beta := q_0^{\langle p_0,\beta\rangle}q_1^{\langle p_1,\beta\rangle}\cdots q_r^{\langle p_r,\beta\rangle}\in\Lambda_{\mathcal{E}}.$$

The τ-deformed orbifold quantum product is defined as follows:

$$\alpha\bullet_\tau\beta=\sum_{d\in NE(X)_{\mathbb{Z}}}\sum_{l\geq 0}\sum_{k=1}^{N}\frac{1}{l!}\langle\alpha,\beta,\tau,\dots,\tau,\phi_k\rangle^{\mathcal{X}}_{0,l+3,d}q^d\phi^k, \tag{4}$$

the associated quantum cohomology ring is denoted by

$$QH_\tau(\mathcal{X}) := (H(\mathcal{X})\otimes_{\mathbb{Q}}\Lambda_{\mathcal{X}},\bullet_\tau).$$

Definition 2.2. The Seidel element of $\mathcal{X}$ is the class

$$S(\hat\tau) := \sum_\alpha\sum_{\beta\in NE(E)^{sec}_{\mathbb{Z}}}\sum_{l\geq 0}\frac{1}{l!}\langle\mathbf{1},\hat\tau_{tw},\dots,\hat\tau_{tw},\imath_*\phi_\alpha\psi\rangle^{\mathcal{E}}_{0,l+2,\beta}\phi^\alpha e^{\langle\hat\tau_{0,2},\beta\rangle}, \tag{5}$$

in $QH_\tau(\mathcal{X})\otimes_{\Lambda_{\mathcal{X}}}\Lambda_{\mathcal{E}}$. Here $\imath:\mathcal{X}\to\mathcal{E}$ is the inclusion of a fiber, and

$$\imath_*: H^*(\mathcal{I}\mathcal{X};\mathbb{Q})\to H^{*+2}(\mathcal{I}\mathcal{E};\mathbb{Q})$$

is the Gysin map. Moreover,

$$e^{\langle\hat\tau_{0,2},\beta\rangle}=q^\beta=q_0^{\langle p_0,\beta\rangle}\cdots q_r^{\langle p_r,\beta\rangle},$$

where

$$\hat\tau_{0,2}=\sum_{a=0}^{r}p_a log q_a\in H^2(\mathcal{E}) \quad\text{and}\quad \hat\tau=\hat\tau_{0,2}+\hat\tau_{tw}\in H^{\leq 2}_{orb}(\mathcal{E}).$$

The Seidel element can be factorized as

$$S(\hat\tau)=q_0\tilde S(\hat\tau), \quad \text{with} \quad \tilde S(\hat\tau)\in QH_\tau(\mathcal{X}). \tag{6}$$

2.2. **J-functions.** We will explain the relation between the Seidel element and the J-function of the associated bundle $\mathcal{E}$.

Definition 2.3. The J-function of $\mathcal{E}$ is the cohomology valued function

(7)
$$J_{\mathcal{E}}(\hat\tau,z)=e^{\hat\tau_{0,2}/z}\left(1+\sum_\alpha\sum_{(\beta,l)\neq(0,0),\beta\in NE(E)_{\mathbb{Z}}}\frac{e^{\langle\hat\tau_{0,2},\beta\rangle}}{l!}\langle\mathbf{1},\hat\tau_{tw},\dots,\hat\tau_{tw},\frac{\hat\phi_\alpha}{z-\psi}\rangle^{\mathcal{E}}_{0,l+2,\beta}\hat\phi^\alpha\right),$$

where $\frac{\hat\phi_\alpha}{z-\psi}=\sum_{n\geq 0}z^{-1-n}\hat\phi_\alpha\psi^n$.

Note that when $n = 0$, we will have

(i) $\sum_\alpha \langle \mathbf{1}, \hat{\tau}_{tw}, \ldots, \hat{\tau}_{tw}, \hat{\phi}_\alpha \rangle^{\mathcal{E}}_{0,l+2,\beta} \hat{\phi}^\alpha = 0, \quad \text{for} \quad (l, \beta) \neq (1, 0);$

(ii) $\sum_\alpha \langle \mathbf{1}, \hat{\tau}_{tw}, \ldots, \hat{\tau}_{tw}, \hat{\phi}_\alpha \rangle^{\mathcal{E}}_{0,l+2,\beta} \hat{\phi}^\alpha = \hat{\tau}_{tw}, \quad \text{for} \quad (l, \beta) = (1, 0).$

The J-function can be expanded in terms of powers of z^{-1} as follows:

(8)
$$J_{\mathcal{E}}(\hat{\tau}, z) = e^{\sum_{a=0}^{r} p_a log q_a / z} \left(1 + z^{-1} \hat{\tau}_{tw} + z^{-2} \sum_{n=0}^{\infty} F_n(q_1, \ldots, q_r; \hat{\tau}) q_0^n + O(z^{-3}) \right),$$

where

$$F_n(q_1, \ldots, q_r; \hat{\tau}) = \sum_{\alpha=1}^{M} \sum_{d \in NE(X)_{\mathbb{Z}}} \sum_{l \geq 0} \frac{1}{l!} \langle \mathbf{1}, \hat{\tau}_{tw}, \ldots, \hat{\tau}_{tw}, \hat{\phi}_\alpha \psi \rangle^{\mathcal{E}}_{0,l+2,d+n\sigma_0} q^d \hat{\phi}^\alpha \tag{9}$$

Proposition 2.4. *The Seidel element corresponding to the $\mathbb{C}^\times$ action on $\mathcal{X}$ is given by*

$$S(\hat{\tau}) = \imath^* \left(F_1(q_1, \ldots, q_r; \hat{\tau}) q_0 \right). \tag{10}$$

Proof. The proof in here is identical to the proof given in proposition 2.5 of [5] for smooth projective varieties:

Using the duality identity

$$\sum_{\alpha=1}^{M} \hat{\phi}_\alpha \otimes \imath^* \hat{\phi}^\alpha = \sum_{\alpha=1}^{N} \imath_* \phi_\alpha \otimes \phi^\alpha,$$

we can see that

$$\imath^* F_1(q_1, \ldots, q_r; \hat{\tau}) = \sum_{\alpha=1}^{N} \sum_{d \in NE(X)_{\mathbb{Z}}} \sum_{l \geq 0} \frac{1}{l!} \langle \mathbf{1}, \hat{\tau}_{tw}, \ldots, \hat{\tau}_{tw}, \imath_* \phi_\alpha \psi \rangle^{\mathcal{E}}_{0,l+2,d+\sigma_0} q^d \phi^\alpha.$$

Hence, the conclusion follows, i.e.

$$S(\hat{\tau}) = \imath^* (F_1(q_1, \ldots, q_r; \hat{\tau}) q_0).$$

□

3. Seidel elements corresponding to toric divisors

3.1. **A Review of Toric Deligne-Mumford stacks.** In this section, we will define toric Deligne-Mumford stacks following the construction of [1] and [6].

A toric Deligne-Mumford stack is defined by a stacky fan $\mathbf{\Sigma} = (\mathbf{N}, \Sigma, \beta)$, where $\mathbf{N}$ is a finitely generated abelian group, $\Sigma \subset \mathbf{N}_{\mathbb{Q}} = \mathbf{N} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a rational simplicial fan, and $\beta : \mathbb{Z}^m \to \mathbf{N}$ is a homomorphism. We assume β has finite cokernel and the rank of $\mathbf{N}$ is n. The canonical map $\mathbf{N} \to \mathbf{N}_{\mathbb{Q}}$ generates the 1-skeleton of the fan Σ. Let $\bar{b_i}$ be the image of b_i under this canonical map, where b_i is the image under β of the standard basis of $\mathbb{Z}^m$. Let $\mathbb{L} \subset \mathbb{Z}^m$ be the kernel of β. Then the fan sequence is the following exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^m \xrightarrow{\beta} \mathbf{N}. \tag{11}$$

Let $\beta^\vee : (\mathbb{Z}^*)^m \to \mathbb{L}^\vee$ be the Gale dual of β in [1], where $\mathbb{L}^\vee := H^1(Cone(\beta)^*)$ is an extension of $\mathbb{L}^* = Hom(\mathbb{L}, \mathbb{Z})$ by a torsion subgroup. The divisor sequence is the following exact sequence

$$0 \longrightarrow \mathbf{N}^* \xrightarrow{\beta^*} (\mathbb{Z}^*)^m \xrightarrow{\beta^\vee} \mathbb{L}^\vee. \tag{12}$$

By applying $Hom_{\mathbb{Z}}(-, \mathbb{C}^\times)$ to the dual map $\beta^\vee$, we have a homomorphism

$$\alpha : G \to (\mathbb{C}^\times)^m, \quad \text{where} \quad G := Hom_{\mathbb{Z}}(\mathbb{L}^\vee, \mathbb{C}^\times),$$

and we let G act on $\mathbb{C}^m$ via this homomorphism.

The collection of anti-cones $\mathcal{A}$ is defined as follows:

$$\mathcal{A} := \left\{ I : \sum_{i \notin I} \mathbb{R}_{\geq 0} \bar{b}_i \in \Sigma \right\}.$$

Let $\mathcal{U}$ denote the open subset of $\mathbb{C}^m$ defined by $\mathcal{A}$:

$$\mathcal{U} := \mathbb{C}^m \setminus \cup_{I \notin \mathcal{A}} \mathbb{C}^I,$$

where

$$\mathbb{C}^I = \{(z_1, \ldots, z_m) : z_i = 0 \text{ for } i \notin I\}.$$

Definition 3.1. Following [6], the toric Deligne-Mumford stack $\mathcal{X}$ is defined as the quotient stack

$$\mathcal{X} := [\mathcal{U}/G].$$

Remark 3.2. The toric variety X associated to the fan Σ is the coarse moduli space of $\mathcal{X}$ [1].

Definition 3.3 ([6])**.** Given a stacky fan $\mathbf{\Sigma} = (\mathbf{N}, \Sigma, \beta)$, we define the set of box elements $Box(\mathbf{\Sigma})$ as follows

$$Box(\mathbf{\Sigma}) =: \left\{ v \in \mathbf{N} : \bar{v} = \sum_{k \notin I} c_k \bar{b}_k \text{ for some } 0 \leq c_k < 1, I \in \mathcal{A} \right\}$$

We assume that Σ is complete, then the connected components of the inertia stack $\mathcal{IX}$ are indexed by the elements of $Box(\mathbf{\Sigma})$ (see [1]). Moreover, given $v \in Box(\mathbf{\Sigma})$, the age of the corresponding connected component of $\mathcal{IX}$ is defined by $age(v) := \sum_{k \notin I} c_k$.

The Picard group $Pic(\mathcal{X})$ of $\mathcal{X}$ can be identified with the character group $Hom(G, \mathbb{C}^\times)$. Hence

$$\mathbb{L}^\vee = Hom(G, \mathbb{C}^\times) \cong Pic(\mathcal{X}) \cong H^2(\mathcal{X}; \mathbb{Z}). \tag{13}$$

We can also use the extended stacky fans introduced by Jiang [7] to define the toric Deligne-Mumford stacks. Given a stacky fan $\mathbf{\Sigma} = (\mathbf{N}, \Sigma, \beta)$ and a finite set

$$S = \{s_1, \ldots, s_l\} \subset \mathbf{N}_\Sigma := \{c \in \mathbf{N} : \bar{c} \in |\Sigma|\}.$$

The S-extended stacky fan is given by $(\mathbf{N}, \Sigma, \beta^S)$, where $\beta^S : \mathbb{Z}^{m+l} \to \mathbf{N}$ is defined by:

$$\beta^S(e_i) = \begin{cases} b_i & 1 \leq i \leq m; \\ s_{i-m} & m+1 \leq i \leq m+l. \end{cases} \tag{14}$$

Let $\mathbb{L}^S$ be the kernel of $\beta^S : \mathbb{Z}^{m+l} \to \mathbf{N}$. Then we have the following S-extended fan sequence

$$0 \longrightarrow \mathbb{L}^S \longrightarrow \mathbb{Z}^{m+l} \xrightarrow{\beta^S} \mathbf{N}. \tag{15}$$

By the Gale duality, we have the S-extended divisor sequence

$$0 \longrightarrow \mathbf{N}^* \xrightarrow{\beta^*} (\mathbb{Z}^*)^{m+l} \xrightarrow{\beta^{S\vee}} \mathbb{L}^{S\vee}, \tag{16}$$

where $\mathbb{L}^{S\vee} := H^1(Cone(\beta^S)^*)$.

Assumption 3.4. *In the rest of the paper, we will assume the set*

$$\{v \in Box(\mathbf{\Sigma}); age(v) \leq 1\} \cup \{b_1 \dots, b_m\}$$

generates $\boldsymbol{N}$ *over* $\mathbb{Z}$. *And we choose the set*

$$S = \{s_1, \dots, s_l\} \subset Box(\mathbf{\Sigma})$$

such that the set $\{b_1, \dots, b_m, s_1, \dots, s_l\}$ *generates* $\boldsymbol{N}$ *over* $\mathbb{Z}$ *and* $age(s_j) \leq 1$ *for* $1 \leq j \leq l$.

Let D_i^S be the image of the standard basis of $(\mathbb{Z}^*)^{m+l}$ under the map $\beta^{S\vee}$, then there is a canonical isomorphism

$$\mathbb{L}^{S\vee} \otimes \mathbb{Q} \cong (\mathbb{L}^\vee \otimes \mathbb{Q}) \bigoplus_{i=m+1}^{m+l} \mathbb{Q} D_i^S, \tag{17}$$

which can be constructed as follows ([6]):

Since Σ is complete, for $m < j \leq m+l$, the box element s_{j-m} is contained in some cone in Σ. Namely,

$$s_{j-m} = \sum_{i \notin I_j^S} c_{ji} b_i \quad \text{in} \quad \mathbf{N} \otimes \mathbb{Q}, \quad c_{ji} \geq 0, \quad \exists I_j^S \in \mathcal{A}^S,$$

where I_j^S is the "anticone" of the cone containing s_{j-m}.

By the S-extended fan sequence 15 tensored with $\mathbb{Q}$, we have the following short exact sequence

$$0 \longrightarrow \mathbb{L}^S \otimes \mathbb{Q} \longrightarrow \mathbb{Q}^{m+l} \xrightarrow{\beta^S} \mathbf{N} \otimes \mathbb{Q} \longrightarrow 0.$$

Hence, there exists a unique $D_j^{S\vee} \in \mathbb{L}^S \otimes \mathbb{Q}$ such that

$$\langle D_i^S, D_j^{S\vee} \rangle = \begin{cases} 1 & i = j; \\ -c_{ji} & i \notin I_j^S; \\ 0 & i \in I_j^S \setminus \{j\}. \end{cases} \tag{18}$$

These vectors $D_j^{S\vee}$ define a decomposition

$$\mathbb{L}^{S\vee} \otimes \mathbb{Q} = \operatorname{Ker}\left((D_{m+1}^{S\vee}, \dots, D_{m+l}^{S\vee}) : \mathbb{L}^{S\vee} \otimes \mathbb{Q} \to \mathbb{Q}^l \right) \oplus \bigoplus_{j=m+1}^{m+l} \mathbb{Q} D_j^S.$$

We identify the first factor $\operatorname{Ker}(D_{m+1}^{S\vee}, \dots, D_{m+l}^{S\vee})$ with $\mathbb{L}^\vee \otimes \mathbb{Q}$. Via this decomposition, we can regard $H^2(\mathcal{X}, \mathbb{Q}) \cong \mathbb{L}^\vee \otimes \mathbb{Q}$ as a subspace of $\mathbb{L}^{S\vee} \otimes \mathbb{Q}$.

Let D_i be the image of D_i^S in $\mathbb{L}^\vee \otimes \mathbb{Q}$ under this decomposition. Then

$$D_i = 0, \quad \text{for} \quad m+1 \leq i \leq m+l.$$

Let $\mathcal{A}^S$ be the collection of S-extended anti-cones, i.e.

$$\mathcal{A}^S := \left\{ I^S : \sum_{i \notin I^S} \mathbb{R}_{\geq 0} \overline{\beta^S(e_i)} \in \Sigma \right\}.$$

Note that

$$\{s_1, \ldots, s_l\} \subset I^S, \quad \forall I^S \in \mathcal{A}^S.$$

By applying $Hom_{\mathbb{Z}}(-, \mathbb{C}^\times)$ to the S-extended dual map $\beta^\vee$, we have a homomorphism

$$\alpha^S : G^S \to (\mathbb{C}^\times)^{m+l}, \quad \text{where} \quad G^S := Hom_{\mathbb{Z}}(\mathbb{L}^{S\vee}, \mathbb{C}^\times).$$

We define $\mathcal{U}$ to be the open subset of $\mathbb{C}^{m+l}$ defined by $\mathcal{A}^S$:

$$\mathcal{U}^S := \mathbb{C}^{m+l} \setminus \cup_{I^S \notin \mathcal{A}^S} \mathbb{C}^{I^S} = \mathcal{U} \times (\mathbb{C}^\times)^l,$$

where

$$\mathbb{C}^{I^S} = \left\{ (z_1, \ldots, z_{m+l}) : z_i = 0 \text{ for } i \notin I^S \right\}.$$

Let G^S act on $\mathcal{U}^S$ via α^S. Then we obtain the quotient stack $[\mathcal{U}^S/G^S]$. Jiang [7] showed that

$$[\mathcal{U}^S/G^S] \cong [\mathcal{U}/G] = \mathcal{X}.$$

3.2. **Mirror theorem for toric stacks.** In [3], Coates-Corti-Iritani-Tseng defined the S-extended I-function of a smooth toric Deligne-Mumford stack $\mathcal{X}$ with projective coarse moduli space and proved that this I-function is a point of Givental's Lagrangian cone $\mathcal{L}$ for the Gromov-Witten theory of $\mathcal{X}$. In this paper, we will only need this theorem for the weak Fano case. In this case, the mirror theorem will take a particularly simple form which can be stated as an equality of I-function and J-function via a change of variables, called mirror map.

To state the mirror theorem for weak Fano toric Deligne-Mumford stack, we need the following definitions.

We define the S-extended Kähler cone $C^S_{\mathcal{X}}$ as

$$C^S_{\mathcal{X}} := \cap_{I^S \in \mathcal{A}^S} \Sigma_{i \in I^S} \mathbb{R}_{>0} D^S_i$$

and the Kähler cone $C_{\mathcal{X}}$ as

$$C_{\mathcal{X}} := \cap_{I \in \mathcal{A}} \Sigma_{i \in I} \mathbb{R}_{>0} D_i.$$

Let $p^S_1, \ldots, p^S_{r+l}$ be an integral basis of $\mathbb{L}^{S\vee}$, where $r = m - n$, such that p^S_i is in the closure $cl(C^S_{\mathcal{X}})$ of the S-extended Kähler cone $C^S_{\mathcal{X}}$ for all $1 \leq i \leq r + l$ and $p^S_{r+1}, \ldots, p^S_{r+l}$ are in $\sum_{i=m+1}^{m+l} \mathbb{R}_{\geq 0} D^S_i$. We denote the image of p^S_i in $\mathbb{L}^\vee \otimes \mathbb{R}$ by p_i, therefore $p_1, \ldots, p_r$ are nef and $p_{r+1}, \ldots, p_{r+l}$ are zero. We define a matrix (m_{ia}) by

$$D^S_i = \sum_{a=1}^{r+l} m_{ia} p^S_a, \quad m_{ia} \in \mathbb{Z}.$$

Then the class D_i of toric divisor is given by

$$D_i = \sum_{a=1}^{r} m_{ia} p_a.$$

Definition 3.5 ([6], Section 3.1.4)**.** A toric Deligne-Mumford stack $\mathcal{X}$ is called weak Fano if the first Chern class ρ satisfies

$$\rho = c_1(T\mathcal{X}) = \sum_{i=1}^{m} D_i \in cl(C_{\mathcal{X}}),$$

where $C_{\mathcal{X}}$ is the Kähler cone of $\mathcal{X}$.

We will need a slightly stronger condition:

$$\rho^S := D_1^S + \ldots + D_{m+l}^S \in cl(C_{\mathcal{X}}^S),$$

where $C_{\mathcal{X}}^S$ is the S-extended Kähler cone. By lemma 3.3 of [6], we can see that $\rho^S \in cl(C_{\mathcal{X}}^S)$ implies $\rho \in cl(C_{\mathcal{X}})$. Moreover, under assumption 3.4, we will have

$$\rho^S \in cl(C_{\mathcal{X}}^S) \quad \text{if and only if} \quad \rho \in cl(C_{\mathcal{X}}).$$

For a real number r, let $\lceil r \rceil$, $\lfloor r \rfloor$ and $\{r\}$ be the ceiling, floor and fractional part of r respectively.

Definition 3.6. We define two subsets $\mathbb{K}$ and $\mathbb{K}_{\text{eff}}$ of $\mathbb{L}^S \otimes \mathbb{Q}$ as follows:

$$\mathbb{K} := \left\{ d \in L^S \otimes \mathbb{Q}; \{ i \in \{1, \ldots, m+l\}; \langle D_i^S, d \rangle \in \mathbb{Z} \} \in \mathcal{A}^S \right\},$$
$$\mathbb{K}_{\text{eff}} := \left\{ d \in L^S \otimes \mathbb{Q}; \{ i \in \{1, \ldots, m+l\}; \langle D_i^S, d \rangle \in \mathbb{Z}_{\geq 0} \} \in \mathcal{A}^S \right\}.$$

Remark 3.7. We will use $\mathbb{K}_{\mathcal{E}_j}$ and $\mathbb{K}_{\text{eff},\mathcal{E}_j}$ to denote the corresponding sets for the associated bundle $\mathcal{E}_j$, and use $\mathbb{K}_{\mathcal{X}}$ and $\mathbb{K}_{\text{eff},\mathcal{X}}$ to denote the corresponding sets for $\mathcal{X}$.

Definition 3.8 ([6], Section 3.1.3)**.** The reduction function v is defined as follows:

$$\begin{aligned} v : \mathbb{K} &\longrightarrow Box(\mathbf{\Sigma}) \\ d &\longmapsto \sum_{i=1}^{m} \lceil \langle D_i^S, d \rangle \rceil b_i + \sum_{j=1}^{l} \lceil \langle D_{m+j}^S, d \rangle \rceil s_j \end{aligned}$$

By the S-extended fan exact sequence, we have

$$\sum_{i=1}^{m} \langle D_i^S, d \rangle b_i + \sum_{j=1}^{l} \langle D_{m+j}^S, d \rangle s_j = 0 \in \mathbf{N} \otimes \mathbb{Q}.$$

Moreover, by the definition of $\mathbb{K}$, we have

$$\langle D_{m+j}^S, d \rangle \in \mathbb{Z}, \quad \text{for all} \quad d \in \mathbb{K} \quad \text{and} \quad 1 \leq j \leq l.$$

Hence,

$$v(d) = \sum_{i=1}^{m} \{-\langle D_i^S, d \rangle\} b_i + \sum_{j=1}^{l} \{-\langle D_{m+j}^S, d \rangle\} s_j = \sum_{i=1}^{m} \{-\langle D_i^S, d \rangle\} b_i.$$

By abuse of notation, we use D_i to denote the divisor $\{z_i = 0\} \subset \mathcal{X}$ and the cohomology class in $H^2(\mathcal{X}; \mathbb{Z}) \cong \mathbb{L}^\vee$, for $1 \leq i \leq m$.

We consider the $\mathbb{C}^\times$-action fixing a toric divisor D_j, $1 \leq j \leq m$, the action of $\mathbb{C}^\times$ on $\mathbb{C}^m$ is given by

$$(z_1, \ldots, z_m) \mapsto (z_1, \ldots, t^{-1} z_j, \ldots, z_m), \quad t \in \mathbb{C}^\times.$$

We can extend this to the diagonal $\mathbb{C}^\times$-action on $\mathcal{U} \times (\mathbb{C}^2 \setminus \{0\})$ by

$$(z_1, \ldots, z_m, u, v) \mapsto (z_1, \ldots, t^{-1} z_j, \ldots, z_m, tu, tv), \quad t \in \mathbb{C}^\times.$$

The associated bundle $\mathcal{E}_j$ of the $\mathbb{C}^\times$-action on $\mathcal{X}$ is given by

$$\mathcal{E}_j = \mathcal{U} \times (\mathbb{C}^2 \setminus \{0\})/G \times \mathbb{C}^\times.$$

We can also use the S-extended stacky fan of $\mathcal{X}$ to define $\mathcal{E}_j$:

$$\mathcal{E}_j = \mathcal{U}^S \times (\mathbb{C}^2 \setminus \{0\})/G^S \times \mathbb{C}^\times.$$

Therefore $\mathcal{E}_j$ is also a toric Deligne-Mumford stack. We can identify $H^2(\mathcal{E}_j;\mathbb{Z})$ with the lattice of the characters of $G \times \mathbb{C}^\times$:

$$H^2(\mathcal{E}_j;\mathbb{Z}) \cong \mathbb{L}^\vee \oplus \mathbb{Z} \cong H^2(\mathcal{X};\mathbb{Z}) \oplus \mathbb{Z}. \tag{19}$$

Moreover, we have the divisor sequence

$$0 \to \mathbf{N}^* \oplus \mathbb{Z} \to (\mathbb{Z}^*)^{m+2} \to \mathbb{L}^\vee \oplus \mathbb{Z}.$$

And the S-extended divisor sequence

$$0 \to \mathbf{N}^* \oplus \mathbb{Z} \to (\mathbb{Z}^*)^{m+l+2} \to \mathbb{L}^{S\vee} \oplus \mathbb{Z}.$$

Let $\hat{D}_i^S$ be the image of the standard basis of $(\mathbb{Z}^*)^{m+l+2}$ in $\mathbb{L}^{S\vee} \oplus \mathbb{Z}$. Then

$$\hat{D}_i^S = (D_i^S, 0), \text{ for } i \neq j; \quad \hat{D}_j^S = (D_j^S, -1); \quad \hat{D}_{m+l+1}^S = \hat{D}_{m+l+2}^S = (0,1). \tag{20}$$

And,

$$\hat{D}_i = (D_i, 0), \text{ for } i \neq j; \quad \hat{D}_j = (D_j, -1); \quad \hat{D}_{m+1} = \hat{D}_{m+2} = (0,1). \tag{21}$$

The fan Σ_j of $\mathcal{E}_j$ is a rational simplicial fan contained in $N_\mathbb{Q} \oplus \mathbb{Q}$. The 1-skeleton is given by

$$\hat{b}_i = (b_i, 0), \text{ for } 1 \leq i \leq m; \quad \hat{b}_{m+1} = (0,1); \quad \hat{b}_{m+2} = (b_j, -1). \tag{22}$$

We set

$$p_0 := (0,1) = \hat{D}_{m+1} = \hat{D}_{m+2} \in H^2(\mathcal{E}_j;\mathbb{Q}),$$

then a nef integral basis $\{p_1, \ldots, p_r\}$ of $H^2(\mathcal{X};\mathbb{Q})$ can be lifted to a nef integral basis $\{p_0, p_1, \ldots, p_r\}$ of $H^2(\mathcal{E}_j;\mathbb{Q})$, under the splitting (19). Let $p_1^S, \ldots, p_{r+l}^S$ be an integral basis of $\mathbb{L}^{S\vee}$, such that p_i is the image of p_i^S in $\mathbb{L}^\vee \otimes \mathbb{R}$. Let $p_0^S, p_1^S, \ldots, p_{r+l}^S$ be an integral basis of $\mathbb{L}^{S\vee} \oplus \mathbb{Z}$ and p_0 is the image of

$$p_0^S = \hat{D}_{m+l+1}^S = \hat{D}_{m+l+2}^S$$

in $(\mathbb{L}^\vee \oplus \mathbb{Z}) \otimes \mathbb{R}$. Note that $p_{r+1}, \ldots, p_{r+l}$ are zero. We have

$$C_{\mathcal{E}_j}^S = C_{\mathcal{X}}^S + \mathbb{R}_{>0} p_0^S, \quad \rho_{\mathcal{E}_j}^S = \rho_{\mathcal{X}}^S + p_0^S.$$

The following result is straightforward.

Lemma 3.9. *If $\rho_{\mathcal{X}}^S \in cl(C_{\mathcal{X}}^S)$, then $\rho_{\mathcal{E}_j}^S \in cl(C_{\mathcal{E}_j}^S)$, for $1 \leq j \leq m$.*

Definition 3.10. The I-function of $\mathcal{X}$ is the $H_{orb}^*(\mathcal{X})$-valued function:

$$I_{\mathcal{X}}(y,z) = e^{\sum_{i=1}^{r} p_i log y_i/z} \sum_{d \in \mathbb{K}_{\text{eff},\mathcal{X}}} \prod_{i=1}^{m+l} \left(\frac{\prod_{k=\lceil \langle D_i^S, d \rangle \rceil}^{\infty} \left(D_i + \left(\langle D_i^S, d\rangle - k\right) z\right)}{\prod_{k=0}^{\infty} \left(D_i + \left(\langle D_i^S, d\rangle - k\right) z\right)} \right) y^d \mathbf{1}_{v(d)}, \tag{23}$$

where $y^d = y_1^{\langle p_1^S, d\rangle} \cdots y_{r+l}^{\langle p_{r+l}^S, d\rangle}$. Similarly, The I-function of $\mathcal{E}$ is the $H^*_{orb}(\mathcal{E})$-valued function:

(24)

$$I_{\mathcal{E}_j}(y,z) = e^{\sum_{i=0}^{r} p_i log y_i/z} \sum_{\beta\in\mathbb{K}_{\text{eff},\mathcal{E}_j}} \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil\langle \hat{D}_i^S,\beta\rangle\rceil}^{\infty} \left(\hat{D}_i + \left(\langle \hat{D}_i^S,\beta\rangle - k\right) z\right)}{\prod_{k=0}^{\infty} \left(\hat{D}_i + \left(\langle \hat{D}_i^S,\beta\rangle - k\right) z\right)} \right) y^\beta \mathbf{1}_{v(\beta)},$$

where $y^\beta = y_0^{\langle p_0^S, \beta, \rangle} y_1^{\langle p_1^S, \beta\rangle} \cdots y_{r+l}^{\langle p_{r+l}^S, \beta\rangle}$.

Following section 4.1 of [6], The I-functions of $\mathcal{X}$ and $\mathcal{E}_j$ can be rewritten in the form:

(25)

$$I_{\mathcal{X}}(y,z) = e^{\sum_{i=1}^{r} p_i log y_i/z} \sum_{d\in\mathbb{K}_{\mathcal{X}}} \prod_{i=1}^{m+l} \left(\frac{\prod_{k=\lceil\langle D_i^S,d\rangle\rceil}^{\infty} \left(D_i + \left(\langle D_i^S,d\rangle - k\right) z\right)}{\prod_{k=0}^{\infty} \left(D_i + \left(\langle D_i^S,d\rangle - k\right) z\right)} \right) y^d \mathbf{1}_{v(d)},$$

and

(26)

$$I_{\mathcal{E}_j}(y,z) = e^{\sum_{i=0}^{r} p_i log y_i/z} \sum_{\beta\in\mathbb{K}_{\mathcal{E}_j}} \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil\langle \hat{D}_i^S,\beta\rangle\rceil}^{\infty} \left(\hat{D}_i + \left(\langle \hat{D}_i^S,\beta\rangle - k\right) z\right)}{\prod_{k=0}^{\infty} \left(\hat{D}_i + \left(\langle \hat{D}_i^S,\beta\rangle - k\right) z\right)} \right) y^\beta \mathbf{1}_{v(\beta)},$$

respectively, because the summand with $d \in \mathbb{K} \setminus \mathbb{K}_{\text{eff}}$ vanishes. We refer to [6] for more details.

Theorem 3.11 ([6], Conjecture 4.3)**.** *Assume that $\rho^S \in cl(C^S_{\mathcal{X}})$. Then the I-function and the J-function satisfy the following relation:*

$$I_{\mathcal{X}}(y,z) = J_{\mathcal{X}}(\tau(y), z) \tag{27}$$

where

$$\tau(y) = \tau_{0,2}(y) + \tau_{tw}(y) = \sum_{i=1}^{r} (log y_i) p_i + \sum_{j=m+1}^{m+l} y^{D_j^{S\vee}} \mathfrak{D}_j + h.o.t. \in H^{\leq 2}_{orb}(\mathcal{X}), \tag{28}$$

with

$$\tau_{0,2}(y) \in H^2(\mathcal{X}), \quad \tau_{tw}(y) \in H^{\leq 2}_{orb}(\mathcal{X}) \setminus H^2(\mathcal{X}),$$
$$\mathfrak{D}_j = \prod_{i\notin I_j} D_i^{\lfloor c_{ji}\rfloor} \mathbf{1}_{v(D_j^{S\vee})} \in H^*_{orb}(\mathcal{X}).$$

and h.o.t. stands for higher order terms in z^{-1}. Furthermore, $\tau(y)$ is called the mirror map and takes values in $H^{\leq 2}_{orb}(\mathcal{X})$.

For $\tau_{0,2}(y) = \sum_{a=1}^{r} p_a log q_a \in H^2(\mathcal{X})$, we have

$$log q_i = log y_i + g_i(y_1, \ldots, y_{r+l}), \text{ for } i = 1, \ldots, r,$$

where g_i is a (fractional) power series in $y_1, \ldots, y_{r+l}$ which is homogeneous of degree zero with respect to the degree $deg y^d = 2\langle \rho^S_{\mathcal{X}}, d\rangle$.

By lemma 3.9, under the assumption of theorem 3.11, we can also apply the mirror theorem to the associated bundle $\mathcal{E}_j$, hence we have

$$I_{\mathcal{E}_j}(y,z) = J_{\mathcal{E}_j}(\tau^{(j)}(y), z),$$

where

$$\tau^{(j)}(y) = \tau^{(j)}_{0,2} + \tau^{(j)}_{tw}(y) \in H^2(\mathcal{E}_j) \oplus \left(H^{\leq 2}_{orb}(\mathcal{E}_j) \setminus H^2(\mathcal{E}_j) \right)$$

Since $\tau^{(j)}_{0,2}(y) = \sum\limits_{a=0}^{r} p_a log q_a \in H^2(\mathcal{E}_j)$, therefore

$$log q_i = log y_i + g^{(j)}_i(y_0, \ldots, y_{r+l}), \text{ for } i = 0, \ldots, r,$$

where $g^{(j)}_i$ is a (fractional) power series in $y_0, y_1, \ldots, y_{r+l}$ which is homogeneous of degree zero with respect to the degree $deg y^{\beta} = 2\langle \rho^S_{\mathcal{E}_j}, \beta\rangle$.

3.3. Seidel elements and mirror maps.

Proposition 3.12. *The function $g^{(j)}_i$ does not depend on y_0 and we have*

$$g^{(j)}_i(y_0, \ldots, y_{r+l}) = g_i(y_1, \ldots, y_{r+l}), \quad for \quad i = 1, \ldots, r.$$

Proof. The functions g_i is the coefficients of $z^{-1}p_i$ in the expansion of $I_{\mathcal{X}}$:

$$I_{\mathcal{X}}(y, z) = e^{\sum_{i=1}^{r} p_i log y_i/z} \left(1 + z^{-1} \left(\sum_{i=1}^{r} g_i(y) p_i + \tau_{tw} \right) + O(z^{-2}) \right).$$

The functions $g^{(j)}_i$ is the coefficients of $z^{-1}p_i$ in the expansion of $I_{\mathcal{E}_j}$:

$$I_{\mathcal{E}_j}(y, z) = e^{\sum_{i=0}^{r} p_i log y_i/z} \left(1 + z^{-1} \left(\sum_{i=0}^{r} g^{(j)}_i(y) p_i + \tau^{(j)}_{tw} \right) + O(z^{-2}) \right).$$

Following the proof of lemma 3.5 of [5], we obtain the conclusion of this proposition. □

We will prove $\tau^{(j)}_{tw}$ is also independent from y_0. To begin with, the following lemma implies that $\tau^{(j)}_{tw}(y)$ is an (integer) power series in y_0.

Lemma 3.13. *For any $\beta \in \mathbb{K}_{\mathcal{E}_j}$, we have $\langle p^S_0, \beta\rangle \in \mathbb{Z}$. Furthermore, for any $\beta \in \mathbb{K}_{eff,\mathcal{E}_j}$, we have $\langle p^S_0, \beta\rangle \in \mathbb{Z}_{\geq 0}$.*

Proof. Any cone $\sigma \in \Sigma_j$ containing both $\hat{b}_{m+1}$ and $\hat{b}_{m+2}$ should also contain $\hat{b}_j$, this is impossible since the fan Σ_j is simplicial and $\hat{b}_{m+1}$, $\hat{b}_{m+2}$ and $\hat{b}_j$ lie in the same plane. Hence, by the definition of $\mathbb{K}_{\mathcal{E}_j}$ (resp. $\mathbb{K}_{\text{eff},\mathcal{E}_j}$), at least one of $\langle \hat{D}^S_{m+1}, \beta\rangle$ and $\langle \hat{D}^S_{m+2}, \beta\rangle$ has to be integer (resp. non-negative integer), for any $\beta \in \mathbb{K}_{\mathcal{E}_j}$ (resp. $\beta \in \mathbb{K}_{\text{eff},\mathcal{E}_j}$). On the other hand, we have,

$$\langle p^S_0, \beta\rangle = \langle \hat{D}^S_{m+1}, \beta\rangle = \langle \hat{D}^S_{m+2}, \beta\rangle.$$

Therefore, we must have $\langle p^S_0, \beta\rangle \in \mathbb{Z}$ (resp. $\langle p^S_0, \beta\rangle \in \mathbb{Z}_{\geq 0}$). □

As a direct consequence of the above lemma, $\tau^{(j)}_{tw}(y)$ can only contain non-negative integer power of y_0.

Proposition 3.14. *Let $\tau^{(j)}_{tw}(y) = \sum\limits_{n=0}^{\infty} H^{(j)}_n(y) y^n_0$, where $H^{(j)}_n(y)$ is a (fractional) power series in $y_1, \ldots, y_n$. Then*

$$H^{(j)}_n(y) = 0 \quad for \quad n \geq 1,$$

i.e. $\tau^{(j)}_{tw}(y)$ is independent from y_0. Moreover, we have

$$\tau^{(j)}_{tw}(y) = \tau_{tw}(y).$$

Proof. Recall $\tau^{(j)}_{tw}(y)$ is the coefficient of z^{-1} in

(29)
$$e^{-\sum_{i=0}^{r} p_i log y_i/z} I_{\mathcal{E}_j}(y,z) = \sum_{\beta\in\mathbb{K}_{\mathrm{eff},\mathcal{E}_j}} \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil\langle\hat{D}_i^S,\beta\rangle\rceil}^{\infty}\left(\hat{D}_i + \left(\langle\hat{D}_i^S,\beta\rangle - k\right)z\right)}{\prod_{k=0}^{\infty}\left(\hat{D}_i + \left(\langle\hat{D}_i^S,\beta\rangle - k\right)z\right)} \right) y^\beta \mathbf{1}_{v(\beta)},$$

valued in $H^{\leq 2}_{orb}(\mathcal{E}_j)\setminus H^2(\mathcal{E}_j)$. Hence, we only need to consider terms with $v(\beta)\neq 0$, or, equivalently, $v(d)\neq 0$, where d is the natural projection of β on to $\mathbb{K}_{\mathrm{eff},\mathcal{X}}$.

Therefore, it remains to examine the product factor:

$$\begin{aligned}
&\prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil\langle\hat{D}_i^S,\beta\rangle\rceil}^{\infty}\left(\hat{D}_i + \left(\langle\hat{D}_i^S,\beta\rangle - k\right)z\right)}{\prod_{k=0}^{\infty}\left(\hat{D}_i + \left(\langle\hat{D}_i^S,\beta\rangle - k\right)z\right)} \right) \\
&= \frac{\prod_{i:\langle\hat{D}_i^S,\beta\rangle<0}\prod_{\langle\hat{D}_i^S,\beta\rangle\leq k<0}\left(\hat{D}_i + \left(\langle\hat{D}_i^S,\beta\rangle - k\right)z\right)}{\prod_{i:\langle\hat{D}_i^S,\beta\rangle>0}\prod_{0\leq k<\langle\hat{D}_i^S,\beta\rangle}\left(\hat{D}_i + \left(\langle\hat{D}_i^S,\beta\rangle - k\right)z\right)} \\
&= C_\beta z^{-\left(\sum_{i=1}^{m+l+2}\lceil\langle\hat{D}_i^S,\beta\rangle\rceil + \#\{i:\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z}_{<0}\}\right)} \prod_{i:\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z}_{<0}} \hat{D}_i + h.o.t.,
\end{aligned} \tag{30}$$

where

(31)
$$C_\beta = \prod_{i:\langle\hat{D}_i^S,\beta\rangle<0}\ \prod_{\langle\hat{D}_i^S,\beta\rangle<k<0}\left(\langle\hat{D}_i^S,\beta\rangle - k\right) \prod_{i:\langle\hat{D}_i^S,\beta\rangle>0}\ \prod_{0\leq k<\langle\hat{D}_i^S,\beta\rangle}\left(\langle\hat{D}_i^S,\beta\rangle - k\right)^{-1}.$$

By assumption, we need to have

$$\sum_{i=1}^{m+l+2}\lceil\langle\hat{D}_i^S,\beta\rangle\rceil \geq \sum_{i=1}^{m+l+2}\langle\hat{D}_i^S,\beta\rangle \geq 0.$$

The equality holds if and only if

$$\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z}, \quad \text{for all} \quad 1\leq i\leq m+l+2; \quad \text{and} \quad \sum_{i=1}^{m+l+2}\langle\hat{D}_i^S,\beta\rangle = 0.$$

However, this would imply $v(\beta)=0$, hence we cannot have $\sum_{i=1}^{m+l+2}\lceil\langle\hat{D}_i^S,\beta\rangle\rceil = 0$. Therefore, the expansion (30) would contribute to $H^{(j)}_n$ only when

$$\sum_{i=1}^{m+l+2}\lceil\langle\hat{D}_i^S,\beta\rangle\rceil = 1 \quad \text{and} \quad \#\{i:\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z}_{<0}\} = 0.$$

In this case, if $\langle p_0^S,\beta\rangle\geq 1$, then

$$\sum_{i=1}^{m+l+2}\lceil\langle\hat{D}_i^S,\beta\rangle\rceil \geq \sum_{i=1}^{m+l}\lceil\langle D_i^S,d\rangle\rceil + 1,$$

therefore, we have

$$0 \geq \sum_{i=1}^{m+l}\lceil\langle D_i^S,d\rangle\rceil \geq \sum_{i=1}^{m+l}\langle D_i^S,d\rangle = 0.$$

This implies, when $\langle p_0^S,\beta\rangle\geq 1$, we must have

$$\langle D_i^S,d\rangle\in\mathbb{Z}, \text{ for } 1\leq i\leq m+l.$$

It is a contradiction, since $\hat{\tau}_{tw} \in H^{\leq 2}_{orb}(\mathcal{E}_j) \setminus H^2(\mathcal{E}_j)$ implies $v(d) \neq 0$. Hence

$$H_n^{(j)} = 0 \text{ for all } n > 0$$

and $\tau_{tw}^{(j)}(y)$ is independent from y_0. Moreover, by the expression of I-functions and the identity

$$\imath^* I_{\mathcal{E}_j}\big|_{y_0=0} = I_{\mathcal{X}},$$

we have $\tau_{tw}^{(j)}(y) = \tau_{tw}(y)$.

□

As a direct consequence of the above lemma, we can use the following notation for the Seidel element

$$\tilde{S}_j(\tau(y)) := \tilde{S}_j(\tau^{(j)}(y)), \tag{32}$$

since $\tilde{S}_j(\tau^{(j)}(y))$ does not depend on y_0 or q_0.

3.4. **Seidel Elements in terms of I-functions.** We can rewrite the I-function of the associated bundle $\mathcal{E}_j$ as follows:

$$e^{\sum_{i=0}^{r} p_i log y_i/z}\left(1 + z^{-1}\left(\sum_{i=0}^{r} g_i^{(j)}(y)p_i + \tau_{tw}^{(j)}(y)\right) + z^{-2}\left(\sum_{n=0}^{2} G_n^{(j)}(y)y_0^n\right) + O(z^{-3})\right). \tag{33}$$

Then, $logq_i = logy_i + g_i^{(j)}(y)$ implies

$$I_{\mathcal{E}_j}(y,z) = e^{\sum_{i=0}^{r} p_i log q_i/z}\left(1 + z^{-1}\tau_{tw}^{(j)}(y) + z^{-2}\left(\sum_{n=0}^{2} G_n^{(j)}(y)y_0^n\right) + O(z^{-3})\right), \tag{34}$$

where $G_n^{(j)}(y)$ is a (fractional) power series in $y_1, \ldots, y_{r+l}$ taking values in $H^*_{orb}(\mathcal{E}_j)$.

By proposition (2.4), the Seidel element $\tilde{S}_j(\tau^{(j)}(y))$ is the coefficient of q_0/z^2 in

$$exp\left(-\sum_{i=0}^{r} p_i log q_i/z\right) J_{\mathcal{E}_j}(\tau^{(j)}(y), z),$$

hence $J_{\mathcal{E}_j}(\tau^{(j)}(y), z) = I_{\mathcal{E}_j}(y,z)$ and $logq_0 = logy_0 + g_0^{(j)}(y)$ imply the following result:

Theorem 3.15. *The Seidel element S_j associated to the toric divisor D_j is given by*

$$S_j(\tau^{(j)}(y)) = \imath^*(G_1^{(j)}(y)y_0). \tag{35}$$

Furthermore, we have

$$\tilde{S}_j(\tau(y)) = \tilde{S}_j(\tau^{(j)}(y)) = exp(-g_0^j(y))\imath^*(G_1^{(j)}(y)). \tag{36}$$

3.5. **Computation of $g_0^{(j)}$.** The computation is essentially the same as the proof of lemma 3.16 of [5]. Consider the product factors in $I_{\mathcal{E}_j}$:

$$\prod_{i=1}^{m+l+2}\left(\frac{\prod_{k=\lceil\langle\hat{D}_i^S,\beta\rangle\rceil}^{\infty}\left(\hat{D}_i+\left(\langle\hat{D}_i^S,\beta\rangle-k\right)z\right)}{\prod_{k=0}^{\infty}\left(\hat{D}_i+\left(\langle\hat{D}_i^S,\beta\rangle-k\right)z\right)}\right)y^{\beta}\mathbf{1}_{v(\beta)},$$

these factors contribute to $g_i^{(j)}$ if

$$v(\beta)=\sum_{i=1}^{m+l+2}\{-\langle\hat{D}_i^S,\beta\rangle\}\hat{b}_i=0,$$

then, by the definition of $\mathbb{K}_{\text{eff}}$, we must have

$$\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z},\ \text{for all } 1\le i\le m+l+2.$$

In this case, the product factors can be rewritten as

$$\begin{aligned}&\prod_{i=1}^{m+l+2}\left(\frac{\prod_{k=\lceil\langle\hat{D}_i^S,\beta\rangle\rceil}^{\infty}\left(\hat{D}_i+\left(\langle\hat{D}_i^S,\beta\rangle-k\right)z\right)}{\prod_{k=0}^{\infty}\left(\hat{D}_i+\left(\langle\hat{D}_i^S,\beta\rangle-k\right)z\right)}\right)y^{\beta}\mathbf{1}_{v(\beta)}\\ &=\prod_{i=1}^{m+l+2}\frac{\prod_{k=-\infty}^{0}\left(\hat{D}_i+kz\right)}{\prod_{k=-\infty}^{\langle\hat{D}_i^S,\beta\rangle}\left(\hat{D}_i+kz\right)}y^{\beta}\\ &=\left(C_{\beta}z^{-\sum_{i=1}^{m+l+2}\langle\hat{D}_i^S,\beta\rangle-\#\{i:\langle\hat{D}_i^S,\beta\rangle<0\}}\prod_{i:\langle\hat{D}_i^S,\beta\rangle<0}\hat{D}_i+h.o.t.\right)y^{\beta},\end{aligned}\tag{37}$$

where $h.o.t.$ stands for higher order terms in z^{-1} and

$$C_{\beta}=\prod_{i:\langle\hat{D}_i^S,\beta\rangle<0}(-1)^{-\langle\hat{D}_i^S,\beta\rangle-1}\left(-\langle\hat{D}_i^S,\beta\rangle-1\right)!\prod_{i:\langle\hat{D}_i^S,\beta\rangle\ge0}\left(\langle\hat{D}_i^S,\beta\rangle!\right)^{-1}.\tag{38}$$

They contribute to the z^{-1} term if

$$\sum_{i=1}^{m+l+2}\langle\hat{D}_i^S,\beta\rangle+\#\{i:\langle\hat{D}_i^S,\beta\rangle<0\}\le1.$$

Since we assume $\rho_{\mathcal{X}}^S\in cl(C_{\mathcal{X}}^S)$, hence $\rho_{\mathcal{E}_j}^S\in cl(C_{\mathcal{E}_j}^S)$. So it has to be the following three cases:

- $\begin{cases}\sum_{i=1}^{m+l+2}\langle\hat{D}_i^S,\beta\rangle=0\\ \#\{i:\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z}_{<0}\}=0\end{cases}$
- $\begin{cases}\sum_{i=1}^{m+l+2}\langle\hat{D}_i^S,\beta\rangle=1\\ \#\{i:\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z}_{<0}\}=0\end{cases}$
- $\begin{cases}\sum_{i=1}^{m+l+2}\langle\hat{D}_i^S,\beta\rangle\rceil=0\\ \#\{i:\langle\hat{D}_i^S,\beta\rangle\in\mathbb{Z}_{<0}\}=1\end{cases}$.

In the first case, we have $\langle\hat{D}_i^S,\beta\rangle=0$ for all i, hence $\beta=0$; the second case can not happen, since β has to satisfy $\langle\hat{D}_i^S,\beta\rangle=0$ except for one i and this implies $\beta=0$.

Therefore, the coefficient of z^{-1} is from the third case, where

$$\sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta\rangle = 0 \quad \text{and} \quad \#\{i : \langle \hat{D}_i^S, \beta\rangle < 0\} = 1. \tag{39}$$

By the assumption $\rho^S_{\mathcal{X}} \in cl(C^S_{\mathcal{X}})$, we must have $\sum_{i=1}^{m+l} \langle D_i^S, d\rangle = 0$ and $\langle p_0^S, \beta\rangle = 0$. Moreover, $\langle D_i^S, d\rangle < 0$ for exactly one i in $\{1, \ldots, m\}$. (Note that $\langle D_i^S, d\rangle \geq 0$ for $i \in \{m+1, \ldots, m+l\}$.)

Now $g_0^{(j)}$ is the coefficient corresponding to p_0 and $\hat{D}_j = \langle D_j, -1\rangle = D_j - p_0$ is the only one, among $\hat{D}_1, \ldots, \hat{D}_m$, which contains p_0. By expression (37), we must have $\langle D_j^S, d\rangle < 0$ and $\langle D_i^S, d\rangle \geq 0$ for $i \neq j$. Hence we have

Lemma 3.16. *The coefficient $g_0^{(j)}$ is given by*

$$g_0^j(y_1, \ldots, y_{r+l}) = \sum_{\substack{\langle D_i^S, d\rangle \in \mathbb{Z}, 1 \leq i \leq m+l \\ \langle \rho^S_{\mathcal{X}}, d\rangle = 0 \\ \langle D_j^S, d\rangle < 0 \\ \langle D_i^S, d\rangle \geq 0, \forall i \neq j}} \frac{(-1)^{-\langle D_j^S, d\rangle} \left(-\langle D_j^S, d\rangle - 1\right)!}{\prod_{i\neq j} \langle D_i^S, d\rangle!} y^d. \tag{40}$$

4. Batyrev Elements

In this section, we will extend the definition of the Batyrev elements in [5] to toric Deligne-Mumford stacks and explore their relationships with the Seidel elements.

4.1. Batyrev Elements.

Following [6], consider the mirror coordinates $y_1, \ldots, y_{r+l}$ of the toric Deligne-Mumford stacks $\mathcal{X}$ with $\rho^S_{\mathcal{X}} \in cl(C^S_{\mathcal{X}})$. Set $\mathbb{C}[y^{\pm}] = \mathbb{C}[y_1^{\pm}, \ldots, y_{r+l}^{\pm}]$.

Definition 4.1. The Batyrev ring $B(\mathcal{X})$ of $\mathcal{X}$ is a $\mathbb{C}[y^{\pm}]$-algebra generated by the variables $\lambda_1, \ldots, \lambda_{r+l}$ with the following two relations:

$$\begin{aligned} &\text{(multiplicative):} \quad y^d \prod_{i:\langle D_i^S, d\rangle < 0} \omega_i^{-\langle D_i^S, d\rangle} = \prod_{i:\langle D_i^S, d\rangle > 0} \omega_i^{\langle D_i^S, d\rangle}, \quad d \in \mathbb{L}^{\mathbb{S}}; \\ &\text{(linear):} \quad \omega_i = \sum_{a=1}^{r+l} m_{ai}\lambda_a, \end{aligned} \tag{41}$$

where ω_i is invertible in $B(\mathcal{X})$.

Definition 4.2. We define the element $\tilde{p}_i^S \in H^{\leq 2}_{orb}(\mathcal{X}) \otimes \mathbb{Q}[[y_1, \ldots, y_{r+l}]]$ as

$$\tilde{p}_i^S = \frac{\partial \tau(y)}{\partial log y_i}, \quad i = 1, \ldots, r+l.$$

Recall that

$$D_j^S = \sum_{i=1}^{r+l} m_{ij} p_i^S, \text{ for } 1 \leq j \leq m+l,$$

Then, the Batyrev element associated to D_j^S is defined by

$$\tilde{D}_j^S = \sum_{i=1}^{r+l} m_{ij} \tilde{p}_i^S, \text{ for } 1 \leq j \leq m+l.$$

Proposition 4.3. *The Batyrev elements $\tilde{D}_1^S, \ldots, \tilde{D}_{m+l}^S$ satisfy the multiplicative and linear Batyrev relations for $\omega_j = \tilde{D}_j^S$.*

Proof. We consider the differential operator $\mathcal{P}_d \in \mathbb{C}[z, y^{\pm}, zy(\partial/\partial y)]$ for $d \in \mathbb{L}^S$, introduced by Iritani in [6], section 4.2:

$$\mathcal{P}_d := y^d \prod_{i:\langle D_i^S, d\rangle < 0} \prod_{k=0}^{-\langle D_i^S, d\rangle - 1} (\mathcal{D}_i - kz) - \prod_{i:\langle D_i^S, d\rangle > 0} \prod_{k=0}^{\langle D_i^S, d\rangle - 1} (\mathcal{D}_i - kz), \tag{42}$$

where $\mathcal{D}_i := \sum_{j=1}^{r+l} m_{ij} z y_j \partial/\partial y_j$.

By [6] lemma 4.6, we have

$$\mathcal{P}_d I(y, z) = 0, \quad d \in \mathbb{L}^S.$$

Hence

$$0 = \mathcal{P}_d(z, y, zy\partial/\partial y) I(y, z) = \mathcal{P}_d(z, y, zy\partial/\partial y) J(\tau(y), z).$$

This implies that

$$\mathcal{P}_d(z, y, z\tau^*\nabla)\mathbf{1} = 0,$$

where $\tau^*\nabla_i := \nabla_{\tau_*(y_i(\partial/\partial y_i))}$. Since

$$\tau(y) = \sum_{i=1}^{r} p_i log y_i + \tau_{tw}(y) \quad \text{and} \quad \nabla_{\tau_*(y_i(\partial/\partial y_i))} = \tau_*(y_i(\partial/\partial y_i)) + \frac{1}{z} y_i \frac{\partial \tau(y)}{\partial y_i} \circ_\tau,$$

by setting $z = 0$, we proved that the Batyrev elements satisfy the multiplicative relation.

It is straightforward from the definition that the Batyrev elements satisfy the linear relation. □

Consider the I-function for the bundle $\mathcal{E}_j$ associated to the toric divisor D_j^S, for $1 \le j \le m$.

$$I_{\mathcal{E}_j}(y, z) = e^{\sum_{i=0}^{r} p_i log y_i / z} \sum_{\beta \in \mathbb{K}_{\mathcal{E}_j}} \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta\rangle \rceil}^{\infty} \left(\hat{D}_i + \left(\langle \hat{D}_i^S, \beta\rangle - k\right) z\right)}{\prod_{k=0}^{\infty} \left(\hat{D}_i + \left(\langle \hat{D}_i^S, \beta\rangle - k\right) z\right)} \right) y^\beta \mathbf{1}_{v(\beta)},$$

where $y^\beta = y_0^{\langle p_0^S, \beta, \rangle} y_1^{\langle p_1^S, \beta\rangle} \cdots y_{r+l}^{\langle p_{r+l}^S, \beta\rangle}$. The following lemma is a generalization of lemma 3.11 in [5].

Lemma 4.4. *The I-function $I_{\mathcal{E}_j}$ of the bundle $\mathcal{E}_j$, associated to the toric divisor D_j^S, satisfies the following partial differential equation:*

$$z \frac{\partial}{\partial y_0} \left(y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j} = \left(\sum_{i=1}^{r+l} m_{ij} \left(y_i \frac{\partial}{\partial y_i} \right) - y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j} \tag{43}$$

Proof. Consider the left hand side of the equation (43),

$$z \frac{\partial}{\partial y_0} \left(y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j}$$
$$= e^{\sum_{i=0}^{r} p_i log y_i / z} \sum_{\beta \in \mathbb{K}_{\mathcal{E}_j}} \prod_{i=1}^{m+l+2} \left(\frac{\prod_{k=\lceil \langle \hat{D}_i^S, \beta\rangle \rceil}^{\infty} \left(\hat{D}_i + \left(\langle \hat{D}_i^S, \beta\rangle - k\right) z\right)}{\prod_{k=0}^{\infty} \left(\hat{D}_i + \left(\langle \hat{D}_i^S, \beta\rangle - k\right) z\right)} \right) \left(2p_0 \langle p_0^S, \beta\rangle + \langle p_0^S, \beta\rangle^2 z\right) \left(y^\beta / y_0\right) \mathbf{1}_{v(\beta)},$$

and the right hand side of the equation (43)

$$\left(\sum_{i=1}^{r+l} m_{ij}\left(y_i\frac{\partial}{\partial y_i}\right)-y_0\frac{\partial}{\partial y_0}\right)I_{\mathcal{E}_j}$$

$$=e^{\sum_{i=0}^{r}p_i logy_i/z}\sum_{\beta\in\mathbb{K}_{\mathcal{E}_j}}\prod_{i=1}^{m+l+2}\left(\frac{\prod_{k=\lceil\langle\hat{D}_i^S,\beta\rangle\rceil}^{\infty}\left(\hat{D}_i+\left(\langle\hat{D}_i^S,\beta\rangle-k\right)z\right)}{\prod_{k=0}^{\infty}\left(\hat{D}_i+\left(\langle\hat{D}_i^S,\beta\rangle-k\right)z\right)}\right)\left(\hat{D}_j/z+\langle\hat{D}_j^S,\beta\rangle\right)y^\beta\mathbf{1}_{v(\beta)}.$$

It is suffice to prove the coefficients of $y^\beta\mathbf{1}_{v(\beta)}$ in them are the same, for all $\beta\in\mathbb{K}_{\mathcal{E}_j}$. Note that, we can rewrite the product factor

$$\frac{\prod_{k=\lceil\langle\hat{D}_i^S,\beta\rangle\rceil}^{\infty}\left(\hat{D}_i+\left(\langle\hat{D}_i^S,\beta\rangle-k\right)z\right)}{\prod_{k=0}^{\infty}\left(\hat{D}_i+\left(\langle\hat{D}_i^S,\beta\rangle-k\right)z\right)}=\frac{\prod_{k\leq 0,\{k\}=\{\langle\hat{D}_i^S,\beta\rangle\}}\left(\hat{D}_i+kz\right)}{\prod_{k\leq\langle\hat{D}_i^S,\beta\rangle,\{k\}=\{\langle\hat{D}_i^S,\beta\rangle\}}\left(\hat{D}_i+kz\right)}.$$

Let $\beta^{'}=\beta+[\sigma_0]$, hence we have

$$\langle\hat{D}_j^S,\beta^{'}\rangle=\langle\hat{D}_j^S,\beta\rangle-1;\quad\langle\hat{D}_i^S,\beta^{'}\rangle=\langle\hat{D}_i^S,\beta\rangle\text{ for }1\leq i\leq m+l\text{ and }i\neq j;$$

$$\langle\hat{D}_{m+l+1}^S,\beta^{'}\rangle=\langle\hat{D}_{m+l+1}^S,\beta\rangle+1;\quad\langle\hat{D}_{m+l+2}^S,\beta^{'}\rangle=\langle\hat{D}_{m+l+2}^S,\beta\rangle+1.$$

Note that $\beta\in\mathbb{K}_{\mathcal{E}_j}$ if and only if $\beta^{'}\in\mathbb{K}_{\mathcal{E}_j}$. Moreover,

$$\left(y^{\beta^{'}}/y_0\right)\mathbf{1}_{v(\beta^{'})}=y^\beta\mathbf{1}_{v(\beta)}.$$

Hence the coefficient of $y^\beta\mathbf{1}_{v(\beta)}$ in $z\frac{\partial}{\partial y_0}(y_0\frac{\partial}{\partial y_0})I_{\mathcal{E}_j}$ is

$$e^{\sum_{i=0}^{r}p_i logy_i/z}\prod_{i=1}^{m+l+2}\left(\frac{\prod_{k=\lceil\langle\hat{D}_i^S,\beta\rangle\rceil}^{\infty}\left(\hat{D}_i+\left(\langle\hat{D}_i^S,\beta\rangle-k\right)z\right)}{\prod_{k=0}^{\infty}\left(\hat{D}_i+\left(\langle\hat{D}_i^S,\beta\rangle-k\right)z\right)}\right)\frac{\hat{D}_j+\langle\hat{D}_j^S,\beta\rangle z}{\left(p_0+(\langle p_0^S,\beta\rangle+1)z\right)^2}\bullet$$
$$\bullet\,(2p_0(\langle p_0^S,\beta\rangle+1)+(\langle p_0^S,\beta\rangle+1)^2z)$$
$$=e^{\sum_{i=0}^{r}p_i logy_i/z}\prod_{i=1}^{m+l+2}\left(\frac{\prod_{k=\lceil\langle\hat{D}_i^S,\beta\rangle\rceil}^{\infty}\left(\hat{D}_i+\left(\langle\hat{D}_i^S,\beta\rangle-k\right)z\right)}{\prod_{k=0}^{\infty}\left(\hat{D}_i+\left(\langle\hat{D}_i^S,\beta\rangle-k\right)z\right)}\right)\frac{\hat{D}_j+\langle\hat{D}_j^S,\beta\rangle z}{z}\qquad(\text{since }p_0^2=0).$$

This is exactly the coefficient of $y^\beta\mathbf{1}_{v(\beta)}$ in $\left(\sum_{i=1}^{r+l}m_{ij}\left(y_i\frac{\partial}{\partial y_i}\right)-y_0\frac{\partial}{\partial y_0}\right)I_{\mathcal{E}_j}$,

Hence the I-function of $\mathcal{E}_j$ satisfies the differential equation

$$z\frac{\partial}{\partial y_0}\left(y_0\frac{\partial}{\partial y_0}\right)I_{\mathcal{E}_j}=\left(\sum_{i=1}^{r+l}m_{ij}\left(y_i\frac{\partial}{\partial y_i}\right)-y_0\frac{\partial}{\partial y_0}\right)I_{\mathcal{E}_j}.$$

□

Using the expansion of $I_{\mathcal{E}_j}$, we have

$$I_{\mathcal{E}_j}(y,z)=e^{\sum_{i=0}^{r}p_i logy_i/z}\left(1+z^{-1}\left(\sum_{i=0}^{r}g_i^{(j)}(y)p_i+\tau_{tw}^{(j)}\right)+z^{-2}\left(\sum_{n=0}^{2}G_n^{(j)}(y)y_0^n\right)+O(z^{-3})\right),$$

where $G_n^{(j)}$ is a (fractional) power series in $y_1, \ldots, y_{r+l}$ taking values in $H^*_{orb}(\mathcal{E}_j)$. Therefore, we obtain

$$\begin{aligned}
y_0\frac{\partial}{\partial y_0}I_{\mathcal{E}_j} =&\frac{p_0}{z}e^{\sum_{i=0}^{r} p_i log y_i/z}\left(1+z^{-1}\left(\sum_{i=0}^{r} g_i^{(j)}(y)p_i+\tau_{tw}^{(j)}\right)+z^{-2}\left(\sum_{n=0}^{2} G_n^{(j)}(y)y_0^n\right)+O(z^{-3})\right)\\
&+e^{\sum_{i=0}^{r} p_i log y_i/z}\left(z^{-2}\left(\sum_{n=1}^{2} G_n^{(j)}(y)ny_0^n\right)+O(z^{-3})\right).
\end{aligned}$$

Therefore, the left hand side of equation (43) is

$$\begin{aligned}
&z\frac{\partial}{\partial y_0}\left(y_0\frac{\partial}{\partial y_0}\right)I_{\mathcal{E}_j}\\
=&\frac{\partial}{\partial y_0}\left(p_0e^{\sum_{i=0}^{r} p_i log y_i/z}\left(1+z^{-1}\left(\sum_{i=0}^{r} g_i^{(j)}(y)p_i+\tau_{tw}^{(j)}\right)+z^{-2}\left(\sum_{n=0}^{2} G_n^{(j)}(y)y_0^n\right)+O(z^{-3})\right)\right)\\
&+\frac{\partial}{\partial y_0}\left(e^{\sum_{i=0}^{r} p_i log y_i/z}\left(z^{-1}\left(\sum_{n=1}^{2} G_n^{(j)}(y)ny_0^n\right)+O(z^{-2})\right)\right)\\
=&p_0e^{\sum_{i=0}^{r} p_i log y_i/z}\left(O(z^{-2})\right)+\frac{p_0}{y_0z}e^{\sum_{i=0}^{r} p_i log y_i/z}\left(z^{-1}\left(\sum_{n=1}^{2} G_n^{(j)}(y)ny_0^n\right)+O(z^{-2})\right)\\
&+e^{\sum_{i=0}^{r} p_i log y_i/z}\left(z^{-1}\left(\sum_{n=1}^{2} G_n^{(j)}n^2y_0^{n-1}+O(z^{-2})\right)\right)\\
=&e^{\sum_{i=0}^{r} p_i log y_i/z}\left(z^{-1}\left(\sum_{n=1}^{2} G_n^{(j)}n^2y_0^{n-1}\right)+O(z^{-2})\right).
\end{aligned}$$

On the other hand, the pull-back of the right hand side of equation (43) by $\imath^*$ is

$$\begin{aligned}
&\imath^*\left(\sum_{i=1}^{r+l} m_{ij}\left(y_i\frac{\partial}{\partial y_i}\right)-y_0\frac{\partial}{\partial y_0}\right)I_{\mathcal{E}_j}\\
=&\left(\sum_{i=1}^{r+l} m_{ij}\left(y_i\frac{\partial}{\partial y_i}\right)-y_0\frac{\partial}{\partial y_0}\right)\imath^*I_{\mathcal{E}_j}\\
=&\left(\sum_{i=1}^{r+l} m_{ij}\left(y_i\frac{\partial}{\partial y_i}\right)\right)(I_{\mathcal{X}}+O(y_0))\\
=&z^{-1}\left(\sum_{i=1}^{r+l} m_{ij}\left(y_i\frac{\partial}{\partial y_i}\right)\tau(y)\right)+O(z^{-2})+O(y_0).
\end{aligned}$$

Hence we conclude the following lemma.

Lemma 4.5. *The Batyrev element $\tilde{D}_j(y)$ is given by*

$$\tilde{D}_j(y)=\imath^*G_1^{(j)}(y), \quad \textit{for} \quad 1\leq j\leq m+l. \tag{44}$$

Hence, the following theorem is a direct consequence of the above lemma and theorem 3.15.

Theorem 4.6. *The Seidel element $\tilde{S}_j$ corresponding to the toric divisor D_j is given by*

$$\tilde{S}_j(\tau(y)) = exp(-g_0^j(y))\tilde{D}_j(y). \tag{45}$$

4.2. **The computation of $\tilde{D}_j$.** Using the expansion

$$\left(\sum_{i=1}^{r+l} m_{ij}\left(y_i\frac{\partial}{\partial y_i}\right)\right) I_{\mathcal{X}} = e^{\sum_{i=1}^r p_i log y_i/z}\left(z^{-1}\tilde{D}_j + O(z^{-2})\right),$$

we see that $\tilde{D}_j$ is the coefficient of z^{-1} in the expansion of

$$e^{-\sum_{i=1}^r p_i log y_i/z}\left(\sum_{i=1}^{r+l} m_{ij}\left(y_i\frac{\partial}{\partial y_i}\right)\right) I_{\mathcal{X}}.$$

And, by direct computation

$$\left(\sum_{i=1}^{r+l} m_{ij}\left(y_i\frac{\partial}{\partial y_i}\right)\right) I_{\mathcal{X}} =$$
$$e^{\sum_{i=1}^{r} p_i log y_i/z} \sum_{d\in\mathbb{K}_{\mathrm{eff},\mathcal{X}}} \prod_{i=1}^{m+l}\left(\frac{\prod_{k=\lceil\langle D_i^S,d\rangle\rceil}^{\infty}\left(D_i+\left(\langle D_i^S,d\rangle-k\right)z\right)}{\prod_{k=0}^{\infty}\left(D_i+\left(\langle D_i^S,d\rangle-k\right)z\right)}\right)\left(\frac{D_j}{z}+\langle D_j^S,d\rangle\right)y^d\mathbf{1}_{v(d)}.$$

Hence, to compute the Batyrev element $\tilde{D}_j$, it remains to examine the expansion of the product factor

$$\frac{\prod_{k=\lceil\langle D_i^S,d\rangle\rceil}^{\infty}\left(D_i+\left(\langle D_i^S,d\rangle-k\right)z\right)}{\prod_{k=0}^{\infty}\left(D_i+\left(\langle D_i^S,d\rangle-k\right)z\right)} = C_d z^{-\left(\sum_{i=1}^{m+l}\lceil\langle D_i^S,d\rangle\rceil+\#\{i:\langle D_i^S,d\rangle\in\mathbb{Z}_{<0}\}\right)}\prod_{i:\langle D_i^S,d\rangle\in\mathbb{Z}_{<0}} D_i + h.o.t.,$$

where

$$C_d = \prod_{i:\langle D_i^S,d\rangle<0}\ \prod_{\langle D_i^S,d\rangle<k<0}\left(\langle D_i^S,d\rangle-k\right)\prod_{i:\langle D_i^S,d\rangle>0}\ \prod_{0\le k<\langle D_i^S,d\rangle}\left(\langle D_i^S,d\rangle-k\right)^{-1} \tag{46}$$

The summand indexed by $d\in\mathbb{K}_{\mathrm{eff},\mathcal{X}}$ contributes to the coefficient of z^{-1} if and only if

$$\sum_{i=1}^{m+l}\lceil\langle D_i^S,d\rangle\rceil+\#\{i:\langle D_i^S,d\rangle\in\mathbb{Z}_{<0}\}\le 1.$$

It happens only in the following three cases:

- $\sum_{i=1}^{m+l}\lceil\langle D_i^S,d\rangle\rceil+\#\{i:\langle D_i^S,d\rangle\in\mathbb{Z}_{<0}\}=0$
- $\begin{cases}\sum_{i=1}^{m+l}\lceil\langle D_i^S,d\rangle\rceil=0\\ \#\{i:\langle D_i^S,d\rangle\in\mathbb{Z}_{<0}\}=1\end{cases}$
- $\begin{cases}\sum_{i=1}^{m+l}\lceil\langle D_i^S,d\rangle\rceil=1\\ \#\{i:\langle D_i^S,d\rangle\in\mathbb{Z}_{<0}\}=0\end{cases}$.

The first case happens if and only if $d=0$. If the second case happens, then

$$\sum_{i=1}^{m+l}\lceil\langle D_i^S,d\rangle\rceil=\sum_{i=1}^{m+l}\langle D_i^S,d\rangle=\langle\rho_{\mathcal{X}}^S,d\rangle=0.$$

In particular,

$$\langle D_i^S, d\rangle \in \mathbb{Z}, 1 \le i \le m+l.$$

Hence we obtain the following lemma:

Lemma 4.7. *For $1 \le j \le m+l$, the Batyrev element $\tilde{D}_j$ is given by*

(47)
$$\tilde{D}_j = D_j + \sum_{i=1}^{m} D_i \sum_{\substack{\langle \rho^S_{\mathcal{X}}, d\rangle = 0 \\ \langle D_i^S, d\rangle \in \mathbb{Z}_{<0} \\ \langle D_k^S, d\rangle \in \mathbb{Z}_{\ge 0}, \forall k \ne i}} C_d \langle D_j^S, d\rangle y^d + \sum_{\substack{\sum_{i=1}^{m+l} \lceil \langle D_i^S, d\rangle \rceil = 1 \\ \langle D_i^S, d\rangle \notin \mathbb{Z}_{<0}, \forall i}} C_d \langle D_j^S, d\rangle y^d \mathbf{1}_{v(d)},$$

where C_d is given by equation (46).

5. Seidel elements corresponding to Box elements

Consider the box element $s_j \in Box(\mathbf{\Sigma})$, such that

$$\bar{s}_j = \sum_{i=1}^{m} c_{ji} \bar{b}_i \in \mathbf{N}_{\mathbb{Q}}, \quad \text{for some} \quad 0 \le c_{ji} < 1.$$

Let n_j be the least common denominator of $\{c_{ji}\}_{i=1}^m$, we define a $\mathbb{C}^\times$-action on $\mathcal{U}^S \times (\mathbb{C}^2 \setminus \{0\})$ by

$$(z_1, \dots, z_{m+l}, u, v) \mapsto (t^{-c_{j1} n_j} z_1, \dots, t^{-c_{jm} n_j} z_m, z_{m+1}, \dots, z_{m+l}, t^{n_j} u, t^{n_j} v), \quad t \in \mathbb{C}^\times.$$

Hence we have an associated bundle

$$\mathcal{E}_{m+j} = \mathcal{U}^S \times (\mathbb{C}^2 \setminus \{0\}) / G^S \times \mathbb{C}^\times$$

over $\mathbb{CP}^1 \times B\mu_{n_j}$ with $\mathcal{X}$ being the fiber. Furthermore, $\mathcal{E}_{m+j}$ can also be considered as a bundle over $\mathbb{CP}^1$, since there is a natural projection

$$\mathbb{CP}^1 \times B\mu_{n_j} \to \mathbb{CP}^1.$$

We can identify $H^2(\mathcal{E}_{m+j}; \mathbb{Z})$ with $H^2(\mathcal{X}; \mathbb{Z}) \oplus \mathbb{Z}$, where the second summand

$$\mathbb{Z} \cong Pic(\mathbb{CP}^1 \times B\mu_{n_j}),$$

and we have the following short exact sequence from remark 5.5 of [4]:

(48)
$$0 \longrightarrow Pic(\mathbb{CP}^1) \longrightarrow Pic(\mathbb{CP}^1 \times B\mu_{n_j}) \longrightarrow \mathbb{Z}/n_j\mathbb{Z} \longrightarrow 0$$

We identify an element of $Pic(\mathbb{CP}^1)$ with its image in $Pic(\mathbb{CP}^1 \times B\mu_{n_j})$ under the above map. Then the weights of $G^S \times \mathbb{C}^\times$ defining $\mathcal{E}_{m+j}$ are given by

$$\hat{D}_i^S = (D_i^S, -c_{ji} n_j), \quad \text{for} \quad 1 \le i \le m; \quad \hat{D}^S_{m+j} = (D^S_{m+j}, 0) \quad \text{for} \quad 1 \le j \le l;$$

$$\hat{D}^S_{m+l+1} = \hat{D}^S_{m+l+2} = (0, n_j).$$

The fan of $\mathcal{E}_{m+j}$ is contained in $N_{\mathbb{Q}} \oplus \mathbb{Q}$. The 1-skeleton is given by

(49)
$$\hat{b}_i = (b_i, 0), \text{ for } 1 \le i \le m; \quad \hat{b}_{m+1} = (0, 1); \quad \hat{b}_{m+2} = (s_j, -1).$$

Let E_{m+j} be the coarse moduli space of $\mathcal{E}_{m+j}$. Then E_{m+j} is an X-bundle over $\mathbb{CP}^1$. The Seidel element is defined as in equation (5).

We set

$$p_0 := (0, 1) \in H^2(E_{m+j}) \cong H^2(X) \oplus Pic(\mathbb{CP}^1),$$

a nef integral basis $\{p_1,\ldots,p_r\}$ of $H^2(X;\mathbb{Q})$ can be lifted to a nef integral basis $\{p_0,p_1,\ldots,p_r\}$ of $H^2(E_{m+j};\mathbb{Q})$ such that the lift of p_i vanishes on the section class $[\sigma_0]$. There is an isomorphism between $H^2(E_{m+j};\mathbb{Q})$ and $H^2(\mathcal{E}_{m+j};\mathbb{Q})$, by abuse of notation, we identify p_i with its image in $H^2(\mathcal{E}_{m+j};\mathbb{Q})$, for $0\leq i\leq r$. Let $p_1^S,\ldots,p_{r+l}^S$ be an integral basis of $\mathbb{L}^{S\vee}$, such that p_i is the image of p_i^S in $\mathbb{L}^\vee\otimes\mathbb{Q}$, under the canonical splitting of (17). Let $p_0^S,p_1^S,\ldots,p_{r+l}^S$ be an integral basis of $\mathbb{L}^{S\vee}\oplus\mathbb{Z}$ and p_0 be the image of

$$p_0^S=\hat{D}^S_{m+l+1}=\hat{D}^S_{m+l+2}$$

in $(\mathbb{L}^\vee\oplus\mathbb{Z})\otimes\mathbb{R}$. Therefore $p_{r+1},\ldots,p_{r+l}$ are zero.

As in the toric divisor case, we have the following expansion of the I-function:

(50)

$$I_{\mathcal{E}_{m+j}}(y,z)=$$
$$e^{\sum\limits_{i=0}^{r}p_i logy_i/z}\left(1+z^{-1}\left(\sum_{i=0}^{r}g_i^{(m+j)}(y)p_i+\tau_{tw}^{(m+j)}(y)\right)+z^{-2}\left(\sum_{n=0}^{2}G_n^{(m+j)}(y)y_0^n\right)+O(z^{-3})\right),$$

and use the same argument as in lemma 3.12 and lemma 3.14, we can show that $g_i^{(m+j)}(y)$ and $\tau_{tw}^{(m+j)}(y)$ are independent from y_0, for $1\leq i\leq r$ and $1\leq j\leq l$. Moreover, for each $j\in\{1,\ldots,l\}$, we have

$$g_i^{(m+j)}(y_0,\ldots,y_{r+l})=g_i(y_1,\ldots,y_{r+l})\quad\text{for}\quad i=1,\ldots,r.$$

And

$$\tau_{tw}^{(m+j)}(y)=\tau_{tw}(y).$$

We will also obtain the following theorem.

Theorem 5.1. *The Seidel element $\tilde{S}_{m+j}$ associated to the box element s_j is given by*

$$\tilde{S}_{m+j}(\tau(y)):=\tilde{S}_{m+j}(\tau^{(m+j)}(y))=exp\left(-g_0^{(m+j)}(y)\right)\iota^*(G_1^{(m+j)}(y)). \tag{51}$$

Using the same computation as in the toric divisor case, we can compute the correction coefficient $g_0^{(m+j)}$:

Lemma 5.2. *The function $g_0^{(m+j)}$ is given by*

(52)

$$g_0^{(m+j)}(y_1,\ldots,y_{r+l})=\sum_{1\leq k\leq m,k\notin I_j^S}\ \sum_{\substack{\langle D_i^S,d\rangle\in\mathbb{Z},1\leq i\leq m+l\\ \langle\rho^S_{\mathcal{X}},d\rangle=0\\ \langle D_k^S,d\rangle<0\\ \langle D_i^S,d\rangle\geq 0,\forall i\neq k}}c_{jk}\frac{(-1)^{-\langle D_k^S,d\rangle}\left(-\langle D_k^S,d\rangle-1\right)!}{\prod_{i\neq k}\langle D_i^S,d\rangle!}y^d,$$

where I_j^S is the "anticone" of the cone containing s_j.

Proof. The argument is almost the same as the argument in section 3.5. The only change we need to make is the paragraph above lemma 3.16:

In this case, $g_0^{(m+j)}$ is the coefficient corresponding to p_0 and elements in $\{\hat{D}_1,\ldots,\hat{D}_m\}$ that contain p_0 are precisely these elements:

$$\hat{D}_k=\langle D_k,-c_{jk}n_j\rangle=D_k-c_{jk}p_0,\quad\text{for}\quad 1\leq k\leq m\quad\text{and}\quad k\notin I_j^S.$$

Therefore, by expression (37) and (39), we must have $\langle D_k^S, d\rangle < 0$ for exactly one k in $\{k \in \mathbb{Z} | 1 \le k \le m \text{ and } k \notin I_j^S\}$. □

Moreover, by mimicking the computation in lemma 4.4, we have

Lemma 5.3. *the I-function of $\mathcal{E}_{m+j}$ satisfies the following differential equation:*

$$z\frac{\partial}{\partial y_0}\left(y_0\frac{\partial}{\partial y_0}\right)I_{\mathcal{E}_j} = y^{-D^{S\vee}_{m+j}}\left(\sum_{i=1}^{r+l} m_{ij}\left(y_i\frac{\partial}{\partial y_i}\right) - y_0\frac{\partial}{\partial y_0}\right)I_{\mathcal{E}_j}, \tag{53}$$

where $D^{S\vee}_{m+j} \in \mathbb{L}^S \otimes \mathbb{Q}$ is defined by (18).

Proof. The proof is almost identical to the proof of lemma 4.4, except, this time, we will need to choose $\beta' = \beta + [\sigma_0] - D^{S\vee}_{m+j}$. Then everything else follows. □

Using this lemma, following the argument in the toric divisor case, we conclude

Theorem 5.4. *The Seidel element $\tilde{S}_{m+j}$ corresponding to the box element s_j, with*

$$\bar{s}_j = \sum_{i=1}^{m} c_{ji}\bar{b}_i, \quad \textit{for some} \quad 0 \le c_{ji} < 1,$$

is given by

$$\tilde{S}_{m+j}(\tau^{(m+j)}(y)) = exp\left(-g_0^{(m+j)}\right)y^{-D^{S\vee}_{m+j}}\tilde{D}_{m+j}(y), \tag{54}$$

where $\tilde{D}_{m+j}(y)$ is the corresponding Batyrev element. Moreover,

(55)

$$\tilde{D}_{m+j} = \sum_{i=1}^{m} D_i \sum_{\substack{\langle \rho^S_{\mathcal{X}}, d\rangle = 0 \\ \langle D_i^S, d\rangle \in \mathbb{Z}_{<0} \\ \langle D_k^S, d\rangle \in \mathbb{Z}_{\ge 0}, \forall k \ne i}} C_d\langle D^S_{m+j}, d\rangle y^d + \sum_{\substack{\sum_{i=1}^{m+l}\lceil\langle D_i^S, d\rangle\rceil = 1 \\ \langle D_i^S, d\rangle \notin \mathbb{Z}_{<0}, \forall i}} C_d\langle D^S_{m+j}, d\rangle y^d \boldsymbol{1}_{v(d)},$$

and

(56)

$$C_d = \prod_{i:\langle D_i^S, d\rangle < 0}\ \prod_{\langle D_i^S, d\rangle < k < 0}\left(\langle D_i^S, d\rangle - k\right)\prod_{i:\langle D_i^S, d\rangle > 0}\ \prod_{0 \le k < \langle D_i^S, d\rangle}\left(\langle D_i^S, d\rangle - k\right)^{-1}.$$

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